NEW INDUCTION RELATIONS FOR HOMOGENEOUS FUNCTIONS IN JUCYS-MURPHY ELEMENTS

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ABSTRACT. The problem of computing the class expansion of some symmetric function evaluated in Jucys-Murphy elements appears in different contexts, for instance in the computation of matrix integrals. Recently, M. Lassalle gave a unified algebraic method to obtain some induction relations on the coefficients in this kind of expansion. In this paper, we give a simple purely combinatorial proof of its result. Using the same type of argument, we also obtain new simpler formulas. We also prove an analogous formula in the double class algebra and use it to prove a conjecture of S. Matsumoto on the subleading term of orthogonal Weingarten function. Finally, we formulate a conjecture for a continuous interpolation between the two problems.

1. INTRODUCTION

1.1. **Background.** The Jucys-Murphy elements J_i are elements of the symmetric group algebra $\mathbb{Z}[S_n]$, introduced separately by A. Jucys [Juc66, Juc74] and G. Murphy [Mur81]. They play a quite important role in representation theory because they act diagonally on the Young basis of any irreducible representation V_λ: the eigenvalue of J_i on an element e_T of this basis (T is a standard tableau of shape λ) is simply given by the content (*i.e.* the difference between the index of the column-index and the index of the row-index) of the box of T containing i.

In fact, representation theory of symmetric can be constructed entirely using this property (see [OV96]). We also refer to papers of Biane [Bia98] and Okounkov [Oko00] for nice applications of Jucys-Murphy elements to asymptotic representation theory.

A fundamental property, already observed by Jucys and Murphy, is that elementary symmetric functions evaluated in the J_i's have a very nice expression (this evaluation is well-defined because Jucys-Murphy elements commute with each other). More precisely, if $\kappa(\sigma)$ denotes the number of cycles of σ, then

$$e_k(J_1,\dots,J_n) = \sum_{\substack{\sigma\in S_n\\ \kappa(\sigma)=n-k}} \sigma.$$

As this is a central element in the group algebra, all symmetric functions evaluated in Jucys-Murphy elements are also central. Therefore it is natural to ask of their class expansion. In other terms, given some symmetric function F, can we compute the coefficients a_λ^F defined by:

$$F(J_1, \dots, J_n) = \sum_{\lambda \vdash n} a_\lambda^F \mathcal{C}_\lambda,$$

where $\mathcal{C}_\lambda$ denotes the sum of all permutations of cycle-type λ? This problem may seem anecdotic, but it in fact appears in different domains of mathematics:

- When F is a power sum p_k, it is linked with mathematical physics via vertex operators and Virasoro algebra (see [LT01]).
- When F is a complete symmetric function h_k, the coefficients appearing are exactly the coefficients in the asymptotic expansion of unitary Weingarten functions. The latter is the elementary brick to compute polynomial integrals over the unitary group (see [Nov10, ZJ10]).
- The inverse problem (how can we write a given conjugacy class $\mathcal{C}_\lambda$ as a symmetric function in Jucys-Murphy element) is equivalent to express character values as a symmetric functions of the content. The latter has been studied in some papers [CGS04, Las08a] but never using the combinatorics of Jucys-Murphy elements.

1.2. **Previous and new results.** As mentioned in the paragraph above, the class expansion of elementary functions in Jucys-Murphy elements is very simple and was first established by A. Jucys. The next result of this kind was obtained by A. Lascoux and J.-Y. Thibon via an algebraic method: they give the coefficients of the class expansion of power sums in Jucys-Murphy elements as some coefficients of an explicit series [LT01].

Then S. Matsumoto and J. Novak [MN09] computed the coefficients of the permutations of maximal absolute length in any monomial function in Jucys-Murphy elements. Their proof is purely combinatorial but does not seem to be extendable to all coefficients. As the monomial form a linear basis of symmetric functions, one can deduce from their result a formula for the top coefficients for any symmetric function, in particular for complete functions (see also [Mur04, CM09]). To be comprehensive, let us add that the authors also obtain all coefficients of cycles in complete symmetric functions using character theory (their method works only for cycles, but gives all coefficients, not only the top one).

Recently, M. Lassalle [Las10] gave a unified method to obtain some induction relations for the coefficients of the class expansion of several families of symmetric functions in Jucys-Murphy elements. These induction relations allow to compute any coefficient quite quickly and he shows that we can recover using them the results of Jucys, Lascoux and Thibon and also the top component of complete symmetric functions. So the work of Lassalle unifies most of the results obtained until now on the subject. To do that, he uses an involved algebraic machinery: he begins by translate the problem in terms of shifted symmetric function and then introduces some relevant differential operator which increases the degree of functions.

In this paper, we give a simple combinatorial proof of his induction formulas. Our method of proof can also be adapted to find other formulas. The latter are new and simpler than the existing ones. An example of application is the following:

using Matsumoto's and Novak's result on cycles, we are able to compute more coefficients.

1.3. **Generalizations.** An analogous problem can be considered in the so-called double class algebra. All the definitions will be given in section 3. If we look at complete symmetric functions in this context, there still is a relation with integrals over group of matrices, but the complex unitary group should be replaced by the real orthogonal group (see [ZJ10, Mat10]).

The only previous work on this question is due to S. Matsumoto [Mat10], who has computed the coefficients of permutations of maximal length in monomial symmetric functions (hence obtaining an analog of its result with Jonathan Novak). Our new induction formula can also be extended quite easily to this case. This allows us to prove a conjecture of S. Matsumoto about the subleading term of orthogonal Weingarten function [Mat10, Conjecture 9.4].

In fact, one can even define a generalization of the problem with a parameter α which interpolates between the expansion of symmetric function in JM elements (which corresponds to $\alpha = 1$) and the analog in double class algebra (which corresponds to $\alpha = 2$). We recall this construction in section 4. A very interesting point in Lassalle's method to obtain induction formulas is that it can be extended almost without changing anything to this generalization [Las10, section 11]. Unfortunately, we are not able yet to extend our work to this general setting. However, computer exploration seems to indicate that some of the results still old in the general case and we present a conjecture in this sense in section 4.

1.4. **Organization of the paper.** In section 2, we present our results in the symmetric group algebra.

Then, in section 3, we look at the analogous problem in the double class algebra.

Finally, in section 4, we present a conjecture for the continuous deformation between these two models.

2. INDUCTION RELATIONS

2.1. **Definitions and notations.** Let us denote by S_n the symmetric group of size n and by $\mathbb{Z}[S_n]$ its group algebra over the integer ring.

Definition 2.1. The Jucys-Murphy elements J_i (for $1 \leq i \leq n$) are defined by:

$$J_i = (1\ i) + (2\ i) + \cdots + (i-1\ i) \quad \in \mathbb{Z}[S_n]$$

Note that $J_1 = 0$ but we include it in our formulas for esthetic reasons.

Proposition 2.2.
- *Jucys-Murphy elements commute with each other.*
- *If F is a symmetric function, $F(J_1, J_2, \ldots, J_n)$ belongs to the center of the symmetric group algebra $Z(\mathbb{Z}[S_n])$.*

We recall that the cycle-type of a permutation in S_n is by definition the non-increasing sequence of the lengths of its cycle. This is an integer partition, which determines the conjugacy class of the permutation in S_n. Hence a basis of the center of the group algebra $Z(\mathbb{Z}[S_n])$ is given by the sums of the conjugacy classes, that is

$$\mathcal{C}_\lambda = \sum_{\substack{\sigma \in S_n \\ \sigma \text{ has cycle-type } \lambda}} \sigma \quad \text{for } \lambda \vdash n$$

Therefore, there exists some integer numbers a_λ^F such that:

$$F(J_1, \dots, J_n) = \sum_{\lambda \vdash n} a_\lambda^F \mathcal{C}_\lambda$$

We will here focus on the case where F is a complete symmetric function (so $a_\lambda^{h_k}$ will be denoted a_λ^k) because of the link with some integrals over unitary groups mentioned in introduction. All the results of this section could be easily adapted to elementary and power-sum symmetric functions. Unfortunately, we are not able to deal with a linear basis of symmetric functions.

Example 2.3. As an illustration, let us look at the case $k = 2$ and $n = 3$:

$$\begin{aligned} h_2(J_1, J_2, J_3) &= (1\ 2)^2 + \big((1\ 3) + (2\ 3)\big)^2 + (1\ 2) \cdot \big((1\ 3) + (2\ 3)\big) \\ &= \mathrm{Id} + 2\,\mathrm{Id} + (1\ 2\ 3) + (1\ 3\ 2) + (1\ 2\ 3) + (1\ 3\ 2) \\ &= 3\mathcal{C}_{1^3} + 2\mathcal{C}_3 \end{aligned}$$

Note that the coefficients of a permutation at the end of the computation does depend only on its cycle-type, although 1, 2 and 3 play different roles in the computation.

In other terms, we have computed the following coefficients:

$$a^2_{(1^3)} = 3, \quad a^2_{(2\ 1)} = 0, \quad a^2_{(3)} = 2.$$

2.2. **A combinatorial proof of Lassalle's formula.**

Theorem 2.4 (Lassalle, [Las10])**.** *The coefficients a_λ^k are determined by the following inductions formulas. For any partition ρ, one has:*

$$a^{(k)}_{\rho \cup 1} = a^{(k)}_\rho + \sum_i \rho_i a^{(k-1)}_{\rho \setminus (\rho_i) \cup (\rho_i + 1)}; \tag{1}$$

$$\begin{aligned} \sum_i \rho_i a^{(k)}_{\rho \setminus (\rho_i) \cup (\rho_i + 1)} &= \sum_{i \neq j} \rho_i \rho_j a^{(k-1)}_{\rho \setminus (\rho_i, \rho_j) \cup (\rho_i + \rho_j + 1)} \\ &\quad + \sum_i \sum_{\substack{r+s = \rho_i + 1 \\ r,s \geq 1}} a^{(k-1)}_{\rho \setminus (\rho_i) \cup (r,s)}. \end{aligned} \tag{2}$$

We refer to [Las10, end of page 13] for an explanation of why these equations characterize the numbers a_λ^k together with initial conditions:

$$a^{(0)}_\rho = \begin{cases} 1 \text{ if } \rho \text{ has only parts equal to } 1; \\ 0 \text{ else;} \end{cases}$$

$$a^k_{(1)} = \delta_{k,0}.$$

Proof. We start from the obvious induction relation:

$$h_k(J_1,\dots,J_{n+1}) = h_k(J_1,\dots,J_n) + J_{n+1}h_{k-1}(J_1,\dots,J_{n+1}) \tag{3}$$

and we apply to it the following operator:

$$\mathbb{E}: \begin{array}{ccc} \mathbb{Z}[S_{n+1}] & \to & \mathbb{Z}[S_n] \\ \sigma & \mapsto & \begin{cases} \sigma/\{1,\dots,n\} \text{ if } \sigma(n+1)=n+1; \\ 0 \text{ else.} \end{cases} \end{array}$$

We look at the coefficient of a permutation σ of type ρ (denoted $[\sigma]\dots$) in both sides:

$$\begin{aligned} [\sigma]\mathbb{E}\big(h_k(J_1,\dots,J_{n+1})\big) &= [\sigma']h_k(J_1,\dots,J_{n+1}) = a^{(k)}_{\rho\cup 1}, \\ [\sigma]\mathbb{E}\big(h_k(J_1,\dots,J_n)\big) &= [\sigma]h_k(J_1,\dots,J_n) = a^{(k)}_{\rho}, \\ [\sigma]\mathbb{E}\big(J_{n+1}h_{k-1}(J_1,\dots,J_{n+1})\big) &= \sum_{j\le n}[(j\ n+1)\sigma']h_{k-1}(J_1,\dots,J_{n+1}) \\ &= \sum_i \rho_i a^{(k-1)}_{\rho\setminus(\rho_i)\cup(\rho_i+1)}, \end{aligned}$$

where σ' is the image of σ by the canonical embedding of S_n into S_{n+1} (we add $n+1$ as fixed point). The third equality comes from the fact that, if j belongs to a cycle of σ of length ρ_i then $(j\ n+1)\sigma'$ has cycle-type $\rho\setminus(\rho_i)\cup(\rho_i+1)$.

Using equation (3), this finishes the proof of Lassalle's first induction relation.

The second equality is obtained the same way except that we multiply equation (3) by J_{n+1} before applying the operator $\mathbb{E}$. One obtains:

$$\begin{aligned} \mathbb{E}\big(J_{n+1}h_k(J_1,\dots,J_{n+1})\big) = \mathbb{E}\big(J_{n+1}h_k(J_1,\dots,J_n)\big) \\ + \mathbb{E}\big(J^2_{n+1}h_{k-1}(J_1,\dots,J_{n+1})\big) \end{aligned} \tag{4}$$

□

The coefficient of σ' in the left-hand side has already been computed and is equal to

$$\sum_i \rho_i a^{(k)}_{\rho\setminus(\rho_i)\cup(\rho_i+1)}.$$

Note that all permutations τ in $h_k(J_1,\dots,J_n)$ fixes $n+1$. Thus $n+1$ can not be a fixed point of $(j\ n+1)\tau$ and

$$\mathbb{E}\big(J_{n+1}h_k(J_1,\dots,J_n)\big) = 0$$

For the last term, we write:

$$\mathbb{E}\big(J^2_{n+1}h_{k-1}(J_1,\dots,J_{n+1})\big) = \sum_{j_1,j_2\le n}[(j_1\ n+1)\cdot(j_2\ n+1)\cdot\sigma']h_{k-1}(J_1,\dots,J_{n+1}) \tag{5}$$

We split the sum in two parts, depend on whether j_1 and j_2 are in the same cycle of σ or not:

- Suppose that they are in the same cycle of length ρ_i, *i.e.* $j_2 = \sigma^m(j_1)$ with $1 \leq m \leq \rho_i$ (eventually $j_1 = j_2$, which corresponds to $m = \rho_i$). Then $(j_1\ n+1) \cdot (j_2\ n+1) \cdot \sigma'$ has the same cycles as σ except for the one containing j_1 and j_2, as well as two other cycles:

  $$\big(j_1, \sigma(j_1), \dots \sigma^{m-1}(j_1)\big) \text{ and } \big(j_2, \sigma(j_2), \dots \sigma^{\rho_i - m-1}(j_2), n+1\big).$$

  Thus it has cycle type $\rho \setminus (\rho_i) \cup (m, \rho_i - m + 1)$.
- If j_1 and j_2 are not in the same cycle of σ, then $(j_1\ n+1) \cdot (j_2\ n+1) \cdot \sigma'$ has the same cycles as σ except for the ones containing respectively j_1 and j_2, as well as one new cycle:

  $$\big(j_1, \sigma(j_1), \dots \sigma^{\rho_{i_1}-1}(j_1), n+1, j_1, \sigma(j_2), \dots \sigma^{\rho_{i_2}-1}(j_2)\big),$$

  where ρ_{i_1} and ρ_{i_2} are length of the cycles of σ containing j_1 and j_2. Thus $(j_1\ n+1) \cdot (j_2\ n+1) \cdot \sigma'$ has cycle-type $\rho \setminus (\rho_{i_1}, \rho_{i_2}) \cup (\rho_{i_1} + \rho_{i_2} + 1)$.

Putting everything together, we obtain the two terms in the right-hand side of (2).

Remark 2.5. This method is in fact closer to Lassalle's one that it seems at first sight. More details are given in appendix.

2.3. **New relations.** Let us come back to equation (3). Instead of applying $\mathbb{E}$, one can directly look at the coefficient of a given permutation σ of type $\rho \vdash n+1$ in both sides of (3). In the left-hand side, by definition:

$$[\sigma] h_k(J_1, \dots, J_{n+1}) = a_\rho^k.$$

In the right-hand side, this coefficient depends on the length of the cycle containing $n+1$. This length is a part ρ_{i_0} of partition ρ. The coefficients of σ in the two terms in the right-hand side of (3) are given by:

$$\begin{aligned}
[\sigma] h_k(J_1, \dots, J_n) &= \delta_{\rho_{i_0},1} a^k_{\rho \setminus 1};\\
[\sigma] J_{n+1} h_{k-1}(J_1, \dots, J_{n+1}) &= \sum_{j \leq n} [\sigma (j\ n+1)] h_{k-1}(J_1, \dots, J_{n+1})\\
&= \sum_{\substack{1 \leq i \leq \ell(\rho)\\ i \neq i_0}} \rho_i a^{k-1}_{\rho \setminus (\rho_i, \rho_{i_0}) \cup (\rho_i + \rho_{i_0})} + \sum_{\substack{r+s = \rho_{i_0}\\ r,s \geq 1}} a^{k-1}_{\rho \setminus (\rho_{i_0}) \cup (r,s)}.
\end{aligned}$$

The second equation has been obtained as above by looking separately at the cycle-type of $\sigma(j\ n+1)$. The latter depends on whether j and $n+1$ are in the same cycle of σ or not. We do not give all the details. Finally one obtain,

$$a^k_\rho = \delta_{\rho_{i_0},1} a^k_{\rho \setminus 1} + \sum_{\substack{1 \leq i \leq \ell(\rho)\\ i \neq i_0}} \rho_i a^{k-1}_{\rho \setminus (\rho_i, \rho_{i_0}) \cup (\rho_i + \rho_{i_0})} + \sum_{\substack{r+s = \rho_{i_0}\\ r,s \geq 1}} a^{k-1}_{\rho \setminus (\rho_{i_0}) \cup (r,s)}$$

As one can choose any permutation σ of type ρ to compute the coefficients a^k_ρ, we do not have any condition on i_0 in the previous equation (we use the fact that although we know that $h_k(J_1, \dots, J_{n+1})$ is central, $n+1$ plays a particular role in the definition). Therefore, we have proved:

Theorem 2.6. *For any partition ρ and positive integers k, m one has:*

$$a^k_{\rho\cup(m)} = \delta_{m,1} a^k_\rho + \sum_{1\leq i\leq \ell(\rho)} \rho_i a^{k-1}_{\rho\setminus(\rho_i)\cup(\rho_i+m)} + \sum_{\substack{r+s=m\\ r,s\geq 1}} a^{k-1}_{\rho\cup(r,s)} \tag{6}$$

Note that the case $m = 1$ corresponds to Lassalle's first equation (1). His second equation (2) can be easily recovered from (6) by linear combination, but the converse is not true.

2.4. **Taking care of the dependance in n.** As mentioned by Lassalle [Las10, paragraph 2.7], the coefficients $a^k_{\rho\cup 1^{n-|\rho|}}$, seen as functions of n, have a very nice structure. More precisely define c^k_λ, where λ is a partition, by induction by the formula:

$$a^k_\rho = \sum_{i=0}^{m_1(\rho)} c^k_{\bar{\rho}\cup 1^i} \binom{m_1(\rho)}{i}, \tag{7}$$

where $m_1(\rho)$ is the number of parts equal to 1 in ρ and $\bar{\rho}$ is obtained from ρ by erasing its parts equal to 1. The interesting fact now is that c^k_ρ is equal to 0 as soon as $|\rho| - \ell(\rho) + m_1(\rho)$ is bigger than k, while, for a given k, one has infinitely many non-zero a^k_ρ (see an explanation below). As a consequence, coefficients c are convenient to comute simulteneously the class expansion of $h_k(J_1, \dots, J_n)$ for all positive integers n (the integer k being fixed): see Example 2.8 at the end of this paragraph.

Using equation (7), one can translate Theorems 2.4 and 2.6 into relations over the c's, but it is rather technical (see [Las10, section 12]). We prefer here to explain the combinatorial meaning of the coefficients c's and derive directly relations over the c's using this interpretation.

The good tool for that are partial permutations introduced by Ivanov and Kerov in [IK99]. Let $\mathcal{B}_\infty$ be the following $\mathbb{Z}$-algebra:

- as a $\mathbb{Z}$-module, $\mathcal{B}_\infty$ is freely generated by partial permutations *i.e.* pairs (d, σ) where d is a finite set of positive integers and σ a permutation of d.
- the product on the basis elements is given by:

  $$(d_1, \sigma_1) \cdot (d_2, \sigma_2) = (d_1 \cup d_2, \tilde{\sigma_1} \cdot \tilde{\sigma_2}),$$

  where $\tilde{\sigma_1}$ (resp. $\tilde{\sigma_2}$) is the canonical continuation of σ_1 (resp. σ_2) to $d_1 \cup d_2$ (*i.e.* we add fixed points, we will use this notation throughout the paper).

The infinite symmetric group S_∞ acts naturally on $\mathcal{B}_\infty$: if τ belong to S_∞, that is τ is a permutation of $\mathbb{N}^\star$ with finite support, we define

$$\tau \bullet (d, \sigma) = (\tau(d), \tau\sigma\tau^{-1}).$$

The invariants by the action of S_∞ form a subalgebra $\mathcal{A}_\infty$ of $\mathcal{B}_\infty$. A basis of this subalgebra is

$$\big(\mathcal{PC}_\lambda\big)_{\lambda \text{ partition}} \text{ where } \mathcal{PC}_\lambda = \sum_{\substack{d\subset\mathbb{N}^\star,\ |d|=|\lambda|\\ \sigma\in S_d,\ \text{cycle-type}(\sigma)=\lambda}} (d, \sigma).$$

The nice property of this construction is that there exists morphisms φ_n from $\mathcal{B}_\infty$ to every symmetric group algebra $\mathbb{Z}[S_n]$ defined by:

$$\varphi_n(d,\sigma) = \begin{cases} \tilde{\sigma} \text{ if } d \subset \{1,\dots,n\}; \\ 0 \text{ else.} \end{cases}$$

These morphisms restrict to morphisms $\mathcal{A}_\infty \to Z(\mathbb{Z}[S_n])$. The image of vectors of the basis is given by:

$$\varphi_n(\mathcal{PC}_\lambda) = \binom{n-|\lambda|+m_1(\lambda)}{m_1(\lambda)} \mathcal{C}_{\lambda \cup 1^{n-|\lambda|}}$$

It has been observed in [Fér09] that if we define natural analogs of Jucys-Murphy elements in $\mathcal{B}_\infty$ by

$$X_i = \sum_{j<i} \big(\{j,\ i\},\ (j\ i)\big) \quad \text{for } i \geq 1,$$

- they still commute with each other;
- the evaluation $F(X_1, X_2, X_3, \dots)$ of any symmetric function F in partial Jucys-Murphy elements lies in $\mathcal{A}_\infty$.

Therefore there exist coefficients c_λ^k such that

$$h_k(X_1, X_2, X_3, \dots) = \sum_\lambda c_\lambda^k \mathcal{PC}_\lambda.$$

Applying φ_n, we see that this definition of c_λ^k is coherent with the previous one (equation (7)). Note that with this construction, it is obvious that the c's are non-negative integers (fact which was observed numerically by Lassalle, private communication). The fact that c_ρ^k is equal to 0 as soon as $|\rho| - \ell(\rho) + m_1(\rho)$ is bigger than k is also natural because

$$\deg(d,\sigma) = |d| - \#\text{ cycles of } \sigma + \#\text{ fixed points of } \sigma$$

defines a filtration of $\mathcal{B}_\infty$, for which each X_i is of degree 1. We can also obtain induction relations on the c's like we did with the a's:

Theorem 2.7. *For any partition μ and positive integers m and k, one has*

$$c_{\mu\cup 1}^k = \sum_i \mu_i c_{\mu\setminus(\mu_i)\cup(\mu_i+1)}^{k-1};$$
$$c_{\mu\cup 2}^k = \sum_i \mu_i c_{\mu\setminus(\mu_i)\cup(\mu_i+2)}^{k-1} + c_{\mu\cup(1,1)}^{k-1} + 2c_{\mu\cup(1)}^{k-1} + c_\mu^{k-1};$$
$$c_{\mu\cup m}^k = \sum_i \mu_i c_{\mu\setminus(\mu_i)\cup(\mu_i+m)}^{k-1} + \sum_{\substack{r+s=m \\ r,s\geq 1}} c_{\mu\cup(r,s)} + 2c_{\mu\cup(m-1)}^{k-1} \textit{ if } m \geq 3.$$

Proof. Let $n+1 = |\mu| + m$ and fix a partial permutation (d,σ) with:

- $d = \{1,\dots,n+1\}$;
- σ has cycle-type $\mu \cup (m)$ and $n+1$ is in a cycle of length m.

Let us look at the coefficient $c^k_{\mu\cup m}$ of (d,σ) in $h_k(X_1, X_2, \dots)$. As $n+1$ is the biggest element in d, it implies that every monomials in the X_i's contributing to the coefficient of (d,σ) contains no X_i with $i > n+1$ and contains at least one X_{n+1}. Thus:

$$\begin{aligned} c^k_{\mu\cup m} &= [(d,\sigma)] h_k(X_1, X_2, \dots) = [(d,\sigma)] X_{n+1} h_{k-1}(X_1, \dots, X_{n+1}); \\ &= [(d,\sigma)] \sum_{j<n+1} \sum_{\nu} \sum_{\substack{(d',\tau),\ |d'|=|\mu| \\ \text{cycle-type}(\tau)=\nu}} c^{k-1}_\nu \cdot (d' \cup \{j, n+1\}, (j\ n+1)\tilde{\tau}). \end{aligned}$$

We have already discussed in the previous paragraph the possible cycle-types of $\tilde{\tau} = (j\ n+1)\sigma$. The only new thing we have to take care of is the fact that if j or/and $n+1$ are fixed points for $\tilde{\tau}$, they may not belong to d'. This explains the two last terms in the second equation and the last one in the third equation. Details are left to the reader. □

Example 2.8. Here are the non-zero values of c^k_ρ for small values of k ($k \leq 3$). It is immediate that $c^1_{(2)}$ is equal to 1, while all other c^1_μ are 0. Then Theorem 2.7 allows to compute:

$$\begin{aligned} c^2_{(1,1)} &= 1 \cdot c^1_{(2)} = 1; \\ c^2_{(2,2)} &= 2c^1_{(4)} + c^1_{(2,1,1)} + 2c^1_{(2,1)} + c^1_{(2)} = 1; \\ c^2_{(3)} &= 2c^1_{(2,1)} + 2c^1_{(2)} = 2; \\ c^3_{(2)} &= c^2_{(1,1)} = 1; \\ c^3_{(2,1)} &= 2c^2_{(3)} = 4; \\ c^3_{(2,1,1)} &= 2c^2_{(3,1)} + c^2_{(2,2)} = 1; \\ c^3_{(2,2,2)} &= c^2_{(2,2)} = 1; \\ c^3_{(3,2)} &= c^2_{(3)} = 2; \\ c^3_{(4)} &= c^2_{(2,2)} + 2c^2_{(3)} = 5. \end{aligned}$$

Using equation (7), we can compute all coefficients a^k_ρ for $k = 2, 3$ and we find the following class expansion (true for any $n \geq 1$):

$$\begin{aligned} h_2(J_1, \dots, J_n) &= \delta_{n\geq 3}\ 2\mathcal{C}_{(3,1^{n-3})} + \delta_{n\geq 4}\ \mathcal{C}_{(2,2,1^{n-4})} + \binom{n}{2}\mathcal{C}_{1^n}; \\ h_3(J_1, \dots, J_n) &= \delta_{n\geq 4}\ 5\mathcal{C}_{(4,1^{n-4})} + \delta_{n\geq 5}\ 2\mathcal{C}_{(3,2,1^{n-5})} + \delta_{n\geq 6}\ \mathcal{C}_{(2,2,2,1^{n-6})} \\ &\quad + \delta_{n\geq 2}\ \left(\binom{n-2}{2} + 4\binom{n-2}{1} + \binom{n-2}{0}\right)\mathcal{C}_{2,1^{n-2}}. \end{aligned}$$

This type of result could also be obtained with Theorem 2.6 (the initial conditions for the a's are the following: $a^1_{(2,1^{n-2})} = 1$ for any $n \geq 2$ and all other a^1_μ are equal to 0), but the computation are a little harder (it involves discrete integrals of polynomials).

2.5. **Generating series for some coefficients.** S. Matsumoto and J. Novak computed, using character theory, the following generating function [MN09, Theorem 6.7].

Theorem 2.9 (Matsumoto, Novak, 2009)**.** *For any integer $n \geq 2$, one has:*

$$\sum_k a^k_{(n)} z^k = \frac{\mathrm{Cat}_{n-1}\, z^{n-1}}{(1-1^2z^2)(1-2^2z^2)\dots(1-(n-1)^2z^2)}, \tag{8}$$

where $\mathrm{Cat}_{n-1} = \frac{1}{n}\binom{2(n-1)}{n-1}$ *is the usual Catalan number.*

As $a^k_{(n)} = c^k_{(n)}$, the same result is true on the c's. Unfortunately, we are not able to find a proof of their formula *via* Theorem 2.7, but we can use it to derive new results of the same kind.

For instance, with $\mu = (n-1)$ and $m = 1$, our induction relation writes:

$$\begin{aligned}\sum_k c^k_{(n-1,1)} z^k = z \sum_k (n-1) c^{k-1}_{(n)} z^{k-1} \\ = \frac{(n-1)\,\mathrm{Cat}_{n-1}\, z^n}{(1-1^2z^2)(1-2^2z^2)\dots(1-(n-1)^2z^2)}.\end{aligned}$$

In terms of a's, this result implies:

$$\begin{aligned}\sum_k a^k_{(n-1,1)} z^k &= \sum_k \left(c^k_{(n-1,1)} + c^k_{(n-1)}\right) z^k \\ &= \frac{(n-1)\,\mathrm{Cat}_{n-1}\, z^n + (1-(n-1)^2z^2)\,\mathrm{Cat}_{n-2}\, z^{n-2}}{(1-1^2z^2)(1-2^2z^2)\dots(1-(n-1)^2z^2)}. \end{aligned} \tag{9}$$

This expression is simpler than the one obtained by Matsumoto and Novak for the same quantity [MN09, Proposition 6.9] and their equivalence is not obvious at all.

If we want to go further and compute other generating series, one has to solve linear systems. For instance, denoting $F_\mu = \sum_k c^k_\mu z^k$, Theorem 2.7 gives:

$$\begin{aligned}F_{(n-2,1,1)} &= z\left((n-2)F_{(n-1,1)} + F_{(n-2,2)}\right);\\ F_{(n-2,2)} &= z\left((n-2)F_{(n)} + F_{(n-2,1,1)} + F_{(n-2,1)} + F_{(n-2)}\right).\end{aligned}$$

After resolution, one has:

$$\begin{aligned}F_{(n-2,1,1)} &= \frac{z^2\left(n(n-2)F_{(n)} + z(n-2)F_{(n-1)} + F_{(n-2)}\right)}{1-z^2};\\ F_{(n-2,2)} &= \frac{z\left((n-2)F_{(n)} + F_{(n-2,1)} + F_{(n-2)}\right) + z^2(n-2)F_{(n-1,1)}}{1-z^2}.\end{aligned}$$

Using results above, one can deduce an explicit generating series for the c's which can be easily transformed into series for the a's.

3. ANALOGUES IN THE DOUBLE-CLASS ALGEBRA

In this section, we consider a slightly different problem, which happens to be the analog of the one of the previous section. It was first considered recently by S. Matsumoto [Mat10] in connection with integrals over orthogonal groups.

3.1. **Double class algebra.** The results of this section are quite classical. A good survey, with a more representation-theoretical point of view, can be found in I.G. Macdonald's book[Mac95, Chapter 7].

Let us consider the symmetric group of even size S_{2n}, whose elements are seen as permutations of $\{1, \bar{1}, \dots, n, \bar{n}\}$. It contains the hyperoctahedral group which is the subgroup formed by permutations $\sigma \in S_{2n}$ such that $\overline{\sigma(i)} = \sigma(\bar{i})$ (by convention, $\bar{\bar{i}} = i$). We are interested in the double cosets $H_n \backslash S_{2n} / H_n$, *i.e.* the equivalence classes for the relation:

$$\sigma \equiv \tau \text{ if and only if } \exists\, h, h' \in H_n \text{ s.t. } \sigma = h \tau h'.$$

As conjugacy classes in the symmetric group algebra can be characterized easily using cycle-types, one can characterize these double coset classes *via* coset-types.

Definition 3.1. Let σ be a permutation of S_{2n}. Consider the following graph G_σ:

- its $2n$ vertices are labelled by $\{1, \bar{1}, \dots, n, \bar{n}\}$;
- we put a solid edge between i and $\bar{i}$ and a dashed one between $\sigma(i)$ and $\sigma(\bar{i})$ for each i.

Forgetting the types of the edges, we obtain a graph with only vertices of valence 2. Thus, it is a collection of cycles. Moreover, due to the bicoloration of edges, it is easy to see that all these cycles have an even length.

We call coset-type of σ the partition μ such that the lengths of the cycles of G_σ are equal to $2\mu_1, 2\mu_2, \dots$

Example 3.2. Let $n = 4$ and σ be the following permutation:

$$1 \mapsto 3,\ \bar{1} \mapsto 1,\ 2 \mapsto \bar{4},\ \bar{2} \mapsto \bar{3},\ 3 \mapsto \bar{2},\ \bar{3} \mapsto 2,\ 4 \mapsto 4,\ \bar{4} \mapsto \bar{1}.$$

The corresponding graph G_σ is drawn on figure 1.

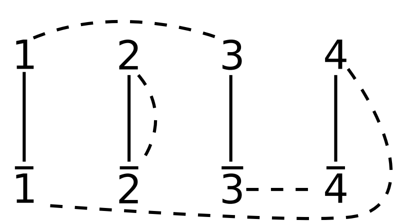

FIGURE 1. Example of graph G_σ

This graph is the disjoint union of one cycle of length 6 $(1, 3, \bar{3}, \bar{4}, 4, \bar{1})$ and one cycle of length 2 $(2, \bar{2})$. Thus the coset-type of σ is the integer partition $(3, 1)$.

Proposition 3.3. [Mac95, section 7.1] *Two permutations are in the same double class if and only if their coset-types are the same.*

If μ is a partition of n, we denote

$$\mathcal{C}_\mu^{(2)} = \sum_{\substack{\sigma \in S_{2n} \\ \text{coset-type}(\sigma)=\mu}} \sigma \ \in \mathbb{Z}[S_{2n}].$$

It is immediate that the elements $\mathcal{C}_\mu^{(2)}$, when μ runs over partitions of n span linearly a subalgebra $Z^{(2)}$ of $\mathbb{Z}[S_{2n}]$. Equivalently, one can define $Z^{(2)}$ as the algebra of functions on S_{2n}, invariant by left and right multiplication by an element of H_n, endowed with the convolution product

$$f \star g(\sigma) = \sum_{\tau_1 \tau_2 = \sigma} f(\tau_1) g(\tau_2).$$

One can prove using representation theory [Mac95, section 7.2] that this algebra is commutative (in other terms, (S_{2n}, H_n) is a Gelfand pair).

3.2. **Odd Jucys-Murphy elements.** In this section we will look at symmetric functions in odd-indexed Jucys-Murphy elements in S_{2n}. Rewriting as permutations on the set $\{1, \bar{1}, 2, \bar{2}, \dots, n, \bar{n}\}$ (ordered by $1 < \bar{1} < 2 < \bar{2} < \dots < n < \bar{n}$), these elements are:

$$J_i^{(2)} = \sum_{j=1,\bar{1},\dots,i-1,\overline{i-1}} (j\ i)$$

They were first considered by S. Matsumoto in the paper [Mat10], where he proved some analogs of results of Jucys. To state them, we need to define the following element in S_{2n}:

$$p_n = \frac{1}{|H_n|} \sum_{h \in H_n} h.$$

Then the following result holds.

Proposition 3.4 (Matsumoto, 2010)**.** *If F is a symmetric function, then:*

$$x_{n,F} := F(J_1^{(2)}, \dots, J_n^{(2)}) p_n = p_n F(J_1^{(2)}, \dots, J_n^{(2)})$$

Moreover $x_{n,F}$ belongs to the algebra $Z^{(2)}$.

Sketch of proof. It is easy to prove by induction that

$$e_k(J_1^{(2)}, \dots, J_n^{(2)}) p_n = p_n e_k(J_1^{(2)}, \dots, J_n^{(2)}) = \sum_{\substack{\mu \vdash n \\ |\mu| - \ell(\mu) = k}} \mathcal{C}_\mu^{(2)}.$$

The result follows for all F by multiplication and linear combination. □

As in section 2, we may look at the class expansion of $x_{n,F}$, *i.e.* the coefficients b_μ^F such that:

$$F(J_1^{(2)}, \dots, J_n^{(2)}) p_n = \sum_{\mu \vdash n} b_\mu^F \mathcal{C}_\mu^{(2)}.$$

As seen in the sketch of proof for the proposition above, the b's are easy to compute in the case of elementary functions.

In the following paragraph, we will establish some induction relations for the b's in the case of complete symmetric functions . We focus on this case (and thus use the short notation $b^k_\mu = b^{h_k}_\mu$) because these coefficients appear in the computation of the asymptotic expansion of some integrals over the orthogonal group [Mat10, Theorem 7.3].

3.3. **A simple induction relation.** In this paragraph, using the same method as in subsection 2.3, we prove the following induction formula for the b's.

Theorem 3.5. *For any partition ρ and positive integers k, m one has:*

$$b^k_{\rho\cup(m)} = \delta_{m,1} b^k_\rho + 2 \sum_{1\le i\le \ell(\rho)} \rho_i b^{k-1}_{\rho\setminus(\rho_i)\cup(\rho_i+m)} + \sum_{\substack{r+s=m \\ r,s\ge 1}} b^{k-1}_{\rho\cup(r,s)} + (m-1) b^{k-1}_\rho. \tag{10}$$

Proof. As before, the starting point of our proof is an induction relation on complete symmetric functions:

$$h_k(J_1^{(2)},\dots,J_n^{(2)},J_{n+1}^{(2)}) = h_k(J_1^{(2)},\dots,J_n^{(2)}) + J_{n+1}^{(2)} h_{k-1}(J_1^{(2)},\dots,J_n^{(2)},J_{n+1}^{(2)}). \tag{11}$$

Multiplying both sides by p_{n+1}, one has:

$$h_k(J_1^{(2)},\dots,J_n^{(2)},J_{n+1}^{(2)})p_{n+1} = h_k(J_1^{(2)},\dots,J_n^{(2)})p_n \cdot p_{n\setminus n+1} + J_{n+1}^{(2)} h_{k-1}(J_1^{(2)},\dots,J_n^{(2)},J_{n+1}^{(2)})p_{n+1},$$

where $p_{n\setminus n+1} = 1 + (n+1\ \overline{n+1}) + \sum_{i=1,\bar{1},\dots,n,\bar{n}} (n+1\ i)(\overline{n+1}\ \bar{i})$ (it obviously fulfills $p_{n+1} = p_n \cdot p_{n\setminus n+1}$).

Let us look first at the case $m=1$. We choose a permutation $\sigma \in S_{2n}$ of coset-type μ and we denote σ' its image by the canonical embedding $S_{2n} \hookrightarrow S_{2n+2}$. Then

$$[\sigma']h_k(J_1^{(2)},\dots,J_n^{(2)},J_{n+1}^{(2)})p_{n+1} = b^k_{\rho\cup 1}$$
$$[\sigma']h_k(J_1^{(2)},\dots,J_n^{(2)})p_n \cdot p_{n\setminus n+1} = [\sigma]h_k(J_1^{(2)},\dots,J_n^{(2)})p_n = b^k_\rho.$$

Indeed $h_k(J_1^{(2)},\dots,J_n^{(2)})p_n$ lies in the algebra $\mathbb{Z}[S_{2n}] \subset \mathbb{Z}[S_{2n+2}]$ and hence is a linear combination of permutations fixing $n+1$ and $\overline{n+1}$. For such permutations τ, neither $\tau(n+1\ \overline{n+1})$ nor $\tau(n+1\ i)(\overline{n+1}\ \bar{i})$ can be equal to σ (these two permutations do not fix $n+1$ and $\overline{n+1}$). This explains the second equality above.

We still have to compute:

$$\begin{aligned}[\sigma']J_{n+1}^{(2)} h_{k-1}(J_1^{(2)},\dots,J_n^{(2)},J_{n+1}^{(2)})p_{n+1} &= \sum_{i=1,\bar{1},\dots,n,\bar{n}} [(n+1\ i)\sigma']h_{k-1}(J_1^{(2)},\dots,J_n^{(2)},J_{n+1}^{(2)})p_{n+1} \\ &= \sum_{i=1,\bar{1},\dots,n,\bar{n}} b^{k-1}_{\text{coset-type}((n+1\ i)\sigma')}.\end{aligned} \tag{12}$$

Let us look at the coset-type of $(n+1\ i)\sigma'$. Denote by d_i (resp. d_{n+1}) the other extremity of the dashed edge of extremity i (resp. $n+1$) in $G_{\sigma'}$ (see definition 3.1). Then the graph $G_{(n+1\ i)\sigma'}$ has exactly the same edges as $G_{\sigma'}$, except for (i, d_i) and $(n+1, d_{n+1})$ which are replaced by (i, d_{n+1}) and $(n+1, d_i)$.

As $(n+1, \overline{n+1})$ is a loop of length 2 in $G_{\sigma'}$, if we assume that i was in a loop of size $2\rho_j$, then these two loops are replaced by a loop of size $2\rho_j+2$ in $G_{(n+1\ i)\sigma'}$ (it is a particular case of the phenomena drawn on Figure 2).

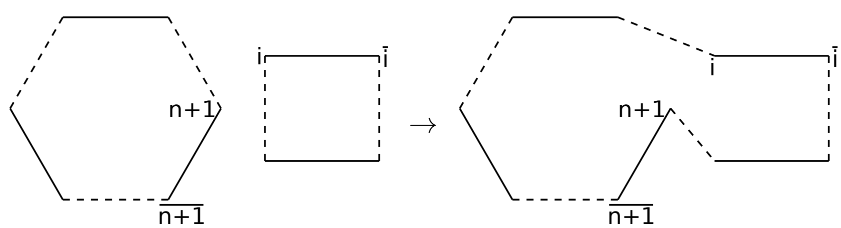

FIGURE 2. G_σ and $G_{(n+1\ i)\sigma}$ in the first case

Therefore $(n+1\ i)\sigma'$ has coset-type $\rho \backslash \rho_j \cup (\rho_j+1)$. Finally

$$[\sigma']J^{(2)}_{n+1} h_{k-1}(J^{(2)}_1, \dots, J^{(2)}_n, J^{(2)}_{n+1}) p_{n+1} = 2 \sum_{1 \le j \le \ell(\rho)} \rho_j b^{k-1}_{\rho \backslash (\rho_j) \cup (\rho_j+1)}$$

and this ends the proof of the case $m = 1$.

Let us consider now the case $m > 1$. We choose a permutation $\sigma \in S_{2m}$ of coset-type $\rho \cup (m)$ such that $n+1$ is in a loop of size $2m$ in G_σ. As $m > 1$, this implies that the $\overline{\sigma^{-1}(n+1)} \neq \sigma^{-1}(\overline{n+1})$ and consequently:

$$[\sigma]h_k(J^{(2)}_1, \dots, J^{(2)}_n) p_{n+1} = 0$$

Hence one has

$$b^k_{\rho\cup(m)} = [\sigma]h_k(J^{(2)}_1, \dots, J^{(2)}_n, J^{(2)}_{n+1}) = [\sigma]J^{(2)}_{n+1} h_{k-1}(J^{(2)}_1, \dots, J^{(2)}_n, J^{(2)}_{n+1}) p_{n+1},$$

and we have to look at the possible coset types of $(n+1\ i)\sigma$ (equation (12) is still true).

As before, the graph $G_{(n+1\ i)\sigma}$ is obtained from G_σ by replacing edges (i, d_i) and $(n+1, d_{n+1})$ by (i, d_{n+1}) and $(n+1, d_i)$. But here, we have to consider different cases:

- If i lies in G_σ in a loop of size $2\rho_j$ different from the loop containing $n+1$, then these two loops of G_σ are replaced in $G_{(n+1\ i)\sigma}$ by a loop of size $2(\rho_j+m)$ (see figure 2).

  In this case, $(n+1\ i)\sigma$ has coset-type $\rho\backslash\rho_j \cup (\rho_j+m)$.
- If i lies in G_σ in the same loop as $n+1$ and if the distance between these two nodes is odd (as i can not be equal to $\overline{n+1}$, there is $m-1$ possible values for i in this case), then they still lie in the same loop of $G_{(n+1\ i)\sigma}$ of size $2m$.(see figure 3).

  In this case, $(n+1\ i)\sigma$ has the same coset-type as σ that is $\rho \cup (m)$.

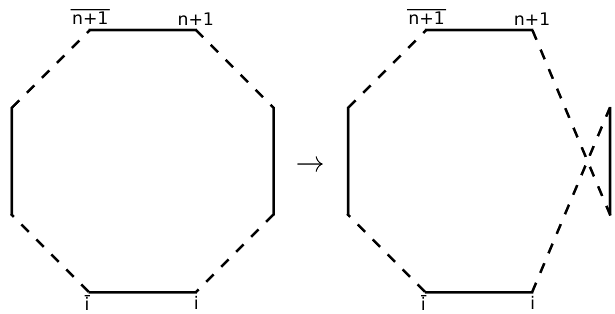

FIGURE 3. G_σ and $G_{(n+1\ i)\sigma}$ in the second case

- If i lies in G_σ in the same loop as $n+1$ and if the distance between this two nodes (in an arbitrary direction) is equal to $2r$ ($1 \leq r \leq m-1$), then their loop in G_σ is replaced in $G_{(n+1\ i)\sigma}$ by two loops of length $2r$ and $2(m-r)$.(see figure 4).

  In this case, $(n+1\ i)\sigma$ has coset-type $\rho \cup (r, m-r)$.

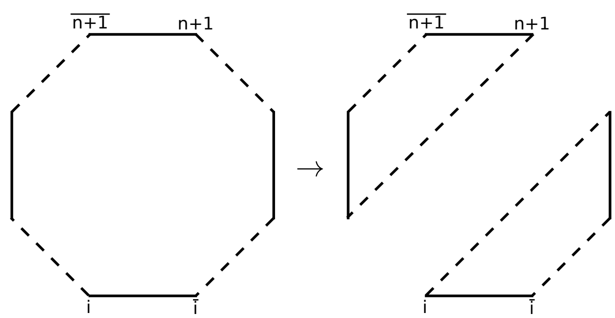

FIGURE 4. G_σ and $G_{(n+1\ i)\sigma}$ in the third case

Finally

$$[\sigma]J^{(2)}_{n+1}h_{k-1}(J^{(2)}_1, \ldots, J^{(2)}_n, J^{(2)}_{n+1})p_{n+1}$$
$$= 2 \sum_{1 \leq i \leq \ell(\rho)} \rho_i b^{k-1}_{\rho\setminus(\rho_i)\cup(\rho_i+m)} + \sum_{\substack{r+s=m \\ r,s\geq 1}} b^{k-1}_{\rho\cup(r,s)} + (m-1)b^{k-1}_\rho$$

and this ends the proof of the theorem. □

Remark 3.6. As in section 2, if we define coefficients d^k_ρ as solutions of the sparse triangular system

$$b^k_\rho = \sum_{i=0}^{m_1(\rho)} d^k_{\bar{\rho}\cup 1^i} \binom{m_1(\rho)}{i}, \tag{13}$$

then, for a given k, only finitely many d^k_ρ are non-zero. But, unfortunately, we have no combinatorial interpretation in this case to obtain directly induction relation on d. This raises the question of the existence of a partial double-coset algebra, out of the scope of this article.

A result similar to Theorem 3.5 could be obtained for power sum symmetric functions.

3.4. **Subleading term.** The induction relation proved in the previous paragraph is a good tool to study the leading and subleading terms of $h_k(J_1^{(2)},\dots,J_n^{(2)})p_n$, that is the coefficients b^k_ρ with $|\rho|-\ell(\rho)=k$ or $k-1$. Indeed, an immediate induction shows that if the degree condition $|\rho|-\ell(\rho)\leq k$ is not satisfied, then $b^k_\rho=0$. We can also recover the following result proved by S. Matsumoto [Mat10, Theorem 5.4].

Proposition 3.7. *If ρ is a partition and k an integer such that $|\rho|-\ell(\rho)=k$, then*

$$b^k_\rho=\prod \mathrm{Cat}_{\rho_i-1}\,.$$

But our induction allows us to go further and to compute the subleading term (case $|\rho|-\ell(\rho)=k-1$), proving this way a conjecture of S. Matsumoto [Mat10, Conjecture 9.4] corresponding to the case where ρ is a hook.

Before stating and proving our result (in paragraph 3.4.2), we need a few definitions and basic lemmas on the total area of Dyck paths (paragraph 3.4.1).

3.4.1. *Area of Dyck paths.*

Definition 3.8. If $I=(i_1,\dots,i_r)$ is a weak composition (*i.e.* a sequence of non-negative integers), let us define $\mathcal{P}_I$ as the set of Dyck paths of length $k=i_1+\dots+i_r$ whose height after i_1, i_1+i_2, …steps is zero (such a path is the concatenation of Dyck paths of lengths i_1, i_2,…).

If C is a subset of Dyck paths of a given length, denote by $\mathfrak{A}_C$ the sum over the paths c in C of the area under c. In the case $C=\mathcal{P}_I$, we shorten the notation and denote $\mathfrak{A}_I=\mathfrak{A}_{\mathcal{P}_I}$.

For a weak composition I of length 1, the set $\mathcal{P}_I$ is the set of all Dyck paths of lengths k. The area $\mathfrak{A}_k$ in this case has a closed form (see [MSV96]), namely

$$\mathfrak{A}_k=4^k-\binom{2k+1}{k}.$$

The general case can be deduced easily, thanks to the following trivial lemma:

Lemma 3.9. *Let C_1 and C_2 be respectively subsets of the set of Dyck paths of length $2m$ and $2n$. Define $C\simeq C_1\times C_2$ the set of Dyck paths of length $2(m+n)$ which are the concatenation of a path in C_1 and a path in C_2. Then*

$$\mathfrak{A}_C=\mathfrak{A}_{C_1}\cdot|C_2|+|C_1|\cdot\mathfrak{A}_{C_2}.$$

With an immediate induction, we obtain, the following corollary.

Corollary 3.10. *For any weak composition I of length r, one has:*

$$\mathfrak{A}_I = \sum_{j=1}^{r} \mathfrak{A}_{i_j} \prod_{k \neq j} \mathrm{Cat}_{i_k}\, .$$

One will also need the following induction relation in the next paragraph.

Lemma 3.11. *If m is a positive integers, one has:*

$$\mathfrak{A}_{m-1} = (m-1)\,\mathrm{Cat}_{m-1} + \sum_{\substack{r+s=m \\ r,s \geq 1}} \left(\mathfrak{A}_{r-1}\,\mathrm{Cat}_{s-1} + \mathfrak{A}_{s-1}\,\mathrm{Cat}_{r-1}\right).$$

Proof. This is a consequence of the usual first return decomposition of Dyck paths. Indeed, let c be a Dyck path of length $2(m-1)$. We denote $2r$ the x-coordinate of the first point where the path touches the x-axis and $s = m - r$. Then c is the concatenation of one climbing step, a Dyck path c_1 of length $2(r-1)$, a down step and a Dyck path c_2 of length $2(s-1)$ and this decomposition is of course bijective.

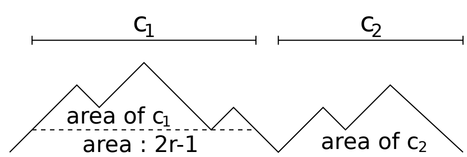

FIGURE 5. First-passage decomposition of a Dyck path

The area under c is the sum of the areas under c_1 and c_2, plus $2r-1$ (see figure 5). Using the lemma above,

$$\mathfrak{A}_{m-1} = \sum_{\substack{r+s=m \\ r,s \geq 1}} \left[\mathfrak{A}_{r-1}\,\mathrm{Cat}_{s-1} + \mathfrak{A}_{s-1}\,\mathrm{Cat}_{r-1} + (2r-1)\,\mathrm{Cat}_{s-1}\,\mathrm{Cat}_{r-1}\right].$$

The last part of the sum may be symmetrized in r and s:

$$\begin{aligned}\sum_{\substack{r+s=m \\ r,s \geq 1}} (2r-1)\,\mathrm{Cat}_{s-1}\,\mathrm{Cat}_{r-1} &= \sum_{\substack{r+s=m \\ r,s \geq 1}} \frac{1}{2}(2r-1+2s-1)\,\mathrm{Cat}_{s-1}\,\mathrm{Cat}_{r-1} \\ &= (m-1) \sum_{\substack{r+s=m \\ r,s \geq 1}} \mathrm{Cat}_{s-1}\,\mathrm{Cat}_{r-1} = (m-1)\,\mathrm{Cat}_{m-1},\end{aligned}$$

which ends the proof of the lemma. □

3.4.2. *Proof of a conjecture of Matsumoto.*

Theorem 3.12. *Let μ be a partition and $k = |\mu| - \ell(\mu) + 1$. Then*

$$b_\mu^k = \mathfrak{A}_{\mu-1}.$$

Proof. We make a double induction, first on the size n of partition μ and then on its smallest part m.

Let us suppose that the theorem is true for all partitions of size smaller than n. If $\mu = \mu' \cup (1)$ is a partition of n with smallest part 1, then, using Theorem 3.5, one has:

$$b_\mu^k = b_{\mu'}^k$$

Indeed, the other term is equal to zero because of the degree condition. The theorem is thus true for μ by induction.

We now look at the case where $\mu = \mu' \cup (m)$ is a partition of n with smallest part $m > 1$ and suppose that the theorem is true for partitions of n with smallest part smaller than m. In this case, using the degree condition and the value of the leading term, Theorem 3.5 becomes:

$$b_\mu^k = (m-1) \prod \operatorname{Cat}_{\mu_i - 1} + \sum_{\substack{r+s=m \\ r,s \geq 1}} b_{\mu' \cup (r,s)}^{k-1}$$

By induction,

$$\begin{aligned} b_{\mu' \cup (r,s)}^{k-1} = \mathfrak{A}_{(\mu' \cup (r,s))-1} = \operatorname{Cat}_{r-1} \operatorname{Cat}_{s-1} \left(\sum_i \mathfrak{A}_{\mu'_i - 1} \prod_{j \neq i} \operatorname{Cat}_{\mu'_j - 1} \right) \\ + \mathfrak{A}_{s-1} \operatorname{Cat}_{r-1} \prod_i \operatorname{Cat}_{\mu'_i - 1} + \mathfrak{A}_{r-1} \operatorname{Cat}_{s-1} \prod_i \operatorname{Cat}_{\mu'_i - 1} \end{aligned}$$

Putting it in the previous equation, we obtain:

$$\begin{aligned} b_\mu^k = \operatorname{Cat}_{m-1} \left(\sum_i \mathfrak{A}_{\mu'_i - 1} \prod_{j \neq i} \operatorname{Cat}_{\mu'_j - 1} \right) \\ + \big((m-1) \operatorname{Cat}_{m-1} + \mathfrak{A}_{s-1} \operatorname{Cat}_{r-1} + \mathfrak{A}_{r-1} \operatorname{Cat}_{s-1} \big) \prod_i \operatorname{Cat}_{\mu'_i - 1} \end{aligned}$$

Therefore, using Lemma 3.11, one has:

$$b_\mu^k = \sum_i \mathfrak{A}_{\mu_i - 1} \prod_{j \neq i} \operatorname{Cat}_{\mu_j - 1} = \mathfrak{A}_\mu \qquad \square$$

S. Matsumoto established a deep connection between the coefficients b_μ^k and the asymptotic expansion of orthogonal Weingarten functions [Mat10, Theorem 7.3]. In particular, Theorem 3.12 gives the subleading term of some matrix integrals over orthogonal group when the dimension of the group goes to infinity.

4. TOWARDS A CONTINUOUS DEFORMATION?

The questions studied in sections 2 and 3 may seem quite different at first sight but there exists a continuous deformation from one to the other.

We denote by $\mathcal{Y}_n$ the set of all Young diagrams (or partition) of size n. For any $\alpha > 0$, we consider two families of functions on $\mathcal{Y}_n$.

- First, we call α-content of a box of the Young diagram λ the difference between α times its column index and its row index. If $A^{(\alpha)}_\lambda$ stands for the multiset of the α-contents of λ, one can look at the evaluation of complete symmetric functions $h_k(A^{(\alpha)}_\lambda)$.
- Second, we consider Jack polynomials, which is the basis of symmetric function ring indexed by partitions and depending of a parameter α (they are deformations of Schur functions). The expansion of Jack polynomials on the power sum basis

$$J^{(\alpha)}_\lambda = \sum_\mu \Theta^{(\alpha)}_\mu(\lambda) p_\mu$$

  defines functions $\Theta^{(\alpha)}_\mu(\lambda)$ (we use the same normalization and notation as in [Mac95, Chapter 6] for Jack polynomials).

The functions $\Theta^{(\alpha)}_\mu$ spans the algebra $Z^{(\alpha)}_n$ of functions over $\mathcal{Y}_n$ because Jack polynomials form a basis of symmetric functions. Therefore one has coefficients $a^{k,(\alpha)}_\mu$ such that:

$$h_k(A^{(\alpha)}_\lambda) = \sum_\mu a^{k,(\alpha)}_\mu \Theta_\mu(\lambda),$$

For $\alpha = 1$, using the action of Jucys-Murphy element on the Young basis [Juc66] and the discrete Fourrier transform of S_n, one can see that $a^{k,(1)}_\mu = a^k_\mu$.

For $\alpha = 2$, using the identification between Jack polynomials for this special value of the parameter and zonal polynomials for the Gelfand pair (S_{2n}, H_n) [Mac95, Chapter 7], as well as the spherical expansion of $h_k(J^{(2)}_1, \dots, J^{(2)}_n) p_n$ established by S. Matsumoto [Mat10, Theorem 4.1], one has $a^{k,(2)}_\mu = b^k_\mu$.

It is natural to wonder if there are results similar to Theorems 2.6 and 3.5 in the general setting. Computer exploration using Sage [S$^+$10] leads to the following conjecture:

Conjecture 4.1. *The coefficients $a^{k,(\alpha)}_\rho$ fulfill the linear relation:*

$$a^{(k)}_{\rho\cup\rho_i} = \sum_{\substack{r+s=\rho_i \\ r,s\geq 1}} a_{\rho\cup(r,s)} + \alpha \sum_{r\in\rho} r a^{k-1,(\alpha)}_{\rho\backslash r\cup(\rho_i+r)} + (\alpha-1)\cdot(\rho_i-1)\, a^{k-1,(\alpha)}_\rho \tag{14}$$

Unfortunately, as we do not have a combinatorial description of the algebra $Z^{(\alpha)}_n$ (and of a bigger algebra containing deformation of Jucys-Murphy elements), we are not able to prove it. With Lassalle's algebraic approach, one can prove a generalization of Theorem 2.4 (see [Las10, Section 11]) which is weaker as Conjecture 4.1, but has been used in our numerical exploration.

A motivation for this conjecture is that it is a hint towards the existence of combinatorial constructions for other values of the parameter α (like the conjectures of papers [GJ96, Las08b, Las09]).

APPENDIX A. LINK WITH LASSALLE'S METHOD

In this appendix, we establish a link between Lassalle's algebraic method and our combinatorial one for his theorem. In fact, the main difference between them is that we are using elements in the center of the symmetric group algebra, while Lassalle is making computation with their eigenvalue in irreducible representations of the symmetric group. Indeed, the left-hand side of equations (4.6) and (4.7) in [Las10] correspond to the normalized character value of the left-hand sides of (4) and (5). More generally, the following proposition holds:

Proposition A.1. *For any partition λ of n, non-negative integer ℓ and element $x \in Z(\mathbb{Z}[S_{n+1}])$, one has:*

$$\hat{\chi}^\lambda(\mathbb{E}(J_{n+1}^\ell x)) = \sum_{\substack{i \text{ inner} \\ \text{corner of } \lambda}} \frac{\dim(\lambda^{(i)})}{(n+1)\dim(\lambda)} c(i)^\ell \hat{\chi}^{\lambda^{(i)}}(x). \tag{15}$$

Here, $\hat{\chi}^\lambda$ stands for the normalized (i.e. divided by the dimension $\dim(\lambda)$) character value of the irreducible representation indexed by λ, while $\lambda^{(i)}$ is the Young diagram obtained from λ by adding a box at the corner i (the inner corners of λ are exactly the places where we can add a box to obtain a new Young diagram of size $n+1$) and $c(i)$ is the (1-)content of i.

The case $x = \mathrm{Id}_{n+1}$ was proved by P. Biane [Bia98, Proposition 3.3]. Our proof follows roughly the same guideline.

Proof. As it is not central in this paper, we assume in this proof that the reader is familiar with the representation theory of symmetric group.

We consider the central idempotent $\pi_\lambda \in \mathbb{C}[S_n]$. Left multiplication by π_λ is the projection $p : \mathbb{C}[S_n] \to \mathbb{C}[S_n]$ on the isotypic component of type λ. As $\mathbb{C}[S_n] \subset \mathbb{C}[S_{n+1}]$, one can consider the subspace $E = \pi_\lambda \mathbb{C}[S_{n+1}]$ (note that the fact that π_λ is acting on the left is important because π_λ is not central in $\mathbb{C}[S_{n+1}]$). We will compute in two different ways the trace of the left multiplication by $J_{n+1}^\ell x$ on E.

It is well known that, as a representation of $S_{n+1} \times S_{n+1}$ (acting by left and right multiplication), one has the isomorphism:

$$\mathbb{C}[S_{n+1}] \simeq \bigoplus_{\Lambda \vdash n+1} V_\Lambda \otimes V_\Lambda.$$

Using the branching rule, we know that, as representation of S_n

$$V_\Lambda = \bigoplus_{\substack{\lambda \vdash n \text{ s.t.} \\ \exists\ i \text{ s.t. } \Lambda = \lambda^{(i)}}} V_\lambda.$$

Moreover, the restriction $V_\lambda \subset V_\Lambda$ of the left multiplication by J_{n+1} is an homothetic transformation of ration $c(i)$ where i is the inner corner of λ such that $\Lambda = \lambda^{(i)}$ (this is a classical result on Jucys-Murphy elements, see [OV96] for example). Finally, as a representation of $S_n \times S_{n+1}$, one has:

$$\mathbb{C}[S_{n+1}] \simeq \bigoplus_{\substack{\lambda,\Lambda \\ \lambda^{(i)}=\Lambda}} V_\lambda \otimes V_\Lambda$$

$$\text{Therefore,}\quad E = \pi_\lambda \mathbb{Z}[S_{n+1}] \simeq \bigoplus_{\Lambda,\Lambda=\lambda^{(i)}} V_\lambda \otimes V_\Lambda$$

As mentioned before, the left multiplication by J_{n+1} coincides on each component with the multiplication by $c(i)$, while the left multiplication x coincides with multiplication by $\hat{\chi}^\Lambda(x)$ (indeed, $V_\lambda \otimes V_\Lambda$ is a subspace of the isotypic component $V_\Lambda \otimes V_\Lambda$, on which the central element x acts as an homothetic transformation). Thus the trace of the left multiplication by $J_{n+1}^\ell x$ on E is

$$\sum_{\Lambda,\Lambda=\lambda^{(i)}} \dim(\lambda)\dim(\Lambda) c(i)^\ell \hat{\chi}^\Lambda(x) = \dim(\lambda) \sum_{\substack{i \text{ inner} \\ \text{corner of } \lambda}} \dim(\lambda^{(i)}) c(i)^\ell \hat{\chi}^{\lambda^{(i)}}(x).$$

To compute the same number in a second way, let us consider the following decomposition of $\mathbb{C}[S_{n+1}]$ (by convention, $(n+1\ n+1)$ is the identity permutation):

$$\mathbb{C}[S_{n+1}] = \bigoplus_{i=1}^{n+1} \mathbb{C}[S_n](i\ n+1).$$

Of course, this implies (the sum is direct because each component $\pi_\lambda \mathbb{C}[S_n](i\ n+1)$ is contained in $\mathbb{C}[S_n](i\ n+1)$)

$$E = \pi_\lambda \mathbb{C}[S_{n+1}] = \bigoplus_{i=1}^{n+1} \pi_\lambda \mathbb{C}[S_n](i\ n+1).$$

But there exist some $y_i \in \mathbb{C}[S_n]$ such that:

$$J_{n+1}^\ell x = \mathbb{E}(J_{n+1}^\ell x) + \sum_{i=1}^{n} y_i (i\ n+1).$$

The matrix of the left multiplication by $\mathbb{E}(J_{n+1}^\ell x)$ on E is block diagonal (with respect to the decomposition above) and each diagonal block corresponds to the left multiplication by element $\mathbb{E}(J_{n+1}^\ell x)$ on $\pi_\lambda \mathbb{C}[S_n]$, that is an homothetic transformation of ratio $\hat{\chi}^\lambda(\mathbb{E}(J_{n+1}^\ell x))$ on a space of dimension $\dim(\lambda)^2$.

In addition, if we write the block decomposition of the multiplication by $y_i(i\ n+1)$, the diagonal blocks are equal to zero. Indeed, for any $\sigma \in \mathbb{C}[S_n]$ and any $j \leq n+1$,

$$\begin{aligned} y_i\ (i\ n+1)\ p_\lambda\ \sigma\ (j\ n+1) &= \sum_{k<n+1} z_k (k\ n+1)(j\ n+1) \\ &= \delta_{j\neq n+1} z_j + \sum_{\substack{k<n+1 \\ k\neq j}} z_k (k\ j)(k\ n+1) \in \bigoplus_{\substack{i\leq n+1 \\ i\neq j}} \pi_\lambda \mathbb{C}[S_n](i\ n+1). \end{aligned}$$

Finally, one has:

$$\mathrm{Tr}_E(J_{n+1}^\ell x) = \mathrm{Tr}_E(\mathbb{E}(J_{n+1}^\ell x)) = (n+1)\dim(\lambda)^2 \hat{\chi}^\lambda(\mathbb{E}(J_{n+1}^\ell x)).$$

The proposition follows by comparing the two expressions for $\mathrm{Tr}_E(J_{n+1}^\ell x)$. $\square$

As particular cases of this proposition, one obtains a new proof of [Las10, Theorem 4.1], but also of a recent result of J. Gallovich [Gal10, Theorem 3].

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