On generalized Frame-Stewart numbers

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Abstract

For the multi-peg Tower of Hanoi problem with $k \geq 4$ pegs, so far the best solution is obtained by the Stewart's algorithm [15] based on the the following recurrence relation:

$$\mathrm{S}_k(n) = \min_{1\leq t\leq n}\left\{2\cdot \mathrm{S}_k(n-t) + \mathrm{S}_{k-1}(t)\right\},\quad \mathrm{S}_3(n) = 2^n - 1.$$

In this paper, we generalize this recurrence relation to

$$\mathrm{G}_k(n) = \min_{1\leq t\leq n}\left\{p_k\cdot \mathrm{G}_k(n-t) + q_k\cdot \mathrm{G}_{k-1}(t)\right\},\quad \mathrm{G}_3(n) = p_3\cdot \mathrm{G}_3(n-1) + q_3,$$

for two sequences of arbitrary positive integers $(p_i)_{i\geq 3}$ and $(q_i)_{i\geq 3}$ and we show that the sequence of differences $(\mathrm{G}_k(n) - \mathrm{G}_k(n-1))_{n\geq 1}$ consists of numbers of the form $(\prod_{i=3}^{k} q_i)\cdot(\prod_{i=3}^{k} {p_i}^{\alpha_i})$, with $\alpha_i \geq 0$ for all i, lined in the increasing order. We also apply this result to analyze recurrence relations for the Tower of Hanoi problems on several graphs.

Keywords: multi-peg Tower of Hanoi, Tower of Hanoi on graphs, Frame-Stewart numbers, generalized Frame-Stewart numbers, recurrence relations, smooth numbers.

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1 Introduction

The Tower of Hanoi problem was introduced by Édouard Lucas in 1883 [9] for the case of 3 pegs and n disks of different sizes. Initially, n disks are placed on one of the 3 pegs with the largest at the bottom. Then, at each time one of the topmost disks is moved to a peg with a larger disk on the top. The goal of the problem is to transfer all the disks from the initial peg to the peg of destination with the minimum number of moves. A simple recursive argument shows that $2^n - 1$ moves are necessary and sufficient to carry out this task. This

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Tower of Hanoi problem was then extended to the case of 4 pegs by Dudeney in 1907 [3] and to arbitrary $k \geq 3$ pegs by Stewart in 1939 [14]. In 1941, Frame [5] and Stewart [15] independently proposed algorithms which achieve the same numbers of moves for the k-peg Tower of Hanoi problem with $k \geq 4$ pegs. Klavžar et al.[7] showed that seven different approaches to the k-peg Tower of Hanoi problem, including those by Frame and Stewart, are all equivalent, that is, achieve the same numbers of moves. Thus, these numbers are called the *Frame-Stewart numbers* [8].

Somewhat surprisingly, the optimal solution for the multi-peg Tower of Hanoi problem with $k \geq 4$ pegs is not known yet. So far, the best upper bounds are achieved by the Frame-Stewart numbers and the best lower bounds are obtained by Chen et al.[2]. Since the upper bounds are believed to be optimal, they are called the "presumed optimal" solution.

The Stewart's recursive algorithm for the k-peg Tower of Hanoi problem is summarized as follows. For integer t such that $1 \leq t \leq n$,

1. recursively transfer a pile of $n-t$ smallest disks from the first peg to a temporary peg using k pegs;
2. transfer the remaining pile of t largest disks from the first peg to the final peg using $k-1$ pegs, ignoring the peg occupied by the $n-t$ smallest disks;
3. recursively transfer the pile of $n-t$ smallest disks from the temporary peg to the final peg using k pegs.

The algorithm chooses the integer t such that the number of moves $2 \cdot \mathrm{S}_k(n-t) + \mathrm{S}_{k-1}(t)$ is minimized. Thus, the Frame-Stewart numbers $\mathrm{S}_k(n)$ satisfy the following recurrence relations:

$$\mathrm{S}_k(n) = \min_{1 \leq t \leq n} \left\{ 2 \cdot \mathrm{S}_k(n-t) + \mathrm{S}_{k-1}(t) \right\}, \text{ for } n \geq 1,\ k \geq 4,$$

$$\mathrm{S}_3(n) = 2^n - 1, \text{ for } n \geq 1, \text{ and } \mathrm{S}_k(0) = 0, \text{ for } k \geq 3.$$

When $k = 4$ for instance, $\mathrm{S}_4(n)$ is obtained by the following simple formula:

$$\mathrm{S}_4(n) - \mathrm{S}_4(n-1) = 2^{i-1}, \text{ for } \binom{i}{2} < n \leq \binom{i+1}{2},$$

where $\binom{i}{2}$ is the binomial coefficient equal to $i(i-1)/2$. In the general case $k \geq 4$, $\mathrm{S}_k(n)$ is obtained by several different approaches, e.g., [5, 7, 8, 10, 15].

In [11], the following general recurrence relation was considered to clarify the combinatorial structure latent in the recurrence relation for $\mathrm{S}_k(n)$ and to cope with the recurrence relations for the Tower of Hanoi *on graphs* in which pegs are placed on vertices of a given graph and disks are only moved along the edges:

$$\mathrm{T}(n) = \min_{1 \leq t \leq n} \left\{ \alpha \cdot \mathrm{T}(n-t) + \beta \cdot (2^t - 1) \right\}, \text{ for } n \geq 1, \text{ and } \mathrm{T}(0) = 0,$$

where α and β are arbitrary positive integers. It was shown that the sequence of differences $(\mathrm{T}(n) - \mathrm{T}(n-1))_{n \geq 1}$ consists of numbers of the form $\beta \cdot 2^i \cdot \alpha^j$, with $i, j \geq 0$, lined in the increasing order. When $\alpha = 3$, $2^i \cdot \alpha^j$ increases as $1, 2, 3, 2^2, 2 \cdot 3, 2^3, 3^2, 2^2 \cdot 3, 2^4, 2 \cdot 3^2, \cdots$. These numbers are called "3-smooth numbers"[13] and have been studied extensively in number theory, in relation to the distribution of prime numbers [6] and to new number representations [1, 4]. The formulation and analysis of $\mathrm{T}(n)$, however, has some defects such that (i) it is only focused on the 4-peg case with no consideration for the general case $k \geq 3$; and (ii) even in the 4-peg case, term $2^i \cdot \alpha^j$ consists of constant 2 and parameter α, which might admit further generalization.

In this paper, we fully generalize the recurrence relations for the previous $\mathrm{S}_k(n)$ and $T(n)$ and obtain the exact formulas. Namely, we define the following recurrence relations for two sequences of arbitrary positive integers $(p_i)_{i\geq 3}$ and $(q_i)_{i\geq 3}$:

$$\mathrm{G}_k(n) = \min_{1\leq t\leq n} \left\{ p_k \cdot \mathrm{G}_k(n-t) + q_k \cdot \mathrm{G}_{k-1}(t) \right\}, \text{ for } n \geq 1,\ k \geq 4,$$

$$\mathrm{G}_3(n) = p_3 \cdot \mathrm{G}_3(n-1) + q_3, \text{ for } n \geq 1, \text{ and } \mathrm{G}_k(0) = 0, \text{ for } k \geq 3.$$

Then, we show that the sequence of differences $(\mathrm{G}_k(n) - \mathrm{G}_k(n-1))_{n\geq 1}$ consists of numbers of the form $(\prod_{i=3}^k q_i) \cdot (\prod_{i=3}^k {p_i}^{\alpha_i})$, with $\alpha_i \geq 0$ for all i, lined in the increasing order. In other words, we show the following theorem.

Theorem 1. *For every positive integer n and for two sequences of arbitrary positive integers $(p_i)_{i\geq 3}$ and $(q_i)_{i\geq 3}$, we have*

$$\mathrm{G}_k(n) = q \cdot \sum_{j=1}^{n} u_j^k$$

where $q = \prod_{i=3}^k q_i$ and u_j^k is the jth term of the sequence $\left(u_j^k\right)_{j\geq 1}$ of integers $\prod_{i=3}^k {p_i}^{\alpha_i}$, with $\alpha_i \geq 0$ for all i, lined in the increasing order.

We call $\mathrm{G}_k(n)$ the *generalized Frame-Stewart numbers.* Note that $\mathrm{G}_k(n)$ is equal to $\mathrm{S}_k(n)$ when $(p_i, q_i) = (2, 1)$ for all $i \geq 3$ and is equal to $\mathrm{T}(n)$ when $(p_3, q_3) = (2, 1)$ and $(p_4, q_4) = (\alpha, \beta)$.

The remaining of the paper is organized as follows. In Section 2, we show some basic properties of the sequence $\left(u_j^k\right)_{j\geq 1}$ defined from $(p_i)_{i\geq 3}$. In Section 3, we prove Theorem 1, the main result of this paper. In Section 4, application of these numbers in obtaining upper bounds of the number of moves for the Tower of Hanoi problem on several graphs is provided.

2 Basic results on smooth numbers sequences

Let $(p_i)_{i\geq 3}$ be a sequence of positive integers. We consider the sequence $\left(u_j^k\right)_{j\geq 1}$ of all the integers of the form $\prod_{i=3}^k {p_i}^{\alpha_i}$, where $\alpha_i \geq 0$ for all i, lined in the increasing order. For instance, for $(p_3, p_4) = (2, 2)$ and $(p_3, p_4) = (2, 3)$, the first few terms of $(u_j^4)_{j\geq 1}$ are $(1, 2, 2, 2^2, 2^2, 2^2, 2^3, \cdots)$ and $(1, 2, 3, 2^2, 2 \cdot 3, 2^3, 3^2, \cdots)$, respectively. When there is some i_0 such that p_{i_0} is equal to 1, then by definition $\left(u_j^k\right)_{j\geq 1}$ is the constant sequence of 1's, for every $k \geq i_0$. We note that $\left(u_j^k\right)_{j\geq 1}$ is closely related to *smooth numbers* which have been explored extensively in number theory. A positive integer is called *B-smooth* if none of its prime factors are greater than a positive integer B. The sequence $\left(u_j^k\right)_{j\geq 1}$ then consists of B-smooth numbers for $B = \max_{3\leq i\leq k}\{p_i\}$.

In this section, we restrict to the case where all the p_i's are greater than 1 and prove a simple lemma on a certain "recursive" structure of the smooth numbers sequence $\left(u_j^k\right)_{j\geq 1}$, which will be used to prove Theorem 1 in the next section.

Lemma 1. *Let $k \geq 4$ and let $(k_j)_{j\geq 1}$ be the sequence of positive integers defined by $k_1 = 1$ and $k_j = \min\left\{l > k_{j-1} \mid u_l^k = u_j^{k-1}\right\}$ for $j \geq 2$. Then, for every integer n such that $k_j < n < k_{j+1}$, we have $u_n^k = p_k \cdot u_{n-j}^k$.*

Proof. If $k_{j+1} = k_j + 1$, then the lemma is trivial. Suppose now that $k_{j+1} - k_j \geq 2$ and let n be a positive integer such that $k_j < n < k_{j+1}$. First, consider a term $\prod_{i=1}^{k} {p_i}^{\alpha_i}$ of the sequence $(u_l^k)_{l\geq 1}$. If $\alpha_k = 0$, then $\prod_{i=1}^{k} {p_i}^{\alpha_i} = \prod_{i=1}^{k-1} {p_i}^{\alpha_i}$ belongs to $(u_{k_l}^k)_{l\geq 1}$ by definition of $(k_l)_{l\geq 1}$. Otherwise, if $\alpha_k \geq 1$, then $\prod_{i=1}^{k} {p_i}^{\alpha_i} = p_k \cdot \left({p_k}^{\alpha_k - 1} \cdot \prod_{i=1}^{k-1} {p_i}^{\alpha_i}\right)$ belongs to $(p_k \cdot u_l^k)_{l\geq 1}$. Now, since $k_j < n < k_{j+1}$, it follows that $u_{k_j}^k \leq u_n^k < u_{k_{j+1}}^k$ by the growth of the sequence $(u_l^k)_{l\geq 1}$. So the first n terms of $(u_l^k)_{l\geq 1}$ exactly contains the first j terms of $(u_{k_l}^k)_{l\geq 1}$. We already know that a term of $(u_l^k)_{l\geq 1}$ belongs to $(u_{k_l}^k)_{l\geq 1}$ or to $(p_k \cdot u_l^k)_{l\geq 1}$. This leads to the decomposition

$$\left\{u_l^k \mid 1 \leq l \leq n\right\} = \left\{u_{k_l}^k \mid 1 \leq l \leq j\right\} \bigcup \left\{p_k \cdot u_l^k \mid 1 \leq l \leq n - j\right\}$$

and to the equality $u_n^k = p_k \cdot u_{n-j}^k$, by the maximality of u_n^k. □

Lemma 1 can be also used for computing $\left(u_j^k\right)_{j\geq 1}$ explicitly for special sequences $(p_i)_{i\geq 3}$. Here, we compute $\left(u_j^k\right)_{j\geq 1}$ in the simple case $p_i = p \geq 2$ for all $i \geq 3$ (we note that when $p = 2$, $\left(u_j^k\right)_{j\geq 1}$ is the sequence for the original k-peg Tower of Hanoi problem).

Proposition 1. *Let $p_i = p \geq 1$ for all $3 \leq i \leq k$. Then, for all integers $j \geq 0$ and $n \geq 1$ such that $\binom{k+j-3}{k-2} < n \leq \binom{k+j-2}{k-2}$, we have $u_n^k = p^j$.*

Proof. For $p = 1$, the result is clear. Suppose now that $p \geq 2$ and that the result is verified for $i = k-1$ and $n \geq 1$, and for $i = k$ and $n \leq \binom{k+j_0-3}{k-2}$ for some $j_0 \geq 1$. By hypothesis of recurrence, we know that

$$p^{j_0} = u_{\binom{k+j_0-4}{k-3}+l_1}^{k-1}, \quad \text{for } 1 \leq l_1 \leq \binom{k+j_0-4}{k-4},$$

and

$$p^{j_0-1} = u_{l_2}^k, \quad \text{for } \binom{k+j_0-4}{k-2} < l_2 \leq \binom{k+j_0-3}{k-2}.$$

By definition of the sequence $(k_l)_{l\geq 1}$, we have

$$k_{\binom{k+j_0-4}{k-3}+l_1} = \binom{k+j_0-3}{k-2} + l_1, \quad \text{for } 1 \leq l_1 \leq \binom{k+j_0-4}{k-4}.$$

Moreover,

$$k_{\binom{k+j_0-4}{k-3}+\binom{k+j_0-4}{k-4}+1} = k_{\binom{k+j_0-3}{k-3}+1} \quad \text{and} \quad u^k_{k_{\binom{k+j_0-3}{k-3}+1}} = u^{k-1}_{\binom{k+j_0-3}{k-3}+1} = p^{j_0+1}.$$

By Lemma 1, we know that, for every positive integer n such that $k_{\binom{k+j_0-3}{k-3}} < n < k_{\binom{k+j_0-3}{k-3}+1}$, the equality

$$p^{j_0} = u^k_n = p_k \cdot u^k_{n-\binom{k+j_0-3}{k-3}} = p \cdot u^k_{n-\binom{k+j_0-3}{k-3}}$$

holds. This leads to

$$u^k_{n-\binom{k+j_0-3}{k-3}} = p^{j_0-1}, \quad \text{for} \quad k_{\binom{k+j_0-3}{k-3}} < n < k_{\binom{k+j_0-3}{k-3}+1}.$$

Since $u^k_{l_2} = p^{j_0-1}$ if and only if $\binom{k+j_0-4}{k-2} < l_2 \leq \binom{k+j_0-3}{k-2}$ by hypothesis of recurrence, it follows that

$$k_{\binom{k+j_0-3}{k-3}+1} = \binom{k+j_0-3}{k-2} + \binom{k+j_0-3}{k-3} + 1 = \binom{k+j_0-2}{k-2} + 1.$$

Therefore,

$$u^k_n = p^{j_0}, \quad \text{for } \binom{k+j_0-3}{k-2} < n \leq \binom{k+j_0-2}{k-2}, \quad \text{and} \quad u^k_{\binom{k+j_0-2}{k-2}+1} = p^{j_0+1}.$$

This completes the proof. □

3 Proof of Theorem 1

Let $\mathrm{G}^1_k(n)$ denotes the special case of $\mathrm{G}_k(n)$ associated with arbitrary sequence $(p_i)_{i\geq 3}$ and with the constant sequence $(q_i)_{i\geq 3}$ with $q_i = 1$ for $i \geq 3$. There exists a simple relationship between numbers $\mathrm{G}_k(n)$ and $\mathrm{G}^1_k(n)$.

Proposition 2. *For every nonnegative integer n and for every sequence of integers $(q_i)_{i\geq 3}$, we have*

$$\mathrm{G}_k(n) = q \cdot \mathrm{G}^1_k(n),$$

where $q = \prod_{i=3}^k q_i$.

Proof. By recurrence on k and n. For $k = 3$, we can prove by simple induction on n that $\mathrm{G}_3(n) = q_3 \cdot \mathrm{G}^1_3(n)$ for all n. Suppose the result is true for $k-1$ and all $n \geq 0$, and k and all l such that $l \leq n-1$. By the recursive definition of $\mathrm{G}_k(n)$ and by the assumption of induction, we obtain

$$\begin{aligned}
\mathrm{G}_k(n) &= \min_{1\leq t\leq n}\left\{p_k \cdot \mathrm{G}_k(n-t) + q_k \cdot \mathrm{G}_{k-1}(t)\right\} \\
&= \min_{1\leq t\leq n}\left\{p_k \cdot \prod_{i=3}^{k} q_i \cdot \mathrm{G}^1_k(n-t) + q_k \cdot \prod_{i=3}^{k-1} q_i \cdot \mathrm{G}^1_{k-1}(t)\right\} \\
&= \prod_{i=3}^{k} q_i \cdot \min_{1\leq t\leq n}\left\{p_k \cdot \mathrm{G}^1_k(n-t) + \mathrm{G}^1_{k-1}(t)\right\} \\
&= q \cdot \mathrm{G}^1_k(n).
\end{aligned}$$

□

By Proposition 2, it is sufficient to prove Theorem 1 for $\mathrm{G}^1_k(n)$ instead of $\mathrm{G}_k(n)$. Now, we show at which argument $\mathrm{G}^1_k(n) = \min_{1\leq t\leq n}\left\{p_k \cdot \mathrm{G}^1_k(n-t) + \mathrm{G}^1_{k-1}(t)\right\}$ takes its minimum.

Lemma 2. *Let n be a positive integer. Suppose that $p_i > 1$ for all $3 \leq i \leq k$. Suppose also that $\Delta\mathrm{G}^1_i(l) = \mathrm{G}^1_i(l) - \mathrm{G}^1_i(l-1) = u^i_l$ for $3 \leq i \leq k-1$ and $l \geq 1$ and that $\Delta\mathrm{G}^1_k(l) = u^k_l$ for $1 \leq l \leq n-1$. Let j be the integer such that $k_j \leq n < k_{j+1}$. Then, for $1 \leq t \leq n$, $\mathrm{G}^1_{k,n}(t) = p_k \cdot \mathrm{G}^1_k(n-t) + \mathrm{G}^1_{k-1}(t)$ takes its minimum at $t = j$.*

Proof. Since

$$\begin{aligned}
\mathrm{G}^1_{k,n}(t+1) - \mathrm{G}^1_{k,n}(t) &= p_k \cdot \mathrm{G}^1_k(n-t-1) + \mathrm{G}^1_{k-1}(t+1) - p_k \cdot \mathrm{G}^1_k(n-t) - \mathrm{G}^1_{k-1}(t) \\
&= -p_k \cdot (\mathrm{G}^1_k(n-t) - \mathrm{G}^1_k(n-t-1)) + (\mathrm{G}^1_{k-1}(t+1) - \mathrm{G}^1_{k-1}(t)) \\
&= 1 - p_k \cdot \Delta\mathrm{G}^1_k(n-t) + \Delta\mathrm{G}^1_{k-1}(t+1)
\end{aligned}$$

for every $1 \leq t \leq n-1$, it follows by hypothesis that

$$\mathrm{G}^1_{k,n}(t+1) - \mathrm{G}^1_{k,n}(t) = -p_k \cdot u^k_{n-t} + u^{k-1}_{t+1} \quad \text{for} \quad 1 \leq t \leq n-1.$$

First, when $1 \leq t \leq j-1$, the growth of the sequences $\left(u^k_l\right)_{l\geq 1}$ and $\left(u^{k-1}_l\right)_{l\geq 1}$ yields the following inequalities

$$u^k_{n-t} \geq u^k_{n-j+1} \geq u^k_{k_j-j+1}, \quad u^{k-1}_{t+1} \leq u^{k-1}_j = u^k_{k_j}.$$

Let $m = \min\left\{l \geq 0 \mid k_{j+l+1} - k_{j+l} \geq 2\right\}$. Such m always exists. By definition of k_{j+l}, we have $k_{j+l} = k_j + l$ for $0 \leq l \leq m$ and $k_{j+m} < k_j + m + 1 < k_{j+m+1}$. So we deduce from Lemma 1 that

$$u^k_{k_j+m+1} = p_k \cdot u^k_{(k_j+m+1)-(j+m)} = p_k \cdot u^k_{k_j-j+1}.$$

Thus,

$$\mathrm{G}^1_{k,n}(t+1) - \mathrm{G}^1_{k,n}(t) = -p_k \cdot u^k_{n-t} + u^{k-1}_{t+1} \leq -u^k_{k_j+m+1} + u^k_{k_j} \leq 0$$

for $1 \leq t \leq j-1$. Therefore, $\mathrm{G}^1_{k,n}(t) \geq \mathrm{G}^1_{k,n}(j)$ for all $1 \leq t \leq j$.

Similarly, when $j \leq t \leq n-1$, we have

$$u^k_{n-t} \leq u^k_{n-j} \leq u^k_{k_{j+1}-j-1}, \quad u^{k-1}_{t+1} \geq u^{k-1}_{j+1} = u^k_{k_{j+1}}.$$

Let $m = \min\{l \geq 0 \mid k_{j-l+1} - k_{j-l} \geq 2\}$. If such m does not exist, then $n = k_j = j$ and we already know that $\mathrm{G}^1_{k,n}(t)$ takes its minimum at $t = j$. Suppose now that the integer m exists. By definition of k_{j-l+1}, we have $k_{j-l+1} = k_{j+1} - l$ for $0 \leq l \leq m$ and $k_{j-m} < k_{j+1} - m - 1 < k_{j-m+1}$. So we deduce from Lemma 1 that

$$u^k_{k_{j+1}-m-1} = p_k \cdot u^k_{(k_{j+1}-m-1)-(j-m)} = p_k \cdot u^k_{k_{j+1}-j-1}.$$

Thus,

$$\mathrm{G}^1_{k,n}(t+1) - \mathrm{G}^1_{k,n}(t) = -p_k \cdot u^k_{n-t} + u^{k-1}_{t+1} \geq -u^k_{k_{j+1}-m-1} + u^k_{k_{j+1}} \geq 0$$

for $j \leq t \leq n-1$. Therefore, $\mathrm{G}^1_{k,n}(t) \geq \mathrm{G}^1_{k,n}(j)$ for all $j \leq t \leq n$.

Consequently, $\mathrm{G}^1_{k,n}(t)$ takes its minimum at $t = j$. □

We are now ready to prove the main result of this paper.

Proof of Theorem 1. From Proposition 1, it is sufficient to prove that

$$\mathrm{G}^1_k(n) = \sum_{j=1}^{n} u^k_j$$

for every positive integer n.

First, suppose that $p_i > 1$ for all integers $3 \leq i \leq k$. We proceed by induction on k and n. It is clear that for all $k \geq 3$, $\mathrm{G}^1_k(1) = 1 = u^k_1$. It is also clear that $\Delta\mathrm{G}^1_3(n) = \mathrm{G}^1_3(n) - \mathrm{G}^1_3(n-1) = p_3^{n-1} = u^3_n$ for all $n \geq 1$. Now assume that $\Delta\mathrm{G}^1_i(l) = u^i_l$ for all $3 \leq i \leq k-1$ and all $l \geq 1$ and that $\Delta\mathrm{G}^1_k(l) = u^k_l$ for all $1 \leq l \leq n-1$. Then, we show that $\Delta\mathrm{G}^1_k(n) = u^k_n$ holds. For n, there exists some $j \geq 1$ such that $k_j \leq n < k_{j+1}$. It is divided into two cases: when $n = k_j$ (Case 1) and when $k_j < n < k_{j+1}$ (Case 2).

Case 1. When $n = k_j$, we obtain

$$\begin{aligned}
\Delta\mathrm{G}^1_k(n) &= \mathrm{G}^1_k(k_j) - \mathrm{G}^1_k(k_j - 1) \\
&= \mathrm{G}^1_{k,k_j}(j) - \mathrm{G}^1_{k,k_j-1}(j-1) \quad \text{(since } k_{j-1} \leq k_j - 1 < k_j \text{ and by Lemma 2)} \\
&= p_k \cdot (\mathrm{G}^1_k(k_j - j) - \mathrm{G}^1_k((k_j-1)-(j-1))) + \big(\mathrm{G}^1_{k-1}(j) - \mathrm{G}^1_{k-1}(j-1)\big) \\
&= \Delta\mathrm{G}^1_{k-1}(j) \\
&= u^{k-1}_j \quad \text{(by assumption of induction)} \\
&= u^k_{k_j} \quad \text{(by definition of } k_j\text{)} \\
&= u^k_n.
\end{aligned}$$

Thus, the proof is shown in this case.

Case 2. When $k_j < n < k_{j+1}$, we obtain

$$\begin{aligned}\Delta\mathrm{G}_k^1(n) &= \mathrm{G}_k^1(n) - \mathrm{G}_k^1(n-1)\\ &= \mathrm{G}_{k,n}^1(j) - \mathrm{G}_{k,n-1}^1(j) \quad \text{(since } k_j \le n-1 < k_{j+1} \text{ and by Lemma 2)}\\ &= p_k \cdot (\mathrm{G}_k^1(n-j) - \mathrm{G}_k^1(n-1-j)) + \big(\mathrm{G}_{k-1}^1(j) - \mathrm{G}_{k-1}^1(j)\big)\\ &= p_k \cdot \Delta\mathrm{G}_k^1(n-j)\\ &= p_k \cdot u_{n-j}^k \quad \text{(by assumption of induction)}\\ &= u_n^k \quad \text{(by Lemma 1).}\end{aligned}$$

Thus, the proof is shown in this case, too.

Next, suppose that $p_i = 1$ for some integer $i \le k$. When $p_3 = 1$, it is clear that $\mathrm{G}_3^1(n) = n$ for all $n \ge 0$. Suppose now, without loss of generality, that $p_{i_0} = 1$ for some $4 \le i_0 \le k$ and $p_i > 1$ for all $3 \le i \le i_0 - 1$. We proceed by induction on n. Assume that $\mathrm{G}_{i_0}^1(l) = l$ for $0 \le l \le n-1$. For n, by definition,

$$\mathrm{G}_{i_0}^1(n) = \min_{1\le t\le n}\left\{\mathrm{G}_{i_0}^1(n-t) + \mathrm{G}_{i_0-1}^1(t)\right\} = \min_{1\le t\le n}\left\{(n-t) + \mathrm{G}_{i_0-1}^1(t)\right\}.$$

Since $p_i > 1$ for all $3 \le i \le i_0 - 1$, we know that $\mathrm{G}_{i_0-1}^1(l) = \sum_{m=1}^{l} u_m^{i_0-1}$ for $l \ge 1$. It is clear that $u_m^{i_0-1} \ge 1$ for all $1 \le m \le l$. Therefore we have $\mathrm{G}_{i_0-1}^1(l) \ge l$ for $l \ge 1$. So $\mathrm{G}_{i_0,n}^1(t) = (n-t) + \mathrm{G}_{i_0-1}^1(t)$ takes its minimum at $t = 1$ and $\mathrm{G}_{i_0}^1(n) = (n-1) + 1 = n$ as announced. Finally, suppose that, for some integer $i \ge 3$, $\mathrm{G}_i^1(l) = l$ for all $l \ge 0$ and $\mathrm{G}_{i+1}^1(l) = l$ for all $1 \le l \le n-1$. For n, we obtain

$$\mathrm{G}_{i+1}^1(n) = \min_{1\le t\le n}\left\{\mathrm{G}_{i+1}^1(n-t) + \mathrm{G}_i^1(t)\right\} = \min_{1\le t\le n}\left\{(n-t) + t\right\} = n.$$

This concludes the proof of Theorem 1. □

Corollary 1. *Let $k \ge 4$ and $j \ge 1$. For every integer n such that $k_j \le n < k_{j+1}$,*

$$\mathrm{G}_k(n) = p_k \cdot \mathrm{G}_k(n-j) + q_k \cdot \mathrm{G}_{k-1}(j).$$

Proof. From Proposition 2, Theorem 1 and Lemma 2. □

We end this section in considering the special case where $p_i = p \ge 1$ for all i.

Proposition 3. *Let $p_i = p \ge 1$ for all $3 \le i \le k$. Then, for all integers $j \ge 0$ and $n \ge 1$ such that*

$$\binom{k+j-3}{k-2} < n \le \binom{k+j-2}{k-2},$$

$\mathrm{G}_k^1(n)$ can be computed as follows:

$$\mathrm{G}_k^1(n) = \sum_{m=0}^{j-1}\binom{k+m-3}{k-3}p^m + \left(n - \binom{k+j-3}{k-2}\right)p^j.$$

Proof. By induction on n. First, we know by Proposition 1 that $u_n^k = p^j$. Moreover, $\mathrm{G}_k^1(n) = \mathrm{G}_k^1(n-1) + u_n^k$ from Theorem 1.

When $n = \binom{k+j-3}{k-2} + 1$, we obtain by the assumption of induction

$$\begin{aligned}
\mathrm{G}_k^1(n) &= \mathrm{G}_k^1(n-1) + p^j \\
&= \sum_{m=0}^{j-2} \binom{k+m-3}{k-3} p^m + \left((n-1) - \binom{k+j-4}{k-2} \right) p^{j-1} + p^j \\
&= \sum_{m=0}^{j-2} \binom{k+m-3}{k-3} p^m + \binom{k+j-4}{k-3} p^{j-1} + p^j \\
&= \sum_{m=0}^{j-1} \binom{k+m-3}{k-3} p^m + \left(n - \binom{k+j-3}{k-2} \right) p^j.
\end{aligned}$$

When $\binom{k+j-3}{k-2} + 1 < n \leq \binom{k+j-2}{k-2}$, we obtain

$$\begin{aligned}
\mathrm{G}_k^1(n) &= \mathrm{G}_k^1(n-1) + p^j \\
&= \sum_{m=0}^{j-1} \binom{k+m-3}{k-3} p^m + \left((n-1) - \binom{k+j-3}{k-2} \right) p^j + p^j \\
&= \sum_{m=0}^{j-1} \binom{k+m-3}{k-3} p^m + \left(n - \binom{k+j-3}{k-2} \right) p^j.
\end{aligned}$$

□

4 Application: the Tower of Hanoi on graphs

Let $G = (V, E)$ be a simple graph with the set of vertices $V = \{v_1, \ldots, v_k\}$ and the set of edges E. A k-peg Tower of Hanoi problem can be considered on G: the k pegs are placed on the vertices $v_1, \ldots, v_k$ and transfer of disks is allowed between the pegs v_i and v_j only if there is an edge between v_i and v_j. The original k-peg Tower of Hanoi problem then corresponds to the Tower of Hanoi problem on the complete graph K_k. The cases of $k = 3$ and $k = 4$ are illustrated in Figure 1.

The main application of the generalized Frame-Stewart numbers is in giving upper bounds of the number of moves for the Tower of Hanoi problem on some simple graphs. For the Tower of Hanoi problem on the complete graph with $k \geq 3$ vertices and $n \geq 0$ disks, we retrieve the Frame-Stewart numbers $\mathrm{S}_k(n)$ stated in Section 1. In the sequel of this section, we consider other special cases where G is the path graph P_3 or the star graph S_k.

4.1 On the path graph P_3

The following theorem shows that the optimal number of moves for the Tower of Hanoi problem on the path graph P_3 is given by the generalized Frame-Stewart numbers.

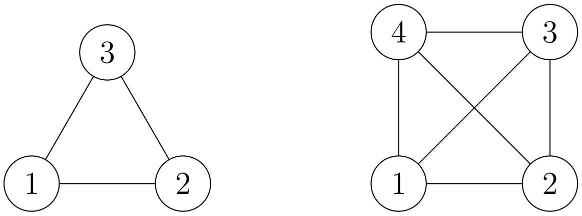

Figure 1: The original Tower of Hanoi problem with 3 pegs (K_3) and 4 pegs (K_4).

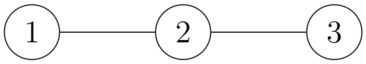

Figure 2: The path graph P_3.

Theorem 2. *Consider the Tower of Hanoi problem on* P_3*, as depicted in Figure 2. The minimum number of moves to transfer* $n \geq 1$ *disks*

- *from peg 1 to peg 3 is* $G_3(n) = 2 \cdot \sum_{i=0}^{n-1} 3^i$*, where* $(p_3, q_3) = (3, 2)$*;*
- *from peg 1 to peg 2 is* $G_3^1(n) = \sum_{i=0}^{n-1} 3^i$*, where* $(p_3, q_3) = (3, 1)$*.*

Though the fact of this theorem is rather well-known (e.g., see [12]), we present a short proof to see the connection with the generalized Frame-Stewart numbers.

Proof. We begin with the transfer between peg 1 and peg 3. In order to move the biggest disk from peg 1 to peg 3, we have to first move it from peg 1 to peg 2 and so the $n-1$ smallest disks must be on peg 3. The $n-1$ smallest disks are transferred from peg 1 to peg 3 in $G_3(n-1)$ moves. Then, we move the biggest disk from peg 1 to peg 2. In order to move this disk to peg 3, we transfer the $n-1$ smallest disks from peg 3 to peg 1 in $G_3(n-1)$ moves. Finally, we put the biggest disk from peg 2 to peg 3 in 1 move and the $n-1$ smallest disks from peg 1 to peg 3 in $G_3(n-1)$ moves. The total number of moves for n disks is then $3 \cdot G_3(n-1) + 2$, which corresponds to $G_3(n)$ as announced. Since this is the best possible, $G_3(n)$ is the optimal number of moves.

For the transfer between peg 1 and peg 2, as before, in order to move the biggest disk from peg 1 to peg 2, we have to first transfer the $n-1$ smallest disks from peg 1 to peg 3. As proved above, the minimum number of moves to do this is $G_3(n-1)$. Moreover, we know that $G_3(n-1) = 2 \cdot G_3^1(n-1)$ by Proposition 2. Then, after moving the biggest disk from peg 1 to peg 2, the $n-1$ smallest disks are transferred from peg 3 to peg 2. It is done in $G_3^1(n-1)$ moves. Thus, we conclude that the minimum number of moves for transferring n disks from peg 1 to peg 2 is $3 \cdot G_3^1(n-1) + 1$ as announced. □

4.2 On the star graph S_k

We end this section by considering the Tower of Hanoi problem on the star graph S_k with $k+1$ vertices and k edges. For $k = 2$, the graph S_2 corresponds to the path graph P_3. The star graphs for $k = 3$ and $k = 4$ are depicted in Figure 3.

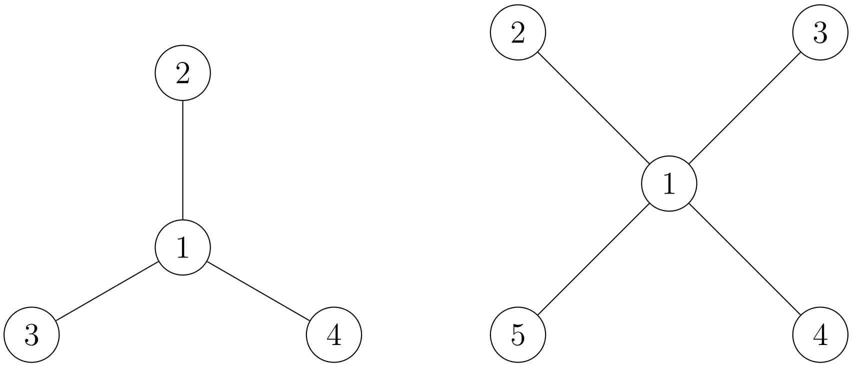

Figure 3: The star graphs S_3 and S_4.

Stockmeyer [16] considered the Tower of Hanoi problem on the star graph S_3, where all the n disks are transferred from one leaf of the graph to another leaf (for instance, from peg 2 to peg 3 in Figure 3). He described a recursive algorithm which achieved a good (seemingly the best) upper bound; thus, called it the “presumed optimal” algorithm. Here, we generalize this algorithm to the star graph S_k for arbitrary $k \geq 2$ and show that the number of moves for this problem is obtained exactly by the generalized Frame-Stewart numbers.

Theorem 3. *Let $k \geq 2$ be an integer. Consider the Tower of Hanoi problem on the star graph S_k in which $n \geq 1$ disks are transferred from one leaf of the graph to another leaf. Then, an upper bound on the number of moves to solve this problem is given by the generalized Frame-Stewart number $\mathrm{G}_{k+1}(n)$, where $(p_3, q_3) = (3, 2)$ and $(p_i, q_i) = (2, 1)$ for $4 \leq i \leq k+1$.*

Proof. By induction on k of S_k. When $k = 2$, as noted before, the star graph S_2 corresponds to the path graph P_3. So by Theorem 2, $\mathrm{G}_3(n)$, where $(p_3, q_3) = (3, 2)$, is the minimum number of moves to transfer n disks from peg 2 to peg 3. Suppose now that the result is true for any number of disks up to S_{k-1} and until $n-1$ disks for S_k. n disks are then recursively transferred from peg 2 to peg 3 as follows. For some integer t such that $1 \leq t \leq n$,

- transfer the $n-t$ smallest disks from peg 2 to peg $k+1$ in $\mathrm{G}_{k+1}(n-t)$ moves;
- consider the remaining k pegs and the subgraphs obtained after deleting the vertex of peg $k+1$, which is the star graph S_{k-1}, and transfer the t largest disks from peg 2 to peg 3 in $\mathrm{G}_k(t)$ moves;
- transfer the $n-t$ smallest disks from peg $k+1$ to peg 3 in $\mathrm{G}_{k+1}(n-t)$ moves.

We choose the integer t such that the number of moves $2 \cdot \mathrm{G}_{k+1}(n-t) + \mathrm{G}_k(t)$ is minimized. Thus, the algorithm satisfies the following recurrence relation:

$$\mathrm{G}_{k+1}(n) = \min_{1 \le t \le n} \bigl\{ 2 \cdot \mathrm{G}_{k+1}(n-t) + \mathrm{G}_k(t) \bigr\}.$$

By this equation with the assumption of induction up to $k-1$, the number of moves of this algorithm is given by the generalized Frame-Stewart number with $(p_3, q_3) = (3, 2)$ and $(p_i, q_i) = (2, 1)$ for $4 \le i \le k+1$. □

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