

# TIME-DEPENDENT MECHANICS AND LAGRANGIAN SUBMANIFOLDS OF PRESYMPLECTIC AND POISSON MANIFOLDS

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ABSTRACT. A description of time-dependent Mechanics in terms of Lagrangian submanifolds of presymplectic and Poisson manifolds is presented. Two new Tulczyjew triples are discussed. The first one is adapted to the restricted Hamiltonian formalism and the second one is adapted to the extended Hamiltonian formalism.

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## 1. INTRODUCTION

It is well-known that the phase space of velocities of a mechanical system may be identified with the tangent bundle  $TQ$  of the configuration space  $Q$ . Under this identification, the Lagrangian

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function is a real  $C^\infty$ -function  $L$  on  $TQ$  and the Euler-Lagrange equations are

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = 0, \quad i = 1, \dots, n = \dim Q$$

where  $(q^i, \dot{q}^i)$  are local fibred coordinates on  $TQ$ , which represent the positions and the velocities of the system, respectively.

If the Lagrangian function is hyperregular one may define the Hamiltonian function  $H : T^*Q \rightarrow \mathbb{R}$  on the phase space of momenta  $T^*Q$  and the Euler-Lagrange equations are equivalent to the Hamilton equations for  $H$

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}, \quad i = 1, \dots, n.$$

Here,  $(q^i, p_i)$  are local fibred coordinates on  $T^*Q$  which represent the positions and the momenta of the system, respectively.

Solutions of the previous Hamilton equations are just the integral curves of the Hamiltonian vector field  $X_H$  on  $T^*Q$  which is characterized by the condition

$$\iota_{X_H} \Omega_Q = dH,$$

$\Omega_Q$  being the canonical symplectic structure of  $T^*Q$  (for more details see, for instance, [1, 13]). Lagrangian (Hamiltonian) Mechanics may be also formulated in terms of Lagrangian submanifolds of symplectic manifolds (see [16, 17]).

In fact, the complete lift  $\Omega_Q^c$  of  $\Omega_Q$  to  $T(T^*Q)$  defines a symplectic structure on  $T(T^*Q)$  and, if on  $T^*(TQ)$  we consider the canonical symplectic structure  $\Omega_{TQ}$ , the canonical Tulczyjew diffeomorphism  $A_Q : T(T^*Q) \rightarrow T^*(TQ)$  is a symplectic isomorphism. Moreover,  $S_L = A_Q^{-1}(dL)$  is a Lagrangian submanifold of the symplectic manifold  $(T(T^*Q), \Omega_Q^c)$  and the local equations defining  $S_L$  as a submanifold of  $T(T^*Q)$  are just the Euler-Lagrange equations for  $L$ .

On the other hand, if  $H : T^*Q \rightarrow \mathbb{R}$  is a Hamiltonian function and  $b_{\Omega_Q} : T(T^*Q) \rightarrow T^*(T^*Q)$  is the vector bundle isomorphism induced by  $\Omega_Q$  then  $b_{\Omega_Q}$  is an anti-symplectic isomorphism (when on  $T^*(T^*Q)$  we consider the canonical symplectic structure  $\Omega_{T^*Q}$ ). In addition,  $S_H = b_{\Omega_Q}^{-1}(dH)$  is a Lagrangian submanifold of  $T(T^*Q)$  and the local equations defining  $S_H$  as a submanifold of  $T(T^*Q)$  are just the Hamilton equations for  $H$ . Figure 1 illustrates the situation

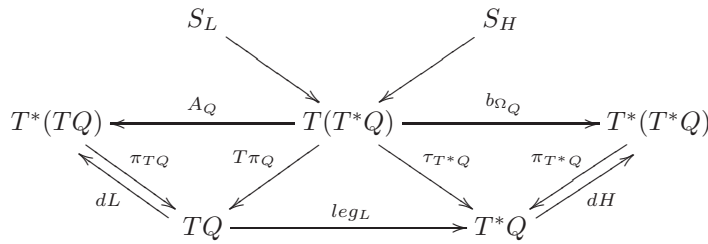


FIGURE 1. *Tulczyjew triple for time-independent Mechanics*

If the Lagrangian function  $L$  is hyperregular then the Legendre transformation  $leg_L : TQ \rightarrow T^*Q$  is a global diffeomorphism and  $S_L = S_H$ .

We remark that in the previous construction the following properties hold:

- (1) The three spaces  $T^*(TQ)$ ,  $T(T^*Q)$  and  $T^*(T^*Q)$  involved in the Tulczyjew triple are of the same type, namely, symplectic manifolds.
- (2) The two maps  $A_Q$  and  $b_{\Omega_Q}$  involved in the construction are a symplectic isomorphism and an anti-symplectic isomorphism, respectively.
- (3) The Lagrangian and the Hamiltonian functions are not involved in the definition of the triple. In this sense, the triple is canonical.
- (4) The dynamical equations (Euler-Lagrange and Hamilton equations) are the local equations defining the Lagrangian submanifolds  $S_L$  and  $S_H$  of  $T(T^*Q)$ .
- (5) The construction may be applied to an arbitrary Lagrangian function (not necessarily regular).

On the other hand, for time-dependent mechanical systems the role of  $TQ$  and  $T^*Q$  is played by the space of 1-jets  $J^1\pi$  of local sections of a fibration  $\pi : M \rightarrow \mathbb{R}$  (in the Lagrangian formalism) and for the dual bundle  $V^*\pi$  to the vertical bundle  $V\pi$  to  $\pi$  (in the restricted Hamiltonian formalism) or for the cotangent bundle  $T^*M$  to  $M$  (in the extended Hamiltonian formalism). For more details on these topics, see [10, 15].

Note that  $V^*\pi$  is not a symplectic manifold, but a Poisson manifold.

Several attempts to extend the Tulczyjew triple for time-dependent mechanical systems have been done. However, although accurate and interesting, they all exhibit some defect if we compare with the original Tulczyjew triple for autonomous mechanical systems. In fact, in [9] the authors described a Tulczyjew triple for the particular case when the fibration  $\pi : M \rightarrow \mathbb{R}$  is trivial, that is,  $M = \mathbb{R} \times Q$  and  $\pi$  is the projection on the first factor. They used the extended formalism and the spaces involved in the construction were too big.

Later, in [11], M. de León et al discussed a Tulczyjew triple for the same fibration  $pr_1 : \mathbb{R} \times Q \rightarrow Q$ . In this case, the Lagrangian and Hamiltonian functions are involved in the definition of the triple. In this construction, they used the notion of the complete lift of a cosymplectic structure.

On the other hand, in [8] the authors proposed a restricted Tulczyjew triple for a general fibration  $\pi : M \rightarrow \mathbb{R}$ . However, the Hamiltonian section is involved in the construction of the triple.

In this paper, we solve the previous problems and deficiencies. In fact, we will propose two new Tulczyjew triples for time-dependent mechanical systems. The first one is adapted to the restricted Hamiltonian formalism and the second one is adapted to the extended Hamiltonian formalism. In this approach, the role of symplectic structures in the original Tulczyjew triple is played by presymplectic and Poisson structures. Then, symplectic (anti-symplectic) isomorphisms are replaced by presymplectic and Poisson (anti-presymplectic and anti-Poisson) isomorphisms. In addition, Lagrangian submanifolds of symplectic manifolds are replaced by Lagrangian submanifolds of presymplectic and Poisson manifolds.

The new Tulczyjew triples follow the same philosophy as the original one (see sections 4, 5 and compare with properties (1), (2), (3), (4) and (5) of the original Tulczyjew triple).

We also remark that our second Tulczyjew's triple has some similarities with the Tulczyjew's triple proposed in [6] although the spaces involved in the definition of the triple in [6] are different and the structural applications between them are not isomorphisms.

The paper is structured as follows. In section 2, we recall some definitions and results on presymplectic and Poisson structures which we will be used in the rest of the paper. The Lagrangian and Hamiltonian formalisms in jet manifolds are discussed in section 3. Sections 4 and 5 contain the results of the paper. In fact, the restricted and extended Tulczyjew triples for time-dependent Lagrangian and Hamiltonian systems are presented in sections 4 and 5, respectively. The paper ends with our conclusions and a description of future research directions.

## 2. PRESYMPLECTIC AND POISSON MANIFOLDS

## 2.1. Presymplectic manifolds.

In this subsection, we will recall some well-known facts on presymplectic manifolds.

**Definition 2.1.** A *presymplectic structure* on a manifold  $M$  is a closed 2-form  $\omega$  on  $M$ .

If  $\omega$  is a presymplectic structure on  $M$ , the couple  $(M, \omega)$  is said to be a *presymplectic manifold*.

Moreover, for each  $x \in M$ , we will denote by  $Ker(\omega(x))$  the subspace of the tangent space  $T_x M$  to  $M$  at  $x$  given by

$$Ker(\omega(x)) = \{v \in T_x M / \iota_v \omega(x) = 0\}.$$

In other words,  $Ker(\omega(x)) = Ker[b_\omega|_{T_x M}]$ , where  $b_\omega : TM \rightarrow T^*M$  is the vector bundle morphism induced by  $\omega$ .

Note that

$$dim[Ker(\omega(x))] = dimM - rank(\omega(x)),$$

$rank(\omega(x))$  being the rank of the 2-form  $\omega(x)$  which is an even number.

In the particular case when

$$rank(\omega(x)) = dimM, \quad \text{for all } x \in M$$

then the dimension of  $M$  is even and the couple  $(M, \omega)$  is a symplectic manifold (see, for instance, [1]).

**Definition 2.2.** A submanifold  $C$  of dimension  $r$  of a presymplectic manifold  $(M, \omega)$  is said to be *Lagrangian* if  $i^* \omega = 0$  and

$$r = \frac{rank(\omega(x))}{2} + dim(T_x C \cap Ker(\omega(x))), \quad \text{for all } x \in C.$$

Here,  $i : C \rightarrow M$  is the canonical inclusion.

We remark that if  $(M, \omega)$  is a symplectic manifold, then one recovers the classical notion of a Lagrangian submanifold of a symplectic manifold (see, for instance, [1]).

The notion of a presymplectic map may be introduced in a natural way.

**Definition 2.3.** A smooth map  $\varphi : M \rightarrow N$  between two presymplectic manifolds  $(M, \omega_M)$  and  $(N, \omega_N)$  is said to be a *presymplectic map* if  $\varphi^* \omega_N = \omega_M$ .

Note that if  $b_{\omega_M} : TM \rightarrow T^*M$  and  $b_{\omega_N} : TN \rightarrow T^*N$  are the bundle maps induced by  $\omega_M$  and  $\omega_N$ , respectively, then  $\varphi$  is a presymplectic map if and only if

$$(b_{\omega_M})|_{T_x M} = (T_x \varphi)^* \circ (b_{\omega_N})|_{T_{\varphi(x)} N} \circ T_x \varphi,$$

for every  $x \in M$ .

**Remark 2.4.** A presymplectic structure  $\omega$  on a manifold  $M$  is a particular example of a Dirac structure (see [3]) in such a way that a Lagrangian submanifold of  $(M, \omega)$  is also a Lagrangian submanifold for the Dirac structure on  $M$  which is induced by the presymplectic form  $\omega$  (see [18]). In addition, a presymplectic map is a backward Dirac map in the sense of Bursztyn et al (see [2]).

## 2.2. Poisson manifolds.

In this subsection, we will recall some well-known facts on Poisson manifolds (see, for instance, [7, 14]).

**Definition 2.5.** A *Poisson structure* on a manifold  $M$  is a 2-vector  $\Lambda$  on  $M$  such that  $[\Lambda, \Lambda] = 0$ , where  $[\cdot, \cdot]$  is the Schouten-Nijenhuis bracket.

If  $\Lambda$  is a Poisson structure on  $M$ , the couple  $(M, \Lambda)$  is said to be a *Poisson manifold*.

A Poisson structure induces a vector bundle morphism  $\Lambda^\sharp : T^*M \longrightarrow TM$  which is given by

$$\Lambda^\sharp(\alpha) = \Lambda(\alpha, -), \quad \text{for } \alpha \in T^*M.$$

Note that  $\Lambda^\sharp$  is a skew-symmetric map and, thus, the dimension of the subspace  $\Lambda^\sharp(T_x^*M)$  is even, for every  $x \in M$ . Moreover, if  $\Lambda^\sharp$  is a vector bundle isomorphism then the inverse morphism  $(\Lambda^\sharp)^{-1} : TM \longrightarrow T^*M$  is just the vector bundle isomorphism induced by a symplectic structure on  $M$ .

**Definition 2.6.** A submanifold  $C$  of a Poisson manifold  $(M, \Lambda)$  is said to be *Lagrangian* if

$$\Lambda(\alpha, \beta) = 0, \quad \text{for all } (\alpha, \beta) \in (\Lambda^\sharp)^{-1}(TC)$$

and

$$\dim(T_x C \cap \Lambda^\sharp(T_x^*M)) = \frac{\dim(\Lambda^\sharp(T_x^*M))}{2}, \quad \text{for all } x \in C.$$

We remark that in the particular case when the map  $\Lambda^\sharp : T^*M \longrightarrow TM$  is a vector bundle isomorphism, that is, the Poisson structure is induced by a symplectic structure on  $M$ , then one recovers the classical notion of a Lagrangian submanifold of a symplectic manifold.

**Definition 2.7.** A smooth map  $\varphi : M \longrightarrow N$  between two Poisson manifolds  $(M, \Lambda_M)$  and  $(N, \Lambda_N)$  is said to be a *Poisson map* if

$$[\Lambda^2(T_x \varphi)](\Lambda_M(x)) = \Lambda_N(\varphi(x)), \quad \text{for each } x \in M.$$

Note that  $\varphi$  is a Poisson map if and only if

$$(\Lambda_N^\sharp)_{|T_{\varphi(x)}N} = T_x \varphi \circ (\Lambda_M^\sharp)_{|T_x M} \circ (T_x \varphi)^*,$$

for each  $x \in M$ .

**Remark 2.8.** A Poisson structure  $\Lambda$  on a manifold  $M$  is a particular example of a Dirac structure (see [3]) in such a way that a Lagrangian submanifold of  $(M, \Lambda)$  is also a Lagrangian submanifold for the Dirac structure on  $M$  which is induced by the Poisson 2-vector  $\Lambda$  (see [18]). In addition, a Poisson map is a forward Dirac map in the sense of Bursztyn et al (see [2]).

## 3. LAGRANGIAN AND HAMILTONIAN FORMALISMS IN JET MANIFOLDS

In this section, we will recall some definitions and results about the Lagrangian and Hamiltonian formalisms of Classical Mechanics in jet manifolds (for more details, see for instance [5, 8, 10, 15]).

### 3.1. The Lagrangian formalism.

Let  $\pi : M \rightarrow \mathbb{R}$  be a fibration, where  $M$  is a manifold of dimension  $n+1$ .

Denote by  $J^1\pi$  the  $(2n+1)$ -dimensional manifold of 1-jets of local sections of  $\pi$ .  $J^1\pi$  is an affine bundle modelled over the vertical bundle  $V\pi$  of  $\pi$ . It can be shown that exists a canonical identification between  $J^1\pi$  and the subset of  $TM$  given by  $\{v \in TM/\eta(v) = 1\}$ , where  $\eta = \pi^*(dt)$ . Thus,  $J^1\pi$  is an embedded submanifold of  $TM$ . In the same way,  $V\pi$  is the vector subbundle of  $TM$  given by  $\{v \in TM/\eta(v) = 0\}$ .

If  $(t, q^i)$  are local coordinates on  $M$  which are adapted to the fibration  $\pi$ , then we can consider the corresponding local coordinates  $(t, q^i, \dot{q}^i)$  on  $J^1\pi$  and  $V\pi$ .

We will denote by  $\pi_{1,0} : J^1\pi \rightarrow M$  and  $\pi_1 : J^1\pi \rightarrow \mathbb{R}$  the canonical projections and by  $\eta_1$  the 1-form on  $J^1\pi$  given by  $\eta_1 = (\pi_1)^*(dt)$ .

Given the fibration  $\pi$ , a *Lagrangian function* is a function  $L \in C^\infty(J^1\pi)$ , that is,  $L : J^1\pi \rightarrow \mathbb{R}$ . Given two points  $x, y \in M$  we define the manifold of infinite piecewise differentiable local sections which connect  $x$  and  $y$  as

$$C^\infty(x, y) = \{c : [0, 1] \rightarrow M/c \text{ is a local section of } \pi, c(0) = x \text{ and } c(1) = y\}.$$

We define the functional  $\mathcal{J} : C^\infty(x, y) \rightarrow \mathbb{R}$  by

$$c \mapsto \mathcal{J}(c) = \int_0^1 L(j^1c(t))dt.$$

Here,  $j^1c : [0, 1] \rightarrow J^1\pi$  is the jet prolongation of the curve  $c$ .

The Hamilton principle states that a curve  $c \in C^\infty(x, y)$  is a motion of the Lagrangian system defined by  $L$  if and only if  $c$  is a critical point on  $\mathcal{J}$ , i.e.,  $d\mathcal{J}(c)(X) = 0$  for all  $X \in T_c C^\infty(x, y)$  which is equivalent to the condition

$$(3.1) \quad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = 0, \forall i.$$

In other words,  $c$  satisfies the Euler-Lagrange equations.

### 3.2. The Hamiltonian formalism.

Denote by  $V^*\pi$  the dual bundle to the vertical bundle to  $\pi$  and by  $\mu : T^*M \rightarrow V^*\pi$  the canonical projection. We have that  $T^*M$  is an affine bundle over  $V^*\pi$  of rank 1 modelled over the trivial vector bundle  $pr_1 : V^*\pi \times \mathbb{R} \rightarrow V^*\pi$  (an AV-bundle in the terminology of [5]).

In this setting, a *Hamiltonian section* is a section  $h : V^*\pi \rightarrow T^*M$  of  $\mu : T^*M \rightarrow V^*\pi$ .

If  $(t, q^i, p, p_i)$  (respectively,  $(t, q^i, p_i)$ ) are local coordinates on  $T^*M$  (respectively,  $V^*\pi$ ) we have that

$$\mu(t, q^i, p, p_i) = (t, q^i, p_i), \quad h(t, q^i, p_i) = (t, q^i, -H(t, q^i, p_i), p_i).$$

Denote by  $\Omega_M$  the canonical symplectic structure of  $T^*M$ . Then, we can obtain a cosymplectic structure  $(\Omega_h, \eta_1^*)$  on  $V^*\pi$ , where

$$\Omega_h = h^*\Omega_M \in \Omega^2(V^*\pi), \quad \eta_1^* = (\pi_1^*)^*(dt) \in \Omega^1(V^*\pi).$$

Here,  $\pi_1^* : V^*\pi \rightarrow \mathbb{R}$  is the canonical projection. Note that

$$\Omega_h = dq^i \wedge dp_i + dH \wedge dt, \quad \eta_1^* = dt.$$

Thus, we can construct the Reeb vector field of  $(\Omega_h, \eta_1^*)$ , which is characterized by the following conditions

$$\iota_{R_h} \Omega_h = 0, \quad \iota_{R_h} \eta_1^* = 1.$$

The local expression of  $R_h$  is

$$(3.2) \quad R_h = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i}$$

and, therefore, the integral curves of  $R_h$  are the solutions of the *Hamilton equations*:

$$(3.3) \quad \frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}, \quad \forall i.$$

This is *the restricted formalism* for time-dependent Hamiltonian Mechanics.

Next, we will present *the extended formalism*.

The AV-bundle  $\mu : T^*M \rightarrow V^*\pi$  is a principal  $\mathbb{R}$ -bundle. We will denote by  $V_\mu \in \mathfrak{X}(T^*M)$  the infinitesimal generator of the action of  $\mathbb{R}$  on  $T^*M$ . Then, there exists a one-to-one correspondence between the space  $\Gamma(\mu)$  of sections of  $\mu$  and the set  $\{F_h \in C^\infty(T^*M) / V_\mu(F_h) = 1\}$ . Thus, the Hamiltonian section  $h : V^*\pi \rightarrow T^*M$  induces a real function  $F_h \in C^\infty(T^*M)$  such that  $V_\mu(F_h) = 1$ . The local expression of  $F_h$  is

$$(3.4) \quad F_h(t, q^i, p, p_i) = p + H(t, q^i, p_i).$$

Note that  $V_\mu = \frac{\partial}{\partial p}$ .

**Remark 3.1.** We remark that  $dF_h$  is invariant under the action of  $\mathbb{R}$  on  $T^*M$  and, thus, it defines a connection 1-form on the principal  $\mathbb{R}$ -bundle  $\mu : T^*M \rightarrow V^*\pi$ .  $\diamond$

Now, we can consider the Hamiltonian vector field  $\mathcal{H}_{F_h}^{\Omega_M}$  of  $F_h$  with respect to the canonical symplectic structure  $\Omega_M$ . The local expression of  $\mathcal{H}_{F_h}^{\Omega_M}$  is

$$(3.5) \quad \mathcal{H}_{F_h}^{\Omega_M} = \frac{\partial}{\partial t} - \frac{\partial H}{\partial t} \frac{\partial}{\partial p} + \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i}.$$

So, it is clear that  $\mathcal{H}_{F_h}^{\Omega_M}$  is  $\mu$ -projectable over  $R_h$ .

In addition, the integral curves of  $\mathcal{H}_{F_h}^{\Omega_M}$  satisfy the following equations

$$(3.6) \quad \frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}, \quad i \in \{1, \dots, m\}$$

and, moreover,

$$(3.7) \quad \frac{dp}{dt} = -\frac{\partial H}{\partial t}$$

(3.6) are the Hamilton equations and using (3.7) we deduce that in time-dependent Mechanics the Hamiltonian energy is not, in general, a constant of the motion (for more details, see the following subsection 3.3).

### 3.3. The equivalence between the Lagrangian and Hamiltonian formalisms.

We are going to introduce the Legendre transformations for the restricted and extended formalisms.

The *extended Legendre transformation*  $Leg_L : J^1\pi \rightarrow T^*M$  is given by  $(Leg_L)(v)(X) = L(v)\eta(X) + \frac{d}{dt}|_{t=0} L(v + t(X - \eta(X)v))$ , for  $v \in J^1\pi$  and  $X \in T_xM$ , with  $x = \pi_{1,0}(v)$ .

The *restricted Legendre transformation*  $leg_L : J^1\pi \rightarrow V^*\pi$  is defined by  $leg_L = \mu \circ Leg_L$ .

The local expression of these transformations is

$$(3.8) \quad Leg_L(t, q^i, \dot{q}^i) = (t, q^i, L - \dot{q}^i \frac{\partial L}{\partial \dot{q}^i}, \frac{\partial L}{\partial \dot{q}^i}), \quad leg_L(t, q^i, \dot{q}^i) = (t, q^i, \frac{\partial L}{\partial \dot{q}^i}).$$

The Lagrangian function  $L$  is said to be *regular* if and only if for each canonical coordinate system  $(t, q^i, \dot{q}^i)$  in  $J^1\pi$ , the Hessian matrix  $W_{ij} = (\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j})$  is non-singular.

From (3.8), we deduce that the following statements are equivalent:

- $L$  is regular.
- $leg_L : J^1\pi \rightarrow V^*\pi$  is a local diffeomorphism.
- $Leg_L : J^1\pi \rightarrow T^*M$  is an immersion.

The Lagrangian function  $L$  is said to be *hyperregular* if the restricted Legendre transformation is a global diffeomorphism. Then, we obtain a Hamiltonian section  $h = Leg_L \circ leg_L^{-1}$ . Moreover, if we consider the vector field  $R_L$  on  $J^1\pi$  given by

$$R_L(v) = (T_{leg_L(v)} leg_L^{-1})(R_h(leg_L(v))), \quad \text{for } v \in J^1\pi,$$

then  $R_L$  is a second order differential equation on  $J^1\pi$  and the trajectories of  $R_L$  are just the solutions of the Euler-Lagrange equations for  $L$ .  $R_L$  is called *the Euler-Lagrange vector field* for  $L$  and its local expression is

$$(3.9) \quad R_L = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + W^{ij} \left( \frac{\partial L}{\partial q^i} - \dot{q}^k \frac{\partial^2 L}{\partial \dot{q}^i \partial q^k} - \frac{\partial^2 L}{\partial t \partial \dot{q}^i} \right) \frac{\partial}{\partial \dot{q}^j},$$

where  $(W^{ij})$  is the inverse matrix of  $(W_{ij}) = (\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j})$ .

Using the above facts, we deduce that if  $\sigma : \mathbb{R} \rightarrow M$  is a solution of the Euler-Lagrange equations for  $L$  then  $leg_L \circ j^1\sigma : \mathbb{R} \rightarrow V^*\pi$  is a solution of the Hamilton equations for  $h$  and, conversely, if  $\tau : \mathbb{R} \rightarrow V^*\pi$  is a solution of the Hamilton equations for  $h$  then  $leg_L^{-1} \circ \tau : \mathbb{R} \rightarrow J^1\pi$  is a prolongation of a solution  $\sigma$  of the Euler-Lagrange equations for  $L$ .

#### 4. RESTRICTED TULCZYJEW'S TRIPLE

##### 4.1. The Lagrangian formalism.

Let  $N$  be a smooth manifold. We will denote by  $A_N : T(T^*N) \rightarrow T^*(TN)$  the canonical Tulczyjew diffeomorphism associated with the manifold  $N$  which is given locally by (see [17])

$$A_N(q^i, p_i; \dot{q}^i, \dot{p}_i) = (q^i, \dot{q}^i; \dot{p}_i, p_i).$$

Here  $(q^i)$  are local coordinates on  $N$  and  $(q^i, p_i)$  (respectively,  $(q^i, p_i; \dot{q}^i, \dot{p}_i)$ ) are the corresponding local coordinates on  $T^*N$  (respectively,  $T(T^*N)$ ).

Now, suppose that  $\pi : M \rightarrow \mathbb{R}$  is a fibration. Then, we may define a smooth map

$$\psi : T^*(J^1\pi) \rightarrow T(V^*\pi)$$

as follows. Let  $\alpha_v$  be a 1-form at the point  $v \in J^1\pi \subseteq TM$ . Then,

$$\psi(\alpha_v) = T\mu(A_M^{-1}(\tilde{\alpha}_v)),$$

with  $\tilde{\alpha}_v \in T_v^*(TM)$  such that  $\tilde{\alpha}_v|_{T_v(J^1\pi)} = \alpha_v$  and  $\mu : T^*M \rightarrow V^*\pi$  being the canonical projection.  $\psi$  is well-defined. In fact, the local expression of  $\psi$  is

$$(4.1) \quad \psi(t, q^i, \dot{q}^i; p_t, p_{q^i}, p_{\dot{q}^i}) = (t, q^i, p_{\dot{q}^i}; 1, \dot{q}^i, p_{q^i}).$$

In particular,  $\psi$  take values in the submanifold  $J^1\pi_1^*$  of  $T(V^*\pi)$ . Thus, we may consider the map

$$\psi : T^*(J^1\pi) \rightarrow J^1\pi_1^*.$$



It is clear that  $\psi$  is not a diffeomorphism (see (4.1)). In order to obtain a diffeomorphism, we consider the vector subbundle  $\langle \eta_1 \rangle$  over  $J^1\pi$  of  $T^*(J^1\pi)$  with rank 1 which is generated by the 1-form  $\eta_1$  and the quotient vector bundle  $T^*(J^1\pi)/\langle \eta_1 \rangle$  over  $J^1\pi$ . Local coordinates on  $T^*(J^1\pi)/\langle \eta_1 \rangle$  are  $(t, q^i, \dot{q}^i; p_{q^i}, p_{\dot{q}^i})$ . In addition, it is easy to prove that there exists a diffeomorphism  $\tilde{\psi} : T^*(J^1\pi)/\langle \eta_1 \rangle \rightarrow J^1\pi_1^*$  such that the following diagram is commutative

$$\begin{array}{ccc} T^*(J^1\pi) & \xrightarrow{\psi} & J^1\pi_1^* \\ \downarrow \pi_{T^*(J^1\pi)} & \nearrow \tilde{\psi} & \\ T^*(J^1\pi)/\langle \eta_1 \rangle & & \end{array}$$

where  $\pi_{T^*(J^1\pi)}$  is the canonical projection. In fact, the local expression of  $\tilde{\psi}$  is

$$\tilde{\psi}(t, q^i, \dot{q}^i; p_{q^i}, p_{\dot{q}^i}) = (t, q^i, p_{q^i}; \dot{q}^i, p_{\dot{q}^i}).$$

We will denote by  $A_\pi : J^1\pi_1^* \rightarrow T^*(J^1\pi)/\langle \eta_1 \rangle$  the inverse of  $\tilde{\psi}$ .  $A_\pi$  will be called *the canonical Tulczyjew diffeomorphism associated with the fibration  $\pi$* . The local expression of  $A_\pi$  is

$$(4.2) \quad A_\pi(t, q^i, p_i; \dot{q}^i, \dot{p}_i) = (t, q^i, \dot{q}^i; \dot{p}_i, p_i).$$

Let  $\Omega_{J^1\pi}$  be the canonical symplectic structure of  $T^*(J^1\pi)$  and  $\Lambda_{J^1\pi}$  be the corresponding Poisson structure.

In local coordinates  $(t, q^i, \dot{q}^i; p_t, p_{q^i}, p_{\dot{q}^i})$  on  $T^*(J^1\pi)$ , we have that

$$\begin{aligned} \Omega_{J^1\pi} &= dt \wedge dp_t + dq^i \wedge dp_{q^i} + d\dot{q}^i \wedge dp_{\dot{q}^i}, \\ \Lambda_{J^1\pi} &= \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial p_t} + \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_{q^i}} + \frac{\partial}{\partial \dot{q}^i} \wedge \frac{\partial}{\partial p_{\dot{q}^i}}. \end{aligned}$$

On the other hand, the vertical bundle of the canonical projection  $\pi_{T^*(J^1\pi)} : T^*(J^1\pi) \rightarrow T^*(J^1\pi)/\langle \eta_1 \rangle$  is generated by the vertical lift  $\eta_1^v$  of the 1-form  $\eta_1$  on  $J^1\pi$ . Note that

$$\eta_1^v = \frac{\partial}{\partial p_t}.$$

Thus, it is clear that

$$\mathcal{L}_{\eta_1^v} \Lambda_{J^1\pi} = 0$$

and, therefore,  $\Lambda_{J^1\pi}$  is  $\pi_{T^*(J^1\pi)}$ -projectable over a Poisson structure  $\tilde{\Lambda}_{J^1\pi}$  on  $T^*(J^1\pi)/\langle \eta_1 \rangle$ . In fact,

$$(4.3) \quad \tilde{\Lambda}_{J^1\pi} = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_{q^i}} + \frac{\partial}{\partial \dot{q}^i} \wedge \frac{\partial}{\partial p_{\dot{q}^i}}.$$

The corank of the Poisson structure  $\tilde{\Lambda}_{J^1\pi}$  is 1.

Now, consider the canonical Poisson structure  $\Lambda_{V^*\pi}$  on  $V^*\pi$ .  $\Lambda_{V^*\pi}$  is characterized by the following conditions

$$\Lambda_{V^*\pi}(d\widehat{X}, d\widehat{Y}) = -[\widehat{X}, \widehat{Y}], \quad \Lambda_{V^*\pi}(d(f \circ \pi_{1,0}^*), d\widehat{Y}) = Y(f) \circ \pi_{1,0}^*, \quad \Lambda_{V^*\pi}(d(f \circ \pi_{1,0}^*), d(g \circ \pi_{1,0}^*)) = 0$$

for  $X, Y$   $\pi$ -vertical vector fields on  $M$  and  $f, g \in C^\infty(M)$ , where  $\pi_{1,0}^* : V^*\pi \rightarrow M$  is the canonical projection. Here,  $\widehat{Z}$  is the linear function on  $V^*\pi$  which is induced by a  $\pi$ -vertical vector field  $Z$  on  $M$ , that is,

$$\widehat{Z}(\alpha) = \alpha(Z(\pi_{1,0}^*(\alpha))), \quad \forall \alpha \in V^*\pi.$$

If  $(t, q^i, p_i)$  are local coordinates on  $V^*\pi$  then

$$\Lambda_{V^*\pi} = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}.$$

Next, let  $\Lambda_{V^*\pi}^c$  be the complete lift of  $\Lambda_{V^*\pi}$  to  $T(V^*\pi)$ .  $\Lambda_{V^*\pi}^c$  is a Poisson structure on  $T(V^*\pi)$ . Note that the local expression of  $\Lambda_{V^*\pi}^c$  is

$$\Lambda_{V^*\pi}^c = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} + \frac{\partial}{\partial \dot{q}^i} \wedge \frac{\partial}{\partial p_i}.$$

On the other hand,  $J^1\pi_1^*$  is a embedded submanifold of  $T(V^*\pi)$ . In fact, if  $(t, q^i, p_i; \dot{t}, \dot{q}^i, \dot{p}_i)$  are local coordinates on  $T(V^*\pi)$  then the local equation defining  $J^1\pi_1^*$  as a submanifold of  $T(V^*\pi)$  is  $\dot{t} = 1$ .

Thus, the restriction  $\Lambda_{J^1\pi_1^*}$  to  $J^1\pi_1^*$  of  $\Lambda_{V^*\pi}^c$  is tangent to  $J^1\pi_1^*$  and, furthermore,  $\Lambda_{J^1\pi_1^*}$  defines a Poisson structure on  $J^1\pi_1^*$ .

If  $(t, q^i, p_i, \dot{q}^i, \dot{p}_i)$  are local coordinates on  $J^1\pi_1^*$ , we have that

$$(4.4) \quad \Lambda_{J^1\pi_1^*} = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} + \frac{\partial}{\partial \dot{q}^i} \wedge \frac{\partial}{\partial p_i}.$$

Therefore,  $\Lambda_{J^1\pi_1^*}$  is a Poisson structure of corank 1.

In addition, from (4.2), (4.3) and (4.4), we deduce

**Theorem 4.1.**  *$A_\pi$  is a Poisson isomorphism between the Poisson manifolds  $(J^1\pi_1^*, \Lambda_{J^1\pi_1^*})$  and  $(T^*(J^1\pi)/\langle\eta_1\rangle, \widetilde{\Lambda}_{J^1\pi})$ .*

The space  $\frac{T^*(J^1\pi)}{\langle\eta_1\rangle}$  is a vector bundle over  $J^1\pi$  with vector bundle projection  $\tilde{\pi}_{J^1\pi} : \frac{T^*(J^1\pi)}{\langle\eta_1\rangle} \rightarrow J^1\pi$ . Moreover, we can consider the jet prolongation  $j^1\pi_{1,0}^* : J^1\pi_1^* \rightarrow J^1\pi$  of the bundle map  $\pi_{1,0}^* : V^*\pi \rightarrow M$ . We have that

$$\tilde{\pi}_{J^1\pi}(t, q^i, \dot{q}^i; p_{q^i}, p_{\dot{q}^i}) = (t, q^i, \dot{q}^i).$$

Therefore, it is clear that  $\tilde{\pi}_{J^1\pi} \circ A_\pi = j^1\pi_{1,0}^*$ .

On the other hand, as we know,  $J^1\pi_1^*$  is an affine bundle over  $V^*\pi$  which is modelled over the vertical bundle to  $\pi_1^* : V^*\pi \rightarrow \mathbb{R}$ . We will denote by  $(\pi_1^*)_{1,0} : J^1\pi_1^* \rightarrow V^*\pi$  the affine bundle projection. It follows that  $(\pi_1^*)_{1,0}(t, q^i, p_i; \dot{q}^i, \dot{p}_i) = (t, q^i, p_i)$ .

The following commutative diagram illustrates the above situation

$$\begin{array}{ccc} T^*(J^1\pi)/\langle\eta_1\rangle & \xleftarrow{A_\pi} & J^1\pi_1^* \\ & \searrow \tilde{\pi}_{J^1\pi} & \swarrow j^1(\pi_{1,0}^*) \\ & & J^1\pi \end{array} \quad \begin{array}{c} \\ \\ \searrow (\pi_1^*)_{1,0} \\ \\ V^*\pi \end{array}$$

Now, suppose that  $L : J^1\pi \rightarrow \mathbb{R}$  is a Lagrangian function. Then, the differential of  $L$  induces a section of the vector bundle  $\tilde{\pi}_{J^1\pi} : T^*(J^1\pi)/\langle\eta_1\rangle \rightarrow J^1\pi$  which we will denote by

$$\widetilde{dL} : J^1\pi \rightarrow T^*(J^1\pi)/\langle\eta_1\rangle.$$

We have that

$$(4.5) \quad \widetilde{dL}(t, q^i, \dot{q}^i) = (t, q^i, \dot{q}^i; \frac{\partial L}{\partial q^i}, \frac{\partial L}{\partial \dot{q}^i}).$$

Furthermore, it is easy to prove that  $\widetilde{dL}(J^1\pi)$  is a Lagrangian submanifold of the Poisson manifold  $(T^*(J^1\pi)/\langle\eta_1\rangle, \widetilde{\Lambda}_{J^1\pi})$ . In fact,

$$(\widetilde{\Lambda}_{J^1\pi}^\sharp)^{-1}(T(\widetilde{dL}(J^1\pi))) = \left\langle \left\{ dp_{q^j} - \frac{\partial^2 L}{\partial q^i \partial q^j} dq^i - \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} d\dot{q}^i, \quad dp_{\dot{q}^k} - \frac{\partial^2 L}{\partial \dot{q}^k \partial q^l} dq^l - \frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^l} d\dot{q}^l \right\} \right\rangle$$

and

$$T(\widetilde{dL}(J^1\pi)) \cap \widetilde{\Lambda}_{J^1\pi}^\sharp \left( T^* \left( \frac{T^*(J^1\pi)}{\langle\eta_1\rangle} \right) \right) = \left\langle \left\{ \frac{\partial}{\partial q^i} + \frac{\partial^2 L}{\partial q^i \partial \dot{q}^j} \frac{\partial}{\partial p_{q^i}} + \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} \frac{\partial}{\partial p_{\dot{q}^i}}, \right. \right. \\ \left. \left. \frac{\partial}{\partial \dot{q}^k} + \frac{\partial^2 L}{\partial \dot{q}^k \partial q^l} \frac{\partial}{\partial p_{q^l}} + \frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^l} \frac{\partial}{\partial p_{\dot{q}^l}} \right\} \right\rangle$$

which implies that

$$\widetilde{\Lambda}_{J^1\pi}(\alpha, \beta) = 0, \quad \forall \alpha, \beta \in (\widetilde{\Lambda}_{J^1\pi}^\sharp)^{-1}(T(\widetilde{dL}(J^1\pi))),$$

$$\dim \left( T_{\widetilde{dL}(z)}(\widetilde{dL}(J^1\pi)) \cap \widetilde{\Lambda}_{J^1\pi}^\sharp \left( T^*_{\widetilde{dL}(z)} \left( \frac{T^*(J^1\pi)}{\langle\eta_1\rangle} \right) \right) \right) = \frac{\dim \left( \widetilde{\Lambda}_{J^1\pi}^\sharp \left( T^*_{\widetilde{dL}(z)} \left( \frac{T^*(J^1\pi)}{\langle\eta_1\rangle} \right) \right) \right)}{2} = 2n,$$

$\forall z \in J^1\pi$ .

Thus, since  $A_\pi$  is a Poisson isomorphism, we deduce that  $S_L = A_\pi^{-1}(\widetilde{dL}(J^1\pi))$  is a *Lagrangian submanifold* of the Poisson manifold  $(J^1\pi_1^*, \Lambda_{J^1\pi_1^*})$ .

On the other hand, we will denote by  $leg_L : J^1\pi \rightarrow V^*\pi$  the restricted Legendre transformation associated with  $L$ . Then, we have the following result.

**Theorem 4.2.** (1) *Let  $\sigma : \mathbb{R} \rightarrow M$  be a local section of  $\pi$ .  $\sigma$  is a solution of the Euler-Lagrange equations for  $L$  if and only if*

$$A_\pi^{-1} \circ \widetilde{dL} \circ j^1\sigma = j^1(leg_L \circ j^1\sigma).$$

(2) *The local equations which define to  $S_L$  as a Lagrangian submanifold of the Poisson manifold  $(J^1\pi_1^*, \Lambda_{J^1\pi_1^*})$  are just the Euler-Lagrange equations for  $L$ .*

**Proof:** A local computation, using (3.1), (3.8) and (4.2) proves the result.  $\square$

Figure 2 illustrates the above situation

#### 4.2. The Hamiltonian formalism.

Let  $\mu : T^*M \rightarrow V^*\pi$  be the AV-bundle associated with the fibration  $\pi : M \rightarrow \mathbb{R}$ .  $\mu$  defines a principal  $\mathbb{R}$ -bundle.

We will denote by  $V_\mu$  the infinitesimal generator of the action of  $\mathbb{R}$  on  $T^*M$  and by

$$b_{\Omega_{T^*M}} : T(T^*M) \rightarrow T^*(T^*M)$$

the vector bundle isomorphism (over the identity of  $T^*M$ ) induced by the canonical symplectic structure  $\Omega_{T^*M}$  of  $T^*M$ .

If  $(t, q^i, p, p_i; \dot{t}, \dot{q}^i, \dot{p}, \dot{p}_i)$  (respectively,  $(t, q^i, p, p_i; p_t, p_{q^i}, p_p, p_{p_i})$ ) are local coordinates on  $T(T^*M)$  (respectively,  $T^*(T^*M)$ ), we have that

$$b_{\Omega_{T^*M}}(t, q^i, p, p_i; \dot{t}, \dot{q}^i, \dot{p}, \dot{p}_i) = (t, q^i, p, p_i, -\dot{p}, -\dot{p}_i, \dot{t}, \dot{q}^i).$$

Now, if  $\widehat{V}_\mu : T^*(T^*M) \rightarrow \mathbb{R}$  is the linear function on  $T^*(T^*M)$  induced by the vector field  $V_\mu$ , we can consider the affine subbundle  $\widehat{V}_\mu^{-1}(1)$  of  $T^*(T^*M)$ , that is,

$$\widehat{V}_\mu^{-1}(1) = \{\gamma \in T^*(T^*M) / \gamma(V_\mu(\pi_{T^*M}(\gamma))) = 1\}$$

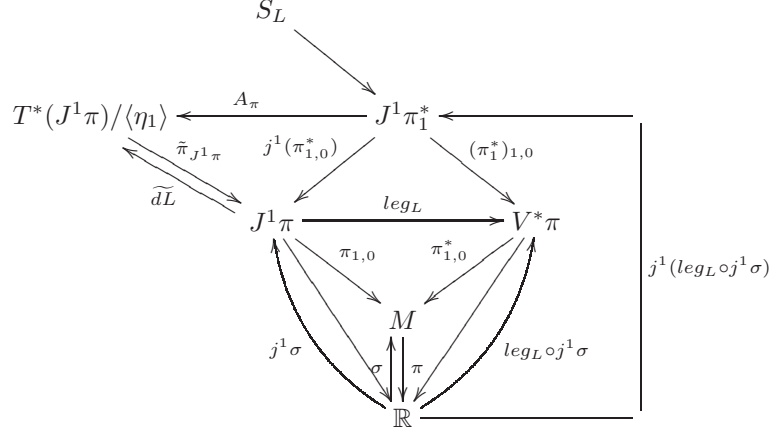


FIGURE 2. The Lagrangian formalism in the restricted Tulczyjew's triple

and the map  $\varphi : \widehat{V}_\mu^{-1}(1) \longrightarrow T(V^* \pi)$  defined by  $\varphi = T\mu \circ b_{\Omega_{T^*M}}^{-1}$ .

Since  $V_\mu = \frac{\partial}{\partial p}$  it follows that  $(t, q^i, p, p_i; p_t, p_{q^i}, p_{p_i})$  are local coordinates on  $\widehat{V}_\mu^{-1}(1)$  and, moreover,

$$\varphi(t, q^i, p, p_i; p_t, p_{q^i}, p_{p_i}) = (t, q^i, p_i; 1, p_{p_i}, -p_{q^i}).$$

Thus,  $\varphi$  takes values in  $J^1 \pi_1^*$  and we can consider the map

$$\varphi : \widehat{V}_\mu^{-1}(1) \longrightarrow J^1 \pi_1^*.$$

The local expression of this map is

$$\varphi(t, q^i, p, p_i; p_t, p_{q^i}, p_{p_i}) = (t, q^i, p_i; p_{p_i}, -p_{q^i}).$$

Therefore, it is clear that  $\varphi$  is not a diffeomorphism. In order to obtain a diffeomorphism, we will proceed as follows.

*First Step:* The cotangent lift of the action of  $\mathbb{R}$  on  $T^*M$  defines an action of  $\mathbb{R}$  on  $T^*(T^*M)$ . In fact, we have that

$$p' \cdot (t, q^i, p, p_i; p_t, p_{q^i}, p_p, p_{p_i}) = (t, q^i, p + p', p_i; p_t, p_{q^i}, p_p, p_{p_i})$$

for  $p' \in \mathbb{R}$  and  $(t, q^i, p, p_i; p_t, p_{q^i}, p_p, p_{p_i}) \in T^*(T^*M)$ .

It is obvious that the affine bundle  $\widehat{V}_\mu^{-1}(1)$  is invariant under this action. Consequently, the space of orbits of this action  $\frac{\widehat{V}_\mu^{-1}(1)}{\mathbb{R}}$  is an affine bundle over  $V^* \pi$  which is modelled over the vector bundle  $\frac{\widehat{V}_\mu^{-1}(0)}{\mathbb{R}}$ .

**Remark 4.3.** The affine bundle  $\frac{\widehat{V}_\mu^{-1}(1)}{\mathbb{R}}$  over  $V^* \pi$  is identified with the *phase bundle*  $P\mu$  associated with the AV-bundle  $\mu : T^*M \longrightarrow V^* \pi$ . The phase bundle associated with an AV-bundle was introduced in [5].  $\diamond$

Note that  $\widehat{V}_\mu^{-1}(0)$  is just the annihilator of the vertical bundle to  $\mu : T^*M \longrightarrow V^* \pi$  and that the quotient vector bundle  $\frac{\widehat{V}_\mu^{-1}(0)}{\mathbb{R}}$  is isomorphic to  $T^*(V^* \pi)$ . So, the affine bundle  $P\mu = \frac{\widehat{V}_\mu^{-1}(1)}{\mathbb{R}}$  is modelled over the vector bundle  $T^*(V^* \pi)$ .

Local coordinates on  $P\mu = \frac{\widehat{V}_\mu^{-1}(1)}{\mathbb{R}}$  are  $(t, q^i, p_i; p_t, p_{q^i}, p_{p_i})$ .

Moreover, there exists a smooth map  $\bar{\varphi} : P\mu \rightarrow J^1\pi_1^*$  such that the following diagram

$$\begin{array}{ccc} \widehat{V}_\mu^{-1}(1) & \xrightarrow{\varphi} & J^1\pi_1^* \\ \pi_{\widehat{V}_\mu^{-1}(1)} \downarrow & \nearrow \bar{\varphi} & \\ P\mu & & \end{array}$$

is commutative, where  $\pi_{\widehat{V}_\mu^{-1}(1)} : \widehat{V}_\mu^{-1}(1) \rightarrow P\mu$  is the canonical projection. The local expression of  $\bar{\varphi}$  is

$$\bar{\varphi}(t, q^i, p_i; p_t, p_{q^i}, p_{p_i}) = (t, q^i, p_i; p_{p_i}, -p_{q^i}).$$

Therefore,  $\bar{\varphi}$  is a surjective submersion.

*Second Step:* Let  $\pi_1^* : V^*\pi \rightarrow \mathbb{R}$  be the canonical projection. Then, the differential of  $\pi_1^*$  is a section of the vector bundle  $\pi_{V^*\pi} : T^*(V^*\pi) \rightarrow V^*\pi$ . Therefore, since  $P\mu$  is an affine bundle modelled over  $T^*(V^*\pi)$ , we may consider the quotient affine bundle  $P\mu/\langle d\pi_1^* \rangle$  over  $V^*\pi$ .  $P\mu/\langle d\pi_1^* \rangle$  is modelled over the quotient vector bundle  $T^*(V^*\pi)/\langle d\pi_1^* \rangle$ .

Local coordinates on  $P\mu/\langle d\pi_1^* \rangle$  are  $(t, q^i, p_i; p_{q^i}, p_{p_i})$ .

Furthermore, there exists a smooth map  $\tilde{\varphi} : P\mu/\langle d\pi_1^* \rangle \rightarrow J^1\pi_1^*$  such that the following diagram

$$\begin{array}{ccc} P\mu & \xrightarrow{\bar{\varphi}} & J^1\pi_1^* \\ \pi_{P\mu} \downarrow & \nearrow \tilde{\varphi} & \\ P\mu/\langle d\pi_1^* \rangle & & \end{array}$$

is commutative, where  $\pi_{P\mu} : P\mu \rightarrow P\mu/\langle d\pi_1^* \rangle$  is the canonical projection. The local expression of  $\tilde{\varphi}$  is

$$\tilde{\varphi}(t, q^i, p_i; p_{q^i}, p_{p_i}) = (t, q^i, p_i; p_{p_i}, -p_{q^i}).$$

Consequently,  $\tilde{\varphi}$  is a diffeomorphism.

We will denote by  $b_\pi : J^1\pi_1^* \rightarrow P\mu/\langle d\pi_1^* \rangle$  the inverse map of  $\tilde{\varphi}$ , that is,  $b_\pi = \tilde{\varphi}^{-1}$ . Then, we have that

$$(4.6) \quad b_\pi(t, q^i, p_i, \dot{q}^i, \dot{p}_i) = (t, q^i, p_i, -\dot{p}_i, \dot{q}^i).$$

Note that  $b_\pi$  is an affine bundle isomorphism over the identity of  $V^*\pi$ .

The following diagram illustrates the situation

$$\begin{array}{ccc} J^1\pi_1^* & \xrightarrow{b_\pi} & P\mu/\langle d\pi_1^* \rangle \\ & \searrow (\pi_1^*)_{1,0} & \swarrow \tilde{\pi}_{P\mu} \\ & & V^*\pi \end{array}$$

Here  $\tilde{\pi}_{P\mu}$  is the affine bundle projection.

$P\mu$  admits a canonical symplectic form  $\Omega_{P\mu}$  (see [5]). In fact, the local expression of  $\Omega_{P\mu}$  is

$$\Omega_{P\mu} = dt \wedge dp_t + dq^i \wedge dp_{q^i} + dp_i \wedge dp_{p_i}.$$

Let  $\Lambda_{P\mu}$  be the Poisson structure on  $P\mu$  associated with  $\Omega_{P\mu}$ . Then,

$$\Lambda_{P\mu} = \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial p_t} + \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_{q^i}} + \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial p_{p_i}}.$$

On the other hand, the vertical lift  $(d\pi_1^*)^v$  to  $P\mu$  of the 1-form  $d\pi_1^*$  on  $V^*\pi$  generates the vertical bundle to the canonical projection from  $P\mu$  on  $P\mu/\langle d\pi_1^* \rangle$ . Note that,

$$(d\pi_1^*)^v = \frac{\partial}{\partial p_t}.$$

Thus,  $\mathcal{L}_{(d\pi_1^*)^v} \Lambda_{P\mu} = 0$  and, therefore,  $\Lambda_{P\mu}$  is projectable to a Poisson structure  $\tilde{\Lambda}_{P\mu}$  on  $P\mu/\langle d\pi_1^* \rangle$ . The local expression of  $\tilde{\Lambda}_{P\mu}$  is

$$(4.7) \quad \tilde{\Lambda}_{P\mu} = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_{q^i}} + \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial p_{p_i}}.$$

Consequently, using (4.4), (4.6) and (4.7), we prove the following result

**Theorem 4.4.**  $b_\pi$  is anti-Poisson isomorphism between the Poisson manifolds  $(J^1\pi_1^*, \Lambda_{J^1\pi_1^*})$  and  $(P\mu/\langle d\pi_1^* \rangle, \tilde{\Lambda}_{P\mu})$ .

Now, let  $h : V^*\pi \rightarrow T^*M$  be a Hamiltonian section and  $F_h$  be the corresponding real function on  $T^*M$  such that  $V_\mu(F_h) = 1$ . Then, one may define a section of the affine bundle  $\widehat{V}_\mu^{-1}(1) \rightarrow T^*M$  as follows

$$\alpha \in T^*M \rightarrow dF_h(\alpha) \in \widehat{V}_\mu^{-1}(1).$$

This section is  $\mathbb{R}$ -equivariant. So, it induces a section  $dh : V^*\pi \rightarrow P\mu$  of the phase bundle  $P\mu$ . We will denote by  $\widetilde{dh} : V^*\pi \rightarrow P\mu/\langle d\pi_1^* \rangle$  the corresponding section of the affine bundle  $P\mu/\langle d\pi_1^* \rangle \rightarrow V^*\pi$ . If the local expression of  $h$  is

$$h(t, q^i, p_i) = (t, q^i, -H(t, q, p), p_i),$$

we have that

$$(4.8) \quad \widetilde{dh}(t, q^i, p_i) = (t, q^i, p_i; \frac{\partial H}{\partial q^i}, \frac{\partial H}{\partial p_i}).$$

Thus,

$$\begin{aligned} (\tilde{\Lambda}_{P\mu}^\#)^{-1}(T(\widetilde{dh}(V^*\pi))) &= \left\langle \left\{ dt, dp_{q^j} - \frac{\partial^2 H}{\partial q^i \partial q^j} dq^i - \frac{\partial^2 H}{\partial p_i \partial q^j} dp_i, dp_{p_j} - \frac{\partial^2 H}{\partial q^i \partial p_j} dq^i - \frac{\partial^2 H}{\partial p_i \partial p_j} dp_i \right\} \right\rangle, \\ (\tilde{\Lambda}_{P\mu}^\#)\left(T_{\widetilde{dh}(\alpha)}^*\left(\frac{P\mu}{\langle d\pi_1^* \rangle}\right)\right) \cap T_{\widetilde{dh}(\alpha)}(\widetilde{dh}(V^*\pi)) &= \left\langle \left\{ \left( \frac{\partial}{\partial q^j} + \frac{\partial^2 H}{\partial q^i \partial q^j} \frac{\partial}{\partial p_{q^i}} + \frac{\partial^2 H}{\partial q^j \partial p_i} \frac{\partial}{\partial p_{p_i}} \right) \Big|_{\widetilde{dh}(\alpha)}, \right. \right. \\ &\quad \left. \left. \left( \frac{\partial}{\partial p_j} + \frac{\partial^2 H}{\partial q^i \partial p_j} \frac{\partial}{\partial p_{q^i}} + \frac{\partial^2 H}{\partial p_i \partial p_j} \frac{\partial}{\partial p_{p_i}} \right) \Big|_{\widetilde{dh}(\alpha)} \right\} \right\rangle, \end{aligned}$$

$\forall \alpha \in V^*\pi$ .

Therefore,

$$\tilde{\Lambda}_{P\mu}(\alpha, \beta) = 0, \quad \forall \alpha, \beta \in (\tilde{\Lambda}_{P\mu}^\#)^{-1}(T(\widetilde{dh}(V^*\pi))),$$

$$\dim\left(T_{\widetilde{dh}(\alpha)}(\widetilde{dh}(V^*\pi)) \cap (\tilde{\Lambda}_{P\mu}^\#)\left(T_{\widetilde{dh}(\alpha)}\left(\frac{P\mu}{\langle d\pi_1^* \rangle}\right)\right)\right) = \frac{\dim\left(\tilde{\Lambda}_{P\mu}^\#\left(T_{\widetilde{dh}(\alpha)}^*\left(\frac{P\mu}{\langle d\pi_1^* \rangle}\right)\right)\right)}{2} = 2n,$$

$\forall \alpha \in V^*\pi$ .

This implies that  $\widetilde{dh}(V^*\pi)$  is a Lagrangian submanifold of the Poisson manifold  $\left(\frac{P\mu}{\langle d\pi_1^* \rangle}, \tilde{\Lambda}_{P\mu}\right)$ .

So, from Theorem 4.4, it follows that  $S_h = b_\pi^{-1}(\widetilde{dh}(V^*\pi))$  is also a Lagrangian submanifold of the Poisson manifold  $(J^1\pi_1^*, \Lambda_{J^1\pi_1^*})$ .

On the other hand, if  $R_h$  is the Reeb vector field of the cosymplectic structure  $(\Omega_h, \eta_1^*)$  on  $V^*\pi$  (see subsection 3.2) then, using (3.2), (4.6) and (4.8), we deduce that

$$S_h = R_h(V^*\pi).$$

Consequently, since the integral curves of  $R_h$  are the solutions of the Hamilton equations for the Hamiltonian section  $h$ , we obtain the following result.

**Theorem 4.5.** (1) *Let  $\tau : \mathbb{R} \rightarrow V^*\pi$  be a local section of the fibration  $\pi_1^* : V^*\pi \rightarrow \mathbb{R}$ . Then,  $\tau$  is a solution of the Hamilton equations for  $h$  if and only if*

$$b_\pi^{-1} \circ \widetilde{dh} \circ \tau = j^1\tau.$$

(2) *The local equations which define to  $S_h$  as a Lagrangian submanifold of the Poisson manifold  $(J^1\pi_1^*, \Lambda_{J^1\pi_1^*})$  are just the Hamilton equations for  $h$ .*

Figure 3 illustrates the situation

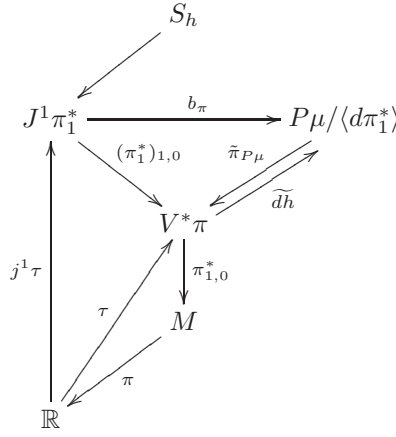


FIGURE 3. *The Hamiltonian formalism in the restricted Tulczyjew's triple*

#### 4.3. The equivalence between the Lagrangian and Hamiltonian formalism.

Let  $L : J^1\pi \rightarrow \mathbb{R}$  be an hyperregular Lagrangian function. Then, the restricted Legendre transformation  $leg_L : J^1\pi \rightarrow V^*\pi$  is a global diffeomorphism and we may consider the Euler-Lagrange vector field  $R_L$  on  $J^1\pi$ . Note that, since  $leg_L^*(\eta_1^*) = \eta_1$  and  $\eta_1(R_L) = 1$ , it follows that  $Tleg_L(R_L(J^1\pi)) \subseteq J^1\pi_1^*$ .

Moreover, using (3.8), (3.9), (4.2) and (4.5), we deduce

**Lemma 4.6.** *The following relation holds*

$$A_\pi \circ Tleg_L \circ R_L = \widetilde{dL}.$$

Now, denote by  $h : V^*\pi \rightarrow T^*M$  the Hamiltonian section associated with the hyperregular Lagrangian function  $L$ , that is,

$$h = Leg_L \circ leg_L^{-1},$$

$Leg_L : J^1\pi \rightarrow T^*M$  being the extended Legendre transformation. Then, using Lemma 4.6 and since  $Tleg_L \circ R_L = R_h \circ leg_L$ , we prove the following result.

**Theorem 4.7.** *The Lagrangian submanifolds  $S_L = A_\pi^{-1}(\widetilde{dL}(J^1\pi))$  and  $S_h = R_h(V^*\pi)$  of the Poisson manifold  $(J^1\pi_1^*, \Lambda_{J^1\pi_1^*})$  are equal.*

The previous result may be considered as the expression of the equivalence between the Lagrangian formalism and the restricted Hamiltonian formalism in the Lagrangian submanifold setting. Figure 4 illustrates the situation

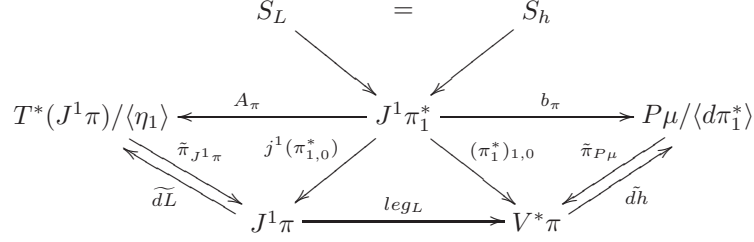


FIGURE 4. *The restricted Tulczyjew's triple for time-dependent Mechanics*

## 5. EXTENDED TULCZYJEW'S TRIPLE

### 5.1. The Lagrangian formalism.

Let  $\tilde{\pi}_M : T^*M \rightarrow \mathbb{R}$  be the fibration from  $T^*M$  on  $\mathbb{R}$ . We consider the space  $J^1\tilde{\pi}_M$  of 1-jets of local sections of  $\tilde{\pi}_M : T^*M \rightarrow \mathbb{R}$ . As we know, there exists a natural embedding from  $J^1\tilde{\pi}_M$  in  $T(T^*M)$ , which we will denote by  $j : J^1\tilde{\pi}_M \rightarrow T(T^*M)$ .

On the other hand, we can consider the 1-jet prolongation  $j^1\pi_M : J^1\tilde{\pi}_M \rightarrow J^1\pi$  of the bundle map  $\pi_M : T^*M \rightarrow M$ .

Then, we may define a smooth map

$$\widetilde{A}_\pi : J^1\tilde{\pi}_M \rightarrow T^*(J^1\pi)$$

as follows:

Let  $\tilde{z}$  be a point of  $J^1\tilde{\pi}_M$  and  $A_M : T(T^*M) \rightarrow T^*(TM)$  be the canonical Tulczyjew diffeomorphism. Then,  $A_M(j(\tilde{z})) \in T_v^*(TM)$ , with  $v \in J^1\pi$ . Indeed, if  $(t, q^i, p, p_i)$  are local coordinates on  $T^*M$ , we have that  $(t, q^i, p, p_i; \dot{q}^i, \dot{p}, \dot{p}_i)$  are local coordinates on  $J^1\tilde{\pi}_M$  and

$$A_M(j(\tilde{z})) = (t, q^i, 1, \dot{q}^i; \dot{p}, \dot{p}_i, p, p_i).$$

Thus,  $A_M(j(\tilde{z})) \in T_v^*(TM)$ , with  $v \in J^1\pi$ . In fact,  $v = (j^1\pi_M)(\tilde{z})$ .

Now, we define

$$\widetilde{A}_\pi(\tilde{z}) = A_M(j(\tilde{z}))|_{T_{j^1\pi_M(\tilde{z})}(J^1\pi)} \in T_{(j^1\pi_M)(\tilde{z})}^*(J^1\pi).$$

Therefore, it follows that

$$(5.1) \quad \widetilde{A}_\pi(t, q^i, p, p_i; \dot{q}^i, \dot{p}, \dot{p}_i) = (t, q^i, \dot{q}^i; \dot{p}, \dot{p}_i, p_i).$$

Consequently,  $\widetilde{A}_\pi$  is a surjective submersion.  $\widetilde{A}_\pi$  is called *the canonical Tulczyjew fibration associated with  $\pi$* .



**Remark 5.1.**  $\widetilde{A}_\pi$  is the bundle projection of a principal  $\mathbb{R}$ -bundle. In fact, if we consider the tangent lift of the principal action of  $\mathbb{R}$  on  $T^*M$ , we have an action of  $\mathbb{R}$  on  $T(T^*M)$ . The local expression of this action is

$$p' \cdot (t, q^i, p, p_i; \dot{t}, \dot{q}^i, \dot{p}, \dot{p}_i) = (t, q^i, p + p', p_i; \dot{t}, \dot{q}^i, \dot{p}, \dot{p}_i)$$

for  $p' \in \mathbb{R}$  and  $(t, q^i, p, p_i; \dot{t}, \dot{q}^i, \dot{p}, \dot{p}_i) \in T(T^*M)$ .

Thus, it is clear that the submanifold  $J^1\widetilde{\pi}_M$  of  $T(T^*M)$  is invariant under the previous action and, from (5.1), it follows that the fibers of  $\widetilde{A}_\pi$  are just the orbits of the action of  $\mathbb{R}$  on  $J^1\widetilde{\pi}_M$ .  $\diamond$

Next, we will denote by  $\Omega_M$  the canonical symplectic structure of  $T^*M$  and by  $\Omega_M^c$  the complete lift of  $\Omega_M$  to  $T(T^*M)$ .  $\Omega_M^c$  defines a symplectic structure on  $T(T^*M)$  and  $j^*(\Omega_M^c) = \Omega_{J^1\widetilde{\pi}_M}$  is a presymplectic form on  $J^1\widetilde{\pi}_M$ .

In fact, the local expressions of these forms are

$$\Omega_M^c = dt \wedge d\dot{p} + d\dot{t} \wedge dp + dq^i \wedge d\dot{p}_i + d\dot{q}^i \wedge dp_i,$$

and

$$(5.2) \quad \Omega_{J^1\widetilde{\pi}_M} = dt \wedge d\dot{p} + dq^i \wedge d\dot{p}_i + d\dot{q}^i \wedge dp_i.$$

Thus,  $\Omega_{J^1\widetilde{\pi}_M}$  is a presymplectic form of corank 1 and the kernel of  $\Omega_{J^1\widetilde{\pi}_M}$  is generated by the restriction to  $J^1\widetilde{\pi}_M$  of the complete lift  $(V_\mu)^c$  of  $V_\mu$  to  $T(T^*M)$ . Note that,

$$(5.3) \quad (V_\mu)^c = \frac{\partial}{\partial p} \quad \text{and} \quad \ker(T\widetilde{A}_\pi) = \langle \{(V_\mu)^c\} \rangle.$$

On the other hand, let  $\Omega_{J^1\pi}$  be the canonical symplectic structure of  $T^*(J^1\pi)$ . Then, if  $(t, q^i, \dot{q}^i; p_t, p_{q^i}, p_{\dot{q}^i})$  are local coordinates on  $T^*(J^1\pi)$ , we have that

$$(5.4) \quad \Omega_{J^1\pi} = dt \wedge dp_t + dq^i \wedge dp_{q^i} + d\dot{q}^i \wedge dp_{\dot{q}^i}.$$

Therefore, using (5.1), (5.2) and (5.4), we deduce the following result.

**Theorem 5.2.** *The canonical Tulczyjew fibration associated with  $\pi$  is a presymplectic map between the presymplectic manifolds  $(J^1\widetilde{\pi}_M, \Omega_{J^1\widetilde{\pi}_M})$  and  $(T^*(J^1\pi), \Omega_{J^1\pi})$ , that is,*

$$\widetilde{A}_\pi^*(\Omega_{J^1\pi}) = \Omega_{J^1\widetilde{\pi}_M}.$$

Now, let  $L : J^1\pi \rightarrow \mathbb{R}$  be a Lagrangian function. Then, it is well-known that  $dL(J^1\pi)$  is a Lagrangian submanifold of the symplectic manifold  $(T^*(J^1\pi), \Omega_{J^1\pi})$ . Consequently, using (5.3) and Theorem 5.2, we obtain that  $\widetilde{S}_L = \widetilde{A}_\pi^{-1}(dL(J^1\pi))$  also is a Lagrangian submanifold of the presymplectic manifold  $(J^1\widetilde{\pi}_M, \Omega_{J^1\widetilde{\pi}_M})$ .

Moreover, if  $\sigma$  is a local section of  $\pi : M \rightarrow \mathbb{R}$  then, from (3.8) and (5.1), we deduce that

$$\widetilde{A}_\pi \circ j^1(\text{Leg}_L \circ (j^1\sigma)(t)) = \left( t, q^i(t), E_L(j^1\sigma)(t), \frac{\partial L}{\partial \dot{q}^i}((j^1\sigma)(t)); \frac{dq^i}{dt}, \frac{d(E_L \circ j^1\sigma)}{dt}, \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \circ j^1\sigma \right) \right)$$

where  $E_L = L - \dot{q}^i \frac{\partial L}{\partial \dot{q}^i}$  and  $\text{Leg}_L : J^1\pi \rightarrow T^*M$  is the extended Legendre transformation (see (3.8)).

We remark that for a solution  $\sigma$  of the Euler-Lagrange equations for  $L$ , we have that

$$\frac{d(E_L \circ j^1\sigma)}{dt} = \frac{\partial L}{\partial t} \circ j^1\sigma.$$

Using the above facts, one may prove the following result.

**Theorem 5.3.** (1) A section  $\sigma : \mathbb{R} \rightarrow M$  is a solution of the Euler-Lagrange equations for  $L$  if and only if

$$dL \circ j^1 \sigma = \widetilde{A}_\pi \circ j^1(\text{Leg}_L \circ j^1 \sigma).$$

(2) The local equations which define to  $\widetilde{S}_L$  as a Lagrangian submanifold of the presymplectic manifold  $(J^1 \widetilde{\pi}_M, \Omega_{J^1 \widetilde{\pi}_M})$  are just the Euler-Lagrange equations for  $L$ .

Figure 5 illustrates the situation

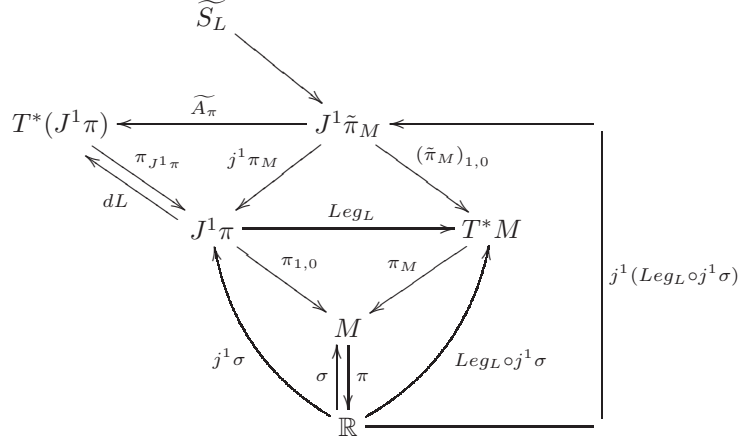


FIGURE 5. The Lagrangian formalism in the extended Tulczyjew's triple

## 5.2. The Hamiltonian formalism.

Let  $\widetilde{\pi}_M : T^*M \rightarrow \mathbb{R}$  be the fibration from  $T^*M$  on  $\mathbb{R}$ . Recall that  $J^1 \widetilde{\pi}_M$  is the space of 1-jets of local sections of  $\widetilde{\pi}_M : T^*M \rightarrow \mathbb{R}$  and that  $j$  is the natural embedding from  $J^1 \widetilde{\pi}_M$  in  $T(T^*M)$ .

Then, we may define a map

$$\widetilde{b}_\pi : J^1 \widetilde{\pi}_M \rightarrow T^*(T^*M)$$

as follows:

Let  $\widetilde{z}$  be a point of  $J^1 \widetilde{\pi}_M$  and  $b_M : T(T^*M) \rightarrow T^*(T^*M)$  the vector bundle isomorphism (over the identity of  $T^*M$ ) induced by the canonical symplectic structure  $\Omega_M$  of  $T^*M$ . Then,  $\widetilde{b}_\pi(\widetilde{z}) = b_M(j(\widetilde{z})) \in T^*(T^*M)$ , with  $\alpha \in T^*M$ . In fact, if  $(t, q^i, p, p_i)$  are local coordinates on  $T^*M$ , we have that  $(t, q^i, p, p_i; \dot{q}^i, \dot{p}, \dot{p}_i)$  are local coordinates on  $J^1 \widetilde{\pi}_M$  and

$$\widetilde{b}_\pi(t, q^i, p, p_i; \dot{q}^i, \dot{p}, \dot{p}_i) = (t, q^i, p, p_i; -\dot{p}, -\dot{p}_i, 1, \dot{q}^i).$$

From the last equation, we observe that the map  $\widetilde{b}_\pi$  takes values on the affine subbundle  $\widehat{V}_\mu^{-1}(1)$  of  $T^*(T^*M)$ . For this reason, we can consider the map

$$\widetilde{b}_\pi : J^1 \widetilde{\pi}_M \rightarrow \widehat{V}_\mu^{-1}(1)$$

which in local coordinates is given by

$$(5.5) \quad \widetilde{b}_\pi(t, q^i, p, p_i; \dot{q}^i, \dot{p}, \dot{p}_i) = (t, q^i, p, p_i; -\dot{p}, -\dot{p}_i, \dot{q}^i).$$

Consequently,  $\widetilde{b}_\pi$  is a diffeomorphism.

**Remark 5.4.** If we consider the cotangent lift of the principal action of  $\mathbb{R}$  on  $T^*M$ , we have an action of  $\mathbb{R}$  on  $T^*(T^*M)$ . The local expression of this action is

$$p' \cdot (t, q^i, p, p_i; p_t, p_{q^i}, p_p, p_{p_i}) = (t, q^i, p + p', p_i; p_t, p_{q^i}, p_p, p_{p_i})$$

for  $p' \in \mathbb{R}$  and  $(t, q^i, p, p_i; p_t, p_{q^i}, p_p, p_{p_i}) \in T^*(T^*M)$ .

Thus, it is clear that the affine subbundle  $\widehat{V}_\mu^{-1}(1)$  of  $T^*(T^*M)$  is invariant under this action. Moreover, if we consider the natural action of  $\mathbb{R}$  on  $J^1\tilde{\pi}_M$  (see Remark 5.1) then, from (5.5), it follows that the diffeomorphism  $\widetilde{b}_\pi$  is equivariant.  $\diamond$

Next, we will denote by  $\Omega_{T^*M}$  the canonical symplectic structure on  $T^*(T^*M)$  and by  $\Phi_{\widehat{V}_\mu^{-1}(1)}$  the 2-form on  $\widehat{V}_\mu^{-1}(1)$  defined by

$$\Phi_{\widehat{V}_\mu^{-1}(1)} = i_{\widehat{V}_\mu^{-1}(1)}^*(\Omega_{T^*M}),$$

where  $i_{\widehat{V}_\mu^{-1}(1)}: \widehat{V}_\mu^{-1}(1) \rightarrow T^*(T^*M)$  is the canonical inclusion.

The local expressions of these forms are

$$\Omega_{T^*M} = dt \wedge dp_t + dq^i \wedge dp_{q^i} + dp \wedge dp_p + dp_i \wedge dp_{p_i},$$

and

$$(5.6) \quad \Phi_{\widehat{V}_\mu^{-1}(1)} = dt \wedge dp_t + dq^i \wedge dp_{q^i} + dp_i \wedge dp_{p_i}.$$

Thus,  $\Phi_{\widehat{V}_\mu^{-1}(1)}$  is a presymplectic form of corank 1 and the kernel of  $\Phi_{\widehat{V}_\mu^{-1}(1)}$  is generated by the restriction to  $\widehat{V}_\mu^{-1}(1)$  of the complete lift  $(V_\mu)^{*c}$  of  $V_\mu$  to  $T^*(T^*M)$ . Note that  $(V_\mu)^{*c}$  is the Hamiltonian vector field of the linear function  $\widehat{V}_\mu: T^*(T^*M) \rightarrow \mathbb{R}$  and, therefore,

$$(5.7) \quad (V_\mu)^{*c} = \frac{\partial}{\partial p}.$$

Consequently, using (5.2), (5.5) and (5.6), we deduce the following result.

**Theorem 5.5.**  $\widetilde{b}_\pi: J^1\tilde{\pi}_M \rightarrow \widehat{V}_\mu^{-1}(1)$  is an anti-presymplectic isomorphism between the presymplectic manifolds  $(J^1\tilde{\pi}_M, \Omega_{J^1\tilde{\pi}_M})$  and  $(\widehat{V}_\mu^{-1}(1), \Phi_{\widehat{V}_\mu^{-1}(1)})$ , that is,

$$\widetilde{b}_\pi^*(\Phi_{\widehat{V}_\mu^{-1}(1)}) = -\Omega_{J^1\tilde{\pi}_M}.$$

Now, let  $h: V^*\pi \rightarrow T^*M$  be a Hamiltonian section and  $F_h: T^*M \rightarrow \mathbb{R}$  be the corresponding real  $C^\infty$ -function on  $T^*M$  satisfying  $V_\mu(F_h) = 1$  (see section 3.2). Then, it is clear that  $dF_h(T^*M) \subseteq \widehat{V}_\mu^{-1}(1) \subseteq T^*(T^*M)$ .

Denote by  $i_{dF_h(T^*M)}: dF_h(T^*M) \rightarrow \widehat{V}_\mu^{-1}(1)$  the canonical inclusion.

Since  $dF_h(T^*M)$  is a Lagrangian submanifold of  $T^*(T^*M)$  and  $\Phi_{\widehat{V}_\mu^{-1}(1)} = i_{\widehat{V}_\mu^{-1}(1)}^*(\Omega_{T^*M})$ , we deduce that

$$(5.8) \quad i_{dF_h(T^*M)}^*(\Phi_{\widehat{V}_\mu^{-1}(1)}) = 0.$$

On the other hand, using (5.7), it is easy to prove that the restriction of  $(V_\mu)^{*c}$  to  $dF_h(T^*M)$  is tangent to  $dF_h(T^*M)$ . Thus,

$$(5.9) \quad \text{Ker}\left(\Phi_{\widehat{V}_\mu^{-1}(1)}(dF_h(\alpha))\right) \subseteq T_{dF_h(\alpha)}(dF_h(T^*M)), \quad \forall \alpha \in T^*M.$$

Therefore, from (5.8) and (5.9), we obtain that  $dF_h(T^*M)$  is a Lagrangian submanifold of the presymplectic manifold  $(\widehat{V}_\mu^{-1}(1), \Phi_{\widehat{V}_\mu^{-1}(1)})$  (see Definition 2.2).

Consequently, using Theorem 5.5, it follows that  $\widetilde{S}_h = \widetilde{b}_\pi^{-1}(dF_h(T^*M))$  is also a Lagrangian submanifold of the presymplectic manifold  $(J^1\widetilde{\pi}_M, \Omega_{J^1\widetilde{\pi}_M})$ .

Next, suppose that  $\tau : \mathbb{R} \rightarrow V^*\pi$  is a section of  $\pi_1^* : V^*\pi \rightarrow \mathbb{R}$ . Then, we have that

$$(dF_h \circ h \circ \tau)(\mathbb{R}) \subseteq \widehat{V}_\mu^{-1}(1)$$

(see (3.4)). Moreover, if  $\tau$  is a solution of the Hamilton equations then, from (3.3), we deduce that

$$\frac{d(H \circ \tau)}{dt} = \frac{\partial H}{\partial t} \circ \tau.$$

Using these facts and (5.5), we may prove the following result.

**Theorem 5.6.** (1) *A section  $\tau : \mathbb{R} \rightarrow V^*\pi$  is a solution of Hamilton equations for  $h$  if and only if*

$$\widetilde{b}_\pi \circ j^1(h \circ \tau) = dF_h \circ h \circ \tau.$$

(2) *The local equations which define  $\widetilde{S}_h$  as a Lagrangian submanifold of  $J^1\widetilde{\pi}_M$  are just the Hamilton equations for  $h$ .*

Figure 6 illustrates the situation

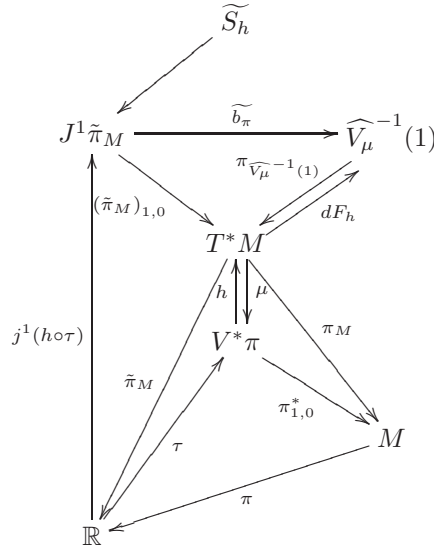


FIGURE 6. *The Hamiltonian formalism in the extended Tulczyjew's triple*

### 5.3. The equivalence between the Lagrangian and Hamiltonian formalism.

Let  $L : J^1\pi \rightarrow \mathbb{R}$  be an hyperregular Lagrangian function. Then, the restricted Legendre transformation  $leg_L : J^1\pi \rightarrow V^*\pi$  is a global diffeomorphism and we may consider the Euler-Lagrange vector field  $R_L$  on  $J^1\pi$ .

Moreover, using (3.8), (3.9) and (5.1), we deduce

**Lemma 5.7.** *The following relation holds*

$$\widetilde{A}_\pi \circ TLeg_L \circ R_L = dL,$$

where  $Leg_L : J^1\pi \rightarrow T^*M$  is the extended Legendre transformation.

Now, denote by  $h : V^*\pi \rightarrow T^*M$  the Hamiltonian section associated with the hyperregular Lagrangian function  $L$ , that is,

$$h = Leg_L \circ leg_L^{-1}.$$

**Theorem 5.8.** *The Lagrangian submanifolds  $\widetilde{S}_L = \widetilde{A}_\pi^{-1}(dL(J^1\pi))$  and  $\widetilde{S}_h = \widetilde{b}_\pi^{-1}(dF_h(T^*M))$  of the presymplectic manifold  $(J^1\widetilde{\pi}_M, \Omega_{J^1\widetilde{\pi}_M})$  are equal.*

Proof: Let  $\tilde{z}$  be a point of  $\widetilde{S}_L$ . Then, since  $\pi_{J^1\pi} \circ \widetilde{A}_\pi = j^1\pi_M$ , it follows that

$$\widetilde{A}_\pi(\tilde{z}) = dL((j^1\pi_M)(\tilde{z})).$$

Thus, using Lemma 5.7 and the fact that  $R_L$  and  $\mathcal{H}_{F_h}^{\Omega_M}$  are  $Leg_L$ -related, we deduce that

$$\widetilde{A}_\pi(\tilde{z}) = \widetilde{A}_\pi(\mathcal{H}_{F_h}^{\Omega_M}(Leg_L(j^1\pi_M)(\tilde{z}))) = \widetilde{A}_\pi(\widetilde{b}_\pi^{-1}(dF_h(Leg_L(j^1\pi_M)(\tilde{z}))).$$

Therefore, from Remark 5.1, we obtain that there exists a unique  $p \in \mathbb{R}$  such that

$$\widetilde{b}_\pi(p \cdot \tilde{z}) = dF_h(Leg_L((j^1\pi_M)(\tilde{z}))).$$

Here,  $\cdot$  denotes the action of  $\mathbb{R}$  on  $J^1\widetilde{\pi}_M$ .

Consequently, using Remarks 3.1 and 5.4, it follows that

$$\widetilde{b}_\pi(\tilde{z}) = dF_h((-p) \cdot Leg_L((j^1\pi_M)(\tilde{z}))) \in dF_h(T^*M).$$

So,  $\tilde{z} \in \widetilde{b}_\pi^{-1}(dF_h(T^*M)) = \widetilde{S}_h$ . This implies that  $\widetilde{S}_L \subseteq \widetilde{S}_h$ .

Proceeding in a similar way, one may prove that  $\widetilde{S}_h \subseteq \widetilde{S}_L$ .  $\square$

The previous result may be considered as the expression of the equivalence between the Lagrangian and extended Hamiltonian formalism in the Lagrangian submanifold setting.

Figure 7 illustrates the situation

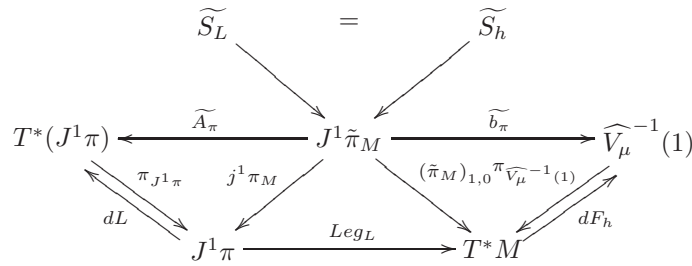
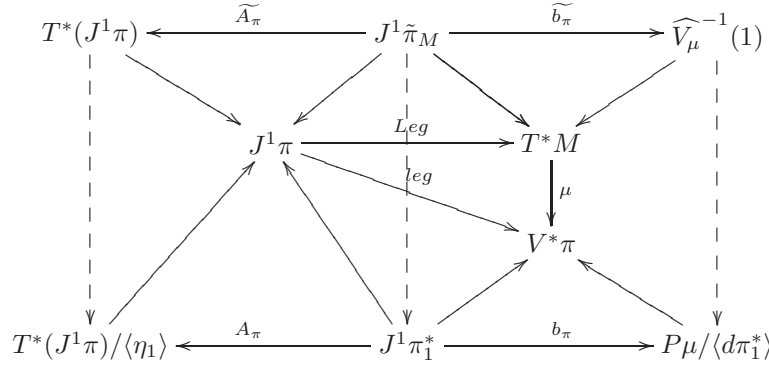


FIGURE 7. *The extended Tulczyjew's triple for time-dependent Mechanics*

Finally, Figure 8 describes both triples. The extended Tulczyjew triple is on the top of the diagram and the restricted Tulczyjew triple is on the bottom.

FIGURE 8. *The restricted and extended Tulczyjew's triples for time-dependent Mechanics*

## 6. CONCLUSIONS AND FUTURE WORK

Using the geometry of presymplectic and Poisson manifolds a new Tulczyjew triple for time-dependent Mechanics is discussed. More precisely, we present two Tulczyjew triples. The first one is adapted to the restricted Hamiltonian formalism for time-dependent mechanical systems and the second one is adapted to the extended Hamiltonian formalism. Our construction solves some problems and deficiencies of previous approaches.

It would be interesting to extend the ideas and results contained in this paper for classical field theories of first order. For this purpose, a suitable higher order generalization of a presymplectic (Poisson) structure must be used. This will be the subject of a forthcoming paper.

Other Tulczyjew triples for classical field theories of first order have been proposed by several authors (see [4, 12]).

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