Composition-Diamond Lemma for Non-associative Algebras over a Commutative Algebra*

Yuqun Chen, Jing Li and Mingjun Zeng
School of Mathematical Sciences, South China Normal University
Guangzhou 510631, P. R. China
yqchen@scnu.edu.cn
yulin_jj@yahoo.com.cn
dearmj@126.com

Abstract: We establish the Composition-Diamond lemma for non-associative algebras over a free commutative algebra. As an application, we prove that every countably generated non-associative algebra over an arbitrary commutative algebra K can be embedded into a two-generated non-associative algebra over K.

Key words: Gröbner-Shirshov basis; non-associative algebra; commutative algebra.

AMS Mathematics Subject Classification(2000): 16S15, 13P10, 17Dxx, 13Axx

1 Introduction

Gröbner bases and Gröbner-Shirshov bases theories were invented independently by A.I. Shirshov [23] for non-associative algebras and commutative (anti-commutative) non-associative algebras [21], for Lie algebras (explicitly) and associative algebras (implicitly) [22], for infinite series algebras (both formal and convergent) by H. Hironaka [19] and for polynomial algebras by B. Buchberger (first publication in [13]). Gröbner bases and Gröbner-Shirshov bases theories have been proved to be very useful in different branches of mathematics, including commutative algebra and combinatorial algebra, see, for example, the books [1, 12, 14, 15, 17, 18], the papers [2, 3, 4, 5, 16], and the surveys [6, 9, 10, 11].

It is well known that every countably generated non-associative algebra over a field k can be embedded into a two-generated non-associative algebra over k. This result follows from Gröbner-Shirshov bases theory for non-associative algebras by A.I. Shirshov [21].

Composition-Diamond lemmas for associative algebras over a polynomial algebra is established by A.A. Mikhalev and A.A. Zolotykh [20], for associative algebras over an associative algebra by L.A. Bokut, Yuqun Chen and Yongshan Chen [7], for Lie algebras over a polynomial algebra by L.A. Bokut, Yuqun Chen and Yongshan Chen [8]. In this paper, we establish the Composition-Diamond lemma for non-associative algebras over

*Supported by the NNSF of China (No.10771077, 10911120389) and the NSF of Guangdong Province (No. 06025062).

a polynomial algebra. As an application, we prove that every countably generated non-associative algebra over an arbitrary commutative algebra K can be embedded into a two-generated non-associative algebra over K, in particular, this result holds if K is a free commutative algebra.

2 Composition-Diamond lemma for non-associative algebras over a commutative algebra

Let k be a field, K a commutative associative $k-$algebra with unit, X a set and $K(X)$ the free non-associative algebra over K generated by X.

Let $[Y]$ denote the free abelian monoid generated by Y, X^* the free monoid generated by X and X^{**} the set of all non-associative words in X. Denote by

$$N = [Y]X^{**} = \{u = u^Y u^X | u^Y \in [Y], u^X \in X^{**}\}.$$

Let kN be a k- linear space spanned by N. For any $u = u^Y u^X,\ v = v^Y v^X \in N$, we define the multiplication of the words as follows

$$uv = u^Y v^Y u^X v^X \in N.$$

It is clear that kN is the free non-associative $k[Y]$-algebra generated by X. Such an algebra is denoted by $k[Y](X)$, i.e., $kN = k[Y](X)$. Clearly,

$$k[Y](X) = k[Y] \otimes k(X).$$

Now, we order the set $N = [Y]X^{**}$.

Let $>$ be a total ordering on X^{**}. Then $>$ is called monomial if

$$(\forall u, v, w \in X^{**})\ \ u > v \Rightarrow wu > wv \ \text{ and } \ \ uw > vw.$$

For example, the deg-lex ordering on X^{**} is monomial: $uv > u_1v_1$, if $deg(uv) > deg(u_1v_1)$, otherwise $u > u_1$ or $u = u_1, v > v_1$. Similarly, we define the monomial ordering on $[Y]$.

Suppose that both $>_X$ and $>_Y$ are monomial orderings on X^{**} and $[Y]$, respectively. For any $u = u^Y u^X, v = v^Y v^X \in N$, define

$$u > v \Leftrightarrow u^X >_X v^X \ \textit{or}\ (u^X = v^X \ \textit{and}\ u^Y >_Y v^Y).$$

It is obvious that $>$ is a monomial ordering on N in the sense of

$$(\forall u, v, w \in [Y]X^{**})\ \ u > v \Rightarrow wu > wv,\ uw > vw \ \text{ and } \ \ w^Y u > w^Y v.$$

We will use this ordering in this paper.

For any polynomial $f \in k[Y](X)$, f has a unique presentation of the form

$$f = \alpha_{\bar{f}} \bar{f} + \sum \alpha_i u_i,$$

where $\bar{f}, u_i \in [Y]X^{**}, \bar{f} > u_i, \alpha_{\bar{f}}, \alpha_i \in k$. $\bar{f}$ is called the leading term of f. f is monic if the coefficient of $\bar{f}$ is 1.

Let $\star \notin X$. By a $\star$-word we mean any expression in $[Y](X \cup \{\star\})^{**}$ with only one occurrence of $\star$. Let u be a $\star$-word and $s \in k[Y](X)$. Then we call $u|_s = u|_{\star \mapsto s}$ an s-word.

It is clear that for s-word $u|_s$, we can express $u|_s = u^Y(asb)$ for some $a, b \in X^*$.

Since $>$ is monomial on $[Y]X^{**}$, we have following lemma.

Lemma 2.1 *Let $s \in k[Y](X)$ be a non-zero polynomial. Then for any s-word $u|_s = u^Y(asb)$, $\overline{u^Y(asb)} = u^Y(a\bar{s}b)$.*

Now, we give the definition of compositions.

Definition 2.2 *Let f and g be monic polynomials of $k[Y](X)$, $w = w^Y w^X \in [Y]X^{**}$ and $a, b, c \in X^*$, where $w^Y = L(\bar{f}^Y, \bar{g}^Y) \triangleq L$ and $L(\bar{f}^Y, \bar{g}^Y)$ is the least common multiple of $\bar{f}^Y$ and $\bar{g}^Y$ in $k[Y]$. Then we have the following compositions.*

1. *X-inclusion*

If $w^X = \bar{f}^X = (a(\bar{g}^X)b)$, then

$$(f, g)_w = \frac{L}{\bar{f}^Y} f - \frac{L}{\bar{g}^Y}(a(g)b)$$

is called the composition of X-inclusion.

2. *Y-intersection only*

If $|\bar{f}^Y| + |\bar{g}^Y| > |w^Y|$ and $w^X = (a(\bar{f}^X)b(\bar{g}^X)c)$, then

$$(f, g)_w = \frac{L}{\bar{f}^Y}(a(f)b(\bar{g}^X)c) - \frac{L}{\bar{g}^Y}(a(\bar{f}^X)b(g)c)$$

is called the composition of Y-intersection only, where for $u \in [Y]$, $|u|$ means the degree of u.

w is called the ambiguity of the composition $(f, g)_w$.

Remark 1.In the case of Y-intersection only in Definition 2.2, $\bar{f}^X$ and $\bar{g}^X$ are disjoint.
Remark 2. By Lemma 2.1, we have $w > \overline{(f, g)_w}$.
Remark 3. In Definition 2.2, the compositions of f, g are the same as the ones in $k(X)$, if $Y = \emptyset$. If this is the case, we have only composition of X-inclusion.

Definition 2.3 *Let S be a monic subset of $k[Y](X)$ and $f, g \in S$. A composition $(f, g)_w$ is said to be* trivial modulo (S, w), *denoted by $(f, g)_w \equiv 0 \mod(S, w)$, if*

$$(f, g)_w = \sum_i \alpha_i u_i|_{s_i},$$

where each $s_i \in S$, $\alpha_i \in k$, $u_i|_{s_i}$ s_i-word and $w > u_i|_{\bar{s}_i}$.

Generally, for any $p, q \in k[Y](X)$, $p \equiv q \mod(S, w)$ if and only if $p - q \equiv 0 \mod(S, w)$.

S is called a Gröbner-Shirshov basis *in $k[Y](X)$ if all compositions of elements in S are trivial modulo S.*

If a subset S of $k[Y](X)$ is not a Gröbner-Shirshov basis then one can add to S all nontrivial compositions of polynomials of S and continue this process repeatedly so that we obtain a Gröbner-Shirshov basis S^c that contains S. Such process is called the Shirshov algorithm.

Lemma 2.4 *Let S be a Gröbner-Shirshov basis in $k[Y](X)$ and $s_1, s_2 \in S$. Let $u_1|_{s_1}$, $u_2|_{s_2}$ be s_1, s_2-words respectively. If $w = u_1|_{\overline{s_1}} = u_2|_{\overline{s_2}}$, then $u_1|_{s_1} \equiv u_2|_{s_2}$ $mod(S, w)$.*

Proof: Clearly, $w^Y = L(\bar{s_1}^Y, \bar{s_2}^Y) \cdot t = L \cdot t$ for some $t \in [Y]$.

There are three cases to consider.

Case 1. X-inclusion.

We may assume that $\bar{s_1}^X = (c(\bar{s_2}^X)d)$ for some $c, d \in X^*$ and $w^X = (a(\bar{s_1}^X)b) = (a(c(\bar{s_2}^X)d)b)$ for some $a, b \in X^*$. Thus,

$$\begin{aligned} u_1|_{s_1} - u_2|_{s_2} &= \frac{L \cdot t}{\bar{s_1}^Y}(a(s_1)b) - \frac{L \cdot t}{\bar{s_2}^Y}(a(c(s_2)d)b) \\ &= t \cdot (a(\frac{L}{\bar{s_1}^Y}s_1 - \frac{L}{\bar{s_2}^Y}(c(s_2)d))b) \\ &= t \cdot (a(s_1, s_2)_{w_1}b) \\ &\equiv 0 \qquad mod(S, w) \end{aligned}$$

where $w_1 = L\overline{s_1}^X$.

Case 2. Y-intersection only.

In this case, $w^X = (a(\bar{s_1}^X)b(\bar{s_2}^X)c)$, $a, b, c \in X^*$ and then

$$\begin{aligned} u_1|_{s_1} - u_2|_{s_2} &= \frac{L \cdot t}{\bar{s_1}^Y}(a(s_1)b(\bar{s_2}^X)c) - \frac{L \cdot t}{\bar{s_2}^Y}(a(\bar{s_1}^X)b(s_2)c) \\ &= t \cdot (s_1, s_2)_{w_1} \\ &\equiv 0 \qquad mod(S, w) \end{aligned}$$

where $w_1 = Lw^X$.

Case 3. Y-disjoint and X-disjoint.

In this case, $L = \bar{s_1}^Y \bar{s_2}^Y$ and $w^X = (a(\bar{s_1}^X)b(\bar{s_2}^X)c)$, $a, b, c \in X^*$. We have

$$\begin{aligned} u_1|_{s_1} - u_2|_{s_2} &= \frac{L \cdot t}{\bar{s_1}^Y}(a(s_1)b(\bar{s_2}^X)c) - \frac{L \cdot t}{\bar{s_2}^Y}(a(\bar{s_1}^X)b(s_2)c) \\ &= t \cdot (\frac{L}{\bar{s_1}^Y}(a(s_1)b(\bar{s_2}^X)c) - \frac{L}{\bar{s_2}^Y}(a(\bar{s_1}^X)b(s_2)c)) \\ &= t \cdot (\bar{s_2}^Y(a(s_1)b(\bar{s_2}^X)c) - \bar{s_1}^Y(a(\bar{s_1}^X)b(s_2)c)) \\ &= t \cdot ((a(s_1)b(\bar{s_2})c) - (a(\bar{s_1})b(s_2)c)) \\ &= t \cdot ((a(s_1)b(\bar{s_2})c) - (a(s_1)b(s_2)c) + (a(s_1)b(s_2)c) - (a(\bar{s_1})b(s_2)c)) \\ &= t \cdot ((a(s_1 - \bar{s_1})b(s_2)c) - (a(s_1)b(s_2 - \bar{s_2})c)) \\ &\equiv 0 \qquad mod(S, w) \end{aligned}$$

since $w = (a(\bar{s_1})b(\bar{s_2})c) > \overline{(a(s_1 - \bar{s_1})b(s_2)c)}$ and $w = (a(\bar{s_1})b(\bar{s_2})c) > \overline{(a(s_1)b(s_2 - \bar{s_2})c)}$.

This completes the proof. □

Lemma 2.5 *Let $S \subseteq k[Y](X)$ with each $s \in S$ monic and $Irr(S) = \{w \in [Y]X^{**} | w \neq u|_{\bar{s}},\ u|_s$ is an s-word, $s \in S\}$. Then for any $f \in k[Y](X)$,*

$$f = \sum_{u_i|_{\overline{s_i}} \leq \bar{f}} \alpha_i u_i|_{s_i} + \sum_{v_j \leq \bar{f}} \beta_j v_j,$$

where $\alpha_i, \beta_j \in k,\ u_i|_{s_i}\ s_i$-word, $s_i \in S$ and $v_j \in Irr(S)$.

Proof. Let $f = \sum_i \alpha_i u_i \in k[Y](X)$, where $0 \neq \alpha_i \in k$ and $u_1 > u_2 > \cdots$. If $u_1 \in Irr(S)$, then let $f_1 = f - \alpha_1 u_1$. If $u_1 \notin Irr(S)$, then there exists an s-word $u|_s$ such that $\bar{f} = u|_{\bar{s}}$. Let $f_1 = f - \alpha_1 u|_s$. In both cases, we have $\bar{f} > \bar{f_1}$. Then the result follows from the induction on $\bar{f}$. □

From the above lemmas, we reach the following theorem:

Theorem 2.6 *(Composition-Diamond lemma for $k[Y](X)$) Let $S \subseteq k[Y](X)$ with each $s \in S$ monic, $>$ the ordering on $[Y]X^{**}$ defined as before and $Id(S)$ the ideal of $k[Y](X)$ generated by S as $k[Y]$-algebra. Then the following statements are equivalent:*

(i) S *is a Gröbner-Shirshov basis in* $k[Y](X)$.

(ii) *If* $0 \neq f \in Id(S)$, *then* $\overline{f} = u|_{\overline{s}}$ *for some s-word* $u|_s,\ s \in S$.

(iii) $Irr(S) = \{w \in [Y]X^{**} | w \neq u|_{\bar{s}}, u|_s$ *is an s-word,* $s \in S\}$ *is a k-linear basis for the factor algebra* $k[Y](X|S) = k[Y](X)/Id(S)$.

Proof: $(i) \Rightarrow (ii)$. Suppose $0 \neq f \in Id(S)$. Then $f = \sum \alpha_i u_i|_{s_i}$ for some $\alpha_i \in k$, s_i-word $u_i|_{s_i}$, $s_i \in S$. Let $w_i = u_i|_{\overline{s_i}}$ and $w_1 = w_2 = \cdots = w_l > w_{l+1} \geq \cdots$. We will prove the result by using induction on l and w_1.

If $l = 1$, then the result is clear. If $l > 1$, then $w_1 = u_1|_{\overline{s_1}} = u_2|_{\overline{s_2}}$. Now, by (i) and Lemma 2.4, $u_1|_{s_1} \equiv u_2|_{s_2}\ \ mod(S, w_1)$. Thus,

$$\begin{aligned} \alpha_1 u_1|_{s_1} + \alpha_2 u_2|_{s_2} &= (\alpha_1 + \alpha_2) u_1|_{s_1} + \alpha_2 (u_2|_{s_2} - u_1|_{s_1}) \\ &\equiv (\alpha_1 + \alpha_2) u_1|_{s_1} \qquad mod(S, w_1). \end{aligned}$$

Therefore, if $\alpha_1 + \alpha_2 \neq 0$ or $l > 2$, then the result follows from the induction on l. For the case $\alpha_1 + \alpha_2 = 0$ and $l = 2$, we use the induction on w_1. Now the result follows.

$(ii) \Rightarrow (iii)$. By Lemma 2.5, $Irr(S)$ generates the factor algebra. Moreover, if $0 \neq h = \sum \beta_j u_j \in Id(S)$, $u_j \in Irr(S), u_1 > u_2 > \cdots$ and $\beta_1 \neq 0$, then $u_1 = \bar{h} = u|_{\bar{s}}$, a contradiction. This shows that $Irr(S)$ is a k-linear basis of the factor algebra.

$(iii) \Rightarrow (i)$. For any $f,\ g \in S$, since $k[Y]S \subseteq Id(S)$, we have $h = (f, g)_w \in Id(S)$. The result is trivial if $(f, g)_w = 0$. Assume that $(f, g)_w \neq 0$. Then, by Lemma 2.5, (iii) and by noting that $w > \overline{(f, g)_w} = \bar{h}$, we have $(f, g)_w \equiv 0\ \ mod(S, w)$.

This shows (i). □

Remark: Theorem 2.6 is the Composition-Diamond lemma for non-associative algebras when $Y = \emptyset$.

3 Applications

Let A be an arbitrary K-algebra and A be presented by generators X and defining relations S

$$A = K(X|S).$$

Let K have a presentation by generators Y and defining relations R

$$K = k[Y|R]$$

as a quotient algebra of the polynomial algebra $k[Y]$ over k.

Then with a natural way, as $k[Y]$-algebras, we have an isomorphism

$$k[Y|R](X|S) \to k[Y](X|S^l, Rx, x \in X), \sum (f_i + Id(R))u_i + Id(S) \mapsto \sum f_i u_i + Id(S'),$$

where $f_i \in k[Y]$, $u_i \in X^{**}$, $S' = S^l \cup \{gx | g \in R,\ x \in X\}$, $S^l = \{\sum f_i u_i \in k[Y](X) | \sum (f_i + Id(R))u_i \in S\}$. Then A has an expression

$$A = k[Y|R](X|S) = k[Y](X|S^l, gx,\ g \in R, x \in X).$$

Theorem 3.1 *Each countably generated non-associative algebra over an arbitrary commutative algebra K can be embedded into a two-generated non-associative algebra over K.*

Proof. Let the notation be as before. Let A be the non-associative algebra over $K = k[Y|R]$ generated by $X = \{x_i | i = 1, 2, \ldots\}$. We may assume that $A = k[Y|R](X|S)$ is defined as above. Then A can be presented as $A = k[Y](X|S^l, gx_i,\ g \in R,\ i = 1, 2, \ldots)$. By Shirshov algorithm, we can assume that, with the deg-lex ordering $>_Y$ on $[Y]$, R is a Gröbner-Shirshov basis in the free commutative algebra $k[Y]$. Let $>_X$ be the deg-lex ordering on X^{**}, where $x_1 > x_2 > \ldots$. We can also assume, by Shirshov algorithm, that with the ordering on $[Y]X^{**}$ defined as before, $S' = S^l \cup \{gx | g \in R,\ x \in X\}$ is a Gröbner-Shirshov basis in $k[Y](X)$.

Let $B = k[Y](X, a, b|S_1\}$ where S_1 consists of

$$\begin{aligned} f_1 &= S^l, \\ f_2 &= \{gx | g \in R,\ x \in X\}, \\ f_3 &= \{a(b^i) - x_i | i = 1, 2, \ldots\}, \\ f_4 &= \{ga | g \in R\}, \\ f_5 &= \{gb | g \in R\}. \end{aligned}$$

Clearly, B is a K-algebra generated by a, b. Thus, to prove the theorem, by using our Theorem 2.6, it suffices to show that with the ordering on $[Y](X \cup \{a, b\})^{**}$ as before, where $a > b > x_i,\ i = 1, 2, \ldots$, S_1 is a Gröbner-Shirshov basis in $k[Y](X, a, b)$.

Denote by $(i \wedge j)_{w_{ij}}$ the composition of the type f_i and type f_j with respect to the ambiguity w_{ij}. Since S' is a Gröbner-Shirshov basis in $k[Y](X)$, we need only to check all compositions related to the following ambiguities w_{ij}:

$1 \wedge 4,\quad w_{14} = L(\bar{f}^Y, \bar{g})(z_1(\bar{f}^X)z_2 a z_3)$;

$1\wedge 5,\quad w_{15}=L(\bar{f}^Y,\bar{g})(z_1(\bar{f}^X)z_2bz_3);$
$2\wedge 4,\quad w_{24}=L(\bar{g'},\bar{g})(z_1xz_2az_3);$
$2\wedge 5,\quad w_{25}=L(\bar{g'},\bar{g})(z_1xz_2bz_3);$
$3\wedge 4,\quad w_{34}=\bar{g}a(b^i);$
$3\wedge 5,\quad w_{35}=\bar{g}a(b^i);$
$4\wedge 1,\quad w_{41}=L(\bar{g},\bar{f}^Y)(z_1az_2(\bar{f}^X)z_3);$
$4\wedge 2,\quad w_{42}=L(\bar{g},\bar{g'})(z_1az_2xz_3);$
$4\wedge 4,\quad w_{44}=L(\overline{g_1},\overline{g_2})a;$
$4\wedge 5,\quad w_{45}=L(\bar{g},\bar{g'})(z_1az_2bz_3);$
$5\wedge 1,\quad w_{51}=L(\bar{g},\bar{f}^Y)(z_1bz_2(\bar{f}^X)z_3);$
$5\wedge 2,\quad w_{52}=L(\bar{g},\bar{g'})(z_1bz_2xz_3);$
$5\wedge 4,\quad w_{54}=L(\bar{g},\bar{g'})(z_1bz_2az_3);$
$5\wedge 5,\quad w_{55}=L(\overline{g_1},\overline{g_2})b;$
where $g,g',g_1,g_2\in R$, $f\in S^l$, $z_1,z_2,z_3\in(X\cup\{a,b\})^*$ and $(z_1v_1z_2v_2z_3)$ is some bracketing.

Now, we prove that all the compositions are trivial.

$1\wedge 4,\quad w_{14}=L(\bar{f}^Y,\bar{g})(z_1(\bar{f}^X)z_2az_3)$, where $f\in S^l,\ g\in R$.

We can write $\bar{f}^X=(uxv)$, where $u,v\in X^*$. Since $S'=\{S^l,Rx,x\in X\}$ is a Gröbner-Shirshov basis in $k[Y](X)$, we have $(f,gx)_w=\sum\alpha_iu_i|_{s_i}$, where $w=L(\bar{f}^Y,\bar{g})\bar{f}^X$, each $\alpha_i\in k,\ s_i\in S',\ u_i\in[Y]X^{**}$ and $w>u_i|_{\overline{s_i}}$. Then

$$\begin{aligned}
(1,4)_{w_{14}} &= \frac{L}{\bar{f}^Y}(z_1fz_2az_3)-\frac{L}{\bar{g}}(z_1(\bar{f}^X)z_2gaz_3)\\
&= \frac{L}{\bar{f}^Y}(z_1fz_2az_3)-\frac{L}{\bar{g}}(z_1(ugxv)z_2az_3)+\frac{L}{\bar{g}}(z_1(ugxv)z_2az_3)-\frac{L}{\bar{g}}(z_1(\bar{f}^X)z_2gaz_3)\\
&= (z_1(\frac{L}{\bar{f}^Y}f-\frac{L}{\bar{g}}(ugxv))z_2az_3)+\frac{L}{\bar{g}}g((z_1(uxv)z_2az_3)-(z_1(\bar{f}^X)z_2az_3))\\
&= (z_1(f,gx)_wz_2az_3)+\frac{L}{\bar{g}}g((z_1(\bar{f}^X)z_2az_3)-(z_1(\bar{f}^X)z_2az_3))\\
&= \sum\alpha_i(z_1u_i|_{s_i}z_2az_3)\\
&\equiv 0\qquad mod(S_1,w_{14}).
\end{aligned}$$

Similarly, $(1,5)_{w_{15}}\equiv 0,\ (4,1)_{w_{41}}\equiv 0,\ (5,1)_{w_{51}}\equiv 0.$

$2\wedge 4,\quad w_{24}=L(\bar{g'},\bar{g})(z_1xz_2az_3)$, where $g,g'\in R$.

If $|\bar{g'}|+|\bar{g}|>|L|$, then since R is a Gröbner-Shirshov basis in $k[Y]$, $(g',g)_w=(\frac{L}{\bar{g'}}g'-\frac{L}{\bar{g}}g)=\sum\alpha_iu_ih_i$, where $w=L(\bar{g'},\ \bar{g})$, each $\alpha_i\in k,u_i\in[Y],\ h_i\in R$ and $w>u_i\overline{h_i}$. Thus

$$\begin{aligned}
(2,4)_{w_{24}} &= \frac{L}{\bar{g'}}(z_1g'xz_2az_3)-\frac{L}{\bar{g}}(z_1xz_2gaz_3)\\
&= (\frac{L}{\bar{g'}}g'-\frac{L}{\bar{g}}g)(z_1xz_2az_3)\\
&= \sum\alpha_iu_ih_i(z_1xz_2az_3)\\
&= \sum\alpha_iu_i(z_1xz_2h_iaz_3)\\
&\equiv 0\qquad mod(S_1,w_{24}).
\end{aligned}$$

.

Similarly, $(2,5)_{w_{25}} \equiv 0$, $(4,2)_{w_{42}} \equiv 0$, $(4,5)_{w_{45}} \equiv 0$, $(5,2)_{w_{52}} \equiv 0$ and $(5,4)_{w_{54}} \equiv 0$.

$3 \wedge 4$, $w_{34} = \bar{g}a(b^i)$, where $g \in R$.

Let $g = \bar{g} + r \in R$. Then

$$\begin{aligned}(3,4)_{w_{34}} &= -\bar{g}x_i - ra(b^i) \\ &\equiv -\bar{g}x_i - rx_i \\ &\equiv gx_i \\ &\equiv 0 \quad mod(S_1, w_{34}).\end{aligned}$$

Similarly, $(3,5)_{w_{35}} \equiv 0$.

$4 \wedge 4$, $w_{44} = L(\overline{g_1}, \overline{g_2})a$, where $g_1, g_2 \in R$.

If $|\bar{g_1}| + |\bar{g_2}| > |L|$, then since R is a Gröbner-Shirshov basis in $k[Y]$, $(g_1, g_2)_w = (\frac{L}{\bar{g_1}}g_1 - \frac{L}{\bar{g_2}}g_2) = \sum \alpha_i u_i h_i$, where $w = L(\bar{g_1}, \bar{g_2})$, each $\alpha_i \in k, u_i \in [Y]$, $h_i \in R$ and $w > u_i\overline{h_i}$. Thus

$$\begin{aligned}(4,4)_{w_{44}} &= \frac{L}{\bar{g_1}}(g_1 a) - \frac{L}{\bar{g_2}}(g_2 a) \\ &= (\frac{L}{\bar{g_1}}g_1 - \frac{L}{\bar{g_2}}g_2)a \\ &= \sum \alpha_i u_i h_i a \\ &\equiv 0 \quad mod(S_1, w_{44}).\end{aligned}$$

.

If $|\bar{g_1}| + |\bar{g_2}| = |L|$, then

$$\begin{aligned}(4,4)_{w_{44}} &= \frac{L}{\bar{g_1}}(g_1 a) - \frac{L}{\bar{g_2}}(g_2 a) \\ &= (\bar{g_2}g_1 - \bar{g_1}g_2)a \\ &\equiv ((g_1 - \bar{g_1})g_2 - (g_2 - \bar{g_2})g_1)a \\ &\equiv 0 \quad mod(S_1, w_{44}).\end{aligned}$$

Similarly, $(5,5)_{w_{55}} \equiv 0$.

Now we have proved that S_1 is a Gröbner-Shirshov basis in $k[Y](X, a, b)$.

The proof is complete. □

A special case of Theorem 3.1 is the following corollary.

Corollary 3.2 *Every countably generated non-associative algebra over a free commutative algebra can be embedded into a two-generated non-associative algebra over a free commutative algebra.*

Acknowledgement. The authors would like to express their deepest gratitude to Professor L.A. Bokut for his kind guidance, useful discussions and enthusiastic encouragement.

References

[1] William W. Adams and Philippe Loustaunau, An introduction to Gröbner bases, Graduate Studies in Mathematics, Vol. 3, American Mathematical Society (AMS), 1994.

[2] G.M. Bergman, The diamond lemma for ring theory, *Adv. Math.*, **29**(1978), 178-218.

[3] L.A. Bokut, Insolvability of the word problem for Lie algebras, and subalgebras of finitely presented Lie algebras, *Izvestija AN USSR (mathem.)*, **36**(6)(1972), 1173-1219.

[4] L.A. Bokut, Imbeddings into simple associative algebras, *Algebra i Logika*, **15**(1976), 117-142.

[5] L.A. Bokut and Yuqun Chen, Gröbner-Shirshov basis for free Lie algebras: after A.I. Shirshov, *Southeast Asian Bull. Math.*, **31**(2007), 1057-1076.

[6] L.A. Bokut and Yuqun Chen, Gröbner-Shirshov bases: Some new results, Proceedings of the Second International Congress in Algebra and Combinatorics, World Scientific, 2008, 35-56.

[7] L.A. Bokut, Yuqun Chen and Yongshan Chen, Composition-Diamond lemma for tensor product of free algebras, Journal of Algebra, **323**(2010), 2520-2537.

[8] L.A. Bokut, Yuqun Chen and Yongshan Chen, Gröbner-Shirshov bases for Lie algebras over a commutative algebra, arXiv:1005.7682

[9] L.A. Bokut, Y. Fong, W.-F. Ke and P.S. Kolesnikov, Gröbner and Gröbner-Shirshov bases in algebra and conformal algebras, *Fundamental and Applied Mathematics*, **6**(3)(2000), 669-706.

[10] L.A. Bokut and P.S. Kolesnikov, Gröbner-Shirshov bases: from their incipiency to the present, *Journal of Mathematical Sciences*, **116**(1)(2003), 2894-2916.

[11] L.A. Bokut and P.S. Kolesnikov, Gröbner-Shirshov bases, conformal algebras and pseudo-algebras, *Journal of Mathematical Sciences*, **131**(5)(2005), 5962–6003.

[12] L.A. Bokut and G. Kukin, Algorithmic and Combinatorial algebra, Kluwer Academic Publ., Dordrecht, 1994.

[13] B. Buchberger, An algorithmical criteria for the solvability of algebraic systems of equations [in German], *Aequationes Math.*, 4, 374-383(1970).

[14] B. Buchberger, G.E. Collins, R. Loos and R. Albrecht, Computer algebra, symbolic and algebraic computation, Computing Supplementum, Vol.4, New York: Springer-Verlag, 1982.

[15] Bruno Buchberger and Franz Winkler, Gröbner bases and applications, London Mathematical Society Lecture Note Series, Vol.251, Cambridge: Cambridge University Press, 1998.

[16] Yuqun Chen and Qiuhui Mo, Artin-Markov normal form for Braid group, *Southeast Asian Bull. Math.*, **33**(2009), 403-419.

[17] David A. Cox, John Little and Donal O'Shea, Ideals, varieties and algorithms: An introduction to computational algebraic geometry and commutative algebra, Undergraduate Texts in Mathematics, New York: Spring-Verlag, 1992.

[18] David Eisenbud, Commutative algebra with a view toward algebraic geometry, Graduate Texts in Math., Vol.150, Berlin and New York: Springer-Verlag, 1995.

[19] H. Hironaka, Resolution of singulatities of an algebraic variety over a field if characteristic zero, I, II, *Ann. Math.*, **79**(1964), 109-203, 205-326.

[20] A. A. Mikhalev and A. A. Zolotykh, Standard Gröbner-Shirshov bases of free algebras over rings, I. Free associative algebras, International Journal of Algebra and Computation, 8(6)(1998), 689-726.

[21] A.I. Shirshov, Some algorithmic problem for ε-algebras, *Sibirsk. Mat. Z.*, **3**(1962), 132-137. (in Russian)

[22] A.I. Shirshov, Some algorithmic problem for Lie algebras, *Sibirsk. Mat. Z.*, **3**(2)(1962), 292-296 (in Russian); English translation in SIGSAM Bull., **33**(2)(1999), 3-6.

[23] Selected works of A.I. Shirshov, Eds Leonid A. Bokut, V. Latyshev, I. Shestakov, E. Zelmanov, Trs M. Bremner, M. Kochetov, Birkhäuser, Basel, Boston, Berlin, 2009.