THE STRUCTURE OF INFINITE 2-GROUPS WITH A UNIQUE 2-ELEMENT SUBGROUP

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Abstract. We prove that each infinite 2-group G with a unique 2-element subgroup is isomorphic either to the quasicyclic 2-group C_{2^∞} or to the infinite group of generalized quaternions Q_{2^∞}. The latter group is generated by the set $C_{2^\infty} \cup Q_8$ in the algebra of quaternions $\mathbb{H}$.

In this paper we describe the structure of 2-groups that contain a unique 2-element subgroup. For finite groups this was done in [2, 5.3.6]: Each finite 2-group with a unique 2-element subgroup is either cyclic or is a group of generalized quaternions.

Let us recall that a group G is called a *2-group* if each element $x \in G$ has order 2^k for some $k \in \mathbb{N}$. The *order* of an element x is the smallest number $n \in \mathbb{N}$ such that $x^n = 1$ where 1 denotes the neutral element of the group. By ω we denote the set of non-negative integer numbers.

For $n \in \omega$ denote by

$$C_{2^n} = \{z \in \mathbb{C} : z^{2^n} = 1\}$$

the cyclic group of order 2^n. The union

$$C_{2^\infty} = \bigcup_{n\in\mathbb{N}} C_{2^n} \subset \mathbb{C}$$

is called the *quasicyclic 2-group*.

The *group of quaternions* is the 8-element subgroup

$$Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$$

in the algebra of quaternions $\mathbb{H}$ (endowed with the operation of multiplication of quaternions). The real algebra $\mathbb{H}$ contains the field of complex numbers $\mathbb{C}$ as a subalgebra.

For $n \in \mathbb{N}$ the subgroup Q_{2^n} of $\mathbb{H}$ generated by the set $C_{2^{n-1}} \cup Q_8$ is called the *group of generalized quaternions*. For $n \geq 3$ this group has a presentatiom

$$\langle x, y \mid x^4 = 1,\ x^2 = y^{2^{n-2}},\ xyx^{-1} = y^{-1}\rangle.$$

The union

$$Q_{2^\infty} = \bigcup_{n\in\mathbb{N}} Q_{2^n}$$

will be called the *infinite group of generalized quaternions*. The quasicyclic group C_{2^∞} has index 2 in Q_{2^∞} and each element $x \in Q_{2^\infty} \setminus C_{2^\infty}$ has order 4.

The main result of this paper is the following extension of Theorem 5.3.6 [2]. It will be essentially used in [1] for describing of the structure of minimal left ideals of the superextensions of twinic groups.

Theorem 1. *Each 2-group with a unique 2-element subgroup is isomorphic to C_{2^n} or Q_{2^n} for some $n \in \mathbb{N} \cup \{\infty\}$.*

As we already know, for finite groups this theorem was proved in [2, 5.3.6]. Let us write this fact as a lemma for the future reference:

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Lemma 1. *Each finite 2-group with a unique 2-element subgroup is isomorphic to C_{2^n} or Q_{2^n} for some $n \in \mathbb{N}$.*

So, it remains to prove Theorem 1 for infinite groups. The abelian case is easy:

Lemma 2. *Each infinite abelian 2-group G with a unique 2-element subgroup is isomorphic to the quasicyclic 2-group C_{2^∞}.*

Proof. Let Z be the unique 2-element subgroup of G and $f : Z \to C_2$ be an isomorphism. Since the group C_{2^∞} is injective, by Baer's Theorem [2, 4.1.2], the homomorphism $f : Z \to C_2 \subset C_{2^\infty}$ extends to a homomorphism $\bar{f} : G \to C_{2^\infty}$. We claim that $\bar{f}$ is an isomorphism. Indeed, the kernel $\bar{f}^{-1}(1)$ of $\bar{f}$ is trivial since it is a 2-group and contains no element of order 2. So, $\bar{f}$ is inejective and then $\bar{f}(G)$ concides with C_{2^∞}, being an infinite subgroup of C_{2^∞}. □

The non-abelian case is a bit more difficult. For two elements a, b of a group G by $\langle a, b\rangle$ we shall denote the subgroup of G generated by the elements a and b. The following lemma gives conditions under which the subgroup $\langle a, b\rangle$ is finite.

Lemma 3. *The subgroup $\langle a, b\rangle$ generated by elements a, b of a group G is finite provided that the following conditions are satisfied:*

(1) $b^2 \in \langle a\rangle$;
(2) $a^2 b \in b \cdot \langle a\rangle$;
(3) *the elements a and ab have finite order.*

Proof. Since the element a has finite order and $b^2 \in \langle a\rangle$, the element b has finite order too. It is clear that the subgroup $H = \langle a, b\rangle$ generated by the elements a, b can be written as the countable union $H = \bigcup_{k\in\omega} H_k$ where $H_0 = \{1\}$ and H_k is the subset of elements of the form $a^{n_1} b^{m_1} \cdots a^{n_k} b^{m_k}$ where $n_i, m_i \geq 0$ for $i \leq k$.

For every $k \in \omega$ consider the set

$$\Pi_k = \{(ab)^i a^j, b(ab)^i a^j : 0 \leq i \leq k,\ j \geq 0\}$$

and observe that $\Pi_k \cdot a = \Pi_k$ and $\{1, a, ab\} \cdot \Pi_k \subset \Pi_{k+1}$.

Claim 1. *$H_k \subset \Pi_k$ for each $k \in \omega$.*

This claim will be proved by induction on k. The inclusion $H_0 = \{1\} \subset \Pi_0$ is trivial. Assume that for some number $k > 0$ the inclusion $H_{k-1} \subset \Pi_{k-1}$ has been proved.

In order to show that $H_k \subset \Pi_k$, take any element $x = a^{n_1} b^{m_1} a^{n_2} b^{m_2} \ldots a^{n_k} b^{m_k} \in H_k$. Since $b^2 \in \langle a\rangle$, we can assume that $m_1 \in \{0, 1\}$. Observe that the product $y = a^{n_2} b^{m_2} \cdots a^{n_k} b^{m_k} \in H_{k-1}$.

If $m_1 = 0$, then $x = a^{n_1+n_2} j^{m_2} \cdots a^{n_k} j^{m_k} \in H_{k-1} \subset \Pi_{k-1} \subset \Pi_k$.

Next, assume that $m_1 = 1$. It follows from $a^2 b \in b \cdot \langle a\rangle$ that for every $n \in \omega$ we get $a^{2n} b \in b \cdot \langle a\rangle$. Write the number n_1 as $n_1 = 2n + \varepsilon$ for some $n \in \omega$ and some $\varepsilon \in \{0, 1\}$. Then $a^{n_1} b = a^\varepsilon a^{2n} b = a^\varepsilon b a^m$ for some $m \in \omega$ and hence

$$x = a^{2n+\varepsilon} b y = a^\varepsilon b a^m y \in a^\varepsilon b a^m \cdot H_{k-1} = a^\varepsilon b \cdot H_{k-1} \subset a^\varepsilon b \cdot \Pi_{k-1} \subset \Pi_k.$$

This completes the proof of the claim.

Since the element ab has finite order, we see that the union $\bigcup_{k\in\omega} \Pi_k$ is finite and so is the subgroup $H = \bigcup_{k\in\omega} \subset \bigcup_{k\in\omega} \Pi_k$. □

The proof of Theorem 1 will be complete as soon as we prove that each infinite non-abelian 2-group G with a unique element of order 2 is isomorphic to Q_{2^∞}. Let 1 denote the neutral element of G and -1 denote the unique element of order 2 in G. It commutes with any other element of G.

Now we prove a series of lemmas and in the final Lemma 12 we shall prove that G is isomorphic to Q_{2^∞}.

Lemma 4. *The group G contains an element of order 8.*

Proof. In the opposite case $x^4 = 1$ for each element $x \in G$. The subgroup $Z = \{1, -1\}$ lies in the center of the group G and hence is normal. Since $x^2 \in Z$ for all $x \in G$, the quotient group G/Z is Boolean in the sense that $y^2 = 1$ for all $y \in G/Z$. Being Boolean, the group G/Z is abelian and locally finite (the latter means that each finite subset of G/Z generates a finite subgroup). Then the group G is locally finite too. Since G is infinite, it contains a finite subgroup H of order $|H| \geq 16$. By Lemma 1, H is isomorphic to C_{2^n} or Q_{2^n} for some $n \geq 4$. In both cases H contains an element of order 8, which contradicts our hypothesis. □

Let $F = \{x \in G : \ x^2 = -1\}$ denote the set of elements of order 4 in the group G.

Lemmas 1 and 2 imply:

Lemma 5. $|F \cap A| \leq 2$ *for each abelian subgroup $A \subset G$.*

Lemma 6. *For each $x \in G$ and $b \in F \setminus \langle x \rangle$ we get $bxb^{-1} = x^{-1}$.*

Proof. This lemma will be proved by induction on the order 2^k of the element x. The equality $bxb^{-1} = x^{-1}$ is true if x has oder ≤ 2 (in which case x is equal to 1 or -1).

Next, we check that the lemma is true if $k = 2$. In this case $b^2 = x^2 = -1 \in \langle x \rangle$ and $xb^2 = xx^2 = x^2x = b^2x \in b^2 \cdot \langle x \rangle$. By Lemma 3, the subgroup $\langle x, b \rangle$ is finite. Now we see that $\langle x, b \rangle$ is a finite 2-group with a single element of order 2, $\langle x, b \rangle$ is generated two elements of order 4 and contains two distinct cyclic subgroups of order 4. Lemma 1 implies that Q_8 is a unique group with these properties. Analyzing the structure of the quaternion group Q_8, we see that $bxb^{-1} = x^{-1}$ (because b and x generate two distinct cyclic subgroups of order 4).

Now assume that for some $n \geq 3$ we have proved that $bxb^{-1} = x^{-1}$ for any element $x \in G$ of order $2^k < 2^n$ such that $b \in F \setminus \langle x \rangle$. Let $x \in G$ be an element of order 2^n and $b \in F \setminus \langle x \rangle$. Then the element x^{-2} has order $2^{n-1} \geq 4$ and $b \in F \setminus \langle x^{-2} \rangle$. By the inductive hypothesis, $bx^{-2}b^{-1} = x^2$, which implies $x^2b = bx^{-2} \in b \cdot \langle x \rangle$. By Lemma 3, the subgroup $\langle x, b \rangle$ is finite. Since $bx^{-2} = x^2b \neq x^{-2}b$, the subgroup $\langle x, b \rangle$ is not abelian and by Lemma 1, it is isomorphic to Q_{2^m} for some m. Now the properties of the group Q_{2^m} imply that $bxb^{-1} = x^{-1}$. □

Lemma 7. *For each maximal abelian subgroup $A \subset G$ of cardinality $|A| > 4$ and each $b \in F \setminus A$, we get $F \setminus A = bA$.*

Proof. By Lemmas 1 and 2, the group A is isomorphic to C_{2^m} for some $3 \leq m \leq \infty$. Take any element $b \in F \setminus A$. To see that $bA \subset F \setminus A$, take any element $x \in A$. The inclusion $bx \in F \setminus A$ is trivial if x has order ≤ 2. So we assume that x has order ≥ 4. Since $b \notin A$, we see that $b \notin \langle x \rangle$. By Lemma 6, $bxb^{-1} = x^{-1}$. Then $bxbx = bxb^{-1}b^2x = x^{-1}(-1)x = -1$, which means that $bx \in F$. Since $x \in A$ and $b \notin A$, we get $bx \in G \setminus A$. Thus $bA \subset F \setminus A$.

To see that $F \setminus A \subset bA$, take any element $c \in F \setminus A$. By Lemma 6, $cxc^{-1} = x^{-1}$ for all $x \in A$. Then for each $x \in A$, $b^{-1}cxc^{-1}b = b^{-1}x^{-1}b = x$, which means that the element $b^{-1}c$ commutes with all elements of A, and thus $b^{-1}c \in A$ by the maximality of A. Then $c = b(b^{-1}c) \in bA$. □

Lemma 8. *For each maximal abelian subgroup $A \subset G$ and each $x \in G \setminus A$ with $x^2 \in A$ we get $x^2 = -1$.*

Proof. Assuming that $x^2 \neq -1$, we conclude that the element x^2 has oder ≥ 4. The maximality of $A \not\ni x$ guarantees that $A \neq \langle x^2 \rangle$. By Lemmas 1 or 2, A is cyclic or quasicyclic, which allows us to find an element $a \in A$ with $a^2 = x^2$. Observe that $x^2 = a^2 \in \langle a \rangle$ and $ax^2 = x^2a \in x^2 \cdot \langle a \rangle$. By Lemma 3, the subgroup $\langle a, x \rangle$ is finite and by Lemma 1, it is isomorphic to C_{2^n} or Q_{2^n} for some $n \in \mathbb{N}$. Observe that $\langle a \rangle$ and $\langle x \rangle$ are two distinct cyclic subgroups of order ≥ 8, which cannot happen in the groups C_{2^n} and Q_{2^n}. This contradiction completes the proof of the equality $x^2 = -1$. □

Lemma 9. $|F| \geq 10$.

Proof. By Lemma 4, the group G contains an element a of order 8. By Zorn's Lemma, the element a lies in some maximal abelian subgroup $A \subset G$. Since G is non-commutative, there is an element $b \in G \setminus A$. Replacing b by a suitable power b^{2^k}, we can additionally assume that $b^2 \in A$. By Lemma 8, $b^2 = -1$ and thus $b \in F \setminus A$. By Lemma 7, we get $bA = F \setminus A$, which implies that $|F| = |F \cap A| + |bA| \geq 2 + 8 = 10$. □

Lemma 10. *For any $n \geq 3$ the group G contains at most one cyclic subgroup of order 2^n.*

Proof. Assume that a, b be two elements generating distinct cyclic subgroups of order 2^n. First we show that these elements do not commute. Otherwise, the subgroup $\langle a, b\rangle$ is abelian and by Lemma 1 is cyclic and hence contains a unique subgroup of order 2^n. Let $A, B \subset G$ be maximal abelian subgroups containing the elements a, b, respectively.

Observe that the set
$$D = (F \cap A) \cup (F \cap B) \cup (F \cap B)a^{-1}$$
contains at most $2 \cdot 3 = 6$ elements. Since $|F| \geq 10$, we can find an element $c \in F \setminus D$. By Lemma 7, $ca \in cA = F \setminus A$ and $cb \in cB = F \setminus B$. The choice of the element c guarantees that $ca \notin F \cap B$ and hence $ca \in (F \setminus A) \cap (F \setminus B) \subset F \setminus B = cB$. Then $a \in B$ and a commutes with b, which is a contradiction. □

Let $A \subset G$ be a maximal abelian subgroup of cardinality ≥ 8. Such a subgroup exists by Zorn's Lemma and Lemma 4.

Lemma 11. *$G \setminus A = F \setminus A$ and A is a normal subgroup of index 2 in G.*

Proof. The inequality $G \setminus A \neq F \setminus A$ implies the existence of an element $x \in G \setminus A$ of order $2^n \geq 8$. Find a number $k < n$ such that $x^{2^k} \notin A$ but $x^{2^{k+1}} \in A$. By Lemma 8, $x^{2^{k+1}} = -1$ and thus $k = n - 2 \geq 1$. Then the element $z = x^{2^{k-1}}$ has order 8 and does not belong to A as $z^2 \notin A$. By Lemma 10, G contains a unique cyclic subgroup of order 8, which is a subgroup of A. Consequently, $z \in \langle z\rangle \subset A$ and this is a contradiction proving the equality $G \setminus A = F \setminus A$.

By Lemma 7, for any $b \in G \setminus A = F \setminus A$ we get $bA = F \setminus A = G \setminus A$, which means that A has index 2 in G and is normal. □

Lemma 12. *The group G is isomorphic to Q_{2^∞}.*

Proof. The subgroup A is infinite (as a subgroup of finite index in the infinite group G). By Lemma 2, there is an isomorphism $\varphi : A \to C_{2^\infty}$. Given any elements $b \in F \setminus A$ and $\bar\varphi(b) \in Q_{2^\infty} \setminus C_{2^\infty}$, extend φ to an isomorphism $\bar\varphi : G \to Q_{2^\infty}$ letting $\bar\varphi(bx) = \bar\varphi(b)\varphi(x)$ for $x \in A$. Using Lemma 6 it is easy to check that $\bar\varphi : G \to Q_{2^\infty}$ is a well-defined isomorphism between the groups G and Q_{2^∞}. □

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