A further note on the inverse nodal problem and Ambarzumyan problem for the p-Laplacian

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Abstract

In this note, we extend some results in a previous paper on the inverse nodal problem and Ambarzumyan problem for the p-Laplacian to periodic or anti-periodic boundary conditions, and to L^1 potentials.

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1 Introduction

Recently, we studied the p-Laplacian with C^1-potentials and solved the inverse nodal problem and Ambarzumyan problem for Dirichlet boundary conditions [7]. In this note, we want to extend the results to periodic or anti-periodic boundary conditions, and to L^1 potentials.

Consider the equation

$$-\left(y'^{(p-1)}\right)' = (p-1)(\lambda - q(x))y^{(p-1)}\ , \tag{1.1}$$

where $f^{(p-1)} = |f|^{p-1}\mathrm{sgn} f$. Assume that $q(1+x) = q(x)$ for $x \in \mathbb{R}$, then (1.1) can be coupled with periodic or anti-periodic boundary conditions respectively:

$$y(0) = y(1)\ , \quad y'(0) = y'(1) \tag{1.2}$$

or

$$y(0) = -y(1)\ , \quad y'(0) = -y'(1). \tag{1.3}$$

When $p = 2$, the above is the classical Hill's equation. It follows from Floquet theory that there are countably many interlacing periodic and anti-periodic eigenvalues of Hill's operator. However, Floquet theory does not work for the case $p \neq 2$. In 2001, Zhang [11] studied the properties of eigenvalues for $p > 1$ with L^1-potentials. He applied the rotation number function to define the minimal eigenvalue $\underline{\lambda}_n(q)$ and the maximal eigenvalue $\overline{\lambda}_n(q)$ corresponding to eigenfunctions having n zeros in $[0, 1)$, respectively. These numbers $\underline{\lambda}_n(q)$ and $\overline{\lambda}_n(q)$ are called rotational periodic eigenvalues and satisfy

(i) If $n \in \mathbb{N} \cup \{0\}$ is even, then $\underline{\lambda}_n(q)$ and $\overline{\lambda}_n(q)$ are eigenvalues of (1.1) and (1.2); if $n \in \mathbb{N}$ is odd, then $\underline{\lambda}_n(q)$ and $\overline{\lambda}_n(q)$ are eigenvalues of (1.1) and (1.3).

(ii) $\overline{\lambda}_0(q) < \underline{\lambda}_1(q) \leq \overline{\lambda}_1(q) < \underline{\lambda}_2(q) \leq \overline{\lambda}_2(q) < \cdots\cdots.$

Although the above properties are very similar to the linear case, it should be mentioned that the case for the p-Laplacian is much more complicated. For example, for the periodic or anti-periodic boundary conditions, there may exist an infinite sequence of variational eigenvalues and non-variational eigenvalues ([3]). In the same paper, the authors also showed that the minimal periodic eigenvalue is simple and variational, while the minimal anti-periodic eigenvalue is variational but may be not simple.

In 2008, Brown and Eastham [4] derived a sharp asymptotic expansion of eigenvalues of the p-Laplacian with locally integrable and absolutely continuous $(r-1)$ derivative potentials respectively. Below is a version of their theorem for periodic eigenvalues of the p-Laplacian (1.1), (1.2).

Theorem 1.1. *([4, Theorem 3.1]) Let q be 1-periodic and locally integrable in $(-\infty,\infty)$. Then the rotationally periodic eigenvalue $\lambda_{2n}=\underline{\lambda}_{2n}$, or $\overline{\lambda}_{2n}$ satisfies*

$$\lambda_{2n}^{1/p}=2n\widehat{\pi}+\frac{1}{p(2n\widehat{\pi})^{p-1}}\int_0^1 q(t)dt+o(\frac{1}{n^{p-1}}). \tag{1.4}$$

By a similar argument, the asymptotic expansion of the anti-periodic eigenvalue $\lambda_{2n-1}=\underline{\lambda}_{2n-1}$ or $\overline{\lambda}_{2n-1}$, which corresponds to the anti-periodic eigenfunction with $2n-1$ zeros in $[0,1)$, satisfies

$$\lambda_{2n-1}^{1/p}=(2n-1)\widehat{\pi}+\frac{1}{p((2n-1)\widehat{\pi})^{p-1}}\int_0^1 q(t)dt+o(\frac{1}{n^{p-1}}). \tag{1.5}$$

The inverse nodal problem is the problem of understanding the potential function through its nodal data. In 2006, some of us (C.-L.) [5] studied Hill's equation. We first made a translation of the interval by the first nodal length so that the periodic problem is reduced to a Dirichlet problem, and then solved the uniqueness, reconstruction and stability problems using the nodal set of periodic eigenfunctions.

We denote by $\{x_i^{(n)}\}_{i=0}^{n-1}$ the zeros of the eigenfunction corresponding to λ_n, and define the nodal length $\ell_i^{(n)}=x_{i+1}^{(n)}-x_i^{(n)}$ and $j=j_n(x)=\max\{i:\ x_i^{(n)}\le x\}$. Our main theorem is as follows.

Theorem 1.2. *Let $q\in L^1(0,1)$ be 1-periodic. Define $F_n(x)$ as the following:*

(a) For periodic boundary condition, let

$$F_{2n}(x)=p(2n\widehat{\pi})^p[(2n)\ell_j^{(2n)}-1]+\int_0^1 q(t)dt,$$

(b) For the anti-periodic boundary condition, let

$$F_{2n-1}(x) = p((2n-1)\widehat{\pi})^p[(2n-1)\ell_j^{(2n-1)} - 1] + \int_0^1 q(t)dt.$$

Then both $\{F_{2n}\}$ *and* $\{F_{2n-1}\}$ *converges to* q *pointwisely a.e. and in* $L^1(0,1)$.

Thus either one of the sequences $\{F_{2n}\}/\{F_{2n-1}\}$ will give the reconstruction formula for q. Note that here $q \in L^1(0,1)$. Furthermore, the map between the nodal space and the set of admissible potentials are homeomorphic after a partition (cf.[7]). The same idea also works for linear separated boundary value problems with integrable potentials.

Using the eigenvalue asymptotics above, the Ambarzumyan problems for the periodic and anti-periodic boundary conditions can also be solved.

Theorem 1.3. *Let* $q \in L^1(0,1)$ *be periodic of period* 1.

(a) If the spectrum of periodic eigenvalues problem (1.1), (1.2) contains $\{(2n\widehat{\pi})^p : n \in \mathbb{N} \cup \{0\}\}$ *and* 0 *is the least eigenvalue, then* $q = 0$ *on* $[0,1]$.

(b) If the spectrum of anti-periodic eigenvalue problem (1.1), (1.3) contains $\{((2n-1)\widehat{\pi})^p : n \in \mathbb{N}\}$; $\widehat{\pi}^p$ *is the least eigenvalue and* $\int_0^1 q(t)(S_p(\widehat{\pi}t)S_p'(\widehat{\pi}t)^{(p-1)})'dt = 0$, *then* $q = 0$ *on* $[0,1]$.

In section 2, we shall apply Theorem 1.1 to study on periodic and anti-periodic boundary conditions. In section 3, we shall deal with the case of linear separated boundary conditions.

The stability issue of the inverse nodal problem with L^1 potentials associated with perodic/antiperiodic as well as linear separated boundary conditions can also be proved. The proof goes in the same manner as in [7] and is so omitted.

2 Proof of main results

Fix $p > 1$ and assume that $q = 0$ and $\lambda = 1$. Then (1.1) becomes

$$-(y'^{(p-1)})' = (p-1)y^{(p-1)}.$$

Let S_p be the solution satisfying the initial conditions $S_p(0) = 0$, $S_p'(0) = 1$. It is well known that S_p and its derivative S_p' are periodic functions on $\mathbb{R}$ with period $2\widehat{\pi}$, where $\widehat{\pi} = \frac{2\pi}{p\sin(\frac{\pi}{p})}$. The two functions also satisfy the following identities (cf. [4, 7]).

Lemma 2.1. *(a)* $|S_p(x)|^p + |S_p'(x)|^p = 1$ *for any* $x \in \mathbb{R}$;

(b) $(S_p S_p'^{(p-1)})' = |S_p'|^p - (p-1)|S_p|^p = 1 - p|S_p|^p = (1-p) + p|S_p'|^p$.

Next we define a generalized Prüfer substitution using S_p and S_p':

$$y(x) = r(x)S_p(\lambda^{1/p}\theta(x)), \quad y'(x) = \lambda^{1/p} r(x) S_p'(\lambda^{1/p}\theta(x)) \ . \tag{2.1}$$

By Lemma 2.1, one obtains ([7])

$$\theta'(x) = 1 - \frac{q}{\lambda}|S_p(\lambda^{1/p}\theta(x))|^p \ . \tag{2.2}$$

Theorem 2.2. *In the periodic/antiperiodic eigenvalue problem, if* $q \in L^1(0,1)$ *be periodic of period* 1*, then*

$$q(x) = \lim_{n\to\infty} p\lambda_n \left(\frac{\lambda_n^{1/p}\ell_j^{(n)}}{\widehat{\pi}} - 1 \right) \ ,$$

pointwisely a.e. and in $L^1(0,1)$*, where* $j = j_n(x) = \max\{k : x_k^{(n)} \leq x\}$.

The proof below works for both even and odd n's, i.e. for both periodic and antiperiodic problems. Some of the arguments above are motivated by [6]. See also [8].

Proof. First, integrating (2.2) from $x_k^{(n)}$ to $x_{k+1}^{(n)}$ with $\lambda = \lambda_n$, we have

$$\begin{aligned} \frac{\widehat{\pi}}{\lambda_n^{1/p}} &= \ell_k^{(n)} - \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} \frac{q(t)}{\lambda_n}|S_p(\lambda_n^{1/p}\theta(t))|^p dt \ , \\ &= \ell_k^{(n)} - \frac{1}{p\lambda_n}\int_{x_k^{(n)}}^{x_{k+1}^{(n)}} q(t)dt - \frac{1}{\lambda_n}\int_{x_k^{(n)}}^{x_{k+1}^{(n)}} q(t)(|S_p(\lambda_n^{1/p}\theta(t))|^p - \frac{1}{p})dt \ . \end{aligned}$$

Hence,

$$\ell_k^{(n)} = \frac{\widehat{\pi}}{\lambda_n^{1/p}} + \frac{1}{p\lambda_n}\int_{x_k^{(n)}}^{x_{k+1}^{(n)}} q(t)dt + \frac{1}{\lambda_n}\int_{x_k^{(n)}}^{x_{k+1}^{(n)}} q(t)(|S_p(\lambda_n^{1/p}\theta(t))|^p - \frac{1}{p})dt \ . \tag{2.3}$$

and

$$p\lambda_n\left(\frac{\lambda_n^{1/p}\ell_k^{(n)}}{\widehat{\pi}}-1\right)=\frac{\lambda_n^{1/p}}{\widehat{\pi}}\int_{x_k^{(n)}}^{x_{k+1}^{(n)}}q(t)dt+\frac{p\lambda_n^{1/p}}{\widehat{\pi}}\int_{x_k^{(n)}}^{x_{k+1}^{(n)}}q(t)(|S_p(\lambda_n^{1/p}\theta(t))|^p-\frac{1}{p})dt\ . \tag{2.4}$$

Now, for $x\in(0,1)$, let $j=j_n(x)=\max\{k: x_k^{(n)}\le x\}$. Then $x\in[x_j^{(n)},x_{j+1}^{(n)})$ and, for large n,

$$[x_j^{(n)},x_{j+1}^{(n)})\subset B(x,\frac{2\widehat{\pi}}{\lambda_n^{1/p}})\ ,$$

where $B(t,\varepsilon)$ is the open ball centering t with radius ε. That is, the sequence of intervals $\{[x_j^{(n)},x_{j+1}^{(n)}): n$ is sufficiently large$\}$ shrinks to x nicely (cf. Rudin [9, p.140]). Since $q\in L^1(0,1)$ and $\frac{\lambda_n^{1/p}\ell_k^{(n)}}{\widehat{\pi}}=1+o(1)$, we have

$$h_n(x)\equiv\frac{\lambda_n^{1/p}}{\widehat{\pi}}\int_{x_j^{(n)}}^{x_{j+1}^{(n)}}q(t)dt=\frac{\lambda_n^{1/p}\ell_j^{(n)}}{\widehat{\pi}}\frac{1}{\ell_j^{(n)}}\int_{x_j^{(n)}}^{x_{j+1}^{(n)}}q(t)dt$$

converges to $q(x)$ pointwisely a.e. $x\in(0,1)$. Furthermore, since

$$|h_n(x)|\le\frac{\lambda_n^{1/p}}{\widehat{\pi}}\int_{x_j^{(n)}}^{x_{j+1}^{(n)}}|q(t)|dt\equiv g_n(x)\ ,$$

and

$$\int_0^1 g_n(t)dt=\sum_{k=0}^{n-1}\frac{\lambda_n^{1/p}\ell_k^{(n)}}{\widehat{\pi}}\int_{x_k^{(n)}}^{x_{k+1}^{(n)}}|q(t)|dt=(1+o(1))\|q\|_1\ ,$$

we have $h_n(t)\to q(t)$ in $L^1(0,1)$ by Lebesgue dominated convergence theorem. On the other hand, let $q_{k,n}\equiv\frac{1}{\ell_k^{(n)}}\int_{x_k^{(n)}}^{x_{k+1}^{(n)}}q(t)dt$. Then $q_{j,n}$ converges to q pointwisely a.e. $x\in(0,1)$. Let $\phi_n(t)=|S_p(\lambda_n^{1/p}\theta(t))|^p-\frac{1}{p}$. Then

$$\begin{aligned}T_n(x)&\equiv\frac{p\lambda_n^{1/p}}{\widehat{\pi}}\int_{x_j^{(n)}}^{x_{j+1}^{(n)}}q(t)\phi_n(t)dt\ ,\\&=\frac{p\lambda_n^{1/p}}{\widehat{\pi}}\int_{x_j^{(n)}}^{x_{j+1}^{(n)}}(q(t)-q_{j,n})\phi_n(t)dt+\frac{p\lambda_n^{1/p}}{\widehat{\pi}}\int_{x_j^{(n)}}^{x_{j+1}^{(n)}}q_{j,n}\phi_n(t)dt\ ,\\&\equiv A_n+B_n\ .\end{aligned}$$

By Lemma 2.1(b) and (2.2),

$$\begin{aligned} B_n &= \frac{p\lambda_n^{1/p} q_{j,n}}{\widehat{\pi}} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} \left(|S_p(\lambda_n^{1/p}\theta(t))|^p - \frac{1}{p}\right)\left(\theta'(t) + \frac{q(t)}{\lambda_n}|S_p(\lambda_n^{1/p}\theta(t))|^p\right) dt, \\ &= -\frac{pq_{j,n}}{\widehat{\pi}} S_p(\lambda_n^{1/p}\theta(t)) S_p'(\lambda_n^{1/p}\theta(t))^{(p-1)}\Big|_{x_j^{(n)}}^{x_{j+1}^{(n)}} + O(\lambda_n^{-1+1/p}) \ , \\ &= O(\lambda_n^{-1+1/p}) \ . \end{aligned}$$

Also,

$$\begin{aligned} |A_n| &\leq \frac{p\lambda_n^{1/p}}{\widehat{\pi}} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} |q(t) - q_{j,n}|||S_p(\lambda_n^{1/p}\theta(t))|^p - \frac{1}{p}|dt \ , \\ &\leq \frac{(p-1)\lambda_n^{1/p}}{\widehat{\pi}} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} |q(t) - q_{j,n}|dt \ , \end{aligned}$$

which converges to 0 pointwisely a.e. $x \in (0,1)$ because the sequence of intervals $\{[x_j^{(n)}, x_{j+1}^{(n)}) : n$ is sufficiently large$\}$ shrinks to x nicely. We conclude that $T_n(x) \to 0$ a.e. $x \in (0,1)$. Finally, applying Lebesgue dominated convergence theorem as above, $T_n(x) \to 0$ in $L^1(0,1)$. Hence the left hand side of (2.4) converges to q pointwisely a.e. and in $L^1(0,1)$. □

Proof of Theorem 1.2.

By the eigenvalue estimates (1.4) and (1.5), we have

$$p\lambda_{2n}\left(\frac{\lambda_{2n}^{1/p}\ell_{j_{2n}(x)}^{(2n)}}{\widehat{\pi}} - 1\right) = p(2n\widehat{\pi})^p(2n\ell_j^{(2n)} - 1) + 2n\ell_{j_{2n}(x)}^{(2n)} \int_0^1 q(t)dt + o(1) \ . \qquad (2.5)$$

Hence by Theorem 2.2 and the fact that $2n\ell_j^{(2n)} = 1 + o(1)$,

$$F_{2n}(x) \equiv p(2n\widehat{\pi})^p(2n\ell_j^{(2n)} - 1) + \int_0^1 q(t)dt$$

also converges to q pointwisely a.e. and in $L^1(0,1)$. The proof for (b) is the same. □

Proof of Theorem 1.3.

Here we only give the proof of (b). First, since all anti-periodic eigenvalues include $\{((2n-1)\widehat{\pi})^p : n \in \mathbb{N}\}$, we have, by (1.5), $\int_0^1 q(t)dt = 0$.

Moreover, $S_p(\widehat{\pi}x)$ satisfies anti-periodic boundary conditions. So by Lemma 2.1(b),

$$\int_0^1 |S_p'(\widehat{\pi}t)|^p dt - \frac{p-1}{p} = \int_0^1 q(t)|S_p(\widehat{\pi}t)|^p dt = \int_0^1 |S_p(\widehat{\pi}t)|^p dt - \frac{1}{p} = 0 \ .$$

Hence, by the variational principle, we have

$$\widehat{\pi}^p = \lambda_1 \leq \frac{\int_0^1 \widehat{\pi}^p |S_p'(\widehat{\pi}t)|^p dt + (p-1)\int_0^1 q(t)|S_p(\widehat{\pi}t)|^p dt}{(p-1)\int_0^1 |S_p(\widehat{\pi}t)|^p dt} = \widehat{\pi}^p \ .$$

This implies $S_p(\widehat{\pi}x)$ is the first eigenfunction. Therefore $q=0$ on $[0,1]$. □

3 Linear separated boundary conditions

Consider the one-dimensional p-Laplacian with linear separated boundary conditions

$$\begin{cases} y(0)S_p'(\alpha) + y'(0)S_p(\alpha) = 0 \\ y(1)S_p'(\beta) + y'(1)S_p(\beta) = 0 \end{cases} , \tag{3.1}$$

where $\alpha, \beta \in [0, \widehat{\pi})$. Letting λ_n be the nth eigenvalue whose associated eigenfunction has exactly $n-1$ zeros in $(0,1)$, the generalized phase θ_n as given in (2.2) satisfies

$$\theta_n(0) = \frac{-1}{\lambda_n^{1/p}} \widetilde{CT}_p^{-1}\left(-\frac{\widetilde{CT}_p(\alpha)}{\lambda_n^{1/p}}\right); \qquad \theta_n(1) = \frac{1}{\lambda_n^{1/p}}\left(n\widehat{\pi} - \widetilde{CT}_p^{-1}\left(-\frac{\widetilde{CT}_p(\beta)}{\lambda_n^{1/p}}\right)\right) , \tag{3.2}$$

where the function $CT_p(\gamma) := \frac{S_p(\gamma)}{S_p'(\gamma)}$ is an analogue of cotangent function, while $\widetilde{CT}_p(\gamma) := CT_p(\gamma)$ if $\gamma \neq 0$; and $\widetilde{CT}_p(\gamma) := 0$ otherwise. Also $\widetilde{CT}_p^{-1}$ stands for the inverse of $\widetilde{CT}_p$, taking values only in $[0, \widehat{\pi})$.

Let $\phi_n(x) = |S_p(\lambda_n^{1/p}\theta_n(x)|^p - \frac{1}{p}$, where . Below we shall state a general Riemann-Lebesgue lemma, which shows that $\int_0^1 \phi_n g \to 0$ for any $g \in L^1(0,1)$, when λ_n's are associated with a certain linear separated boundary conditions. In the case of periodic boundary conditions, Brown and Eastham [4] used a Fourier series expansion of ϕ_n where $\phi_n(\lambda_n^{1/p}\theta_n(x)) \approx \phi_n(\alpha + 2n\widehat{\pi}x)$ and apply Plancherel Theorem to show convergence.

Lemma 3.1. *Let f_n be uniformly bounded and integrable on $(0,1)$. Suppose for each n, there exists a partition $\{x_0^n = 0 < x_1^n < \cdots < x_n^n = 1\}$ such that $\Delta x_k^n = o(1)$, and*

$F_k^n(x) := \int_{x_k^n}^x f_n(t)\,dt$ *satisfies* $F_k^n(x) = O(\frac{1}{n})$ *for* $x \in (x_k^n, x_{k+1}^n)$ *and* $F_k^n(x_{k+1}^n) = o(\frac{1}{n})$ *uniformly in* $k = 1, \ldots, n-2$*, as* $n \to \infty$*. Then for any* $g \in L^1(0,1)$*,* $\int_0^1 g f_n \to 0$ *as* $n \to \infty$*.*

Proof. Take any $\epsilon > 0$, there is a C^1 function $\tilde{g}$ on $[0,1]$ such that $\int_0^1 |\tilde{g} - g| < \epsilon$. Let $|f_n|, |\tilde{g}| \le M$. Then

$$\int_0^1 g f_n = \int_0^1 (g - \tilde{g}) f_n + \int_0^1 \tilde{g} f_n,$$

where $|\int_0^1 (g - \tilde{g}) f_n| \le M\epsilon$. Also

$$\int_0^1 \tilde{g} f_n = \sum_{k=0}^{n-1} \int_{x_k^n}^{x_{k+1}^n} \tilde{g} f_n = \sum_{k=1}^{n-2} \left(\tilde{g}(x_{k+1}^n) F(x_{k+1}^n) - \int_{x_k^n}^{x_{k+1}^n} \tilde{g}' F_k^n \right) + o(1),$$

where

$$|\int_{x_k^n}^{x_{k+1}^n} \tilde{g}' F_k^n| = O(\frac{1}{n}) \int_{x_k^n}^{x_{k+1}^n} |\tilde{g}'| = o(\frac{1}{n}).$$

Therefore $\int_0^1 \tilde{g} f_n = o(1)$ as $n \to \infty$. □

Corollary 3.2. *Consider the p-Laplacian (1.1) with boundary conditions (3.1). Define* $\phi_n(x) = |S_p(\lambda_n^{1/p} \theta_n(x))|^p - \frac{1}{p}$*, then for any* $g \in L^1(0,1)$*,* $\int_0^1 \phi_n g \to 0$*.*

Proof. Since $\theta_n(0)$ and $\theta_n(1)$ are as given in (3.2), ϕ_n is uniformly bounded on $[0,1]$. Take x_k^n be such that $\theta(x_k^n) = \frac{k\widehat{\pi}}{\lambda_n^{1/p}}$. Also by integrating the phase equation (2.2), $\lambda_n^{1/p} = O(n)$, and

$$\Delta x_n = O(\frac{1}{\lambda_n^{1/p}}) = O(\frac{1}{n}).$$

Hence by Lemma 2.1(b) and (3.1), we have for $k = 1, \ldots, n-2$,

$$\begin{aligned}
\int_{x_k^n}^{x_{k+1}^n} \phi_n(x)\,dx &= \frac{-1}{p\lambda_n^{1/p}} \int_{x_k^n}^{x_{k+1}^n} \frac{1}{\theta_n'(x)} \frac{d}{dx} \left[S_p(\lambda_n^{1/p}\theta_n(x)) S_p'(\lambda_n^{1/p}\theta_n(x))^{(p-1)} \right] dx\ , \\
&= \frac{-1}{p\lambda_n^{1/p}} \left[S_p(\lambda_n^{1/p}\theta_n(x)) S_p'(\lambda_n^{1/p}\theta_n(x))^{(p-1)} \right]_{x_k^n}^{x_{k+1}^n} + O(\frac{1}{\lambda_n})\ , \\
&= O(\frac{1}{\lambda_n}) = o(\frac{1}{n})\ ,
\end{aligned}$$

since $S_p(k\widehat{\pi}) = 0$. It is also clear that $\int_{x_k^n}^x \phi_n(x)\,dx = O(\frac{1}{n})$. Thus we may apply Lemma 3.1 to complete the proof. □

Theorem 3.3. *When $q \in L^1(0,1)$, the eigenvalues λ_n of the Dirichlet p-Laplacian (1.1) satisfies, as $n \to \infty$,*

$$\lambda_n^{1/p} = n\widehat{\pi} + \frac{1}{p(n\widehat{\pi})^{p-1}} \int_0^1 q(t)dt + o(\frac{1}{n^{p-1}}) \ . \tag{3.3}$$

Furthermore, F_n converges to q pointwisely and in $L^1(0,1)$, where

$$F_n(x) := p(n\widehat{\pi})^p(n\ell_j^{(n)} - 1) + \int_0^1 q(t)\, dt.$$

Proof. Integrating (2.2) from 0 to 1, we have

$$\begin{aligned}\lambda_n^{1/p} &= n\widehat{\pi} + \frac{1}{p\lambda_n^{1-1/p}} \int_0^1 q(t)|S_p(\lambda_n^{1/p}\theta(t))|^p dt \ , \\ &= n\widehat{\pi} + \frac{1}{p\lambda_n^{1-1/p}} \int_0^1 q(t)dt + \frac{1}{p\lambda_n^{1-1/p}} \int_0^1 q(t)(|S_p(\lambda_n^{1/p}\theta(t))|^p - \frac{1}{p})dt \ .\end{aligned}$$

Then by Corollary 3.2, we have

$$\int_0^1 q(t)(|S_p(\lambda_n^{1/p}\theta(t))|^p - \frac{1}{p})dt = o(1) \ ,$$

for any $q \in L^1(0,1)$. Hence (3.3) holds. Furthermore, by Theorem 2.2, we can obtain the reconstruction formula with pointwise and L^1 convergence. □

Remark. In the same way, the Ambarzumyan Theorems for Neumann as well as Dirichlet boundary conditions as given in [7, Theorems 1.3 and 5.1] can also be extended to work for L^1 potentials. On the other hand, for general linear separated boundary problems (3.1),

$$\lambda_n^{1/p} = n_{\alpha\beta}\widehat{\pi} + \frac{(\widetilde{CT}_p(\beta))^{(p-1)} - (\widetilde{CT}_p(\alpha))^{(p-1)}}{(n_{\alpha\beta}\widehat{\pi})^{p-1}} + \frac{1}{p(n_{\alpha\beta}\widehat{\pi})^{p-1}} \int_0^1 q(x)\, dx + o(\frac{1}{n^{p-1}}), \tag{3.4}$$

where

$$n_{\alpha\beta} = \begin{cases} n & \text{if } \alpha = \beta = 0 \\ n - 1/2 & \text{if } \alpha > 0 = \beta \text{ or } \beta > 0 = \alpha \\ n - 1 & \alpha, \beta > 0 \end{cases}$$

This is because, after an integration of (2.2),

$$\theta_n(1) - \theta_n(0) = 1 - \frac{1}{\lambda_n} \int_0^1 q(x)|S_p(\lambda_n^{1/p}\theta(x))|^p\, dx + o(\frac{1}{\lambda_n}). \tag{3.5}$$

By (3.2), if $\alpha = 0$, then $\theta_n(0) = 0$. Similarly $\theta_n(1) = 0$ if $\beta = 0$. Now, let $y = CT_p^{-1}(x)$. Then $x = CT_p(y)$ and hence

$$y' = -\frac{1/x^2}{1+\frac{1}{|x|^p}} = \frac{-|x|^{p-2}}{1+|x|^p} = -|x|^{p-2}(1+O(|x|^p)),$$

when $|x|$ is sufficiently small. Since $y(0) = \frac{\widehat{\pi}}{2}$, we have

$$y(x) = \frac{\widehat{\pi}}{2} - \frac{x^{(p-1)}}{p-1} + O(x^{2p-1}) \ .$$

Therefore, when n is sufficiently large,

$$\theta_n(0) = \frac{\widehat{\pi}}{2\lambda_n^{1/p}} + \frac{(CT_p(\alpha))^{(p-1)}}{(p-1)\lambda_n^{(p-1)/p}} + O(\lambda_n^{\frac{1-2p}{p}}).$$

Similarly, when $\beta \neq 0$,

$$\theta_n(1) = \frac{(n-\frac{1}{2})\widehat{\pi}}{\lambda_n^{1/p}} + \frac{(CT_p(\beta))^{(p-1)}}{(p-1)\lambda_n^{(p-1)/p}} + O(\lambda_n^{\frac{1-2p}{p}}).$$

Hence (3.4) is valid. Furthermore, F_n converges to q pointwisely and in $L^1(0,1)$, where

$$F_n(x) := p(n_{\alpha\beta}\widehat{\pi})^p\left[(n_{\alpha\beta} + \frac{(\widetilde{CT}_p(\beta))^{(p-1)} - (\widetilde{CT}_p(\alpha))^{(p-1)}}{(n_{\alpha\beta}\widehat{\pi})^{p-1}})\ell_j^{(n)} - 1\right] + \int_0^1 q(t)\,dt.$$

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