Gröbner-Shirshov Bases and Hilbert Series of Free Dendriform Algebras*

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Abstract: In this paper, we give a Gröbner-Shirshov basis of the free dendriform algebra as a quotient algebra of an L-algebra. As applications, we obtain a normal form of the free dendriform algebra. Moreover, Hilbert series and Gelfand-Kirillov dimension of finitely generated free dendriform algebras are obtained.

Key words: Gröbner-Shirshov basis; L-algebra; dendriform algebra; Hilbert series; Gelfand-Kirillov dimension.

AMS 2000 Subject Classification: 13P10, 16S15, 17D99, 16P90.

1 Introduction

The theories of Gröbner-Shirshov bases and Gröbner bases were invented independently by A.I. Shirshov ([8], 1962) for (commutative, anti-commutative) non-associative alegbras, by H. Hironaka ([4], 1964) for infinite series algebras (both formal and convergent) and by B. Buchberger (first publication in [3], 1965) for polynomial algebras. Gröbner–Shirshov technique is very useful in the study of presentations of many kinds of algebras defined by generators and defining relations.

An L-algebra (see [5]) is a vector space over a field k with two operations $\prec$, $\succ$ satisfying one identity: $(x \succ y) \prec z = x \succ (y \prec z)$. A dendriform algebra (see [6, 7])is an L-algebra with two identities: $(x \prec y) \prec z = x \prec (y \prec z) + x \prec (y \succ z)$ and $x \succ (y \succ z) = (x \succ y) \succ z + (x \prec y) \succ z$.

The Composition-Diamond lemma for L-algebras is established in a recent paper [1]. In this paper, by using the Composition-Diamond lemma for L-algebras in [1], we give a Gröbner-Shirshov basis of the free dendriform algebra as a quotient algebra of a free L-algebra and then a normal form of a free dendriform algebra is obtained. As applications, we obtain the Hilbert series and Gelfand-Kirillov dimension of the free dendriform algebra generated by a finite set.

*Supported by the NNSF of China (No.10771077) and the NSF of Guangdong Province (No.06025062).

2 L-algebras

We first introduce some concepts and results from the literature which are related to the Gröbner-Shirshov bases for L-algebras. We will use some definitions and notations which are mentioned in [1].

Let k be a field, X a set of variables, Ω a set of multilinear operations, and

$$\Omega = \cup_{n\geq 1}\Omega_n,$$

where $\Omega_n = \{\delta_i^{(n)} | i \in I_n\}$ is the set of n-ary operations, $n = 1, 2, \ldots$. Now, we define "Ω-words".

Define

$$(X, \Omega)_0 = X.$$

For $m \geqslant 1$, define

$$(X, \Omega)_m = X \cup \Omega((X, \Omega)_{m-1})$$

where

$$\Omega((X, \Omega)_{m-1}) = \cup_{t=1}^{\infty}\{\delta_i^{(t)}(u_1, u_2, \ldots, u_t)|\ \delta_i^{(t)} \in \Omega_t, u_j \in\ (X, \Omega)_{m-1}\}.$$

Let

$$(X, \Omega) = \bigcup_{m=0}^{\infty}(X, \Omega)_m.$$

Then each element in (X, Ω) is called an Ω-word.

Definition 2.1 *[5] An L-algebra is a k-vector space L equipped with two bilinear operations $\prec, \ \succ: L^{\bigotimes 2} \to L$ verifying the so-called entanglement relation:*

$$(x \succ y) \prec z = x \succ (y \prec z), \forall\ x, y, z \in L.$$

Let $\Omega = \{\prec, \succ\}$. In this case, we call an Ω-word as an L-word.

Definition 2.2 *[1] An L-word u is a normal L-word if u is one of the following:*

i) $u = x,$ *where* $x \in X$.

ii) $u = v \succ w$, *where* v *and* w *are normal L-words.*

iii) $u = v \prec w$ *with* $v \neq v_1 \succ v_2$, *where* $v_1,\ v_2,\ v,\ w$ *are normal L-words.*

We denote u by $[u]$ if u is a normal L-word.

We denote the set of all the normal L-words by N. Then, the free L-algebra has an expression $L(X) = kN = \{\sum \alpha_i u_i \mid \alpha_i \in k,\ u_i \in N\}$ with k-basis N and the operations $\prec,\ \succ$: for any $u, v \in N$,

$$u \prec v = [u \prec v], \quad u \succ v = [u \succ v].$$

Clearly, $[u \succ v] = u \succ v$ and

$$[u \prec v] = \begin{cases} u \prec v & \text{if } u = u_1 \prec u_2, \text{or } u \in X, \\ u_1 \succ [u_2 \prec v] & \text{if } u = u_1 \succ u_2. \end{cases}$$

Now, we order N in the same way as in [1].

Let X be a well ordered set. We denote $\succ$ by δ_1, $\prec$ by δ_2. For any normal L-word u, define

$$wt(u) = \begin{cases} (1, x), & \text{if } u = x \in X; \\ (|u|, \delta_i, u_1, u_2), & \text{if } u = \delta_i(u_1, u_2) \in N, \end{cases}$$

where $|u|$ is the number of $x \in X$ in u. Then we order N as follows:

$$u > v \Longleftrightarrow wt(u) > wt(v) \qquad \textit{lexicographically}$$

by induction on $|u| + |v|$, where $\delta_2 > \delta_1$.

Let $\star \notin X$. By a $\star$-L-word we mean any expression in $(X \cup \{\star\}, \{\prec, \succ\})$ with only one occurrence of $\star$.

Let u be a $\star$-L-word and $s \in L(X)$. Then we call $u|_s = u|_{\star \mapsto s}$ an s-word in $L(X)$.

An s-word $u|_s$ is called a normal s-word if $u|_{\overline{s}} \in N$.

It is shown in [1] that the above ordering on N is monomial in the sense that for any $\star$-L-word w and any $u,\ v \in N$, $u > v$ implies $[w|_u] > [w|_v]$.

Assume that $L(X)$ is equipped with the monomial ordering $>$ as above. For any L-polynomial $f \in L(X)$, let $\bar{f}$ be the leading normal L-word of f. If the coefficient of $\bar{f}$ is 1, then f is called monic.

Definition 2.3 *[1] Let $f,\ g \in L(X)$ are two monic polynomials.*

1) *Composition of right multiplication.*

   If $\overline{f} = u_1 \succ u_2$ for some $u_1,\ u_2 \in N$, then for any $v \in N$, $f \prec v$ is called a composition of right multiplication.

2) *Composition of inclusion.*

   If $w = \overline{f} = u|_{\overline{g}}$ where $u|_g$ is a normal g-word, then

$$(f, g)_w = f - u|_g$$

   is called the composition of inclusion and w is called the ambiguity of the composition $(f, g)_\omega$.

Definition 2.4 *[1] Let the ordering on N be as before, $S \subset L(X)$ a monic set and $f,\ g \in S$.*

1) *The composition of right multiplication $f \prec v$ is called trivial modulo S, denoted by $f \prec v \equiv 0\ mod(S)$, if*

$$f \prec v = \sum \alpha_i u_i|_{s_i},$$

   where each $\alpha_i \in k$, $s_i \in S$, $u_i|_{s_i}$ normal s_i-word, and $u_i|_{\overline{s_i}} \leqslant \overline{f \prec v}$.

2) The composition of inclusion $(f,g)_w$ is called trivial modulo (S,w), denoted by $(f,g)_w \equiv 0 \ mod(S,w)$, if

$$(f,g)_w = \sum \alpha_i u_i|_{s_i},$$

where each $\alpha_i \in k$, $s_i \in S$, $u_i|_{s_i}$ normal s_i-word, and $u_i|_{\overline{s_i}} < w$.

S is called a Gröbner-Shirshov basis in $L(X)$ if any composition of polynomials in S is trivial modulo S (and w).

Theorem 2.5 *[1] (Composition-Diamond lemma for L-algebras) Let $S \subset L(X)$ be a monic set and the ordering on N as before. Let $Id(S)$ be the ideal of $L(X)$ generated by S. Then the following statements are equivalent:*

(I) S is a Gröbner-Shirshov basis in $L(X)$.

(II) $f \in Id(S) \Rightarrow \overline{f} = u|_{\overline{s}}$ for some $s \in S$, where $u|_s$ is a normal s-word.

(III) The set $Irr(S) = \{u \in N |\ u \neq v|_{\overline{s}},\ s \in S,\ v|_s$ is a normal s-word$\}$ is a k-basis of the L-algebra $L(X|\ S) = L(X)/Id(S)$.

3 Gröbner-Shirshov bases for free dendriform algebras

In this section, we give a Gröbner-Shirshov basis of the free dendriform algebra $DD(X)$ generated by X. As an application, we obtain a normal form of $DD(X)$.

Definition 3.1 *[6] A dendriform algebra is a k-vector space DD with two bilinear operations $\prec$, $\succ$ subject to the three axioms below: for any $x,\ y,\ z \in DD$,*

1) $(x \succ y) \prec z = x \succ (y \prec z)$,

2) $(x \prec y) \prec z = x \prec (y \prec z) + x \prec (y \succ z)$,

3) $x \succ (y \succ z) = (x \succ y) \succ z + (x \prec y) \succ z$.

Thus, any dendriform algebra is an L-algebra.

It is clear that the free dendriform algebra generated by X, denoted by $DD(X)$, has an expression

$$\begin{aligned} L(X \quad | \quad & (x \prec y) \prec z = x \prec (y \prec z) + x \prec (y \succ z), \\ & x \succ (y \succ z) = (x \succ y) \succ z + (x \prec y) \succ z,\ x,y,z \in N). \end{aligned}$$

The following theorem gives a Gröbner-Shirshov basis for $DD(X)$.

Theorem 3.2 *Let the ordering on N be as before. Let*

$$\begin{aligned} f_1(x,y,z) &= (x \prec y) \prec z - x \prec (y \prec z) - x \prec (y \succ z), \\ f_2(x,y,z) &= (x \prec y) \succ z + (x \succ y) \succ z - x \succ (y \succ z), \\ f_3(x,y,z,v) &= ((x \succ y) \succ z) \succ v - (x \succ y) \succ (z \succ v) + (x \succ (y \prec z)) \succ v. \end{aligned}$$

Then, $S = \{f_1(x,y,z),\ f_2(x,y,z),\ f_3(x,y,z,v) |\ x,y,z,v \in N\}$ is a Gröbner-Shirshov basis in $L(X)$.

Proof. All the possible compositions of right multiplication in S are as follows.

1) $f_2(x,y,z) \prec u,\ u \in N$. We have

$$\begin{aligned} f_2(x,y,z) \prec u &= ((x \prec y) \succ z) \prec u + ((x \succ y) \succ z) \prec u - (x \succ (y \succ z)) \prec u \\ &= (x \prec y) \succ (z \prec u) + (x \succ y) \succ (z \prec u) - x \succ ((y \succ z) \prec u) \\ &\equiv -(x \succ y) \succ (z \prec u) + x \succ (y \succ (z \prec u)) + (x \succ y) \succ (z \prec u) \\ &\quad -x \succ ((y \succ z) \prec u) \\ &\equiv 0\ mod(S). \end{aligned}$$

2) $f_3(x,y,z,v) \prec u,\ u \in N$. We have

$$\begin{aligned} f_3(x,y,z,v) \prec u &= (((x \succ y) \succ z) \succ v) \prec u - ((x \succ y) \succ (z \succ v)) \prec u \\ &\quad +((x \succ (y \prec z)) \succ v) \prec u \\ &= ((x \succ y) \succ z) \succ (v \prec u) - (x \succ y) \succ (z \succ (v \prec u)) \\ &\quad +(x \succ (y \prec z)) \succ (v \prec u) \\ &\equiv (x \succ y) \succ (z \succ (v \prec u)) - (x \succ (y \prec z)) \succ (v \prec u) \\ &\quad -(x \succ y) \succ (z \succ (v \prec u)) + ((x \succ (y \prec z)) \succ v) \prec u \\ &\equiv 0\ mod(S). \end{aligned}$$

We denote by $f_i \wedge f_j$ an inclusion composition of the polynomials f_i and $f_j,\ i,j = 1,2,3$. All the possible ambiguities in S are listed as follows:

3) $f_1(x,y,z) \wedge f_1(a,b,c)$:

3.1 $w_{3.1} = (x|_{((a\prec b)\prec c)} \prec y) \prec z$,

3.2 $w_{3.2} = (x \prec y|_{((a\prec b)\prec c)}) \prec z$,

3.3 $w_{3.3} = (x \prec y) \prec z|_{((a\prec b)\prec c)}$,

3.4 $w_{3.4} = ((a \prec b) \prec c) \prec z$.

4) $f_1(x,y,z) \wedge f_2(a,b,c)$:

4.1 $w_{4.1} = (x|_{((a\prec b)\succ c)} \prec y) \prec z$,

4.2 $w_{4.2} = (x \prec y|_{((a\prec b)\succ c)}) \prec z$,

4.3 $w_{4.3} = (x \prec y) \prec z|_{((a\prec b)\succ c)}$.

5) $f_1(x,y,z) \wedge f_3(a,b,c,d)$:

5.1 $w_{5.1} = (x|_{(((a\succ b)\succ c)\succ d)} \prec y) \prec z$,

5.2 $w_{3.2} = (x \prec y|_{(((a\succ b)\succ c)\succ d)}) \prec z$,

5.3 $w_{5.3} = (x \prec y) \prec z|_{(((a\succ b)\succ c)\succ d)}$.

6) $f_2(a,b,c) \wedge f_1(x,y,z)$:

6.1 $w_{6.1} = (a|_{((x \prec y) \prec z)} \prec b) \succ c$, 6.2 $w_{6.2} = (a \prec b|_{((x \prec y) \prec z)}) \succ c$,
6.3 $w_{6.3} = (a \prec b) \succ c|_{((x \prec y) \prec z)}$, 6.4 $w_{6.4} = ((x \prec y) \prec z) \succ c$.

7) $f_2(a,b,c) \wedge f_2(x,y,z)$:
7.1 $w_{7.1} = (a|_{((x \prec y) \succ z)} \prec b) \succ c$, 7.2 $w_{7.2} = (a \prec b|_{((x \prec y) \succ z)}) \succ c$,
7.3 $w_{7.3} = (a \prec b) \succ c|_{((x \prec y) \succ z)}$.

8) $f_2(a,b,c) \wedge f_3(x,y,z,v)$:
8.1 $w_{8.1} = (a|_{(((x \succ y) \succ z) \succ v)} \prec b) \succ c$, 8.2 $w_{8.2} = (a \prec b|_{(((x \succ y) \succ z) \succ v)}) \succ c$,
8.3 $w_{8.1} = (a \prec b) \succ c|_{(((x \succ y) \succ z) \succ v)}$.

9) $f_3(x,y,z,v) \wedge f_1(a,b,c)$:
9.1 $w_{9.1} = ((x|_{((a \prec b) \prec c)} \succ y) \succ z) \succ v$, 9.2 $w_{9.2} = ((x \succ y|_{((a \prec b) \prec c)}) \succ z) \succ v$,
9.3 $w_{9.3} = ((x \succ y) \succ z|_{((a \prec b) \prec c)}) \succ v$, 9.4 $w_{9.4} = ((x \succ y) \succ z) \succ v|_{((a \prec b) \prec c)}$.

10) $f_3(x,y,z,v) \wedge f_2(a,b,c)$:
10.1 $w_{10.1} = ((x|_{((a \prec b) \succ c)} \succ y) \succ z) \succ v$, 10.2 $w_{10.2} = ((x \succ y|_{((a \prec b) \succ c)}) \succ z) \succ v$,
10.3 $w_{10.3} = ((x \succ y) \succ z|_{((a \prec b) \succ c)}) \succ v$, 10.4 $w_{10.4} = ((x \succ y) \succ z) \succ v|_{((a \prec b) \succ c)}$,
10.5 $w_{10.5} = (((a \prec b) \succ c) \succ z) \succ v$.

11) $f_3(x,y,z,v) \wedge f_3(a,b,c,d)$:
11.1 $w_{11.1} = ((x|_{(((a \succ b) \succ c) \succ d)} \succ y) \succ z) \succ v$,
11.2 $w_{11.2} = ((x \succ y|_{(((a \succ b) \succ c) \succ d)}) \succ z) \succ v$,
11.3 $w_{11.3} = ((x \succ y) \succ z|_{(((a \succ b) \succ c) \succ d)}) \succ v$,
11.4 $w_{11.4} = ((x \succ y) \succ z) \succ v|_{(((a \succ b) \succ c) \succ d)}$,
11.5 $w_{11.5} = (((a \succ b) \succ c) \succ d) \succ v$,
11.6 $w_{11.6} = ((((a \succ b) \succ c) \succ d) \succ z) \succ v$.

We will prove that all compositions are trivial mod(S,w). Here, for example, we only check Case 6.4, Case 10.5 and Case 11.6. The others are easy to check.

Case 6.4:

$$
\begin{aligned}
& (f_2(a,b,c), f_1(x,y,z))_{w_{6.4}} \\
\equiv\ & (x \prec (y \prec z)) \succ c + (x \prec (y \succ z)) \succ c - (x \prec y) \succ (z \succ c) \\
& +((x \prec y) \succ z) \succ c \\
\equiv\ & x \succ ((y \prec z) \succ c) - (x \succ (y \prec z)) \succ c + x \succ ((y \succ z) \succ c) \\
& -(x \succ (y \succ z)) \succ c - x \succ (y \succ (z \succ c)) + (x \succ y) \succ (z \succ c) \\
& +(x \succ (y \succ z)) \succ c - ((x \succ y) \succ z) \succ c \\
\equiv\ & x \succ (y \succ (z \succ c)) - (x \succ (y \prec z)) \succ c - x \succ (y \succ (z \succ c)) \\
& +(x \succ y) \succ (z \succ c) - ((x \succ y) \succ z) \succ c \\
\equiv\ & 0\ mod(S, w_{6.4}).
\end{aligned}
$$

Case 10.5:

$$
\begin{aligned}
& (f_3(x,y,z,v), f_2(a,b,c))_{w_{10.5}} \\
\equiv\ & ((a \succ (b \succ c)) \succ z) \succ v - (((a \succ b) \succ c) \succ z) \succ v \\
& -((a \prec b) \succ c) \succ (z \succ v) + ((a \prec b) \succ (c \prec z)) \succ v \\
\equiv\ & (a \succ (b \succ c)) \succ (z \succ v) - (a \succ ((b \succ c) \prec z)) \succ v \\
& -((a \succ b) \succ c) \succ (z \succ v) + ((a \succ b) \succ (c \prec z)) \succ v \\
& -(a \succ (b \succ c)) \succ (z \succ v) + ((a \succ b) \succ c) \succ (z \succ v) \\
& +(a \succ (b \succ (c \prec z)) \succ v - ((a \succ b) \succ (c \prec z)) \succ v \\
\equiv\ & 0 \ mod(S, w_{10.5}).
\end{aligned}
$$

Case 11.6:

$$
\begin{aligned}
& (f_3(x,y,z,v), f_3(a,b,c,d))_{w_{11.6}} \\
\equiv\ & (((a \succ b) \succ (c \succ d)) \succ z) \succ v - (((a \succ (b \prec c)) \succ d) \succ z) \succ v \\
& -(((a \succ b) \succ c) \succ d) \succ (z \succ v) + (((a \succ b) \succ c) \succ (d \prec z)) \succ v \\
\equiv\ & ((a \succ b) \succ (c \succ d)) \succ (z \succ v) - ((a \succ b) \succ ((c \succ d) \prec z)) \succ v \\
& -((a \succ (b \prec c)) \succ d) \succ (z \succ v) + ((a \succ (b \prec c)) \succ (d \prec z)) \succ v \\
& -((a \succ b) \succ (c \succ d)) \succ (z \succ v) + ((a \succ (b \prec c)) \succ d) \succ (z \succ v) \\
& +((a \succ b) \succ (c \succ (d \prec z)) \succ v - ((a \succ (b \prec c)) \succ (d \prec z)) \succ v \\
\equiv\ & 0 \ mod(S, w_{11.6}).
\end{aligned}
$$

The proof is complete. □

Definition 3.3 *An L-word u is called a normal DD-word, denoted by $\lceil u \rceil$, if*

1) $u = x$, $x \in X$,

2) $u = x \prec \lceil v \rceil$, $x \in X$,

3) $u = x \succ \lceil v \rceil$, $x \in X$,

4) $u = (x \succ \lceil u_1 \rceil) \succ \lceil u_2 \rceil$, $x \in X$.

Remark From Definition 2.2 and Definition 3.3, we know that any normal DD-word is a normal L-word.

The following corollary follows from Theorem 2.5 and Theorem 3.2.

Corollary 3.4 *The set $Irr(S) = \{u |\ u$ is a normal DD-word $\}$ is a k-basis of the free dendriform algebra $DD(X)$.*

4 Hilbert series and Gelfand-Kirillov dimension of the free dendriform algebra

In this section, we give Hilbert series of the free dendriform algebra $DD(X)$ where $|X|$ is finite. As an application, we prove that Gelfand-Kirillov dimension of the free dendriform algebra $DD(X)$ is infinite.

We introduce some basic definitions and concepts that we will use throughout this section.

Definition 4.1 *Let $V = (V, \prec, \succ)$ be a dendriform algebra. Then V is called a finitely graded algebra if*

$$V = \oplus_{m\geq 1} V_m$$

as k-vector spaces such that

$$dim_k V_m < \infty \ \ and \ \ \ \delta(V_i, V_j) \subseteq V_{i+j} \ \ \ for\ all\ \ i, j \geq 1,\ \delta \in \{\prec, \succ\}.$$

Definition 4.2 *Let $V = \oplus_{m\geq 1} V_m$ be a finitely graded dendriform algebra and $dim_k(V_m)$, the dimension of the vector space V_m. Then the Hilbert series of V is defined to be*

$$\mathcal{H}(V, t) = \sum_{m=1}^{\infty} dim_k(V_m) t^m.$$

Let $X = \{x_1, x_2, \dots, x_n\}$ and DD_m the subspace of $DD(X)$ generated by all normal DD-words in $DD(X)$ of degree m. Then

$$DD(X) = \oplus_{m\geq 1} DD_m$$

is a finitely graded dendriform algebra.

By the definition of normal DD-words, one has

$$dim_k(DD_1) = n,\ dim_k(DD_2) = 2n^2.$$

Assume that for any $m \geq 1$, $dim_k(DD_m) = f(m)n^m$. Then $f(1) = 1,\ f(2) = 2$. For convenience, let $f(0) = 1$.

For any $m > 2$, it is clear that DD_m has a k-basis

$$\begin{aligned}
&\{x \prec \lceil u \rceil | x \in X,\ |u| > 1, \lceil u \rceil \text{ is a normal } DD\text{-word}\}\\
\bigcup&\{x \succ \lceil u \rceil | x \in X,\ |u| > 1 \lceil u \rceil \text{ is a normal } DD\text{-word}\}\\
\bigcup&\{(x \succ \lceil u_1 \rceil) \succ \lceil u_2 \rceil | x \in X,\ |u_1|,\ |u_2| \geq 1,\ \lceil u_1 \rceil, \lceil u_2 \rceil \text{ are normal } DD\text{-words}\}.
\end{aligned}$$

It follows that

$$\begin{aligned}
f(m) &= 2 \times f(m-1) + 1 \times 1 \times f(m-2) + 1 \times f(m-2) \times 1\\
&\quad +1 \times \sum_{i=2}^{m-3} f(i) f(m-3-i)\\
&= \sum_{i=0}^{m-1} f(i) f(m-1-i).
\end{aligned}$$

Therefore, we prove the following lemma.

Lemma 4.3 *Let X be a finite set with $|X| = n$. Then the Hilbert series of the free dendriform algebra $DD(X)$ is*

$$\mathcal{H}(DD(X), t) = \sum_{m\geq 1} f(m)n^m t^m,$$

where $f(m)$ satisfies the recursive relation ($f(0) = 1$):

$$f(m) = \sum_{i=0}^{m-1} f(i)f(m-1-i), \quad m \geq 1.$$

Now, we describe the Hilbert series of $DD(X)$ with another way.

Let A, B, C be the subspaces of $DD(X)$ with k-bases

$$\begin{array}{l}\{x \prec \lceil u\rceil |\ x \in X,\ \lceil u\rceil \text{ is a normal } DD\text{-word}\},\\ \{x \succ \lceil u\rceil |\ x \in X,\ \lceil u\rceil \text{ is a normal } DD\text{-word}\},\\ \{(x \succ \lceil u_1\rceil) \succ \lceil u_2\rceil |\ x \in X,\ \lceil u_1\rceil, \lceil u_2\rceil \text{ are normal } DD\text{-words}\},\end{array}$$

respectively. Assume that their Hilbert series are $\mathcal{H}(A,t),\ \mathcal{H}(B,t),\ \mathcal{H}(C,t)$, respectively. Clearly, we have

$$\mathcal{H}(B,t) = \mathcal{H}(A,t).$$

Noting that A has a k-basis

$$\{x_i \prec x_j |\ x_i,\ x_j \in X\} \bigcup \{x \prec \lceil u\rceil |\ |u| > 1,\ x \in X,\ \lceil u\rceil \text{ is a normal } DD\text{-word}\},$$

we have

$$\begin{aligned}\mathcal{H}(A,t) &= n^2t^2 + nt \times (\mathcal{H}(A,t) + \mathcal{H}(B,t) + \mathcal{H}(C,t))\\ &= n^2t^2 + nt \times (2\mathcal{H}(A,t) + \mathcal{H}(C,t)). \qquad (1)\end{aligned}$$

Since C has a k-basis

$$\begin{array}{l}\{(x_i \succ x_j) \succ x_k |\ x_i, x_j, x_k \in X\}\\ \bigcup \{(x_i \succ x_j) \succ \lceil u\rceil | x_i, x_j \in X, |u| > 1,\ \lceil u\rceil \text{ is a normal } DD\text{-word}\}\\ \bigcup\ \{(x_i \succ \lceil u\rceil) \succ x_j | x_i, x_j \in X, |u| > 1,\ \lceil u\rceil \text{ is a normal } DD\text{-word}\}\\ \bigcup \{(x \succ \lceil u\rceil) \succ \lceil v\rceil |\ |u|, |v| > 1,\ \lceil u\rceil, \lceil v\rceil \text{ are normal } DD\text{-words}\},\end{array}$$

we have

$$\begin{aligned}\mathcal{H}(C,t) &= n^3t^3 + 2n^2t^2 \times (\mathcal{H}(A,t) + \mathcal{H}(B,t) + \mathcal{H}(C,t)) + nt \times (\mathcal{H}(A,t)\\ &\quad + \mathcal{H}(B,t) + \mathcal{H}(C,t))^2\\ &= nt \times (nt + (2\mathcal{H}(A,t) + \mathcal{H}(C,t)))^2 \qquad (2)\end{aligned}$$

From equations (1) and (2), we obtain

$$\mathcal{H}(A,t) = \frac{1 - 2nt \pm \sqrt{1-4nt}}{2}.$$

Since $\mathcal{H}(A,0)=0$, we have

$$\mathcal{H}(A,t)=\frac{1-2nt-\sqrt{1-4nt}}{2}.$$

Therefore,

$$\mathcal{H}(C,t)=\frac{1-(1-2nt)\sqrt{1-4nt}}{2nt}-2+nt.$$

Thus, we have the following theorem.

Theorem 4.4 *Let X be a finite set with $|X|=n$. The Hilbert series of the free dendriform algebra $DD(X)$ is*

$$\mathcal{H}(DD(X),t)=\frac{1-2nt-\sqrt{1-4nt}}{2nt}.$$

We now give an exact expression of the function $\mathcal{H}(DD(X),t)$.

For $t\leq\frac{1}{4n}$, we have

$$\sqrt{1-4nt}=(1+(-4nt))^{\frac{1}{2}}=1+\sum_{i=1}^{\infty}\frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-i+1)}{i!}\times(-1)^i4^in^it^i.$$

From this and Lemma 4.3 we get the following theorem.

Theorem 4.5 *Let X be a finite set with $|X|=n$. Then the Hilbert series of the free dendriform algebra $DD(X)$ is*

$$\begin{aligned}\mathcal{H}(DD(X),t) &= \sum_{m=1}^{\infty}\frac{1\times3\times\cdots(2m-1)\times2^m}{(m+1)!}n^mt^m\\ &= \sum_{m=1}^{\infty}\frac{(2m)!\times n^m\times t^m}{(m+1)!m!}.\end{aligned}$$

Therefore, $dim_k(DD_m)=\frac{(2m)!\times n^m}{(m+1)!m!}$, $m\geq1$.

Now, by using Theorem 4.5, we show that Gelfand-Kirillov dimension of the free dendriform algebra $DD(X)$ is infinite when $|X|$ is finite.

Definition 4.6 *[2] Let R be a finitely presented algebra over a field k and $x_1,x_2,\ldots,x_n$ be its generators. Consider $R=\bigcup_{d\in N}V_{(d)}$, where $V_{(d)}$ is spanned by all the monomials in x_i of length $\leq d$. The quantity*

$$GKR=\overline{\lim_{d\to\infty}}\frac{log\ dim_kV_{(d)}}{log\ d}$$

is called the Gelfand-Kirillov dimension of R.

Theorem 4.7 *Let X be a finite set with $|X| = n$. Then the Gelfand-Kirillov dimension of free dendriform algebra $DD(X)$ is*

$$GKDD(X) = \infty.$$

Proof. For a fixed natural d, let $DD_{(d)}$ be the subspace spanned by all the monomials in x_i of length $\leq d$. Then

$$dimDD_{(d)} = \sum_{i=1}^{d} dim_k(DD_i) \geq dim_k(DD_d).$$

Therefore,

$$\begin{aligned} GKDD(X) &\geq \overline{\lim_{d\to\infty}} \frac{log\ dim_k(DD_d)}{log\ d} = \overline{\lim_{d\to\infty}} \frac{ln\ \frac{(2d)!\times n^d}{(d+1)!d!}}{ln\ d} \\ &= \overline{\lim_{d\to\infty}} \frac{d\ ln\ 2n + \sum_{i=1}^{d} ln(2i-1) - \sum_{i=1}^{d+1} ln\ i}{ln\ d} \\ &= \overline{\lim_{d\to\infty}} d\ ln2n + \overline{\lim_{d\to\infty}} \sum_{i=1}^{d} \frac{ln \frac{2i-1}{i}}{ln\ d} - 1 \\ &= \overline{\lim_{d\to\infty}} (d\ ln2n) + \overline{\lim_{d\to\infty}} (\frac{d}{ln\ d} \sum_{i=1}^{d} \frac{ln\ (2-\frac{1}{i})}{d}) - 1 \\ &= \infty. \end{aligned}$$

□

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