Categorical Non-standard Analysis

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Abstract

In this paper we propose a new framework for Non-standard Analysis ([4]) in terms of category theory ([3],[1]).

1 Introduction

Non-standard Analysis ([4]) and Elementary Theory of the Category of Sets ([3]) are two of the great inventions in so-called "foundation of mathematics". Both of them have given productive viewpoints to organize many kinds of topics in mathematics or related fields. However, integration of two theories still seems to be developed. In the present paper, we propose a new framework for Non-standard Analysis in terms of Elementary Theory of the Category of Sets, in the line of pioneering work of Kock-Mikkelsen ([1]).

First we introduce the notion of duplicated category of sets. This notion is nothing but a categorical counterpart of "transfer principle" and "standardization principle" ([4], [**?**]). Then we proceed to the "compactness" axiom, which is a version of "concurrence"([4]). In the last section, we construct a duplicated category of sets which satisfies the compactness axiom, assuming ZFC.

2 Duplicated category of sets

Definition 2.1. A quintet $(\mathcal{S}_0, \mathcal{S}, {}^{*_0}, {}^{*}, \iota)$ is called a duplicated category of sets if it satisfies the following:

(i) $\mathcal{S}_0$ and $\mathcal{S}$ are categories which satisfy the axioms for $Sets$ ([3]).

(ii) ${}^{*_0} : \mathcal{S}_0 \longrightarrow \mathcal{S}$ is a functor which preserves all finite limits, subobject classifier 2, exponentials and natural number objects.

(iii) ${}^{*} : \mathcal{S}_0 \longrightarrow \mathcal{S}$ is a functor which preserves all finite limits.

(iv) $\iota : {}^{*_0} \longrightarrow {}^{*}$ is a natural transformation which satisfies $\iota_X = id_{^{*_0}(X)} = id_{^{*}(X)}$ for any finite object X in $\mathcal{S}_0$. (Here id denotes the identity map.)

Theorem 2.2. *^{*_0} and * are faithful.*

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Proof. Because they preserve diagonal and complement. ∎

^{*_0} **embeds $\mathcal{S}_0$ into $\mathcal{S}$ as a "subuniverse". We identify $^{*_0}(f)$ with f.**

Remark 2.3. * can also be considered as another embedding, however, it does not preserve exponentials in general. (* is not completely "logical" but "elementary".)

Theorem 2.4. *For any element $x : 1 \longrightarrow X$ in $\mathcal{S}_0$, $\iota(x) = {}^*(x)$.*

Proof. By naturality of ι and $^*(1) = 1$. ∎

Corollary 2.5. *All components of ι are monic.*

From the discussion above, an object X in $\mathcal{S}_0$ is to be considered as a canonical subset of $^*(X)$ in $\mathcal{S}$ through $\iota_X : X \longrightarrow {}^*(X)$.

Theorem 2.6. *Let $f : X \longrightarrow Y$ be a map in $\mathcal{S}_0$. Then $^*(f) \circ \iota = \iota \circ f$.*

Proof. By functoriality of * and theorem 2.4. ∎

From the discussion above, $^*(f)$ in $\mathcal{S}$ can be considered as "the morphism induced from f through ι".

We introduce a family of canonical inclusion other than ι to represent "inducing $^*(f)$ from f thorugh ι" in terms of exponentials (function spaces).

Definition 2.7. We define a family of morphism $\kappa_{A,B} : {}^*(B^A) \longrightarrow {}^*(B)^{^*(A)}$ by the lambda conversion of $^*(ev_{A,B}) : {}^*(A \times B^A) \cong {}^*(A) \times {}^*(B^A) \longrightarrow {}^*(B)$ for all pairs of objects in A, B in $\mathcal{S}_0$. (Here $ev_{A,B}$ denotes the evaluation map.)

Theorem 2.8. *([1],[2]) For any pair of objects A, B in $\mathcal{S}_0$, $\kappa_{A,B}$ is monic.*

By the theorem above, we can consider $^*(B^A)$ as a subset of $^*(B)^{^*(A)}$ thorugh $\kappa_{A,B}$. Since B^A is included to $^*(B^A)$ thorugh ι_{B^A}, it is also considered as a subset of $^*(B)^{^*(A)}$ thorugh $\kappa_{A,B} \circ \iota_{B^A}$. The theorem below means that $\kappa_{A,B} \circ \iota_{B^A}$ represents "inducing $^*(f)$ from f thorugh ι" in terms of exponentials (function spaces).

Theorem 2.9. *Let $f : A \longrightarrow B$ be any map in $\mathcal{S}_0$. Then the following equality holds.*

$$\kappa_{A,B} \circ \iota_{B^A}(\widehat{f}) = \widehat{{}^*(f)} \tag{2.1}$$

(Here $\widehat{\ }$ denotes the lambda conversion operation.)

Proof. Take the (inverse) lambda conversion of the right hand side of the equality to be proved. It is $^*(ev_{A,B}) \circ (id_{^*(A)} \times \iota_{B^A}) \circ (id_{^*(A)} \times \widehat{f})$. By naturality of ι and funtorial properties of *, it is calculated as follows:

$$\begin{aligned}
{}^*(ev_{A,B}) \circ (id_{^*(A)} \times \iota_{B^A}) \circ (id_{^*(A)} \times \widehat{f}) &= {}^*(ev_{A,B}) \circ (id_{^*(A)} \times (\iota \circ \widehat{f})) \\
&= {}^*(ev_{A,B}) \circ (id_{^*(A)} \times {}^*(\widehat{f})) \\
&= {}^*(ev_{A,B}) \circ {}^*(id_A \times \widehat{f}) \\
&= {}^*(ev_{A,B} \circ (id_A \times \widehat{f})) \\
&= {}^*(f)
\end{aligned}$$

∎

Remark 2.10. The monicness of $\kappa_{A,B} \circ \iota_{B^A}$ also follows directly from the faithfulness of *, without using the monicness of $\kappa_{A,B}$.

Definition 2.11. Let A, B be objects in $\mathcal{S}_0$. A map $F : {}^*(A) \longrightarrow {}^*(B)$ is called standard or internal if its lambda conversion $\widehat{F}$ factors thorough $\kappa_{A,B} \circ \iota_{B^A}$ or $\kappa_{A,B}$, respectively. Otherwise, it is called external. If f is internal but not standard, it is is called nonstandard.

Theorem 2.12. *Let A be any object in $\mathcal{S}_0$. Any element $x : {}^*(1) = 1 \longrightarrow {}^*(A)$ is internal. That is, any element in ${}^*(A)$ is either standard or nonstandard.*

Theorem 2.13. *Let A, B be objects in $\mathcal{S}_0$. A map $F : {}^*(A) \longrightarrow {}^*(B)$ in $\mathcal{S}$ is standard if and only if there exist (unique) map $f : A \longrightarrow B$ and $F = {}^*(f)$.*

A standard function ${}^*(f) : {}^*(X) \longrightarrow {}^*(Y)$ in $\mathcal{S}$ will be identified with its preimage $f : X \longrightarrow Y$ in $\mathcal{S}_0$ and often denote as $f : {}^*(X) \longrightarrow {}^*(Y)$ in applications. In addition, we will usually omit ι or κ from now on.

3 The compactness axiom

So far, the possibility that $(\mathcal{S}_0, \mathcal{S}, {}^{*_0}, {}^*, \iota)$ is trivial is not excluded. Hence, it may happen there is no external sets or nonstandard elements. For suitable restriction, we introduce the "compactness" axiom, which is a version of the "concurrence([4])" or "idealization([?])" principle.

Definition 3.1. Let $(\mathcal{S}_0, \mathcal{S}, {}^{*_0}, {}^*, \iota)$ be a duplicated category of sets. It is called to satisfy the compactness axiom if the following condition holds.

(Compactness axiom) Let $A : \Lambda \longrightarrow 2^{{}^*(X)}$ be a map such that Λ, X are in $\mathcal{S}_0$ and every A_λ is standard. If $\bigcap_F A \neq \phi$ for any finite subset $F \hookrightarrow \Lambda$, then $\bigcap_\Lambda A \neq \phi$.

Or dually,

(Compactness axiom) Let $A : \Lambda \longrightarrow 2^{{}^*(X)}$ be a map such that Λ, X are in $\mathcal{S}_0$ and every A_λ is standard. If $\bigcup_\Lambda A = {}^*(X)$, then $\bigcup_F A = {}^*(X)$ for some finite subset $F \hookrightarrow \Lambda$.

The compactness axiom implies existence of nonstandard elements:

Theorem 3.2. *Let $(\mathcal{S}_0, \mathcal{S}, {}^{*_0}, {}^*, \iota)$ be a duplicated category of sets which satisfies the compactness axiom. If X is any infinite object in $\mathcal{S}_0$, ${}^*(X)$ has nonstandard elements.*

The theorem above is situated at the core of non-standard analysis. Moreover, basic theorems and principles in non-standard analysis ([4]) are followed from the compactness axiom ("enlargement"). For details and the discussion on stronger versions of the compactness axiom, we are preparing another paper ([5]).

A duplicated category of sets $(\mathcal{S}_0, \mathcal{S}, {}^{*_0}, {}^*, \iota)$ which satisfies the compactness axiom gives a categorical framework for non-standard analysis. Any entities appeared in ordinary mathematics are represented naturally in $\mathcal{S}_0$, and by the virtue of the compactness axiom, notions are simplified and enriched in $\mathcal{S}$.

4 A model

This section is devoted to prove the existence theorem of duplicated categories of sets which satisfies the compactness axiom.

Theorem 4.1. *There exist duplicated categories of sets which satisfies the compactness axiom.*

Proof. We fix an ZFC universe as an ambient category of sets $\mathcal{S}$ and a small category of sets $\mathcal{S}_0$ generated by $S\mathbb{N}$, a superstructure of $\mathbb{N}$ in $\mathcal{S}$. To define * as a ultrapower functor, we construct the ultrafilter $\mathcal{U}$ over an index set I as follows:

First we define the index set I by the set of all finite sets in $S\mathbb{N}$. Then we take $\{\Gamma_\alpha\}_{\alpha\in I}$ as a filter basis such that each Γ_α is defined as

$$\Gamma_\alpha := \{\beta \in I | \alpha \subset \beta\} \tag{4.1}$$

and fix a ultrafilter $\mathcal{U}$ which includes $\{\Gamma_\alpha\}_{\alpha\in I}$.

We can also define ^{*_0} as the functor associated to * which identifies X in $\mathcal{S}_0$ with the set of all diagonal elements of $^*(X)$ in $\mathcal{S}$, which is a canonical subset of $^*(X)$ and denoted as $^{*_0}(X)$. Then ι is defined as the family of natural inclusion $\iota_X : {}^{*_0}(X) \hookrightarrow {}^*(X)$. Now it is easy to see that the quintet $(\mathcal{S}_0, \mathcal{S}, {}^{*_0}, {}^*, \iota)$ is a duplicated category of sets.

To prove $(\mathcal{S}_0, \mathcal{S}, {}^{*_0}, {}^*, \iota)$ satisfies the compactness axiom, it suffices to see the following:

($S\mathbb{N}$-Compactness) Let $A : \Lambda = S\mathbb{N} \longrightarrow 2^{^*(X)}$ be a map such that X is in $\mathcal{S}_0$ and every A_λ is standard. If $\bigcap_F A \neq \phi$ for any finite subset $F \hookrightarrow \Lambda$, then $\bigcap_\Lambda A \neq \phi$.

Proof of $S\mathbb{N}$-Compactness: First note that A_λ is represented as $A_\lambda = \overline{(\alpha_\lambda)_{\mu\in I}}$ such that each α_λ is an element in 2^X. Then $B_F := \bigcap_F A$ is represented by $\beta_F := \bigcap_{\lambda\in F} \alpha_\lambda$ as $B_F = \overline{(\beta_F)_{\mu\in I}}$.

By assumption, there exist an element $x_F := \overline{(\xi_{F,\mu})_{\mu\in I}}$ in B_F for any F. Here we can assume $\xi_{F,\mu} \in \beta_F$ for any μ without loss of generality.

Now $x_\Delta := \overline{(\xi_{\mu,\mu})_{\mu\in I}}$ is an element of A_λ for any λ because

$$\alpha_\lambda = \beta_{\{\lambda\}} \supset \beta_\mu \ni \xi_{\mu,\mu} \tag{4.2}$$

for any $\mu \in \Gamma_{\{\lambda\}} \in \mathcal{U}$. Hence $x_\Delta \in \bigcap_\Lambda A \neq \phi$. □

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