ON THE ATIYAH PROBLEM FOR THE LAMPLIGHTER GROUPS

ŁUKASZ GRABOWSKI

Abstract. Recently groups giving rise to irrational l^2-Betti numbers have been found. All the examples known so far share a common property: they have one of the lamplighter groups $\mathbb{Z}_{/p} \wr \mathbb{Z}$ as a subgroup. In this paper we prove that in fact already all the groups $\mathbb{Z}_{/p} \wr \mathbb{Z}$ give rise to transcendental l^2-Betti numbers.

Contents

1. Introduction

1-A. Presentation of the result

Let G be a countable discrete group. We will say that a non-negative real number r is an *l^2-Betti number arising from G* if and only if there exists $\theta \in M_m(\mathbb{Q}G)$, a matrix over the rational group ring of G, such that the von Neumann dimension of kernel of θ is r. The motivation for the name is as follows: when G is finitely presented and r is an l^2-Betti number arising from G, then there exists a closed manifold M whose fundamental group is G, and such that one of the l^2-Betti numbers of the universal cover of M is equal to r. We refer to [Eck00] and [Lüc02] for more details.

The following problem is a fine-grained version of a question asked by Atiyah in [Ati76].

Question 1 (The Atiyah problem for a group G). *What is the set of l^2-Betti numbers arising from G?*

Let us call this set the *l^2-complexity of G*, and denote it by $\mathcal{C}(G)$. For a class of groups $\mathbf{G}$ define $\mathcal{C}(\mathbf{G}) = \cup_{G\in\mathbf{G}}\mathcal{C}(G)$.

So far $\mathcal{C}(G)$ has been computed only in cases where $\mathcal{C}(G)$ is a subset of $\mathbb{Q}$. In fact, what has become to be known as the *Atiyah conjecture for torsion-free groups* says that $\mathcal{C}(G) = \mathbb{N}$ for any torsion-free group, and till the article [DS02] of Dicks and Schick it was widely conjectured that $\mathcal{C}(G) \subset \mathbb{Q}$ for every group G. However, Dicks and Schick gave an example of an operator $\theta \in \mathbb{Q}((\mathbb{Z}_{/2} \wr \mathbb{Z})^2)$ together with an heuristic argument showing why $\dim_{vN} \ker\theta$ is probably irrational. Their work was motivated by the article [GŻ01] of Grigorchuk and Żuk.

Only recently Austin has been able to obtain a definite result by proving in [Aus09] that $\mathcal{C}$(Finitely generated groups) is uncountable. This work has been a motivation for much of the following efforts, by showing that computation of $\dim_{vN}$ can be sometimes done by analyzing certain dynamical systems and using Pontryagin duality.

Subsequently it has been shown independently by the author in [Gra10] and by Pichot, Schick and Żuk in [PSZ10] that in fact $\mathcal{C}$(Finitely generated groups) $= \mathbb{R}_{\geq 0}$ and that $\mathcal{C}$(Finitely presented groups) $\nsubseteq \mathbb{Q}$. Moreover, in [Gra10] it is shown that $\mathcal{C}((\mathbb{Z}_{/2}\wr\mathbb{Z})^3) \nsubseteq \mathbb{Q}$.

More recently, Lehner and Wagner showed in [LW10] that $\mathcal{C}(\mathbb{Z}_{/p}\wr F_d)$ contains irrational algebraic numbers, where F_d is the free group on d generators, and $d \geq 2, p \geq 2d-1$.

In all the articles cited above the following is trivial to check: if it is proven that for a given group G it holds that $\mathcal{C}(G) \nsubseteq \mathbb{Q}$ then there exists p such that $\mathbb{Z}_{/p} \wr \mathbb{Z} \subset G$. In other words, according to the current state of knowledge, $\mathbb{Z}_{/p} \wr \mathbb{Z} \subset G$ could be the necessary condition for $\mathcal{C}(G) \nsubseteq \mathbb{Q}$. We prove that it is a sufficient condition. Indeed, it is very easy to see that if $A \subset B$ are groups then $\mathcal{C}(A) \subset \mathcal{C}(B)$ (see for example Corollary 4.2.2 in [Gra10]) and here we prove the following theorem.

Theorem A. *Let $p \geq 2$. Then $\mathcal{C}(\mathbb{Z}_{/p} \wr \mathbb{Z})$ contains transcendental numbers.*

We finish this subsection by stating two related open questions. The first one summarizes the current state of knowledge on irrational l^2-Betti numbers.

Question 2. *Is it the case that $\mathcal{C}(G) \nsubseteq \mathbb{Q}$ is equivalent to $\mathbb{Z}_{/p} \wr \mathbb{Z} \subset G$ for some p?*

As mentioned above, $\mathcal{C}(G)$ has been computed only in cases where in fact $\mathcal{C}(G) \subset \mathbb{Q}$. The "easiest" group known so far for which $\mathcal{C}(G) \nsubseteq \mathbb{Q}$ is $\mathbb{Z}_{/2} \wr \mathbb{Z}$, and hence the following question.

Question 3. *What is $\mathcal{C}(\mathbb{Z}_{/2} \wr \mathbb{Z})$?*

This question contains many interesting subquestions. For example, does $\mathcal{C}(\mathbb{Z}_{/2} \wr \mathbb{Z})$ contain irrational algebraic numbers?

1-B. Outline of the paper

In order to prove Theorem A we need to find an operator in $M_m(\mathbb{Q}(\mathbb{Z}_{/p} \wr \mathbb{Z}))$ whose kernel has transcendental von Neumann dimension. However, Lemma 6.1 says that $|H| \cdot \mathcal{C}(G \times H) = C(G)$, for any group G and any finite group H, so we can as well find such an operator in $\mathbb{Q}(\mathbb{Z}_{/p} \wr \mathbb{Z} \times H)$, where H is some finite group.

In Section 6, *Back to the lamplighter groups*, we see how Pontryagin duality allows us to exchange the above question with a question about existence of an operator in the von Neumann algebra $L^\infty(X) \rtimes \Gamma$ whose kernel has transcendental von Neumann dimension, where $X := {\mathbb{Z}_{/p}}^{\mathbb{Z}} \times {\mathbb{Z}_{/2}}^3$, and $\Gamma := \mathbb{Z} \times GL_3(\mathbb{Z}_{/2})$.

The operator $T \in L^\infty(X) \rtimes \Gamma$, whose dimension we are able to calculate, is defined in Section 4, *Description of the operator*, in terms of another operator S. Our computational tool is the one developed in [Gra10], and we present it in Section 3, *Our computational tool*.

The main idea is as follows: we are given a probability measure space (X, μ), an action $\rho\colon \Gamma \curvearrowright X$ by measure preserving maps, an operator $S \in L^\infty(X) \rtimes \Gamma$, and another operator T which is defined in terms of S. In order to compute $\dim_{vN} \ker T$ we proceed as follows: we decompose X into family of sets, each of which is the set of vertices of certain graph g - this decomposition depends on the operator S. Next, we "restrict" the operator T to an operator T^g defined on the Hilbert space $l^2 g$ spanned by vertices of g (i.e. points of X.) Computing $\dim \ker T^g$ turns out to be relatively easy, and it turns out that to obtain $\dim_{vN} \ker T$ one needs to "integrate" the function $\dim \ker T^g$ over all the graphs g which appear as "subgraphs" of X.

The graphs which appear in the decomposition of X induced by our S are described in Section 2, *Preliminaries on certain graphs*. In Section 5, *Application of the computational tool*, we prove that the graphs described in Section 2 are indeed all the graphs we need to consider. After this we are ready to apply the computational tool: Corollary 5.6 shows what is $\dim_{vN} \ker T$; transcendence of it follows from the work [aT02] of Tanaka.

1-C. Basic notation

The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote respectively the sets $\{0, 1, \ldots\}$, the set of integers, the set of rational numbers, the set of real numbers and the set of complex numbers. We choose one of the two generators of $\mathbb{Z}$ once and for all and denote it by t.

The cyclic group of order p is denoted by $\mathbb{Z}_{/p}$.

For two groups A and B, A^B denotes the set of functions $B \to A$. Usually B will be equal to $\mathbb{Z}$ in which case A-valued functions will be identified with A-valued sequences. $A^{\oplus B}$ denotes the set of finitely supported functions $B \to A$.

The wreath product of a group A with $\mathbb{Z}$ is defined as $A \wr \mathbb{Z} := A^{\oplus \mathbb{Z}} \rtimes_\rho \mathbb{Z}$, where $[\rho(t)((a_i))]_j := a_{j+1}$.

Given a group G, the Hilbert space spanned by the elements of G is denoted by $l^2 G$; elements of the canonical basis of $l^2 G$ are denoted by ζ_g, $g \in G$. Given a field K of complex numbers we often consider the group ring KG of linear combinations of elements of G with coefficients in K. KG acts on $l^2 G$ by the linear extension of the rule $g \cdot \zeta_h := \zeta_{gh}$, $g, h \in A$.

Given a ring R and a positive integer m, $M_m(R)$ denotes the ring of $m \times m$ matrices over R. The elements of the matrix ring $M_m(\mathbb{Q}G) = \mathbb{Q}G \otimes M_m(\mathbb{Q})$ act on the Hilbert space $(l^2 G)^m = l^2 G \otimes \mathbb{C}^m$.

Given $\theta \in \mathbb{Q}G$, we can investigate the kernel $\ker \theta \subset l^2 G$ of θ. The von Neumann dimension $\dim_{vN} \ker \theta$ of kernel of θ is defined as

$$\dim_{vN} \ker \theta := \mathrm{tr}_{vN}(P_\theta),$$

where $P_\theta : l^2 G \to l^2 G$ is the orthogonal projection onto $\ker \theta$, and the von Neumann trace tr_{vN} on a given operator T is defined as $\mathrm{tr}_{vN}(T) := \langle T\zeta_e, \zeta_e \rangle$, with e being the neutral element of G. We proceed similarly when $\theta \in M_m(\mathbb{Q}G)$, by defining the von Neumann trace on $B(l^2 G) \otimes M_m(\mathbb{C})$ as $\mathrm{tr}_{vN} \otimes \mathrm{tr}$, where tr is the standard matrix trace. For details and motivations see [Eck00] or [Lüc02].

1-D. Thanks and acknowledgements

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2. Preliminaries on certain graphs

In this section we consider directed graphs g whose vertices are labeled by the letters A, B, C, D, I (as in *Initial*) and F (as in *Final*), and whose edges are labeled by integers. The sets of vertices and edges are denoted respectively by $V(g)$ and $E(g)$. The labels of an edge e and a vertex v are denoted respectively by $L(e)$ and $L(v)$. The starting and final vertices of e are denoted respectively by $s(e)$ and $t(e)$.

The Hilbert space spanned by the vertices of g is denoted by l^2g; elements of its canonical basis are denoted by ζ_v, $v \in V(g)$. The scalar product in l^2g is denoted by $\langle\zeta_1, \zeta_2\rangle$. The convention about which place is linear and which is conjugate linear is such that for a given vector $\zeta \in l^2g$ and $v \in V(g)$, the coefficient of ζ_v in the representation of ζ in the canonical basis is equal to $\langle\zeta, \zeta_v\rangle$.

We say that a vertex v is *directly smaller* (resp. *directly greater*) than a vertex w, denote it by $v \leftarrow w$ (resp. $v \rightarrow w$), if and only if there exists an outgoing edge from w to v (resp. from v to w.) The denotation $v < w$ will be used for the binary relation generated by the relation $\leftarrow$. The words "greatest" or "smallest" will be used with respect to this relation.

Given a graph g we will consider an operator $T^g : l^2g \to l^2g$ defined in the following way:

$$T^g(\zeta_v) := \sum_{e\in E(g):\ s(e)=v} L(e)\zeta_{t(e)} + \begin{cases} 0 & \text{if } L(v) \in \{I, F\} \\ \zeta_v & \text{otherwise} \end{cases}$$

Sometimes we use the letter T alone when g is understood.

2.1. **Definition.** For a vertex $v \in V(g)$ and $\zeta \in l^2g$ define the *incoming flow at v with respect to ζ* to be

$$\sum_{e\in E(g):t(e)=v} L(e) \cdot \langle\zeta, \zeta_{s(e)}\rangle.$$

The following lemma will be used many times. It follows directly from the definition of T^g.

2.2. **Lemma** ("flow lemma")**.** *If $\zeta \in \ker T$ then for every vertex v with label other than I or F, $-\langle\zeta, \zeta_v\rangle$ is equal to the incoming flow at v. For a vertex with label I or F the incoming flow is* 0.

2-A. The graph $g(k)$

The graph $g(k)$, $k \in \{1, 2, \ldots\}$, is depicted on Figure 1.

We need some notation for vertices. The greatest vertex with label A will be called a_1; for $m < k$ the vertex with label A which is directly smaller than a_m will be called a_{m+1}. The smallest vertex with label B will be called b_1; for $m < k$ the vertex with label B which is directly greater than b_m will be called b_{m+1}.

2.3. **Lemma.** $\dim\ker T^{g(k)} = 0$

Proof. We first check the case $k = 1$, by explicitly writing down the matrix of T.

For $k > 1$ suppose that $\zeta \in l^2(g(k))$ is such that $T(\zeta) = 0$. From the flow lemma we see that $\langle\zeta, \zeta_{a_1}\rangle = \langle\zeta, \zeta_{b_1}\rangle$, and using induction we prove that $\langle\zeta, \zeta_{a_1}\rangle = \langle\zeta, \zeta_{b_k}\rangle$ and finally that $\langle\zeta, \zeta_{a_1}\rangle = \langle\zeta, \zeta_{a_k}\rangle$.

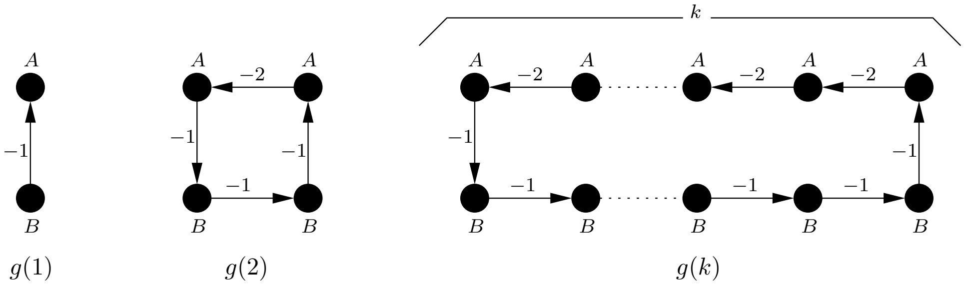

FIGURE 1. The graph $g(k)$

But on the other hand from the flow lemma it follows by induction that $\langle \zeta, \zeta_{a_1} \rangle = 2^{k-1} \langle \zeta, \zeta_{a_k} \rangle$. Since $k > 1$, this proves that $\langle \zeta, \zeta_{a_1} \rangle = 0$, and thus $\langle \zeta, \zeta_{a_i} \rangle = \langle \zeta, \zeta_{b_i} \rangle = 0$. □

2-B. The graph $h(l)$

The graph $h(l)$, $l \in \{1, 2, \ldots\}$, is depicted on Figure 2.

Let the unique vertex with label F be denoted by f. Let the greatest vertex with label C (resp. D) be called c_1 (resp. d_1); for $m < l$ the vertex with label C (resp. D) which is directly smaller than c_m (resp. d_m) will be called c_{m+1} (resp. d_{m+1}).

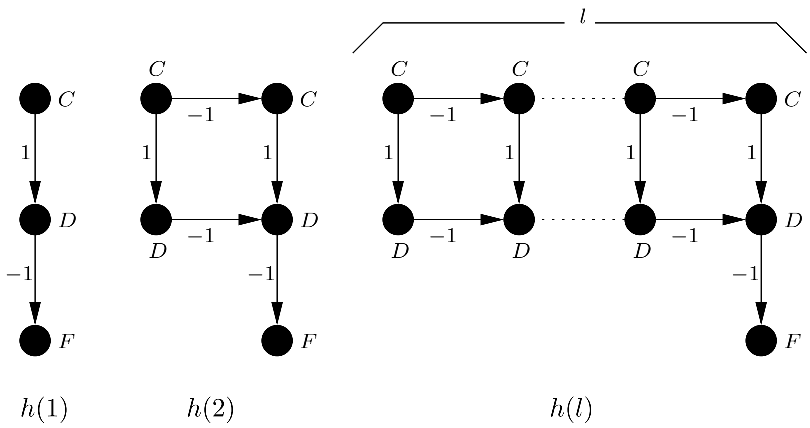

FIGURE 2. The graph $h(l)$

2.4. **Lemma.** $\dim \ker T^{h(l)} = 1$

Proof. Let us consider the matrix of T in the basis $\zeta_{c_1}, \ldots, \zeta_{c_l}, \zeta_{d_1}, \ldots, \zeta_{d_l}, \zeta_f$. This matrix is lower triangular, and the diagonal consists of $2l$ 1's and of one 0 (the one which corresponds to ζ_f.) This shows the lemma. □

2-C. The graph $j(k, l)$

The graph $j(k, l)$, $k, l \in \{1, 2, \ldots\}$ is depicted on Figure 3. It consists of a copy of the graph $g(k)$, a copy of the graph $h(l)$, and one additional vertex with the label I together with three additional edges. The vertex with the label I will be denoted by ι. The rest of the vertices will be denoted in the way described in the two previous subsections.

2.5. **Lemma.** *If* $l = 2^{k-1} - 1$ *then* $\dim \ker T^{j(k,l)} = 2$. *Otherwise* $\dim \ker T^{j(k,l)} = 1$

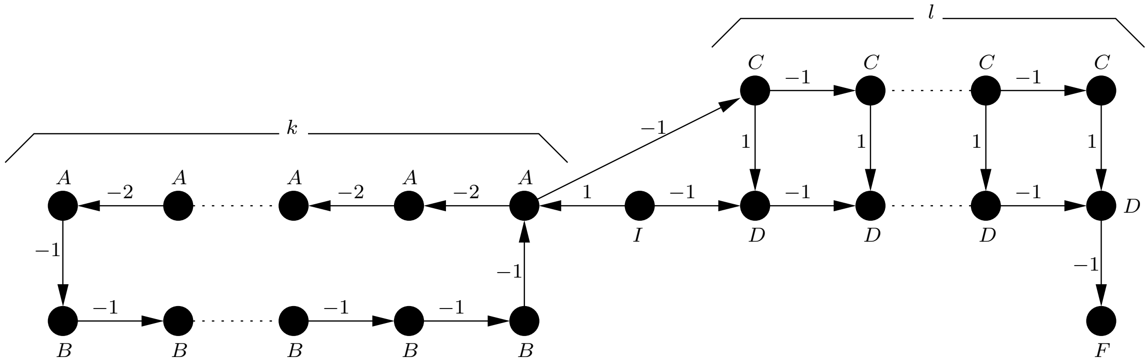

FIGURE 3. The graph $j(k,l)$

Proof. We will focus on the case $k>1$. The arguments in the case $k=1$ are very similar and are left to the reader.

First, let $l=2^{k-1}-1$. The first generator of $\ker T$ is ζ_f, and the coefficients of another generator of $\ker T$ are depicted on Figure 4.

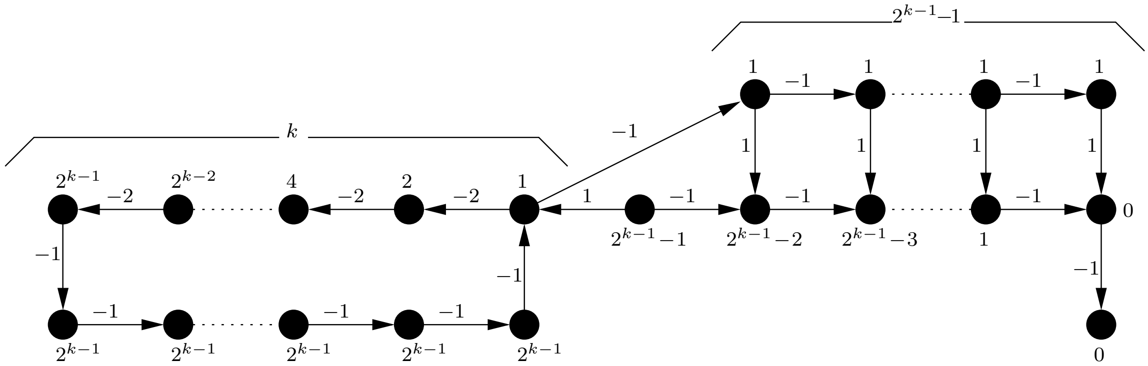

FIGURE 4. Coefficients of the second generator of $\ker T^{j(k,2^{k-1}-1)}$

To see that these two vectors generate the whole $\ker T$ let us prove a general (i.e. valid for all pairs (k,l)) claim:

Claim. *Let $\zeta\in\ker T$ be such that $\langle\zeta,\zeta_f\rangle=0$ and $\langle\zeta,\zeta_{a_1}\rangle=0$. Then $\zeta=0$.*

Proof. First we see from the flow lemma that $\langle\zeta,\zeta_{a_1}\rangle=0$ implies $\langle\zeta,\zeta_\iota\rangle=\langle\zeta,\zeta_{c_1}\rangle=0$ on the one hand, and on the other we see inductively that $\langle\zeta,\zeta_{a_i}\rangle=\langle\zeta,\zeta_{b_i}\rangle=0$ for $i=1,\dots k$.

Now, $\langle\zeta,\zeta_{c_1}\rangle=\langle\zeta,\zeta_\iota\rangle=0$ implies that $\langle\zeta,\zeta_{d_1}\rangle=0$, and $\langle\zeta,\zeta_{c_i}\rangle=0$ gives us $\langle\zeta,\zeta_{c_{i+1}}\rangle=0$. Finally $\langle\zeta,\zeta_{d_i}\rangle=\langle\zeta,\zeta_{c_{i+1}}\rangle=0$ implies $\langle\zeta,\zeta_{d_{i+1}}\rangle=0$ which shows that in fact also $\langle\zeta,\zeta_{c_i}\rangle=\langle\zeta,\zeta_{d_i}\rangle=0$ for all $i=1,\dots,l$; and $\langle\zeta,\zeta_f\rangle$ is equal to 0 by assumption. □

Thus to finish the proof it is enough to show that if $\zeta\in\ker T$ is such that $\langle\zeta,\zeta_{a_1}\rangle=1$ then $l=2^{k-1}-1$.

Indeed, $\langle\zeta,\zeta_{a_1}\rangle=1$ implies $\langle\zeta,\zeta_{a_2}\rangle=2$ and, inductively, $\langle\zeta,\zeta_{a_k}\rangle=2^{k-1}$. This implies that $\langle\zeta,\zeta_{b_k}\rangle=2^{k-1}$, and, by induction, $\langle\zeta,\zeta_{b_1}\rangle=2^{k-1}$.

Now, $\langle\zeta,\zeta_{a_1}\rangle=1$ and $\langle\zeta,\zeta_{b_1}\rangle=2^{k-1}$ imply that $\langle\zeta,\zeta_\iota\rangle=2^{k-1}-1$. On the other hand $\langle\zeta,\zeta_{a_1}\rangle=1$ implies also $\langle\zeta,\zeta_{c_1}\rangle=1$; since $\langle\zeta,\zeta_{c_i}\rangle=1$ clearly implies $\langle\zeta,\zeta_{c_{i+1}}\rangle=1$ we get $\langle\zeta,\zeta_{c_i}\rangle=1$ for $i=1,\dots,l$.

Note that $\langle\zeta,\zeta_\iota\rangle=2^{k-1}-1$ and $\langle\zeta,\zeta_{c_1}\rangle=1$ imply $\langle\zeta,\zeta_{d_1}\rangle=2^{k-1}-2$; but from the flow lemma we see $\langle\zeta,\zeta_{d_{i+1}}\rangle=\langle\zeta,\zeta_{d_i}\rangle-\langle\zeta,\zeta_{c_{i+1}}\rangle=\langle\zeta,\zeta_{d_i}\rangle-1$ so using induction we get that $\langle\zeta,\zeta_{d_l}\rangle=2^{k-1}-l-1$.

Note that $T(\zeta_{d_l}) = \zeta_{d_l} - \zeta_f$, and that $T(\zeta_{d_l}^\perp) \perp \zeta_f$, where $\zeta_{d_l}^\perp$ denotes the orthogonal complement of the subspace spanned by ζ_{d_l}. This means that $0 = \langle T(\zeta), \zeta_f\rangle = \langle\ \langle \zeta, \zeta_{d_l}\rangle T(\zeta_{d_l}), \zeta_f\ \rangle = -\langle \zeta, \zeta_{d_l}\rangle$ and thus $2^{k-1} - l - 1 = 0$. □

3. Our computational tool

In this section (X, μ) can be taken to be any probability measure space, and $\rho\colon \Gamma \curvearrowright X$ any probability measure preserving action.

Let us recall some definitions from Section 5 of [Gra10].

Let $S \in L^\infty(X) \rtimes \Gamma$ be given as $S := \sum_{i=1}^n \theta_i \chi_i$, where θ_i's are elements of the group ring $\mathbb{C}\Gamma$, and χ_i's are characteristic functions of pairwise disjoint measurable sets X_i. The coefficients of θ_i's will be denoted by $\theta_i(\gamma)$, i.e. $\theta_i = \sum_{\gamma\in\Gamma} \theta_i(\gamma)\gamma$.

In what follows we can without a loss of generality assume that the union of the sets X_i is the whole of X, by adding to S an additional summand $0 \cdot \chi_{X - \cup X_i}$.

In a directed graph g whose edges and vertices are labeled, $L(v)$ and $L(e)$ will denote, as in Section 2, respectively the label of a vertex v and of an edge e. The rest of the notation from Section 2 will also be adopted.

3.1. **Definition.** An *S-graph* is a directed graph $\mathbf{g}$ whose vertices are labeled by elements of the set $\{1, \ldots, n\}$, and whose edges are labeled by elements of Γ, in such a way that the following conditions hold.

(1) For every vertex v the labels of the edges starting at v are pairwise different.
(2) For every vertex v and every $\gamma \in \operatorname{supp} \theta_{L(v)}$ there exists an edge starting at v with label γ.

An *X-embedded S-graph* is a pair $(\mathbf{g}, \phi)$, where $\mathbf{g}$ is an S-graph and $\phi : V(\mathbf{g}) \to X$ is an injection such that for every edge $e \in E(\mathbf{g})$ we have that $\phi(t(e)) = \rho(L(e))(s(e))$

A *maximal X-embedded S-graph* is an X-embedded S-graph $(\mathbf{g}, \phi)$ such that if $x \in X_i$ and $\gamma \in \operatorname{supp} \theta_i$ are such that $\rho(\gamma)(x) \in \phi(V(\mathbf{g}))$ then $x \in \phi(V(\mathbf{g}))$.

3.2. *Remark.* In the applications it is often convenient to enumerate the vertices of a given S-graph by the sets X_i (instead of numbers $1, \ldots, n$.)

Given a (not necessarily directed) path p in an S-graph $\mathbf{g}$, one can define the label $L(p)$ of p as the product of labels and inverses of labels of consecutive edges in p, depending on their orientation (see Definition 5.3.5 in [Gra10] for details.)

3.3. **Definition.** We will say that an S-graph $\mathbf{g}$ is *simply connected* if and only if for every closed path p in $\mathbf{g}$ the label $L(p)$ of p is the neutral element of Γ.

There is a natural notion of isomorphism for S-graphs (bijection between the sets of vertices which is Γ-equivariant wherever it can) and maximal X-embedded S-graphs (bijection as before which commutes with the embedding maps) - see Definition 5.3.8 in [Gra10] for details. Let $S\text{-Graphs}_{\text{fin}}$ denote the set of isomorphism classes of those S-graphs $\mathbf{g}$ such that $V(\mathbf{g})$ is finite and such that there exists a maximal X-embedded S-graph $(\mathbf{g}, \phi)$. We will sometimes identify maximal X-embedded S-graphs with finite number of vertices with their images in $S\text{-Graphs}_{\text{fin}}$.

For an element $\mathbf{g} \in S\text{-Graphs}_{\text{fin}}$ define $\mu(\mathbf{g})$ to be equal to $\mu(\{x \in X :$ there exists a maximal X-embedded S-graph $(\mathbf{g}, \phi)$ such that $x \in \phi(V(h))\ \})$. This gives a measure on the countable set $S\text{-Graphs}_{\text{fin}}$.

For an S-graph $\mathbf{g}$ and every $i = 1, \ldots, n$ we define $\chi_i^{\mathbf{g}} : l^2\mathbf{g} \to l^2\mathbf{g}$ on the canonical basis by

$$\chi_i^{\mathbf{g}}(\zeta_v) := \begin{cases} \zeta_v & \text{if } i = L(v) \\ 0 & \text{otherwise.} \end{cases}$$

Similarly for $\gamma \in \Gamma$, let $\gamma^{\mathbf{g}}$ be given by

$$\gamma^{\mathbf{g}}(\zeta_{s(e)}) := \begin{cases} \zeta_{t(e)} & \text{if } L(e) = \gamma \\ 0 & \text{otherwise.} \end{cases}$$

Finally define $\theta_i^{\mathbf{g}} := \sum_{\gamma\in\Gamma} \theta(\gamma) \cdot \gamma^{\mathbf{g}}$ and $S^{\mathbf{g}} := \sum_{i=1}^n \theta_i^{\mathbf{g}} \gamma_i^{\mathbf{g}}$.

Let $T \in L^\infty(X) \rtimes \Gamma$ be a polynomial expression in S and χ_i's. For a given S-graph $\mathbf{g}$ define $T^{\mathbf{g}} \colon l^2\mathbf{g} \to l^2\mathbf{g}$ to be the same polynomial expression in $S^{\mathbf{g}}$ and $\chi_i^{\mathbf{g}}$'s.

3.4. **Theorem.** *Suppose that*

(1) *the measure μ on S-Graphs$_{fin}$ is a probability measure,*
(2) *the elements of S-Graphs$_{fin}$ are simply-connected,*
(3) *the elements of S-Graphs$_{fin}$ do not possess non-trivial automorphisms (as S-graphs.)*

Then

$$\dim_{vN} \ker T = \sum_{\mathbf{g}\in S\text{-}Graphs_{fin}} \frac{\mu(\mathbf{g})}{|V(\mathbf{g})|} \dim \ker T^{\mathbf{g}}.$$

This is a direct consequence of Theorem 5.4.12 in [Gra10].

4. Description of the operator

Let us fix $p \in \{2, 3, \ldots\}$ and let (X, μ) be the compact abelian group ${\mathbb{Z}_{/p}}^{\mathbb{Z}} \times {\mathbb{Z}_{/2}}^3$ with the normalized Haar measure, let Γ be the group $\mathbb{Z} \times GL_3(\mathbb{Z}_{/2})$, and let $\rho \colon \Gamma \curvearrowright X$ be the action of Γ on X by the following measure-preserving group automorphisms: the generator t of $\mathbb{Z}$ acts on ${\mathbb{Z}_{/p}}^{\mathbb{Z}}$ by $[\rho(t)((a_i))]_j = a_{j+1}$, and $GL_3(\mathbb{Z}_{/2})$ acts in the natural way on ${\mathbb{Z}_{/2}}^3$.

We will now describe an operator T in the von Neumann algebra $L^\infty(X) \rtimes \Gamma$. One standard monograph on the subject of von Neumann algebras is [Sak98]. For our notation see Subsection 2.2 of [Gra10].

It is convenient to think of elements of ${\mathbb{Z}_{/2}}^3$ as "labels". Thus let A, B, C, D, F, I, U_1, U_2 (U stands for "unimportant") denote the elements of ${\mathbb{Z}_{/2}}^3$. The only assumption on the bijection between the above letters and the elements of ${\mathbb{Z}_{/2}}^3$ is that the first 6 symbols correspond to non-zero elements of ${\mathbb{Z}_{/2}}^3$.

For every pair $(x, y) \in \{A, B, C, D, F, I\}$, let us fix an automorphism $[xy] \in GL_3(\mathbb{Z}_{/2})$ which sends x to y, in such a way that

$$[xy] = [yx]^{-1} \tag{4.1}$$

and

$$[AC][CD] = [AI][ID]. \tag{4.2}$$

When dealing with subsets of $\mathbb{Z}_{/p}$ and ${\mathbb{Z}_{/p}}^{\mathbb{Z}}$, the symbol 0 will denote the set $\{0\} \subset \mathbb{Z}_{/p}$ and the symbol 1 will denote the set $\{1, 2, 3, \ldots, p-1\} \subset \mathbb{Z}_{/p}$. Let

$$(\varepsilon_{-a}\varepsilon_{-a+1} \ldots \varepsilon_{-1}\underline{\varepsilon_0}\varepsilon_1 \ldots \varepsilon_b, x),$$

where $\varepsilon_i \in \{0, 1\} \subset 2^{\mathbb{Z}_{/p}}$ denote the following subset of X:

$$\{((m_i), y) \in {\mathbb{Z}_{/p}}^{\mathbb{Z}} \times {\mathbb{Z}_{/2}}^3 : m_{-a} \in \varepsilon_{-a}, \ldots, m_b \in \varepsilon_b, y = x\}.$$

Let

$$\chi(\varepsilon_{-a}\varepsilon_{-a+1} \ldots \varepsilon_{-1}\underline{\varepsilon_0}\varepsilon_1 \ldots \varepsilon_b, x)$$

be the characteristic function of $(\varepsilon_{-a}\varepsilon_{-a+1} \ldots \varepsilon_{-1}\underline{\varepsilon_0}\varepsilon_1 \ldots \varepsilon_b, x)$.

Let us define an operator S as the sum of the following summands:

$$(-t[ID] + t^{-1}[IA]) \cdot \chi(1\underline{0}1, I) \tag{4.3}$$

$$(-t^2[AC] - 2t^{-1}) \cdot \chi(1\underline{1}01, A) \tag{4.4}$$

$$-t^2[AC] \cdot \chi(0\underline{1}01, A) \tag{4.5}$$

$$-2t^{-1} \cdot \chi(1\underline{1}00, A) \tag{4.6}$$

$$0 \cdot \chi(0\underline{1}00, A) \tag{4.7}$$

$$-2t^{-1} \cdot \chi(1\underline{1}1, A) \tag{4.8}$$

$$-[AB] \cdot \chi(0\underline{1}1, A) \tag{4.9}$$

$$-t \cdot \chi(\underline{1}1, B) \tag{4.10}$$

$$-[BA] \cdot \chi(\underline{1}0, B) \tag{4.11}$$

$$(-t + [CD]) \cdot \chi(\underline{1}1, C) \tag{4.12}$$

$$+[CD] \cdot \chi(\underline{1}0, C) \tag{4.13}$$

$$-t \cdot \chi(\underline{1}1, D) \tag{4.14}$$

$$-[DF] \cdot \chi(\underline{1}0, D) \tag{4.15}$$

$$0 \cdot \chi(\underline{1}0, F) \tag{4.16}$$

$$0 \cdot \chi(U), \tag{4.17}$$

where U denotes "all the rest", i.e. the complement of the union of the sets $(1\underline{0}1, I)$, $(1\underline{1}01, A)$, $(0\underline{1}01, A)$, $(1\underline{1}00, A)$, $(0\underline{1}00, A)$, $(1\underline{1}1, A)$, $(0\underline{1}1, A)$, $(\underline{1}1, B)$, $(\underline{1}0, B)$, $(\underline{1}1, C)$, $(\underline{1}0, C)$, $(\underline{1}1, D)$, $(\underline{1}0, D)$ and $(\underline{1}0, F)$; and $\chi(U)$ is the characteristic function of U.

The operator T in which we are interested is defined as

$$T := S + (1 - \chi(U) - \chi(1\underline{0}1, I) - \chi(\underline{1}0, F)) \tag{4.18}$$

5. Application of the computational tool

We will now compute $\dim_{vN} \ker T$, where T is the operator from Section 4. First we compute the (countable) measure space $S\text{-Graphs}_{\text{fin}}$ (S is also from Section 4.)

5-A. The trivial S-graph $\mathbf{u}$

The S-graph $\mathbf{u}$ is shown on Figure 5. It consists of a single vertex with label U and no edges.

Figure 5. The S-graph $\mathbf{u}$

5.1. **Lemma.**

(1) $\dim \ker T^{\mathbf{u}} = 1$
(2) *The S-graph $\mathbf{u}$ does not possess non-trivial automorphisms.*
(3) *The S-graph $\mathbf{u}$ is simply-connected.*
(4) $\mu(\mathbf{u}) = \frac{1}{8}(2 + 5\frac{1}{p} + p^3 + 2\frac{p-1}{p}p^2 + \frac{p-1}{p} + (\frac{p-1}{p})^2)$

Proof. Note that properties (1)-(3) concern S-graphs (as opposed to embedded S-graphs.)

(1) is clear since $T^{\mathbf{u}}$ is the 0-endomorphism of a one-dimensional space.

(2) and (3) are also clear.

As to (4), note that for every point x of U we get an embedded S-graph $(\mathbf{u}, \phi)$ by sending the unique vertex of $\mathbf{u}$ to x. We will now check that $(\mathbf{u}, \phi)$ is in fact a maximal embedded S-graph.

Note that $U = (\underline{0}, A) \cup (\underline{0}, B) \cup (\underline{0}, C) \cup (\underline{0}, D) \cup (\cdot, U_1) \cup (\cdot, U_2) \cup (\underline{0}, F) \cup (\underline{1}1, F) \cup (\underline{1}, I) \cup (1\underline{0}0, I) \cup (0\underline{0}1, I) \cup (0\underline{0}0, I)$.

Suppose for example that $x \in (\underline{0}, A)$, and consider for example the summand (4.3) of S, i.e. $(-t[ID] + t^{-1}[IA]) \cdot \chi(1\underline{0}1, I)$. According to Definition 3.1 we need to check that $x \notin \rho(t[ID])((1\underline{0}1, I)) \cup \rho(t^{-1}[IA])((1\underline{0}1, I))$. This is clear since $\rho(t[ID])((1\underline{0}1, I)) \subset (\cdot, D)$ and $\rho(t^{-1}[IA])((1\underline{0}1, I)) = (\underline{1}01, A)$.

All the remaining cases (4.4) - (4.16) are checked in an analogous straight-forward fashion. Similarly when x is an element of another summand of U.

This shows that $\mu(\mathbf{u}) \geq \mu(U)$, which is easily computed to be $\frac{1}{8}(2 + 5\frac{1}{p} + p^3 + 2\frac{p-1}{p}p^2 + \frac{p-1}{p} + (\frac{p-1}{p})^2)$. The opposite inequality is clear since the unique vertex of $\mathbf{u}$ has to be sent to U. $\square$

5-B. The S-graph $\mathbf{g}(k)$

The S-graph $\mathbf{g}(k)$, $k \in \{1, 2, \ldots\}$, is shown on Figure 6.

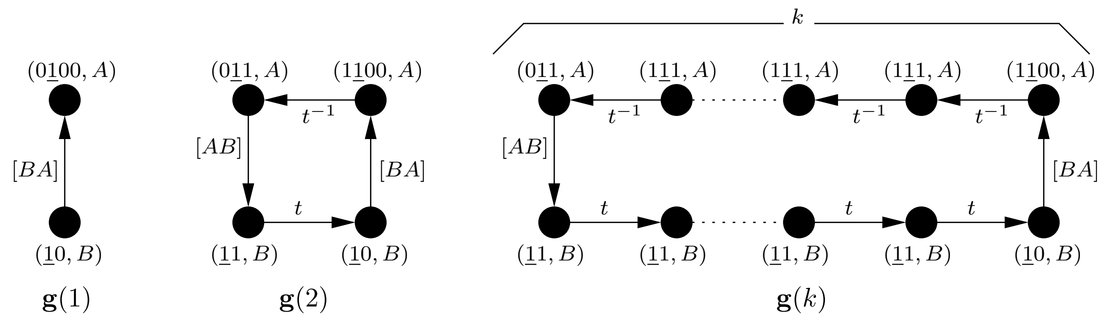

Figure 6. The S-graph $\mathbf{g}(k)$

It is straightforward to see that there is a unique bijection $V(\mathbf{g}(k)) \to V(g(k))$ which induces an isomorphism of directed graphs, and which sends vertices with labels of the form $(..., x)$ to vertices with the label x, for every $x \in \{A, B, C, D, I, F\}$. Note that this bijection induces an isomorphism $l^2\mathbf{g} \to l^2 g$ which intertwines $T^{\mathbf{g}}$ with T^g.

5.2. **Lemma.**

(1) $\dim \ker T^{\mathbf{g}(k)} = 0$

(2) *The S-graphs $\mathbf{g}(k)$ do not possess non-trivial automorphisms.*

(3) *The S-graphs $\mathbf{g}(k)$ are simply-connected.*

(4) $\mu(\mathbf{g}(k)) \geq 2k \cdot \frac{1}{8} \cdot (\frac{1}{p})^3 \cdot (\frac{p-1}{p})^k$

Proof. (1) follows from the existence of an isomorphism $l^2\mathbf{g} \to l^2 g$ intertwining $T^{\mathbf{g}}$ with T^g.

(2) and (3) are straightforward to check using the fact that $[AB][BA] = \mathrm{Id}$ (which follows from equation (4.1).)

As to (4), let x be a fixed element of the set $(01^{k-1}\underline{1}00, A)$, where 1^x denotes x symbols 1. Let us denote x by $(-01^{k-1}\underline{1}00-, A)$. Similarly, for example $t^{-1}(x)$ will be denoted by $(-01^{k-2}\underline{1}100-, A)$.

On Figure 7 we show an embedded S-graph $(\mathbf{g}(k), \phi)$. Label of a given vertex is the value of ϕ on this vertex. In particular, different vertices are mapped to different points of X.

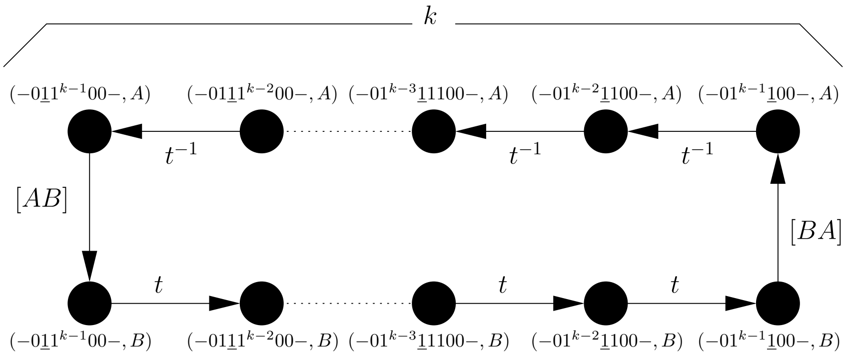

FIGURE 7. The embedded S-graph $(\mathbf{g}(k), \phi)$

As in Lemma 5.1, it is straightforward, although tedious, to check from the definition of S that Figure 7 contains in fact a maximal embedded S-graph. It follows that $\mu(\mathbf{g}(k))$ is at least equal to $|V(\mathbf{g}(k))| \cdot \mu((01^{k-1}\underline{1}00, A)) = 2k \cdot \frac{1}{8} \cdot (\frac{1}{p})^3 (\frac{p-1}{p})^k$. □

5-C. THE S-GRAPH $\mathbf{h}(l)$

The S-graph $\mathbf{h}(l)$ is shown on on Figure 8.

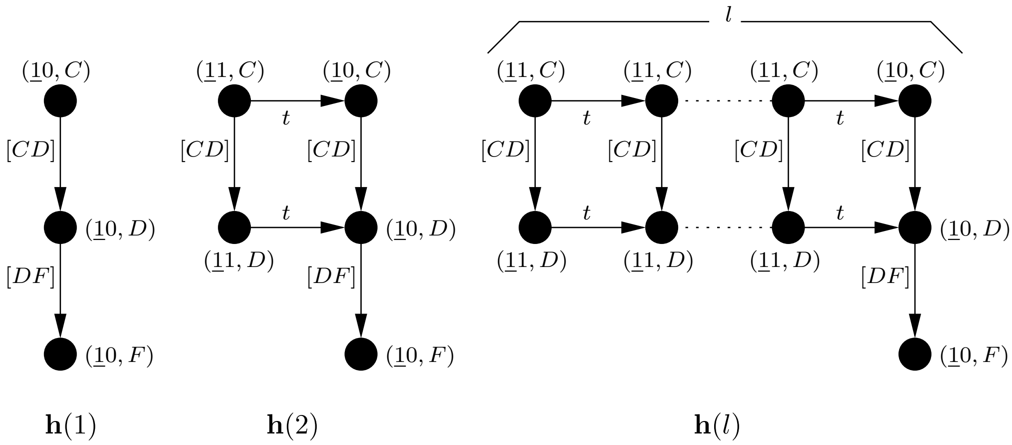

FIGURE 8. The S-graph $\mathbf{h}(l)$

As in Subsection B, note the existence of a bijection $V(\mathbf{h}(l)) \to V(h(l))$ which induces an isomorphism $l^2\mathbf{h}(l) \to l^2 h(l)$ intertwining $T^{\mathbf{h}(l)}$ and $T^{h(l)}$.

5.3. **Lemma.**

(1) $\dim\ker T^{\mathbf{h}(l)} = 1$
(2) *The S-graphs $\mathbf{h}(l)$ do not possess non-trivial automorphisms.*
(3) *The S-graphs $\mathbf{h}(l)$ are simply-connected.*
(4) $\mu(\mathbf{h}(l)) \geq (2l+1) \cdot \frac{1}{8} \cdot (\frac{1}{p})^3 \cdot (\frac{p-1}{p})^l$

Proof. (1), (2) and (3) are proved as in Lemma 5.2.

To prove (4) we proceed also as in Lemma 5.2, and we use analogous notation. Thus let $x = (-00\underline{1}1^{l-1}0-, C)$ be a fixed element of the set $(00\underline{1}1^{l-1}0, C)$. On Figure 9 we show an embedded S-graph $(h(l), \phi)$.

It is again straightforward but tedious to check that Figure 9 contains in fact a maximal embedded S-graph. It follows that $\mu(\mathbf{h}(l))$ is at least equal to $|V(\mathbf{h}(l))| \cdot \mu((00\underline{1}1^{l-1}0, A)) = (2l+1) \cdot \frac{1}{8} \cdot (\frac{1}{p})^3 \cdot (\frac{p-1}{p})^l$ □

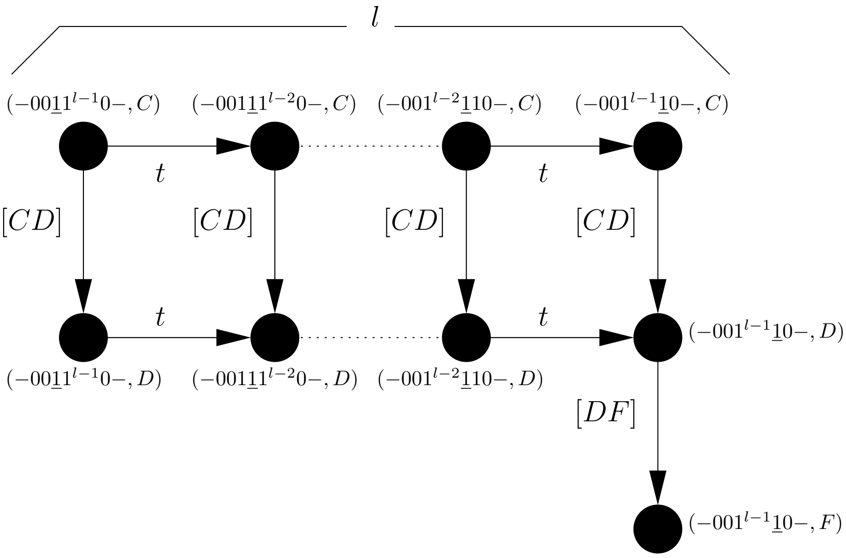

FIGURE 9. The embedded S-graph $(\mathbf{h}(l), \phi)$

5-D. The S-graph $\mathbf{j}(k,l)$

The S-graph $\mathbf{j}(k,l)$ is shown on Figure 10.

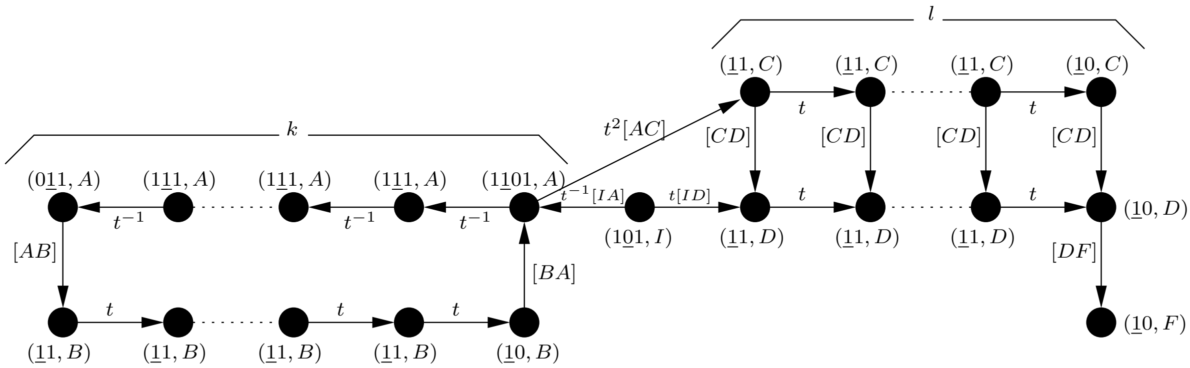

FIGURE 10. The S-graph $\mathbf{j}(k,l)$

As in Subsection B, note the existence of a bijection $V(\mathbf{j}(k,l)) \to V(j(k,l))$ which induces an isomorphism $l^2\mathbf{j}(k,l) \to l^2 j(k,l)$ intertwining $T^{\mathbf{j}(k,l)}$ and $T^{j(k,l)}$.

5.4. **Lemma.**

(1) $\dim\ker T^{\mathbf{j}(k,l)} = \begin{cases} 2 & \textit{if } l = 2^{k-1}-1 \\ 1 & \textit{otherwise} \end{cases}$

(2) *The S-graphs $\mathbf{j}(k,l)$ do not possess non-trivial automorphisms.*

(3) *The S-graphs $\mathbf{j}(k,l)$ are simply-connected.*

(4) $\mu(\mathbf{j}(k,l)) \geq (2k+2l+2)\cdot\frac{1}{8}\cdot(\frac{1}{p})^3\cdot(\frac{p-1}{p})^{k+l}$

Proof. (1), (2) are proved as in Lemma 5.2. (3) follows from the fact that $[AB][BA] = [CD][DC] = 1$ (eq. (4.1)) and $[AC][CD] = [AI][ID]$ (eq. (4.2).)

To prove (4) we proceed also as in Lemma 5.2, and we use analogous notation. Thus let $x = (-01^k\underline{0}1^l0-, I)$ be a fixed element of the set $(01^k\underline{0}1^l0, I)$. On Figure 11 we show an embedded S-graph $(j(k,l), \phi)$.

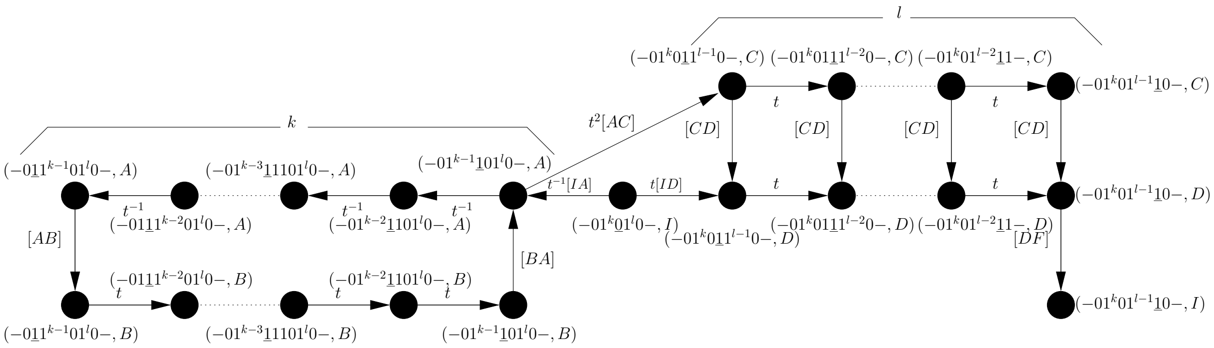

FIGURE 11. The embedded S-graph $(\mathbf{j}(k,l),\phi)$

It is again straightforward and quite tedious to check from the definition of S that Figure 11 contains in fact a maximal embedded S-graph. It follows that $\mu(\mathbf{j}(k,l))$ is at least equal to $|V(\mathbf{j}(k,l))| \cdot \mu((-01^k\underline{0}1^l0-,I)) = (2k+2l+2)\cdot\frac{1}{8}\cdot(\frac{1}{p})^3\cdot(\frac{p-1}{p})^{k+l}$. □

5-E. The measure space S-Graphs$_{\text{fin}}$

In this subsection let $\alpha = \frac{1}{p}$, $\beta = \frac{p-1}{p}$.

5.5. **Corollary.** *The measure space (S-Graphs$_{\text{fin}}$, μ) is a probability measure space. Its only points with non-trivial measure are* $\mathbf{u}$, $\mathbf{g}(k)$, $k \geq 1$, $\mathbf{h}(l)$, $l \geq 1$, *and* $\mathbf{j}(k,l)$, $k,l \geq 1$. *Their measures are as follows:*

$$\begin{aligned}
\mu(\mathbf{u}) &= \frac{1}{8}(2+5\frac{1}{p}+p^3+2\frac{p-1}{p}p^2+\frac{p-1}{p}+(\frac{p-1}{p})^2),\\
\mu(\mathbf{g}(k)) &= 2k\cdot\frac{1}{8}\cdot(\frac{1}{p})^3\cdot(\frac{p-1}{p})^k,\\
\mu(\mathbf{h}(l)) &= (2l+1)\cdot\frac{1}{8}\cdot(\frac{1}{p})^3\cdot(\frac{p-1}{p})^l,\\
\mu(\mathbf{j}(k,l)) &= (2k+2l+2)\cdot\frac{1}{8}\cdot(\frac{1}{p})^3\cdot(\frac{p-1}{p})^{k+l}.
\end{aligned}$$

Proof. We know from Section 5.4 of [Gra10] (see in particular proof of Theorem 5.4.12) that the measure space S-Graphs$_{\text{fin}}$ is always a subspace of a probability measure space. On the other hand we know already that

$$\begin{aligned}
\mu(\mathbf{u}) &\geq \frac{1}{8}(2+5\frac{1}{p}+p^3+2\frac{p-1}{p}p^2+\frac{p-1}{p}+(\frac{p-1}{p})^2),\\
\mu(\mathbf{g}(k)) &\geq 2k\cdot\frac{1}{8}\cdot(\frac{1}{p})^3\cdot(\frac{p-1}{p})^k,\\
\mu(\mathbf{h}(l)) &\geq (2l+1)\cdot\frac{1}{8}\cdot(\frac{1}{p})^3\cdot(\frac{p-1}{p})^l,\\
\mu(\mathbf{j}(k,l)) &\geq (2k+2l+2)\cdot\frac{1}{8}\cdot(\frac{1}{p})^3\cdot(\frac{p-1}{p})^{k+l},
\end{aligned}$$

so to prove the corollary it is enough to check that

$$\begin{aligned}\frac{1}{8}(2+5\frac{1}{p}+p^3+2\frac{p-1}{p}p^2+\frac{p-1}{p}+(\frac{p-1}{p})^2) &\quad + \\ \sum_{k=1}^{\infty}2k\cdot\frac{1}{8}\cdot(\frac{1}{p})^3\cdot(\frac{p-1}{p})^k &\quad + \\ \sum_{l=1}^{\infty}(2l+1)\cdot\frac{1}{8}\cdot(\frac{1}{p})^3\cdot(\frac{p-1}{p})^l &\quad + \\ \sum_{k,l=1}^{\infty}(2k+2l+2)\cdot\frac{1}{8}\cdot(\frac{1}{p})^3\cdot(\frac{p-1}{p})^{k+l} &\quad = \quad 1.\end{aligned}$$

Recall the formula

$$\sum_{n=1}^{\infty}(n+C)x^n = \frac{x}{(1-x)^2}+\frac{Cx}{1-x}$$

for $0 \le x \le 1$. Using this formula we see that

$$\sum_{k\ge1}2k\cdot\frac{1}{8}\cdot\alpha^3\cdot\beta^k = \frac{\alpha^3}{4}\sum_{k\ge1}k\beta^k = \frac{\alpha^3}{4}\cdot\frac{\beta}{\alpha^2} = \frac{\alpha\beta}{4}.$$

Similarly

$$\sum_{l\ge1}(2l+1)\cdot\frac{1}{8}\cdot\alpha^3\beta^l = \frac{\alpha^3}{4}\sum_{l\ge1}l\beta^l+\frac{\alpha^3}{8}\sum_{l\ge1}\beta^l = \frac{\alpha\beta}{4}+\frac{\alpha^3}{8}\frac{\beta}{1-\beta} = \frac{\alpha\beta}{4}+\frac{\alpha^2\beta}{8}.$$

Finally

$$\begin{aligned}\sum_{k,l\ge1}(2k+2l+2)\cdot\frac{1}{8}\cdot\alpha^3\beta^{k+l} &= \frac{\alpha^3}{4}\sum_k\beta^k\sum_l(l+(k+1))\beta^l \\ &= \frac{\alpha^3}{4}\sum_k\beta^k(\frac{(k+1)\beta}{\alpha}+\frac{\beta}{\alpha^2}) \\ &= \frac{\alpha^2\beta}{4}\sum_k(k+1)\beta^k+\frac{\alpha\beta}{4}\sum_k\beta^k \\ &= \frac{\alpha^2\beta}{4}(\frac{\beta}{\alpha^2}+\frac{\beta}{\alpha})+\frac{\alpha\beta}{4}\frac{\beta}{\alpha} \\ &= \frac{\beta^2}{2}+\frac{\alpha\beta^2}{4}.\end{aligned}$$

Putting everything together we get

$$\frac{1}{8}(2+5\alpha+\alpha^3+2\beta\alpha^2+\beta+\beta^2)+\frac{\alpha\beta}{4}+(\frac{\alpha\beta}{4}+\frac{\alpha^2\beta}{8})+(\frac{\beta^2}{2}+\frac{\alpha\beta^2}{4}) =$$

$$= \left(\frac{1}{4}+\frac{5}{8}\alpha+\frac{1}{8}\alpha^3+\frac{1}{4}\alpha^2-\frac{1}{4}\alpha^3+\frac{1}{8}-\frac{1}{8}\alpha+\frac{1}{8}-\frac{1}{4}\alpha+\frac{1}{8}\alpha^2\right)+$$

$$+\left(\frac{1}{4}\alpha-\frac{1}{4}\alpha^2\right)+\left(\frac{1}{4}\alpha-\frac{1}{4}\alpha^2+\frac{1}{8}\alpha^2-\frac{1}{8}\alpha^3\right)+\left(\frac{1}{2}-\alpha+\frac{1}{2}\alpha^2+\frac{1}{4}\alpha-\frac{1}{2}\alpha^2+\frac{1}{4}\alpha^3\right) = 1,$$

as required. □

5.6. **Corollary.** *Let T be the operator defined in Section 4. Then*

$$\dim_{vN}\ker T = \frac{4p^3+3p^2+2p-1}{8p^3}+\frac{1}{8p^2(p-1)}\sum_{k=1}^{\infty}(\frac{p-1}{p})^{k+2^{k-1}},$$

which is a transcendental number.

Proof. As Lemmas 5.1-5.4 and Corollary 5.5 show, we can use Theorem 3.4:

$$\dim_{vN} \ker T = \sum_{\mathbf{g} \in S\text{-Graphs}_{\text{fin}}} \frac{\mu(\mathbf{g})}{|V(\mathbf{g})|} \dim \ker T^{\mathbf{g}}$$

According to Corollary 5.5 the above sum can be written as

$$\begin{aligned} \frac{1}{8}(2+5\alpha+\alpha^3+2\beta\alpha^2+\beta+\beta^2)\cdot \dim\ker T^{\mathbf{u}} \quad &+ \\ \sum_{k=1}^{\infty} \frac{1}{8}\cdot \alpha^3 \cdot \beta^k \cdot \dim\ker T^{\mathbf{g}(k)} \quad &+ \\ +\sum_{l=1}^{\infty} \frac{1}{8}\cdot \alpha^3\beta^l \dim\ker T^{\mathbf{h}(l)} \quad &+ \\ \sum_{k,l=1}^{\infty} \frac{1}{8}\cdot \alpha^3\beta^{k+l} \dim\ker T^{\mathbf{j}(k,l)}. \end{aligned}$$

Substituting the values for $\dim\ker T$'s we get

$$\begin{aligned} \frac{1}{8}(2+5\alpha+\alpha^3+2\beta\alpha^2+\beta+\beta^2) \quad &+ \\ 0 \quad &+ \\ \sum_{l=1}^{\infty} \frac{1}{8}\cdot \alpha^3\beta^l \quad &+ \\ \sum_{k,l=1}^{\infty} \frac{1}{8}\cdot \alpha^3\beta^{k+l} + \sum_{k=1}^{\infty} \frac{1}{8}\cdot \alpha^3\beta^{k+2^{k-1}-1}. \end{aligned}$$

Noting that $\sum_{k,l=1}^{\infty} \beta^{k+l} = \sum_k \beta^k \sum_l \beta^l = (\frac{\beta}{\alpha})^2$ we get

$$\frac{1}{8}(2+5\alpha+\alpha^3+2\beta\alpha^2+\beta+\beta^2) + \frac{1}{8}\alpha^2\beta + \frac{1}{8}\alpha\beta^2 + \frac{1}{8}\frac{\alpha^3}{\beta}\sum_{k=1}^{\infty} \beta^{k+2^{k-1}},$$

which is easily seen to be what we want.

Clearly to prove transcendence od $\dim_{vN} \ker T$ it is enough to prove that $\sum_{k=1}^{\infty} (\frac{p-1}{p})^{k+2^{k-1}}$ is transcendental. This follows directly from Tanaka's Theorem 1 in [aT02]. Although similar series have been investigated already by Mahler in [Mah29], to the author's best knowledge [aT02] is the first work which implies transcendence of $\sum_{k=1}^{\infty} (\frac{p-1}{p})^{k+2^{k-1}}$. □

6. Back to the lamplighter groups

In the previous section we have seen that the operator $T \in L^\infty(\mathbb{Z}_{/p}{}^{\mathbb{Z}} \times \mathbb{Z}_{/2}{}^3) \rtimes (\mathbb{Z} \times Gl_3(\mathbb{Z}_{/2}))$ defined in Section 4 has kernel with transcendental von Neumann dimension. Using Pontryagin duality (see for example Subsection 4.2 of [Gra10] for details) we get an operator $\widehat{T} \in K\Big[\Big(\mathbb{Z}_{/p}{}^{\oplus\mathbb{Z}} \rtimes \mathbb{Z}\Big) \times \Big(\mathbb{Z}_{/2}{}^3 \rtimes Gl_3(\mathbb{Z}_{/2})\Big)\Big]$ with the same dimension of the kernel, where K is the smallest subfield of $\mathbb{C}$ such all the characteristic functions which appear in the definitions of S, i.e. in equations (4.3)-(4.16), and of T, i.e. equation (4.18), are in the image of the Fourier transform

$$K(\mathbb{Z}_{/p}{}^{\oplus\mathbb{Z}} \oplus \mathbb{Z}_{/2}{}^3) \to L^\infty(\mathbb{Z}_{/p}{}^{\mathbb{Z}} \times \mathbb{Z}_{/2}{}^3),$$

where $K(\mathbb{Z}_{/p}{}^{\oplus\mathbb{Z}} \oplus \mathbb{Z}_{/2}{}^3)$ is the group ring over K of the group $\mathbb{Z}_{/p}{}^{\oplus\mathbb{Z}} \oplus \mathbb{Z}_{/2}{}^3$.

We claim that in our case $K = \mathbb{Q}$. Indeed, all the functions in the equations (4.3)-(4.16) and (4.18) are products of functions of two types: (1) functions of the form

$$\mathbb{Z}_{/p}{}^{\mathbb{Z}} \times \mathbb{Z}_{/2}{}^3 \to \mathbb{Z}_{/2} \xrightarrow{f} \mathbb{R},$$

where f is the characteristic function of either the set $\{0\}$ or $\{1\}$; and (2) functions of the form

$$\mathbb{Z}_{/p}{}^{\mathbb{Z}} \times \mathbb{Z}_{/2}{}^3 \to \mathbb{Z}_{/p} \xrightarrow{g} \mathbb{R},$$

where g is either the characteristic function of the set $\{0\}$ or of the set $\{1, 2, \ldots, p-1\}$. Thus our claim follows from functoriality of the Pontryagin duality and the fact that both f and g are in the images of Fourier transforms

$$\mathbb{Q}\widehat{\mathbb{Z}_{/2}} \to L^\infty(\mathbb{Z}_{/2})$$

and, respectively,

$$\mathbb{Q}\widehat{\mathbb{Z}_{/p}} \to L^\infty(\mathbb{Z}_{/p}).$$

Indeed, it is straightforward to check that $\pi := \frac{0+1}{2} \in \mathbb{Q}\widehat{\mathbb{Z}_{/2}}$ is mapped to the characteristic function of $\{0\} \subset \mathbb{Z}_{/2}$, $1 - \pi$ is mapped to the characteristic function of $\{1\} \subset \mathbb{Z}_{/2}$; and $\sigma := \frac{0+1+\ldots+(p-1)}{p} \in \mathbb{Q}\widehat{\mathbb{Z}_{/p}}$ is mapped to the characteristic function of $\{0\} \subset \mathbb{Z}_{/p}$, $1 - \sigma$ is mapped to the characteristic function of $\{1, 2, \ldots, (p-1)\} \subset \mathbb{Z}_{/p}$.

It is clear that to finish the proof of Theorem A it is enough to prove the following lemma.

6.1. **Lemma.** *Let G be a discrete countable group and let H be a finite group. Then $|H| \cdot \mathcal{C}(G \times H) = \mathcal{C}(G)$.*

Proof. Note that $|H| \cdot \mathcal{C}(G \times H) \supseteq \mathcal{C}(G)$, since there is a projection π in $\mathbb{Q}H \subset \mathbb{Q}G \times H$ whose trace is $\frac{1}{|H|}$ and which commutes with $\mathbb{Q}G \subset \mathbb{Q}(G \times H)$. It is easy to check that $|H| \cdot \dim_{vN} \ker((1-\pi) + \pi\theta) = \dim_{vN} \ker \theta$ (see for example proof of Proposition 4.2.7 in [Gra10].)

For the other containment note first that the regular representation of H gives rise to a unital injection of *-algebras $\iota\colon \mathbb{Q}H \hookrightarrow M_{|H|}(\mathbb{Q})$ such that $|H| \operatorname{tr}_H(\theta) = \operatorname{tr}(\iota(\theta))$. This means that the unital *-homomorphism $\widehat{\iota} := \operatorname{Id} \otimes \iota : M_k(\mathbb{Q}(G \times H)) = M_k(\mathbb{Q}G) \otimes \mathbb{Q}H \to M_k(\mathbb{Q}G) \otimes M_{|H|}(\mathbb{Q}) = M_{k+|H|}(\mathbb{Q}G)$ also has the property $|H| \operatorname{tr}_H(\theta) = \operatorname{tr}(\widehat{\iota}(\theta))$.

Now the result follows for example from Lemma 4.2.1 in [Gra10] by taking G there to be equal to $G \times H$ here, and L there to be $M_{k+|H|}(\mathbb{Q}G)$ with normalized trace. ☐

References

[aT02] Taka aki Tanaka. Transcendence of the values of certain series with Hadamard's gaps. *Arch. Math. (Basel)*, 78(3):202–209, 2002.

[Ati76] M. F. Atiyah. Elliptic operators, discrete groups and von Neumann algebras. In *Colloque "Analyse et Topologie" en l'Honneur de Henri Cartan (Orsay, 1974)*, pages 43–72. Astérisque, No. 32–33. Soc. Math. France, Paris, 1976.

[Aus09] Tim Austin. Rational group ring elements with kernels having irrational dimension, 2009. Preprint, available at http://arxiv.org/abs/0909.2360.

[DS02] Warren Dicks and Thomas Schick. The spectral measure of certain elements of the complex group ring of a wreath product. *Geom. Dedicata*, 93:121–137, 2002.

[Eck00] Beno Eckmann. Introduction to l_2-methods in topology: reduced l_2-homology, harmonic chains, l_2-Betti numbers. *Israel J. Math.*, 117:183–219, 2000. Notes prepared by Guido Mislin.

[Gra10] L. Grabowski. On Turing machines, dynamical systems and the Atiyah problem. *ArXiv e-prints*, April 2010. Available at http://arxiv.org/abs/1004.2030.

[GŻ01] Rostislav I. Grigorchuk and Andrzej Żuk. The lamplighter group as a group generated by a 2-state automaton, and its spectrum. *Geom. Dedicata*, 87(1-3):209–244, 2001.

[Lüc02] Wolfgang Lück. *L^2-invariants: theory and applications to geometry and K-theory*, volume 44 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2002.

[LW10] F. Lehner and S. Wagner. Free Lamplighter Groups and a Question of Atiyah. *ArXiv e-prints*, May 2010. Available at http://arxiv.org/abs/1005.2347.

[Mah29] Kurt Mahler. Arithmetische Eigenschaften der Lösungen einer Klasse von Funktionalgleichungen. *Math. Ann.*, 101(1):342–366, 1929.

[PSZ10] M. Pichot, T. Schick, and A. Zuk. Closed manifolds with transcendental L2-Betti numbers. *ArXiv e-prints*, May 2010. Available at http://arxiv.org/abs/1005.1147.

[Sak98] Shôichirô Sakai. *C^*-algebras and W^*-algebras*. Classics in Mathematics. Springer-Verlag, Berlin, 1998. Reprint of the 1971 edition.

Georg-August-Universität Göttingen, Mathematisches Institut, Bunsenstrasse 3-5, D-37073 Göttingen, Germany

E-mail address: graboluk@gmail.com