Solution of the Bosonic and Algebraic Hamiltonians by using AIM

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(Dated: September 2, 2010)

Abstract

We apply the notion of asymptotic iteration method (AIM) to determine eigenvalues of the bosonic Hamiltonians that include a wide class of quantum optical models. We consider solutions of the Hamiltonians, which are even polynomials of the fourth order with the respect to Boson operators. We also demonstrate applicability of the method for obtaining eigenvalues of the simple Lie algebraic structures. Eigenvalues of the multi-boson Hamiltonians have been obtained by transforming in the form of the single boson Hamiltonian in the framework of AIM.

PACS numbers: 03.65.Ge, 03.65.Fd, 02.30.-f

Keywords: Asymptotic Iteration Method, Analytical solution, Boson Hamiltonian

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I. INTRODUCTION

The study of the same problems from different point of view lead to the progress of the science and include a lot of mathematical tastes. An iteration technique [1–7] has recently been suggested to obtain eigenvalues of Schrödinger equation which improves both analytical and numerical determination of the eigenvalues and has been developed for some matrix Hamiltonians, arising from the development of fast computers[8, 9]. Asymptotic iteration method (AIM) is very efficient to establish eigenvalues of the various quantum mechanical systems, because of their simplicity and low round off error. This method has been widely applied for determination of eigenvalues of the Schrödinger type equations. Encouraged by its satisfactory performance through comparisons with the other methods, we feel tempted to develop AIM to obtain eigenvalues of algebraic Hamiltonians. In contrast to the solution of the Schrödinger equation by using AIM including Coulomb, Morse, harmonic oscillator, etc. type potentials, the study of the algebraic Hamiltonians [10–12] has not attracted much attention in the literature. Such Hamiltonians have been found to be useful in the study of electronic properties of semiconductors, quantum dots and quantum wells. It is evident that the formalism can also be developed for solving algebraic equations.

The algebraic techniques have been proven to be useful in the description of the physical problems in a variety of fields [11–18]. In recent years there has been a great deal of interest in quantum optical models which reveal new physical phenomena described by the Hamiltonians expressed as nonlinear functions of Lie algebra generators or boson and/or fermion operators[19–23]. Such systems have often been analyzed by using numerical methods, because the implementation of the Lie algebraic techniques to solve those problems is not very efficient and most of the other analytical techniques do not yield simple analytical expressions. They require tedious calculations. In principle, if a Hamiltonian is expressed by boson operators, one could rely directly on the known formulae of the action of boson operators on a state with a defined number of particles without solving differential equations. Apart from the mentioned method, sometimes the Hamiltonians can be put in a simple form by using the transformation properties of the bosons.

In this article, AIM is suggested and adapted to solve the bosonic Hamiltonians. We note that this has never been done before. As a particular case our model includes the solutions of the Hamiltonian of the multiphoton interactions and the Hamiltonian of the systems of photons and bosons expressed in a single mode form. We briefly discuss the bosonic construction of the various Hamiltonians. These Hamiltonians are not only mathematically interesting but they also have potential interest in physics.

The paper is organized as follows. In section 2, we briefly review the properties of boson and its differential realization. The procedure for solving a bosonic Hamiltonian in the framework of the AIM is presented in this section. Section 3 is devoted to illustrate determination of the eigenvalues of a bosonic Hamiltonian in the framework of the AIM. The bosonization of the physical Hamiltonians whose original forms are given as differential operators is discussed. As a practical example we illustrate the solution of the anharmonic oscillator and multiphoton interaction problems. In section 4, we introduce a technique to obtain eigenvalues of the two mode bosonic Hamiltonians by using AIM. We present the application of the AIM in order to obtain eigenvalues for a class of models describing two-mode multiphoton processes. Finally, we comment on the validity of our method and remark on the possible use of our method in the different fields of the physics.

II. BASIC FORMALISM AND SOLUTION OF SINGLE BOSON HAMILTONIAN

In this section, we illustrate solution of the single boson Hamiltonians, by modifying AIM. The usual differential realization of the annihilation operator a, and creation operator a^+, are given by

$$a^+ = \frac{1}{\sqrt{2}}\left(-\frac{\partial}{\partial x} + x\right) ; \; a = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x} + x\right) \tag{1}$$

and they act on the state $|n\rangle$:

$$a^+ |n\rangle = \sqrt{n+1}\, |n+1\rangle \, ; \; a\, |n\rangle = \sqrt{n}\, |n-1\rangle \tag{2}$$

with the commutation relation

$$\left[a, a^+\right] = 1.$$

A single boson Hamiltonian describing a physical system can be expressed as

$$H = \sum_i \gamma_{i,i} a^i (a^+)^i + \sum_{i,j(i\neq j)} \gamma_{i,j} a^i (a^+)^j \tag{3}$$

where $\gamma_{i,j}$ is a constant. It is obvious that first part of then H is diagonal and exactly solvable. Second part of the Hamiltonian H includes non-diagonal terms and it is usually solved by using various perturbation techniques. Our task is now to develop an AIM to obtain eigenvalues of H. We assume that action of H on the state $|n\rangle$ produce the following three term recurrence relation (or reduced to three term recurrence relation) such that

$$|n+2\rangle = r_n |n+1\rangle + s_n |n\rangle \tag{4}$$

where r_n and $s_n = E - s'_n$ are function of n. From the analogy of the AIM [1] it follows that (4) can be put a more suitable form in order to obtain eigenvalues E of H. Reformulation of (4) provides the following equations:

$$\begin{aligned} n &= 0;\ |2\rangle = r_0 |1\rangle + s_0 |0\rangle = p_0 |1\rangle + q_0 |0\rangle \\ n &= 1;\ |3\rangle = r_1 |2\rangle + s_1 |1\rangle = p_1 |1\rangle + q_1 |0\rangle \\ &\ldots \\ n &= m;\ |m+2\rangle = r_m |m+1\rangle + s_m |m\rangle = p_m |1\rangle + q_m |0\rangle \end{aligned} \tag{5}$$

where p_m and q_m are given by

$$\begin{aligned} p_m &= r_m p_{m-1} + s_m p_{m-2} \\ q_m &= r_m q_{m-1} + s_m q_{m-2} \end{aligned} \tag{6}$$

with the initial conditions

$$p_{-1} = q_{-2} = 1 \text{ and } p_{-2} = q_{-1} = 0.$$

To this end we assume that m is large enough and the states reach their asymptotic values. Thus we can write

$$\begin{aligned} |m+2\rangle &= p_m |1\rangle + q_m |0\rangle \\ |m+3\rangle &= p_{m+1} |1\rangle + q_{m+1} |0\rangle \end{aligned} \tag{7}$$

After all we can concisely write that

$$\frac{p_m}{q_m} = \frac{p_{m+1}}{q_{m+1}} \text{ or } q_m p_{m+1} - q_{m+1} p_m = 0. \tag{8}$$

The last equation can be solved for eigenvalues E, then the last approximation leads to the determination of the eigenvalues of the Hamiltonian H. Before going futher, we note that eigenvalues of the associated problem can be obtained by using the following MATHEMATICA program code. Let us define $|n\rangle = f[n]$ then

```
k = 20; Do[f[n+2] = Simplify[r_n f[n+1]+s_n f[n]], {n, 0, k}]
    (*where k is number of iteration*)
NSolve[Coefficient[f[k + 2], f[0]]*Coefficient[f[k ], f[2]] -
        Coefficient[f[k + 2], f[2]]*Coefficient[f[k ], f[0]] == 0, E1]
    (*E1 is eigenvalues of the H*)
```

In the next sections, we want to illustrate our task on an explicit example.

III. EIGENSTATE OF THE SINGLE BOSON HAMILTONIANS

In this section we study the determination of the single and multi-boson Hamiltonians in the framework of the AIM.

1. Anharmonic oscillator

The solution of the Schrödinger equation including anharmonic potential has attracted a lot of attention, arising its considerable impact on the various branches of physics as well as biology and chemistry. Besides its importance in physics, biology and chemistry, in practice anharmonic oscillator problem is always used to test the accuracy and the efficiency of the unperturbative methods. In this section we take a new look at the solution of the anharmonic oscillator problem through the modified AIM. The equation is described by the Hamiltonian:

$$H = -\frac{d^2}{dx^2} + x^2 + \alpha x^4. \tag{9}$$

where α is a constant. Our task is now to demonstrate that the Hamiltonian (9) can be expressed in terms of the bosons. One way to express the Hamiltonian H with boson operators is to use an appropriate differential realization of bosons. Using the realization (1), the Hamiltonian (9) can be written as:

$$H = a^+a + aa^+ + \frac{\alpha}{4}(a + a^+)^4. \tag{10}$$

When the Hamiltonian (10) acts on the state $|n\rangle$, the eigenvalue equation $H|n\rangle = E|n\rangle$ can be transformed to the following recurrence relation:

$$
\begin{aligned}
(\quad H - E)|n\rangle = (2n+1-E)|n\rangle + \frac{3\alpha}{2}\left(n + n^2 + \frac{1}{2}\right)|n\rangle + \\
\alpha\sqrt{(n+1)(n+2)}\left(n+\frac{3}{2}\right)|n+2\rangle + \alpha\sqrt{n(n-1)}\left(n-\frac{1}{2}\right)|n-2\rangle + \\
\frac{\alpha}{4}\sqrt{(n+1)(n+2)(n+3)(n+4)}\,|n+4\rangle + \frac{\alpha}{4}\sqrt{n(n-1)(n-2)(n-3)}\,|n-4\rangle = 0
\end{aligned}
\tag{11}
$$

Here, the skill is to express the n^{th} *even* state in terms of $|0\rangle$ and $|2\rangle$ states and n^{th} *odd* state in terms of $|1\rangle$ and $|3\rangle$ states. Applying the technique given in the previous section, we can obtain the following expressions:

$$
\begin{aligned}
n &= 0;\ |4\rangle = p_0|0\rangle + q_0|2\rangle \\
n &= 2;\ |6\rangle = p_2|0\rangle + q_2|2\rangle \\
&\cdots \\
n &= m;\ |m+4\rangle = p_m|0\rangle + q_m|2\rangle \\
n &= m+2;\ |m+6\rangle = p_{m+2}|0\rangle + q_{m+2}|2\rangle .
\end{aligned}
\tag{12}
$$

The truncation of the state for large values of m leads to the following relations

$$q_m p_{m+2} - p_m q_{m+2} = 0. \tag{13}$$

Here p_i and q_i can be calculated by using the following MATHEMATICA program code (again we define $|n\rangle = f[n]$)

```
s1 = Collect[Simplify[Solve[(H-E)f[n] == 0, f[n + 4]]], {f[n_]}]
     (*f[n+4] is obtained from (11)*)
k = 20; Do[f[n+4] = Simplify[s1[[1,1,2]]], {n, 0, k}]
     (*where k is number of iteration*)
Solve[Coefficient[f[k + 4], f[0]]*Coefficient[f[k + 2], f[2]] -
        Coefficient[f[k + 4], f[2]]*Coefficient[f[k + 2], f[0]] == 0, E1] /. α → 0.1
     (*gives eigenvalues of the even state*)
Solve[(Coefficient[f[k + 3], f[1]]*Coefficient[f[k + 1], f[3]] -
        Coefficient[f[k + 3], f[3]]*Coefficient[f[k + 1], f[1]]) == 0, E1]/. α → 0.1
     (*gives eigenvalues of the odd states*)
```

It is obvious that, the program can easily be adapted for similar problems. The method introduced here gives accurate results for bosonic Hamiltonian (10). The results are given in Table I. As shown in the Table I our data confirm some previous results. Note that the results are obtained after 20 iteration.

n	$E_{present}$	E [1]	E [28, 29]
0	1.065286	1.065286	1.065286
1	3.306872	3.306871	3.306872
2	5.747959	5.747960	5.747959
3	8.352678	8.352642	8.352678
4	11.09860	11.09835	11.09860
5	13.96993	13.96695	13.96993

TABLE I: The comparison of eigenvalues of anharmonic oscillator computed by the AIM [1], direct numerical integration method [28, 29] and by the present work, ATEM when $\alpha = 0.1$.

In the following subsections, it is shown that this asymptotic approach opens the way to the treatment of single boson quantum optical systems.

2. A simple multiphoton interaction Hamiltonian

Hamiltonian of the single mode coherent light with an optically bistable two photon medium is given by [24–26]

$$H = \omega a^{+}a + \kappa \left(a^{+2} - a^{2}\right) + \Omega a^{+2}a^{2} \tag{14}$$

where ω is frequency, and κ and Ω are real constants. Time development of the Hamiltonian (14) was studied by [24]. Here we study the determination of the eigenstate of the equation $H\left|n\right\rangle = E\left|n\right\rangle$. The action of the Hamiltonian on the state $\left|n\right\rangle$ can be written as

$$\left(\omega n + \Omega n(n-1) - E\right)\left|n\right\rangle + \kappa\left(\sqrt{n(n-1)}\left|n-2\right\rangle - \sqrt{(n+1)(n+2)}\left|n+2\right\rangle\right) = 0. \tag{15}$$

Our task is now to express n^{th} state in terms of $|0\rangle$ and $|1\rangle$ states.

$$\begin{aligned}
n &= 0;\ |2\rangle = p_0\,|0\rangle \\
n &= 1;\ |3\rangle = p_1\,|1\rangle \\
n &= 2;\ |4\rangle = p_2\,|0\rangle \\
&\ \ \cdots \\
n &= m;\ |m+2\rangle = p_m\,|0\rangle \\
n &= m+1;\ |m+3\rangle = p_{m+1}\,|0\rangle\,.
\end{aligned} \tag{16}$$

It is obvious that eigenvalues of (14) can be obtained for even/odd eigenstates setting $p_m = 0/\ p_{m+1} = 0$. In this case we have used the MATHEMATICA program code given in SECTION II. The results are given in Table II. We have checked that the Hamiltonian (14) can exactly be solved when $\Omega = 0$. In this case for $\kappa = \frac{\sqrt{3}}{2}$, eigenvalues, $E = 2n + \frac{1}{2}$, and we have obtained the same result by using the procedure given here.

n	$\kappa = \frac{\sqrt{3}}{2}; \Omega = 0$	$\kappa = 0.1; \Omega = 0.1$	$\kappa = 0.1; \Omega = 0.5$	$\kappa = 0.5; \Omega = 0.1$
0	0.5	0.00903368	0.00665483	0.19828087652
1	2.5	1.02298633	1.011994512	1.52644677404
2	4.5	2.23086041	3.010484295	2.9397418394
3	6.5	3.63572596	6.0102248594	4.47732150123
4	8.5	5.23894189	10.010131290	6.1677784947
5	10.5	7.04117881	15.010086374	8.0292960950

TABLE II: Eigenvalues E of the Hamiltonian (14), for $\omega = 1$ and various values of κ and Ω.

Consequently, we have shown that AIM can be applied to the determination of the eigenstate of the single boson system.

IV. EIGENSTATE OF MULTIBOSON HAMILTONIANS

In this section we present application of the AIM in order to obtain eigenvalues for a class of models describing two-mode multiphoton processes. In addition to the annihilation operator a, and creation operator a^+, we introduce the operators b and b^+ in Hilbert space are given by

$$b^+ = \frac{1}{\sqrt{2}}\left(-\frac{\partial}{\partial y} + y\right); \quad b = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial y} + y\right). \tag{17a}$$

Two boson operator, a and b, obey the usual commutation relations

$$[a, b] = \left[a, b^+\right] = \left[b, a^+\right] = \left[a^+, b^+\right] = 0, \quad \left[a, a^+\right] = \left[b, b^+\right] = 1. \tag{18}$$

Following a similar method which have been developed in the previous section, we try to determine the eigenvalues for a general class of two-mode multiphoton models. Hamiltonian of such system is given by

$$H = r\omega a^+ a + s\omega b^+ b + \kappa(a^{+s} b^r + b^{+r} a^s) \tag{19}$$

where r and k are positive integers.

In this formalism when $r = s$ the Hamiltonian (19) satisfies the $SU(2)$ symmetry with the generators [12, 27]:

$$J_+ = a^+ b, \quad J_- = b^+ a, \quad J_0 = \frac{1}{2}\left(a^+ a - b^+ b\right). \tag{20}$$

These are the Schwinger representation of $su(2)$ algebra and they satisfy the commutation relations

$$[J_+, J_-] = 2J_0 \quad [J_0, J_\pm] = \pm J_\pm \tag{21}$$

The fourth generator is the total boson number operator

$$N = (a^+ a + b^+ b) \tag{22}$$

which commutes with the $su(2)$ generators. The Casimir operator of this structure is given by

$$J = J_- J_+ + J_0(J_0 + 1) = \frac{1}{4}N(N + 2). \tag{23}$$

If we denote the eigenvalues of the operator J by

$$J = j(j + 1) \tag{24}$$

It is obvious that the irreducible representations of $su(2)$ can be characterized by the total boson number $N = 2j$. The application of the realization (20) on a set of $2j + 1$ states leads to the $(2j+1)$-dimensional unitary irreducible representation for each $j = 0, 1/2, 1, ...$ If the basis states are $|j,m\rangle$ $(m = j, j-1, ..., -j)$, then the action of the operators on the basis states are given by:

$$\begin{aligned}
J_0 |j,m\rangle &= m |j,m\rangle \\
J_\pm |j,m\rangle &= \sqrt{(j \mp m)(j \pm m + 1)}\, |j, m \pm 1\rangle \\
C |j,m\rangle &= j(j+1) |j,m\rangle .
\end{aligned} \tag{25}$$

An immediate practical consequence of these representation of $su(2)$ algebra is that the Hamiltonian (19) can easily be expressed as

$$H = \omega s N + \kappa \left(J_+^s + J_-^s \right) . \tag{26}$$

Eigenvalue equation $H|j,m\rangle = E|j,m\rangle$ can be written as

$$\begin{aligned}
&(2\omega s j - E)|j,m\rangle + \\
&\kappa \sqrt{\frac{(-1)^s (m+s-j-1)!(m+s+j)!}{(m-j-1)!(m+j)!}}\, |j, m+s\rangle + \\
&\kappa \sqrt{\frac{(-1)^s (-m+s-j-1)!(-m+s+j)!}{(-m-j-1)!(-m+j)!}}\, |j, m-s\rangle = 0
\end{aligned} \tag{27}$$

where $N = 0, 1, 2, ...$ .In this case the state $|j, m+s\rangle$ can be expressed as follows,

$$\begin{aligned}
m = -j; &\quad |j, -j+s\rangle = p_{-j} |j, -j\rangle + q_{-j} |j, -j-s\rangle \\
m = -j+1; &\quad |j, -j+s+1\rangle = p_{-j+1} |j, -j+1\rangle + q_{-j+1} |j, -j-s+1\rangle \\
&\quad \cdots \\
m = j-1; &\quad |j, j+s-1\rangle = p_{j-1} |j, j-1\rangle + q_{j-1} |j, j-s-1\rangle \\
m = j; &\quad |j, j+s\rangle = p_{j-1} |j, j\rangle + q_{j-1} |j, j-s\rangle ,
\end{aligned} \tag{28}$$

boundary condition $|j, -j-s\rangle = 0$. The Hamiltonian (26) is exactly solvable when $s = 1$ and a after some straightforward treatment we can show that $E = 2j + 2(n-j)\kappa$. For the values $s = 2$ and $j = 3$, the the results are given in Table III.

V. CONCLUSION

The basic feature of our approach is to reformulate AIM for obtaining eigenvalues of the bosonic Hamiltonians. Furthermore the technique given here has been used to determine

m	$\kappa = \frac{1}{10}$;	$\kappa = \frac{1}{5}$;	$\kappa = \frac{1}{2}$;
0	12	12	12
± 1	$\frac{1}{5}\left(57 \pm 2\sqrt{6}\right)$	$\frac{2}{5}\left(27 \pm 2\sqrt{6}\right)$	$\left(9 \pm 2\sqrt{6}\right)$
± 2	$\frac{1}{5}\left(60 \pm 2\sqrt{15}\right)$	$\frac{2}{5}\left(30 \pm 2\sqrt{15}\right)$	$\left(12 \pm 2\sqrt{15}\right)$
± 3	$\frac{1}{5}\left(63 \pm 2\sqrt{6}\right)$	$\frac{2}{5}\left(33 \pm 2\sqrt{6}\right)$	$\left(15 \pm 2\sqrt{6}\right)$

TABLE III: Eigenvalues of the Hamiltonian (26), for $\omega = 1$, $s = 2$ and $j = 3$.

eigenvalues of anharmonic oscillator, multiphoton interaction problem and a class of models describing two-mode multiphoton processes. We have shown that AIM gives accurate results for eigenvalue of bosonic Hamiltonians.

As a further work the method presented here can be developed in various directions. Complete spectrum of the quasi-exactly solvable problems can be obtained in the framework of the method presented here. Since most of the quasi-exactly solvable problems can be expressed in terms of generators of $su(1,1)$ or $su(2)$ Lie algebra, the resulting recurrence relation can easily be solved by using the procedure given in this paper. The suggested approach can also be extended for solving boson-fermion systems. Before ending this work a remark is in order. This extension leads to the determination of eigenvalues of various Hamiltonians; Jahn-Teller Hamiltonians [19], Rabi Hamiltonian [20], Hamiltonians of the Bose-Einstein condensation problems.

VI. ACKNOWLEDGEMENT

The research was supported by the Scientific and Technological Research Council of TURKEY (TÜBİTAK).

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