THREE REMARKS ON A QUESTION OF ACZÉL

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Consider the functional equation

$$f(x^2R) = \frac{k}{2xR}f(x)\ ,\ x > \frac{1}{R}, \tag{0.1}$$

where R, k are positive constants. At the 49th International Symposium on Functional Equations 2009, J. Aczél [1, p. 195] presented the general solution of this equation as

$$f(x) = \frac{(\ln(xR))^c}{xR}p(\log_2(\ln(xR))), \tag{0.2}$$

where $c = \log_2(k/2)$ and p is an arbitrary periodic function of period 1 on $\mathbb{R}$ (for proof define $p(s) = (\frac{2}{k})^s \exp(2^s) f\left(\frac{\exp(s^2)}{R}\right)$), and then asked for the *monotonic* solutions.

A special solution for the functional equation (0.1) is

$$\varphi_c(x) = \frac{(\ln(xR))^c}{xR}, \tag{0.3}$$

obtained by taking $p \equiv 1$ in (0.2). This function is monotonic precisely when $c \leq 0$, that is, when $k \leq 2$. Are all monotonic solutions of (0.1) scalar multiples of φ_c? We show that when $k = 2$ the answer is *yes*, and that it is *no* when $k < 2$. We then consider the effects of imposing continuity and differentiability conditions at $1/R$. Here, too, the results depend on the value of k.

1. Monotonicity

First, assume that $k = 2$. We claim that a function of the form

$$\frac{1}{xR} \cdot p(\log_2(\ln(xR)))$$

is monotonic only if $p \equiv const$. Indeed, if p is not constant, then it takes two distinct values $M > m$. For any x_1, x_2 sufficiently close to $1/R$,

$$M\frac{1}{x_1R} > m\frac{1}{x_2R}.$$

Let x_1 be sufficiently close to $1/R$ which satisfies $p(\log_2(\ln(x_1R))) = M$, and let $x_2 > 1/R$ be smaller than x_1 such that $p(\log_2(\ln(x_2R))) = m$. Then $x_2 < x_1$, but $f(x_1) > f(x_2)$, so f is not monotonic decreasing. If f is an increasing function then $-f$ is a decreasing function, so we conclude that the only monotonic solutions to (0.1) are $f(x) = \frac{\lambda}{xR} = \lambda\varphi_0(x)$, $\lambda \in \mathbb{R}$.

Now we consider the case $k < 2$. We claim that there are infinitely many differentiable functions p such that (0.2) is a monotonic solution. Indeed, differentiating

(0.2) we find

$$f'(x) = \frac{\ln(xR)^{c-1}}{x^2R}\Big((c-\ln(xR))\cdot p(\log_2(\ln(xR)))+\log_2 e\cdot p'(\log_2(\ln(xR)))\Big). \quad (1.1)$$

Keeping in mind that $c<0$ in this case, it is easy to see that, so long as p is bounded away from 0 and $|p'|$ is bounded from above by a small enough number, f' has a constant sign in $(1/R,\infty)$. Thus, there are many monotonic - even differentiable - solutions other than φ_c.

2. Continuity at $1/R$

If the domain in (0.1) is changed to $x\in(0,1/R)$, then the general solutions is

$$f(x) = \frac{(-\ln(xR))^c}{xR}p(\log_2(-\ln(xR))). \quad (2.1)$$

Note that f can be extended to a continuous function on $[1/R,\infty)$ that satisfies (0.1) at $x=1/R$ only if $k=2$ or $f(1/R)=0$. None of the solutions for $k<2$ can be extended to a continuous solution on $[1/R,\infty)$.

On the other hand, every solution for $k>2$ can be extended to $x=1/R$ by defining $f(1/R)=0$. Pasting these solutions with (2.1) for $0<x<1/R$, we see that there are many continuous solutions for (0.1) when $k>2$ on the interval $(0,\infty)$. All of these solutions vanish at 0.

Finally, when $k=2$, we see that for every $\lambda\in\mathbb{R}$, the function $\lambda\varphi_0$ is the unique continuous solution for (0.1) that satisfies $f(1/R)=\lambda$. Of course, this is a solution of (0.1) on the entire interval $(0,\infty)$.

3. Continuous differentiability

The only case worth discussing in the setting of contionuous differentiablity is $k>2$. In this case, we saw that the functional equation has continuous solutions on the the interval $(0,\infty)$ given by (0.2) and (2.1). We ask: which of these solutions is continuously differentiable? Examining (0.2), we see that we may limit the discussion to solutions of the form (0.2) with p continuously differentiable.

When $k\in(2,4]$, so $c\in(0,1]$, we see by examining (1.1) that f' will have a limit at $x=1/R$ only if

$$\lim_{x\to 1/R^+}(c-\ln(xR))\cdot p(\log_2(\ln(xR)))+\log_2 e\cdot p'(\log_2(\ln(xR))) = L,$$

and L must 0 when $c<1$. But since p is periodic, this is possible only if p satisfies the differential equation

$$p' + \frac{c}{\log_2 e}p = L/\log_2 e.$$

But the only periodic solutions of this differential equation are constants, thus any continuously differentialbe solution when $k\in(2,4]$ is multiple of φ_c.

When $k>4$ any continuously differentiable periodic p will give rise to a continuously differentiable solution.

References

1. *The Forty-seventh International Symposium on Functional Equations, June 14-20, 2009, Gargnano, Italy*, Report of Meeting, Aequationes Math. 79 (2010), 173-202.

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