A TOPOLOGICAL DEGREE COUNTING FOR SOME LIOUVILLE SYSTEMS OF MEAN FIELD EQUATIONS

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Abstract. Let $A = (a_{ij})_{n\times n}$ be an invertible matrix and $A^{-1} = (a^{ij})_{n\times n}$ be the inverse of A. In this paper, we consider the generalized Liouville system:

$$\Delta_g u_i + \sum_{j=1}^n a_{ij}\rho_j \left(\frac{h_j e^{u_j}}{\int h_j e^{u_j}} - 1\right) = 0 \quad \text{in } M, \tag{0.1}$$

where $0 < h_j \in C^1(M)$ and $\rho_j \in \mathbb{R}^+$, and prove that, under the assumptions of (H_1) and (H_2) (see Introduction), the Leray-Schauder degree of (0.1) is equal to

$$\frac{(-\chi(M)+1)\cdots(-\chi(M)+N)}{N!}$$

if $\rho = (\rho_1, \cdots, \rho_n)$ satisfies

$$8\pi N \sum_{i=1}^n \rho_i < \sum_{1\le i,j\le n} a_{ij}\rho_i\rho_j < 8\pi(N+1)\sum_{i=1}^n \rho_i.$$

Equation (0.1) is a natural generalization of the classic Liouville equation and is the Euler-Lagrangian equation of Nonlinear function Φ_ρ:

$$\Phi_\rho(u) = \frac{1}{2}\int_M \sum_{1\le i,j\le n} a^{ij}\nabla_g u_i \cdot \nabla_g u_j + \sum_{i=1}^n \int_M \rho_i u_i - \sum_{i=1}^n \rho_i \log \int_M h_i e^{u_i}.$$

The Liouville system (0.1) has arisen in many different research areas in mathematics and physics. Our counting formulas are the first result in degree theory for Liouville systems.

1. Introduction

Let (M, g) be a compact Riemann surface with volume 1, $h_1, ..., h_n$ be positive C^1 functions on M, $\rho_1, .., \rho_n$ be nonnegative constants. In this article we consider the following Liouville system defined on (M, g):

$$\Delta_g u_i + \sum_{j=1}^n \rho_j a_{ij}\left(\frac{h_j e^{u_j}}{\int_M h_j e^{u_j} dV_g} - 1\right) = 0, \quad i \in I := \{1, .., n\} \tag{1.1}$$

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where dV_g is the volume form, Δ_g is the Laplace-Beltrami operator, in local coordinates it is of the form:

$$\Delta_g = \sum_{i,j=1}^{2} \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x_i}(\sqrt{\det(g)} g^{ij} \frac{\partial}{\partial x_j}), \quad (g^{ij})_{2\times 2} = (g_{ij})^{-1}_{2\times 2}.$$

When $n = 1$, equation (1.1) is the mean field equation of the Liouville type:

$$\Delta_g u + \rho \left(\frac{he^u}{\int_M he^u dV_g} - 1 \right) = 0 \quad \text{in } M \tag{1.2}$$

when $a_{11} = 1$. Therefore, the Liouville system (1.1) is a natural extension of the classical Liouville equation, which has profound connection with geometry and physics, and has been extensively studied for the past three decades.

If u is a solution of (1.1), then after adding a constant, $u + c$ is also a solution of (1.1). Hence, we can always assume $u \in \mathring{H}^1(M)$, where

$$\mathring{H}^1(M) = \left\{ u \in L^2(M) \,\middle|\, |\nabla_g u| \in L^2(M) \text{ and } \int_M u \, dV_g = 0 \right\}.$$

For any $\rho = (\rho_1, \cdots, \rho_n)$, $\rho_i > 0$, let Φ_ρ be a nonlinear functional defined in $\mathring{H}^{1,n} = \mathring{H}^1(M) \times \cdots \times \mathring{H}^1(M)$ by

$$\Phi_\rho(u) = \frac{1}{2} \sum_{i,j \in I} a^{ij} \int_M \nabla_g u_i \cdot \nabla_g u_j dV_g - \sum_{j \in I} \rho_j \log \int_M h_j e^{u_j} dV_g$$

where $(a^{ij})_{n\times n}$ is the inverse of $A = (a_{ij})_{n\times n}$. It is easy to see that equation (1.1) is the Euler-Lagrangian equation of Φ_ρ.

For a bounded smooth domain Ω in $\mathbb{R}^2$, we are also interested in the following system of equations:

$$\begin{cases} \Delta u_i + \sum_{j=1}^n a_{ij} \rho_j \frac{h_j e^{u_j}}{\int_\Omega h_j e^{u_j} dx} = 0 \quad \text{in } \Omega, \\ u_i|_{\partial\Omega} = 0, \quad i \in I \end{cases} \tag{1.3}$$

where h_j are positive C^1 function on $\overline{\Omega}$.

The Liouville equation (1.2) or systems (1.1) and (1.3) have appeared in many different disciplines in mathematics. In conformal geometry, when $\rho = 8\pi$ and M is the sphere $\mathbb{S}^2$, equation (1.2) is equivalent to the famous Nirenberg problem. For a bounded domain in $\mathbb{R}^2$ and $n = 1$, (1.3) can be derived from the mean field limit of Euler flows or spherical Onsager vortex theory, as studied by Caglioti, Lions, Marchioro and Pulvirenti [7, 8], and Kiessling [26], Chanillo and Kiessling [9]. In classical gauge field theory, equation (1.1) is closely related to the Chern-Simons-Higgs equation for the abelian case, see [6, 22, 23, 40]. Various Liouville systems are also used to describe models in the theory of Chemotaxis [16, 25], in the physics of charged particle beams [4, 19, 27, 28], in the non-abelian Chern-Simons-Higgs theory [20, 24, 40] and other gauge field models [21, 29]. For recent developments of these subjects or related Liouville systems in more general settings, we refer the readers to [1, 2, 3, 5, 12, 13, 14, 15, 17, 18, 30, 31, 32, 33, 35, 36, 37, 38, 39, 41, 42] and the references therein.

For a bounded smooth domain Ω in $\mathbb{R}^2$, Chipot-Shafrir-Wolansky [17] considered equation (1.3), where the constant matrix $A = (a_{ij})_{n\times n}$ satisfies the following condition:

(H1): A is symmetric, nonnegative, irreducible and invertible.

Here A is called nonnegative if $a_{ij} \geq 0$ for all $i, j \in I = \{1, 2, \cdots, n\}$, and is called irreducible if there is no subset J of I such that

$$a_{ij} = 0 \quad \text{for all } i \in J \text{ and } j \in I \setminus J.$$

In another word, equation (1.3) can not be written as two de-coupled subsystems. In [17], the authors introduced nonlinear functions $\Lambda_J(\rho)$ of $\rho = (\rho_1, \cdots, \rho_n)$ defined by

$$\Lambda_J(\rho) = 8\pi \sum_{i\in J} \rho_i - \sum_{i,j\in J} a_{ij}\rho_i\rho_j$$

for any non-empty subset J of $I = \{1, 2, \cdots, n\}$. Let

$$\Gamma = \{(\rho_1, \cdots, \rho_n) \,|\, \rho_i > 0,\ \Lambda_J(\rho) > 0 \text{ for all } \emptyset \subsetneq J \subsetneq I\}.$$

Among other things, Chipot-Shafrir-Wolansky [17] proved the following theorem.

Theorem A. *Suppose A satisfies* **(H1)**, *$h_1, .., h_n$ are positive C^1 functions on $\overline{\Omega}$, and $\rho = (\rho_1, \cdots, \rho_n)$ satisfies*

$$\rho \in \Gamma. \tag{1.4}$$

Then equation (1.3) *possesses a solution.*

We note that in [17, 18], the authors also proved that the sufficient condition (1.4) in Theorem A is also a necessary condition for the existence of equation (1.3) when Ω is a ball. When $n = 1$, equation (1.3) with the parameter $\rho \in \Gamma$ is equivalent to the Liouville equation with $\rho < 8\pi$. From various known results of Liouville equations, we expect that solutions of equation (1.3) with $\rho \notin \Gamma$, should have Morse index bigger than 1. Therefore, the classical Leray-Schauder degree theory is a suitable tool to be applied for studying equation (1.1) or (1.3) when $\rho \notin \Gamma$.

To apply the degree theory, we should first prove the a priori bound for non-critical parameter ρ, or equivalently, study the asymptotic behavior of bubbling solutions. In [33], we have proved that near each blow up point, the behavior of these bubbling solutions can be controlled well by the standard bubble, under the assumption that all the components u_i of solutions have to blow up simultaneously i.e., a suitable scaling of u_i should converge to some entire solutions of Liouville system:

$$\begin{cases} \Delta U_i + \sum_{j\in I} a_{ij} e^{U_j} = 0 \quad \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^{U_j} dx < +\infty, \quad i \in I. \end{cases} \tag{1.5}$$

This blow-up is called type 1. However, there might be situations that at some blow-up points, only part of the components of u_i $(i \in I)$ blows up, but the remaining part does not blow up. This blow-up is called type 2. If both type 1 and type 2 occur simultaneously for a sequence of bubbling solution, then the set of critical parameters could be very complicated. For example $n = 2$, the critical parameter might be

$$\begin{aligned}&\big\{(\rho_1, \rho_2) \,|\, \rho_1 = \tilde\rho_1 + 8\pi m, \rho_2 = \tilde\rho_2 + 8\pi l \text{ and } (\tilde\rho_1, \tilde\rho_2) \text{ satisfies}\\ &8\pi k(\tilde\rho_1 + \tilde\rho_2) = \sum_{i,j} a_{ij}\tilde\rho_1\tilde\rho_2,\, m.l, k \in \mathbb{N}\big\},\end{aligned}$$

where $a_{11} = a_{22} = 1$ is assumed, $\mathbb{N}$ is the set of all natural numbers. Thus, the topological degree would be very difficult to compute. In this paper, we will prove that this complexity can be avoided if we assume the coefficient matrix A to satisfy **(H1)** and the following **(H2)** condition:

(H2): $a^{ii} \le 0$ for $i \in I$, $a^{ij} \ge 0$ for $i \ne j$, $i, j \in I$,
and $\sum_{j\in I} a^{ij} \ge 0$ for $i \in I$.

Throughout the paper, we assume that A satisfies both **(H1)** and **(H2)**. For $n = 2$, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22}\end{pmatrix}$ satisfies **(H1)** and **(H2)** if and only if $a_{ij} \ge 0$, $\max(a_{11}, a_{22}) \le a_{12}^2$ and $\det A \ne 0$. For $n = 3$, assume $A = \begin{pmatrix} 0 & a_1 & a_2 \\ a_1 & 0 & a_3 \\ a_2 & a_3 & 0\end{pmatrix}$. Then A satisfies **(H1)** and **(H2)** if and only if $a_i > 0$ and $a_i + a_j \ge a_k$ for $i \ne j \ne k$.

To state our results, we begin with equation (1.1).

Theorem 1.1. *Let $A = (a_{ij})_{n\times n}$ satisfy* **(H1)** *and* **(H2)***, and N be a nonnegative integer and*

$$\begin{aligned}\mathcal{O}_N = \big\{(\rho_1, \cdots, \rho_n) \,\big|\, \rho_i \ge 0, i \in I \text{ and}\\ 8\pi N \sum_{i\in I}\rho_i < \sum_{i,j\in I} a_{ij}\rho_i\rho_j < 8\pi(N+1)\sum_{i\in I}\rho_i\big\}.\end{aligned}$$

Suppose $h_i \in C^1(M)$ is positive and K is a compact subset of $\mathcal{O}_N$. Then there exists a constant C such that for any solution $u = (u_1, \cdots, u_n)$ of (1.1) *with $\rho \in K$, we have*

$$|u_i(x)| \le C \quad \text{for } i \in I \text{ and } x \in M.$$

Note that the set $\mathcal{O}_N$ is bounded if $a_{ii} > 0$ for all i, and $\mathcal{O}_N$ is unbounded if $a_{ii} = 0$ for some i. By Theorem 1.1, the critical parameter set for (1.1) is the set Γ_N, where

$$\Gamma_N = \left\{\rho \,\Big|\, 8\pi N \sum_{i\in I}\rho_i = \sum_{i,j\in I} a_{ij}\rho_i\rho_j\right\}.$$

After Theorem 1.1, for $\rho \notin \Gamma_N$ for any positive integer N, we can define the nonlinear map $T_\rho = (T^1, \cdots, T^n)$ from $\mathring{H}^{1,n} = \mathring{H}^1(M) \times \cdots \times \mathring{H}^1(M)$ to $\mathring{H}^{1,n}$ by

$$T^i = -\Delta_g^{-1}\left(\sum_{j\in I} a_{ij}\rho_j\left(\frac{h_j e^{u_j}}{\int_M h_j e^{u_j}} - 1\right)\right), \quad i \in I.$$

Obviously, T_ρ is compact from $\mathring{H}^{1,n}$ to itself. Then we can define the Leray-Schauder degree of equation (1.1) by

$$d_\rho = \deg\,(I - T_\rho\,;\, B_R, 0),$$

where R is sufficiently large and $B_R = \{u \,|\, u \in \mathring{H}^{1,N} \text{ and } \|u\| < R\}$. By the homotopic invariance and Theorem 1.1, d_ρ satisfies the following properties:

(i) d_ρ is a constant for $\rho \in \mathcal{O}_N$,

(ii) d_ρ is independent of $h = (h_1, h_2, \cdots, h_n)$.

The following result is the formula for computing d_ρ.

Theorem 1.2. *d_ρ be the Leray-Schauder degree for (1.1) for $\rho \in \mathcal{O}_{N-1}$, $N \in \mathbb{N}$. Then*

$$d_\rho = \begin{cases} 1 & \text{if } \rho \in \mathcal{O}_0 \\ \frac{1}{N!}\Big((-\chi_M+1)...(-\chi_M+N)\Big) & \text{if } \rho \in \mathcal{O}_N. \end{cases} \tag{1.6}$$

where χ_M is the Euler characteristic of M.

Since $\chi_M = 2 - 2g_e$ where g_e is the genus of M, the following existence theorem is implied by Theorem 1.2:

Theorem 1.3. *(Main Theorem) Let M be a compact Riemann surface with genus greater than 0 and $h_1, .., h_n$ be positive C^1 functions on M. Then (1.1) always has a solution for $\rho \notin \Gamma_N$ for any $N \in \mathbb{N}$.*

Similarly, for equation (1.3), we have the following result:

Theorem 1.4. *Let $(h_1, .., h_n)$ be positive C^1 functions on $\bar{\Omega}$. Then the Leray-Schauder degree d_ρ for (1.3) is*

$$d_\rho = \begin{cases} 1, & \rho \in \mathcal{O}_0 \\ \frac{1}{N!}\Big((-\chi+1)...(-\chi+N)\Big), & \rho \in \mathcal{O}_N, \quad N \in \mathbb{N} \end{cases}$$

where $\chi = 1 - g_e$, g_e is the number of holes inside Ω. In particular if Ω is not simply connected, (1.3) always has a solution for ρ.

The organization of this paper is as follows. In section 2 we mainly address entire solutions of (1.5) and some important properties implied by the assumption **(H2)**. Then in section 3 we give a detailed description on the asymptotic behavior of blowup solutions near a blowup point. In [33] the authors have proved that if the system all converges to an entire system of n equations around each blow-up, then all the blow-up solutions converge to the same system after scaling. In this section we consider the case of the type 2 blow-up, and give a sharper estimate for the bubbling solution near the blow-up point. In section 4 by using the sharper estimates in section 3, we will prove that type 1 and type 2 blow-up can not occur simultaneously in any sequence of bubbling solutions, which leads to the proof of Theorem 1.1 and Theorem 1.2 in section 5. Finally in section 6 we prove Theorem 1.4.

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2. On entire solutions

In this section, we discuss the entire solutions of the Liouville system

$$\begin{cases} \Delta u_i + \sum_{j\in I} a_{ij} e^{u_j} = 0, \quad \mathbb{R}^2, \quad i \in I \\ \int_{\mathbb{R}^2} e^{u_i} < \infty, \quad i \in I. \end{cases} \tag{2.1}$$

System (2.1) is closely related to the following system of equations

$$\begin{cases} \Delta v_i + \mu_i e^{\sum_{j\in I} a_{ij} e^{v_j}} = 0 \quad \mathbb{R}^2, \quad i \in I. \\ \int_{\mathbb{R}^2} \mu_i e^{\sum_{j\in I} a_{ij} e^{v_j}} < \infty, \quad i \in I, \end{cases} \tag{2.2}$$

which was studied initially by Chanillo and Kiessling [10], and by Chipot-Shafrir-Wolansky [17, 18]. Obviously, these two systems are equivalent if the coefficient matrix $A = (a_{ij})$ is invertible. In this section, the coefficient matrix A is assumed to be symmetric and nonnegative, but not necessarily invertible. After a permutation of rows, A can be written as

$$\begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{pmatrix} \tag{2.3}$$

where each $A_l := (a_{ij})_{i,j\in I_l}\,(l = 1, \cdots, k)$ is irreducible and $I = \bigcup_{l=1}^k I_l$. For a positive vector $\sigma = (\sigma_1, \cdots, \sigma_n)$ (which means each σ_i is positive), we define

$$\Lambda_J(\sigma) = 4\sum_{i\in J} \sigma_i - \sum_{i,j\in J} a_{ij}\sigma_i\sigma_j \quad \text{for any } \emptyset \subsetneq J \subset I.$$

Then Chipot-Shafrir-Wolansky prove the following theorem in [17]:

Theorem B. *(Theorem 1.4 of* [17]*): Let A be a nonnegative, symmetric matrix that satisfies (2.3). Let $\sigma = (\sigma_1, \cdots, \sigma_n)$ be a positive vector such* *that*

$$\Lambda_{I_l}(\sigma) = 0, \quad \Lambda_J(\sigma) > 0, \quad \emptyset \subsetneq J \subsetneq I_l, \quad l = 1, \cdots, k. \tag{2.4}$$

Then there exist $v = (v_1, \cdots, v_n)$ and $\mu = (\mu_1, \cdots, \mu_n)$ $(\mu_i > 0)$ such that (2.2) is satisfied and

$$\sigma_i = \frac{1}{2\pi} \int_{\mathbb{R}^2} \mu_i e^{\sum_{j \in I} a_{ij} v_j}, \quad i \in I. \tag{2.5}$$

Conversely, for an entire solution v to (2.2), σ defined by (2.5) satisfies (2.4).

We note that if one submatrix A_l is zero, then I_l consists of only one element because A_l is irreducible. In this case, no positive σ satisfies $\Lambda_{I_l}(\sigma) = 4\sigma = 0$. Therefore (2.2) has no solution for any positive $\mu = (\mu_1, \cdots, \mu_n)$.

Let σ_i $(i \in I)$ be positive such that $\sigma = (\sigma_1, \cdots, \sigma_n)$ satisfies (2.4). Then by Theorem B, there is a solution $v = (v_1, \cdots, v_n)$ of (2.2) such that (2.5) holds. In [17], Chipot-Shafrir-Wolansky proved the asymptotic behavior of v at ∞:

$$v_i(x) = -\sigma_i \log |x| + O(1), \quad i \in I, \tag{2.6}$$

and

$$\sum_{j \in I} a_{ij} \sigma_j > 2, \tag{2.7}$$

due to $\exp\left(\sum_{j \in I} a_{ij} v_j\right) \in L^1(\mathbb{R}^n)$. For any solution v of (2.2), set

$$u_i = \sum_{j \in I} a_{ij} v_j + \log \mu_i. \tag{2.8}$$

Then u_i is a solution of the system (2.1) with

$$u_i = -m_i \log |x| + O(1) \quad \text{for large } |x|, \tag{2.9}$$

and

$$m_i > 2, \tag{2.10}$$

where

$$m_i = \sum_{j \in I} a_{ij} \sigma_j. \tag{2.11}$$

Conversely, let $u = (u_1, \cdots, u_n)$ be a solution of (2.1) and set

$$\sigma_i = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{u_i} dx \text{ is finite}, \quad i \in I. \tag{2.12}$$

As did in [17], u_i satisfies the asymptotic behavior of (2.9) where m_i is defined by (2.11). By the fact $e^{u_i} \in L^1(\mathbb{R}^2)$ and by using the Brezis-Merle type argument (see Lemma 4.1 in [33]), it can be proved that $u_i \in L^\infty_{loc}(\mathbb{R}^2)$

and then $u_i \in C^\infty(\mathbb{R}^2)$ by further applications of standard elliptic regularity theorems. Set v_i to be

$$v_i(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\frac{|x|}{|x-y|} e^{u_i(x)} dx. \tag{2.13}$$

Then $u_i - \sum_{j\in I} a_{ij}v_j$ is a harmonic function in $\mathbb{R}^2$ and is bounded from above by $c\log|x|$ for large x. Thus, $u_i(x) = \sum_{j\in I} a_{ij}v_j(x) + c_i$ for some constant c_i. Clearly, $(v_1, \cdots, v_n)$ satisfies (2.2):

$$-\Delta v_i = e^{u_i} = \mu_i e^{\sum_{j\in I} a_{ij}v_j}$$

for $\mu_i = e^{c_i}$. Therefore, $\sigma = (\sigma_1, \cdots, \sigma_n)$ satisfies (2.4).

We note that in [10] and [17], it has been shown that any component u_i of a solution u of (2.1) with $e^{u_i} \in L^1(\mathbb{R}^2)$ must be radially symmetric with respect to some point in $\mathbb{R}^2$, in paticular, if A is irreducible, then u_i, $i \in I$, are radially symmetric with respect to one common point. If A is not irreducible, then by (2.3), u_i, $i \in I_l$, is symmetric with respect to some p_l for $l = 1, 2, \cdots, k$. By replacing u_i by $u_i(x + p_l)$, $i \in I_l$, we conclude that for any σ satisfying (2.4), there exists a radial solution u_i of (2.1) such that $\sigma_i = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{u_i}dx$. In summary, Theorem B can be written as the following result for system (2.1).

Theorem C: Let A be the same as in Theorem B. Then a solution $u = (u_1, \cdots, u_n)$ of (2.1) exists with $\sigma_i = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{u_i}dx$ if and only if σ satisfies (2.4). Furthermore, after a translation, the solution u is radially symmetric with respect to the origin.

Set

$$\mathcal{E} := \{\sigma; \sigma_i > 0,\ i \in I;\ \Lambda_I(\sigma) = 0;\ \Lambda_J(\sigma) > 0\ \forall\, \emptyset \subsetneq J \subsetneq I.\}. \tag{2.14}$$

Let u be a radial solution with $\sigma \in \mathcal{E}$, where σ is given by (2.12). Then it is clear to see that

$$\tilde{u}_i(x) = u_i(\delta x) + 2\log\delta, \quad i \in I,$$

is also a solution of (2.1) with the same σ_i. Thus, without loss of generality, we may assume u satisfies $u_i(0) = \alpha_i$, $u_n(0) = 0$, $1 \le i \le n-1$, and let $\mathcal{B} \subseteq \mathbb{R}^{n-1}$ be the set of initial values of solutions in (2.1). Assume (**H1**) holds. In [33], the authors proved

$$\mathcal{B} \text{ is open and is homeomorphic to } \mathcal{E} \quad \text{(defined in (2.14))}. \tag{2.15}$$

In particular $\mathcal{E}$ is an open set in $\{\sigma;\ \Lambda_I(\sigma) = 0\ \}$. If $a_{ii} > 0$ for all $i \in I$, then it is not difficult to see $\mathcal{B} = \mathbb{R}^{n-1}$. In general $\mathcal{B}$ might not be equal to $\mathbb{R}^{n-1}$, it is even not known whether $\mathcal{B} \ne \emptyset$.

In the following Lemma 2.1 and Theorem 2.1 we show that with (**H2**), all submatrices of A are irreducible, $\mathcal{B} \ne \emptyset$ and the condition $\Lambda_J(\sigma) > 0$ is automatically satisfied if $\Lambda_I(\sigma) = 0$.

Lemma 2.1. *Let A satisfy* (**H1**) *and* (**H2**)*, then $a_{ij} > 0$ and $\max(a_{ii}, a_{jj}) \le a_{ij}$ for all $i \ne j$ in I. In particular, all submatrices of A are irreducible.*

Proof of Lemma 2.1:

Step one: If $a_{i_0i_0}>0$ for some i_0, then $a_{i_0j}\ge a_{i_0i_0}$ for all $j\ne i_0$.

Suppose $a_{11}>0$, let $\sigma_1=1,\sigma_2=..=\sigma_n=0$ and $m_i=\sum_{j=1}^n a_{ij}\sigma_j$. Then $m_j=a_{1j}$ for all $j\in I$. Let $m=\min\{m_2,...,m_n\}$. We want to show that $m>0$. Indeed, if $m=0$, we let

$$J=\{i\in I;\quad m_i=0.\}.$$

Clearly $1\notin J$, so for any $i\in J$, $\sigma_i=0$, which reads

$$0=\sigma_i=\sum_{j\in J}a^{ij}m_j+\sum_{j\notin J}a^{ij}m_j=\sum_{j\notin J}a^{ij}m_j,\quad \forall i\in J.$$

Since $a^{ij}\ge 0$ for all $i\ne j$ (see (**H2**)), we have $a^{ij}=0$ for all $i\in J$ and $j\notin J$. After a permutation of the rows of A^{-1} (therefore the same permutation on the columns) A^{-1} is of the form:

$$\begin{pmatrix} \mathbf{B_1} & \mathbf{0}\\ \mathbf{0} & \mathbf{B_2}\end{pmatrix}$$

which means A is of the form

$$\begin{pmatrix} \mathbf{B_1^{-1}} & \mathbf{0}\\ \mathbf{0} & \mathbf{B_2^{-1}}\end{pmatrix}$$

after a permutation of its rows and columns. This is a contradiction to the irreducibility of A. Therefore we have proved $m>0$. Next we claim $m\ge m_1$. If this is not true, we have $m<m_1$ and we let

$$J:=\{i\in I;\quad m_i=m\quad\}.$$

Then by our assumption, $J\ne\emptyset$ and $1\notin J$. Thus, for any $i\in J$, the fact $\sigma_i=0$ yields

$$\begin{aligned}0 &= \sigma_i=\sum_{j\in J}a^{ij}m_j+\sum_{j\notin J}a^{ij}m_j\\ &\ge m\sum_{j\in J}a^{ij}+m\sum_{j\notin J}a^{ij}\\ &= m\sum_{j\in I}a^{ij}\ge 0,\quad \forall i\in J.\end{aligned}\tag{2.16}$$

Note that we have used (**H2**) in both inequalities. We see that the second line of (2.16) is a strict inequality unless $a^{ij}=0$ for all $i\in J$ and $j\notin J$. Thus, (2.16) yields $a^{ij}=0$ for $i\in J$ and $j\notin J$, a contradiction to the irreducibility of A. Step one is established.

Step two: $a_{ij}>0$ for all $i\ne j$

We prove step two by contradiction. Suppose there exist $i\ne j$ such that $a_{ij}=0$. By step one, we have $a_{ii}=a_{jj}=0$. Without loss of generality we assume $i=1$ and $j=2$. We can also assume $a_{13}>0$, because the invertibility of A implies $a_{1i}>0$ for some $i\ge 3$. We can apply a permutation on A to move the positive entry to the third row. Thus the matrix A is of the following form after a permutation of rows and columns:

$$\begin{pmatrix} 0 & 0 & a_{13} & \dots \\ 0 & 0 & a_{23} & \dots \\ a_{13} & a_{23} & a_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Let $\sigma_1 = 1, \sigma_2 = 0, \sigma_3 = 1$ and $\sigma_4 = .. = \sigma_n = 0$ and $m_i = \sum_{j=1}^n a_{ij}\sigma_j$. Clearly $m_1 = a_{13}$, $m_3 \ge a_{13}$. Let $m = \min\{m_2, m_4, .., m_n\}$. We first claim $m > 0$. Suppose this is not the case, let $J = \{i \in I; \quad m_i = 0\}$. Obviously $1, 3 \notin J$, which implies $\sigma_j = 0$ for all $j \in J$. Using the same argument as in step one, we have $a^{ij} = 0$ for all $i \in J$ and all $j \notin J$, thus a contradiction to the irreducibility of A.

Next we claim $m \ge m_1$. Suppose this is not true, then $m < m_1$. Observe that $m_3 \ge m_1$ so $m < m_3$. Let

$$J = \{i \in I; \quad m_i = m\}.$$

Then $1, 3 \notin J$, so $\sigma_i = 0$ for all $i \in J$. Then (2.16) yields

$$0 = \sigma_i \ge m \sum_{j \in I} a^{ij} + 2(m_1 - m)a^{13} > 0,$$

which is a contradiction. Therefore $m \ge m_1$ is proved. In particular, $m_2 = a_{23} \ge m_1$, which gives $a_{23} \ge a_{13}$. Since we can switch the first two rows of A the same argument gives $a_{13} \ge a_{23}$, consequently $a_{13} = a_{23}$. We use a_{13} to represent any nonzero entry on the first row, so we have proved that the first two rows are identical, a contradiction to the invertibility of A. Lemma 2.1 is established. □

Theorem 2.1. *Let A satisfy* (**H1**) *and* (**H2**). *Suppose $\sigma = (\sigma_1, .., \sigma_n)$ has positive components and $\Lambda_I(\sigma) = 0$. Then $\Lambda_J(\sigma) > 0$ for all $\emptyset \subsetneq J \subsetneq I$.*

Proof of Theorem 2.1:

First we assume $\mathcal{E} \neq \emptyset$ (which will be proved in Lemma 2.2) and prove

$$\text{(2.17)} \quad \text{If } \tilde\sigma \in \partial\mathcal{E}, \text{ then } \exists J \subsetneq I, \quad \text{such that } \tilde\sigma_i > 0 \quad \forall i \in J; \\ \sigma_i = 0 \ \forall i \notin J; \quad \text{and } \tilde\sigma \text{ satisfies } \ \Lambda_J(\tilde\sigma) = 0.$$

Proof of (2.17):

Let σ^k be a sequence of points in $\mathcal{E}$ that tends to $\tilde\sigma \in \partial\mathcal{E}$. Let $u^k = (u_1^k, .., u_n^k)$ be global, radial solutions that correspond to σ^k. Without loss of generality we assume $u_1^k(0) = \max_{i \in I} u_i^k(0) = 0$. Since $e^{u_i^k} \le 1$, by the standard elliptic estimates, there exists a subsequence of u^k (still denoted by u^k) such that parts of u^k converge in $C^\infty_{loc}(\mathbb{R}^2)$. There are two cases of the convergence of u^k to be discussed separately.

Case one: u^k converges to a global solution $v = (v_1, .., v_n)$ which satisfies the system (2.1) of n equations.

We claim this case can not happen. Indeed, let

$$\sigma_i^k := \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{u_i^k} \text{ and } \sigma_i := \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{v_i}, \quad i \in I.$$

Clearly

$$\tilde{\sigma}_i := \lim_{k\to\infty} \sigma_i^k \geq \sigma_i. \tag{2.18}$$

Note that $\tilde{\sigma} = (\tilde{\sigma}_1, .., \tilde{\sigma}_n)$ and $\sigma := (\sigma_1, .., \sigma_n)$ both satisfy $\Lambda_I = 0$ and we also have $\sum_{j\in I} a_{ij}\sigma_j > 2$ for $i \in I$ because $\int_{\mathbb{R}^2} e^{v_i} < \infty$. Taking the difference on the two equations $\Lambda_I(\tilde{\sigma}) = 0$ and $\Lambda_I(\sigma) = 0$, we arrive at

$$\sum_{j\in I}(\sum_{i\in I} a_{ij}\tilde{\sigma}_i - 2)(\tilde{\sigma}_j - \sigma_j) + \sum_{j\in I}(\sum_{i\in I} a_{ij}\sigma_i - 2)(\tilde{\sigma}_j - \sigma_j) = 0. \tag{2.19}$$

For each $i \in I$, $\sum_{j\in I} a_{ij}\tilde{\sigma}_j \geq \sum_{j\in I} a_{ij}\sigma_j > 2$. Combining this fact with (2.18) and (2.19) we have $\tilde{\sigma}_i = \sigma_i$ for all $i \in I$. Thus $\tilde{\sigma} \in \mathcal{E}$, which is an open subset of the hypersurface $\Lambda_I = 0$ (see (2.15)), a contradiction to the assumption that $\tilde{\sigma} \in \partial\mathcal{E}$.

Case two: There exists $K \subsetneq I$ such that u_j^k converges for $j \in K$ and $u_j^k(r) \to -\infty$ uniformly in any bounded set of $[0, \infty)$ for $j \notin K$.

Let $l = |K|$, clearly u_1^k must converge, so without loss of generality, we assume that the first l components of u^k converge to $v = (v_1, .., v_l)$ which satisfies

$$-\Delta v_i = \sum_{j=1}^{l} a_{ij} e^{v_j}, \quad i = 1, .., l, \quad \text{in } \mathbb{R}^2 \tag{2.20}$$

and it is easy to show

$$\tilde{\sigma}_i := \lim_{k\to\infty} \sigma_i^k \geq \sigma_i := \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{v_i}, \quad i = 1, .., l.$$

Since $\int_{\mathbb{R}^2} e^{v_i} < \infty$, by (2.7), we have

$$\sum_{j=1}^{l} a_{ij}\sigma_j > 2, \quad i = 1, .., l. \tag{2.21}$$

Let $\sigma := (\sigma_1, .., \sigma_l, 0, .., 0)$ and $\tilde{\sigma} = (\tilde{\sigma}_1, .., \tilde{\sigma}_n)$ be the limit of σ^k. We have proved that $\tilde{\sigma}_i \geq \sigma_i$ for $1 \leq i \leq l$. Although $(a_{ij})_{l\times l}$ may not be invertible, we still have the Pohozaev identity:

$$\sum_{i,j=1}^{l} a_{ij}\sigma_i\sigma_j = 4\sum_{i=1}^{l} \sigma_i. \tag{2.22}$$

See Theorem C. Now we claim

$$\tilde{\sigma}_i = \sigma_i, \quad i = 1, .., l; \quad \tilde{\sigma}_i = 0 \quad \text{for} \quad i > l. \tag{2.23}$$

Clearly (2.17) follows from (2.23) and (2.22). The proof of (2.23) relies heavily on (**H2**). Set

$$m_i = \sum_{j=1}^{n} a_{ij}\sigma_j, \quad i \in I.$$

We want to prove

$$m_i > 2 \quad \text{for all } \; i \in I. \tag{2.24}$$

Obviously this is true for $i = 1, .., l$, so we need to prove $m_i > 2$ for $i > l$. To see this, we first observe that $m_i > 0$ for all i because $a_{ij} > 0$, $i \neq j$, by Lemma 2.1.

The proof of $m_i > 2$ can be obtained by the similar argument of Lemma 2.1. Let

$$m = \min\{m_{l+1}, m_{l+2}, .., m_n\}.$$

Suppose $m \leq 2$, then let $J = \{i \in I; \, m_i = m\}$ and $m^* = \min_{i \in J^c}\{m_i\}$. Same as before, $1, .., l \notin J$, and $\sigma_i = 0$ for all $i \in J$. So, (2.16) yields

$$0 = \sigma_i \geq m \sum_{j \in I} a^{ij} + (m^* - m) \sum_{j \notin J^c} a^{ij} \geq 0,$$

and then $a^{ij} = 0$ for $i \in J$ and $j \notin J^c$, which yields a contradiction to the irreducibility of A. Hence, $m_i > 2$ is proved.

Now we finish the proof of (2.23). Certainly, (2.22) can be written as $\Lambda_I(\sigma) = 0$ (the last $n - l$ components of σ are 0). $\tilde{\sigma}$ also satisfies $\Lambda(\tilde{\sigma}) = 0$. Besides we have $\tilde{\sigma}_i \geq \sigma_i$ and $\tilde{m}_i, m_i > 2$ for all $i \in I$. Using (2.19), we obtain $\tilde{\sigma}_i = \sigma_i$ for all i. (2.23) and (2.17) are established.

Finally, for any σ on the surface $\Lambda_I = 0$ with all the components positive, we claim that $\sigma \in \mathcal{E}$. Suppose $\sigma \notin \mathcal{E}$. Let $\sigma_E \in \mathcal{E}$, since $\Lambda_I(\sigma) = 0$ is connected, we can find a path $\Gamma(t)(0 \leq t \leq 1)$ on $\Lambda_I = 0$ that connects σ and σ_E ($\Gamma(0) = \sigma_E, \Gamma(1) = \sigma$). Here we require all the components of $\Gamma(t)$ ($0 \leq t \leq 1$) be positive. Because $\sigma \notin \mathcal{E}$, there exists $t_0 \in [0, 1]$ such that $\Gamma(t_0) \in \partial\mathcal{E}$. But it yields a contradiction to (2.17) because no component of $\Gamma(t_0)$ is zero. Theorem 2.1 is established. □

In the proof of Theorem 2.1 we assumed $\mathcal{E} \neq \emptyset$, which is established in the following lemma.

Lemma 2.2. *$\mathcal{E}$ is not empty.*

Proof of Lemma 2.2:

Let $\xi_i = \sum_{j=1}^{n} a^{ij}$ for $i \in I$. From (**H2**) we know $\xi_i \geq 0$ for all $i \in I$. First consider the case that all ξ_i are positive:

Case one: $\xi_i > 0$ for all $i \in I$

Using the properties of inverse matrices

$$\sum_{j=1}^{n} a_{ij}\xi_j = \sum_{j=1}^{n} a_{ij}\sum_{k=1}^{n} a^{jk} = \sum_{k=1}^{n}\delta_{ik} = 1. \tag{2.25}$$

Let $\sigma_i = 4\xi_i$ for $i \in I$. Then by the assumption in this case $\sigma_i > 0$ for all $i \in I$ and direct computation shows

$$\begin{aligned} \Lambda_I(\sigma) &= 4\sum_{i\in I}\sigma_i - \sum_{i,j\in I} a_{ij}\sigma_i\sigma_j \\ &= 16(\sum_{i\in I}\xi_i - \sum_{i,j\in I} a_{ij}\xi_i\xi_j) = 16\sum_{i\in I}\xi_i(1-\sum_{j=1}^{n} a_{ij}\xi_j) = 0. \end{aligned} \tag{2.26}$$

For any nonempty $J \subsetneq I$, without loss of generality $J = 1,..,l$ for some $l < n$, easy to see

$$\begin{aligned} \Lambda_J &= 4\sum_{i=1}^{l}\sigma_i - \sum_{i,j=1}^{l} a_{ij}\sigma_i\sigma_j \\ &= 16(\sum_{i=1}^{l}\xi_i - \sum_{i,j=1}^{l} a_{ij}\xi_i\xi_j) \\ &= 16\sum_{i=1}^{l}\xi_i(1-\sum_{j=1}^{l} a_{ij}\xi_j) = 16\sum_{i=1}^{l}(\sum_{j=l+1}^{n} a_{ij}\xi_j)\xi_i. \end{aligned}$$

Clearly $\Lambda_J \geq 0$. Since by Lemma 2.1 $a_{ij} > 0$ for $1 \leq i \leq l$ and $l+1 \leq j \leq n$ we have $\Lambda_J > 0$. Therefore $\sigma \in \mathcal{E}$ and $\mathcal{E} \neq \emptyset$.

Case Two: There exists $\xi_i = 0$

First we observe that it is not possible to have all $\xi_i = 0$ because otherwise adding all the rows of A^{-1} to the first row would make all the entries of the first row 0, a contradiction to the invertibility of A^{-1}. Without loss of generality we assume $\xi_{l+1} = ... = \xi_n = 0$ and $\xi_1,..,\xi_l > 0$. Let $J = \{1,..,l\}$ then we claim

$$\Lambda_J(\sigma) = 0 \quad \text{and} \quad \Lambda_{J_1} > 0 \quad \forall \emptyset \subsetneq J_1 \subsetneq J. \tag{2.27}$$

From (2.26) we see easily $\Lambda_J = 0$. For $\emptyset \subsetneq J_1 \subsetneq J$, without loss of generality we assume $J_1 = \{1,..,l_1\}$ with $l_1 < l$. Similar to case one (using $\sum_{j=1}^{l} a_{ij}\xi_j = 1$ and $\sigma_i = 4\xi_i$ for all $i = 1,..,l$)

$$\Lambda_{J_1} = 16\sum_{i=1}^{l_1}(\sum_{j=l_1+1}^{l} a_{ij}\xi_j)\xi_i.$$

Thus $\Lambda_{J_1} > 0$ is an immediate consequence of Lemma 2.1. (2.27) is established.

Let $\tilde{A}=(a_{ij})_{i,j\in J}$. Although $\tilde{A}$ may not be invertible, Theorem B can still be applied to conclude that there exists a radially symmetric solution $u(r)=(u_1(r),..,u_l(r))$ to

$$\left\{\begin{array}{l}\Delta u_i+\sum_{j=1}^l a_{ij}e^{u_j}=0,\quad \mathbb{R}^2,\quad i=1,..,l,\\ \\ \frac{1}{2\pi}\int_{\mathbb{R}^2}e^{u_i}=\sigma_i=4\xi_i,\quad i=1,..,l.\end{array}\right. \tag{2.28}$$

Since $e^{u_i}\in L^1(\mathbb{R}^2)$, we have

$$\sum_{j=1}^l a_{ij}\sigma_j>2,\quad i=1,..,l. \tag{2.29}$$

It has been discussed in the proof of Theorem 2.1 that (**H2**) implies

$$\sum_{j=1}^l a_{ij}\sigma_j>2,\quad \forall i\in I. \tag{2.30}$$

which leads to

$$\frac{1}{2\pi}\int_{B_R}\sum_{j=1}^l a_{ij}e^{u_j}>2+\delta,\quad i\in I \tag{2.31}$$

for some $\delta>0$ and $R>1$. Now we construct a sequence of functions $u^\epsilon=(u_1^\epsilon,..,u_n^\epsilon)$ as follows:

$$\left\{\begin{array}{l}(u_i^\epsilon)''(r)+\frac{1}{r}(u_i^\epsilon)'(r)+\sum_{j=1}^n a_{ij}e^{u_j^\epsilon(r)}=0,\quad 0<r<\infty,\quad i\in I,\\ \\ u_i^\epsilon(0)=u_i(0),\quad i=1,...,l,\\ \\ u_i^\epsilon(0)=\log\epsilon,\quad i=l+1,..,n.\end{array}\right.$$

By standard ODE existence theory, solution u^ϵ is well defined for all $r>0$. Since $u_i^\epsilon(r)$ are decreasing functions, it is easy to see that as ϵ tends to 0, u_i^ϵ $(i>l)$ tends to $-\infty$ over $[0,R]$ for any fixed $R>0$. Therefore, the first l components of u^ϵ converge uniformly to the corresponding components of u over any fixed $[0,R]$. In particular for the R in (2.31) we have, for ϵ sufficiently small

$$\frac{1}{2\pi}\int_{B_R}\sum_{j=1}^n a_{ij}e^{u_j^\epsilon}\ge\frac{1}{2\pi}\int_{B_R}\sum_{j=1}^l a_{ij}e^{u_j^\epsilon}>2+\delta/2,\quad \forall i\in I, \tag{2.32}$$

which implies

$$(u_i^\epsilon)'(r)=-\frac{1}{r}\int_0^r\sum_{j=1}^n a_{ij}e^{u_j^\epsilon(s)}sds<-\frac{2+\delta/2}{r},\quad r>R.$$

Thus

$$u_i^\epsilon(r)\le-(2+\delta/2)\log r+O(1),\qquad \text{for } r>R$$

where $O(1)$ is a constant independent of r but may depend on ϵ. Hence for $\epsilon > 0$ small, $\int_{\mathbb{R}^2} e^{u_i^\epsilon} < \infty$ for all $i \in I$. Thus $\mathcal{E} \neq \emptyset$. Case two and Lemma 2.2 are established. □

Remark 2.1. *For the* 2×2 *case, let* $A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$ *be an non-negative and irreducible matrix (which means* $c > 0$*). It can be proved that the conclusion of Theorem 2.1 holds for* A *if and only if* $\max(a, b) \leq 2c$*. However,* A *satisfies (***H1***) and (***H2***) if and only if* $\max(a, b) \leq c$ *and* $c^2 \neq ab$*.*

3. The case of partial blowup

In this section we consider the following case: Let $u^k = \{u_1^k, .., u_n^k\}$ be a sequence of solutions of

$$-\Delta u_i^k = \sum_{j=1}^n a_{ij} h_j^k(x) e^{u_j^k}, \quad B_1, \quad i \in I \tag{3.1}$$

where $h^k = (h_1^k, .., h_n^k)$ is a sequence of positive functions with uniformly bounded C^1 norm:

$$\frac{1}{C} \leq h_i^k(x) \leq C, \quad |\nabla h_i^k|(x) \leq C, \quad \forall x \in B_1, \quad \forall i \in I. \tag{3.2}$$

Suppose that u^k blows up only at 0:

$$M_k = \max_{i \in I, x \in B_1} u_i^k(x) \to \infty, \quad \max_{i \in I} u_i^k \leq C(K) \ \forall K \subset\subset B_1 \setminus \{0\}. \tag{3.3}$$

Let

$$\sigma_i^k = \frac{1}{2\pi} \int_{B_1} h_i^k e^{u_i^k}; \quad m_i^k = \sum_{j=1}^n a_{ij} \sigma_j^k, \quad i \in I. \tag{3.4}$$

Without loss of generality, we assume $u_1^k(0) = M_k$, and set

$$v_i^k(y) = u_i^k(\delta_k y) + 2 \log \delta_k, \quad \text{where } \delta_k = e^{-\frac{1}{2} u_1^k(0)}. \tag{3.5}$$

By elliptic estimates, we can show that there is $J \subseteq I$ such that $\{v_i^k\}_{i \in J}$ converges, in $C^2_{loc}(\mathbb{R}^2)$, to $\{v_i\}_{i \in J}$. We may assume $J = \{1, 2, \cdots, l\}$. Then $v = (v_1, \cdots, v_l)$ solves the following subsystem:

$$-\Delta v_i = \sum_{j=1}^l a_{ij} \mathcal{H}_j e^{v_j}, \quad i = 1, \cdots, l, \quad \text{in } \mathbb{R}^2, \tag{3.6}$$

where $\mathcal{H}_i = \lim_{k \to \infty} h_i^k(0)$. In this case, we have $v_i^k(x) \to -\infty$ for $i \notin J$ as $k \to +\infty$ in any compact set of $\mathbb{R}^2$.

We also assume u^k to have bounded oscillation on ∂B_1 :

$$|u_i^k(x) - u_i^k(y)| \leq C, \quad \forall x, y \in \partial B_1, \quad i \in I \tag{3.7}$$

and uniformly bounded energy in B_1.

$$\int_{B_1} h_i^k e^{u_i^k} \le C, \quad i \in I. \tag{3.8}$$

When $l = n$, it was proved by the authors [33] that there exists an entire solution $U^k = (U_1^k, \cdots, U_n^k)$ of

$$\Delta U_i^k + \sum a_{ij} \mathcal{H}_j e^{U_i^k} = 0 \quad \text{in } \mathbb{R}^2$$

such that

$$|u_i^k(x) - U_i^k(x)| \le C \quad \text{for } |x| \le \frac{1}{2} \quad \text{and} \quad i \in I.$$

An immediate consequence is the following estimate:

$$u_i^k(x) = -\frac{m_i^k - 2}{2} M_k + O(1) \quad \text{for } |x| = \frac{1}{2}. \tag{3.9}$$

In this section, we want to extend the estimate (3.9) to the case when there are only l components of v^k, $l < n$, which converge to an entire solution of (3.6).

Proposition 3.1. *Let $h^k = (h_1^k, \cdots, h_n^k)$ satisfy (3.2), and let u^k be solutions of (3.1) such that (3.3), (3.6), (3.7) and (3.8) hold. Let $\sigma_i = \lim_{k\to+\infty} \sigma_i^k$ and $m_i = \lim_{k\to+\infty} m_i^k$, where σ_i^k and m_i^k are given by* (3.4). *Then*

1. $4\sum_{i=1}^l \sigma_i = \sum_{i,j=1}^l a_{ij}\sigma_i\sigma_j$.
2. $\sigma_1, .., \sigma_l > 0$, $\sigma_{l+1} = ... = \sigma_n = 0$.
3. $m_i > 2 \quad \forall i \in I$.
4. *For* $1 \le i \le l$

$$|u_i^k(x) + \frac{m_i^k - 2}{2} M_k| \le C(\epsilon), \quad \forall x \in B_1 \setminus \bar{B}_\epsilon, \tag{3.10}$$

5. *For* $l + 1 \le i \le n$

$$|u_i^k(x) + \frac{m_i^k - 2}{2} M_k + (M_k - u_i^k(0))| \le C(\epsilon), \quad \forall x \in B_1 \setminus \bar{B}_\epsilon. \tag{3.11}$$

Proof of Proposition 3.1:

This proof is divided into a few steps.

Step one: $m_i > 2$ for all $i \in I$.

Let

$$\sigma_{iv} = \frac{1}{2\pi} \int_{\mathbb{R}^2} \mathcal{H}_i(0) e^{v_i}, \quad i = 1, 2, .., l.$$

Then Theorem C and (2.7) imply

$$\begin{cases} \sum_{i,j=1}^l a_{ij}\sigma_{iv}\sigma_{jv} = 4\sum_{i=1}^l \sigma_{iv} \\ \\ \sum_{j=1}^l a_{ij}\sigma_{jv} > 2, \quad i = 1, ..., l. \end{cases} \tag{3.12}$$

Obviously, $\sigma_i \geq \sigma_{iv}$ for $i = 1, .., l$. Let $\sigma_{iv} = 0$ for $i = l+1, .., n$ and

$$m_{iv} = \sum_{j=1}^{n} a_{ij}\sigma_{jv}, \quad i = 1, .., n.$$

We claim that $m_{iv} > 2$ for all $i = 1, .., n$. For $i \leq l$, this is already known by (3.12). The proof of $m_{iv} > 2$ for $i > l$ is the same as (2.24) in the proof of Theorem 2.1. Since $\sigma_i \geq \sigma_{iv}$, we also have $m_i > 2$.

Step two: $u_i^k(x) \to -\infty$ over any compact subset of $B_1 \setminus \{0\}$ for all $i = 1, 2, .., n$.

We first show that u_i^k tends to $-\infty$ on ∂B_1. Let $G(x, y)$ be the Green's function with respect to the Dirichlet condition on ∂B_1, the Green's representation formula gives

$$u_i^k(x) \geq \int_{B_1} G(x,\eta)\left(\sum_{j=1}^{l} a_{ij} h_j^k(\eta) e^{u_j^k(\eta)}\right) d\eta + \min_{\partial B_1} u_i^k.$$

If $\min_{\partial B_1} u_i^k \geq -C$ for some $C > 0$, we use

$$\sum_{j=1}^{l} a_{ij} h_j^k e^{u_j^k} \to 2\pi m_i \delta_0 \quad \text{in measure}$$

where δ_0 is the Dirac mass at 0, and

$$G(x,\eta) \geq -\frac{1}{2\pi} \log|x-\eta| - C_1 \quad \text{for } |x| \leq \frac{1}{2}$$

to obtain

$$\lim_{k\to\infty} e^{u_i^k(x)} \geq C_2 |x|^{-2-\epsilon_1} \quad \text{for } 0 < |x| \leq \frac{1}{2} \quad \text{and for some } \epsilon_1 > 0,$$

where C_1 and C_2 are two positive constants. This is a contradiction to $\int_{B_1} e^{u_i^k} \leq C$. Therefore we have proved $\min_{\partial B_1} u_i^k \to -\infty$. Since u_i^k is bounded from above over any compact subset of $B_1 \setminus \{0\}$, standard elliptic estimate implies that $u_i^k \to -\infty$ uniformly in any compact subset of $B_1 \setminus \{0\}$.

Step three: $\sigma_{l+1} = ... = \sigma_n = 0$.

Let $w_i^k = \sum_{j\in J} a^{ij} u_j$. Then w_i^k satisfies

$$-\Delta w_i^k = h_i^k e^{\sum_{j\in I} a_{ij} w_j^k}.$$

We can apply the Pohozaev identity to the above equation, i.e., multiplying $x \cdot \nabla(\sum a_{ij} w_j^k)$ and applying integration by parts, we obtain by passing $k \to +\infty$,

$$\sum_{i,j=1}^{n} a_{ij}\sigma_i\sigma_j = 4\sum_{i=1}^{n} \sigma_i.$$

On the other hand, $\sigma_v = (\sigma_{1v}, ..., \sigma_{lv}, 0, .., 0)$ also satisfies the equality above. Besides, we also have

$$\sigma_i \ge \sigma_{iv}, \quad i = 1, .., l, \quad \sigma_i \ge 0, \quad i = l+1, .., n.$$

By taking the difference of the above equations satisfied by σ and σ_v, we arrive at the same as (2.19):

$$\sum_{j \in I} \left(\sum_{i \in I} a_{ij}\sigma_i - 2 \right)(\sigma_j - \sigma_{jv}) + \sum_{j \in I}\sum_{i \in I}(a_{ij}\sigma_{iv} - 2)(\sigma_j - \sigma_{jv}) = 0.$$

Recall $m_i = \sum a_{ij}\sigma_j > 2$ and $m_{iv} = \sum a_{ij}\sigma_{jv} > 2$. Then the above identity leads to $\sigma = \sigma_v$, i.e., $\sigma_i = \sigma_{iv}$ for $i \le l$ and $\sigma_i = 0$ for $i > l$.

Step four: $u_i^k(x) + 2\log|x| \le C$

By scaling, it is equivalent to proving

$$v_i^k(y) + 2\log|y| \le C, \quad |y| \le \delta_k^{-1}. \tag{3.13}$$

Since only $\{v_1^k, .., v_l^k\}$ converges to $\{v_1, .., v_l\}$ in $C^2_{loc}(\mathbb{R}^2)$, it implies that $v_{l+1}^k, ..., v_n^k$ tend to $-\infty$ in any compact subset of $\mathbb{R}^2$. Suppose (3.13) does not hold, there exists $y_k \to \infty$ such that, along a subsequence

$$v_{i_0}^k(y_k) + 2\log|y_k| = \max_{|y| \le \delta_k^{-1}, i \in I}(v_i^k(y) + 2\log|y|) \to +\infty. \tag{3.14}$$

It is easy to see that $|y_k|\delta_k \to 0$ as u_i^k is bounded above in any compact subset of $B_1 \setminus \{0\}$. Consider $y \in B(y_k, |y_k|/2)$, from

$$v_i^k(y) + 2\log|y| \le v_{i_0}^k(y_k) + 2\log|y_k|, \quad i \in I$$

we have

$$v_i^k(y) \le v_{i_0}^k(y_k) + 2\log 2 \quad \text{for } y \in B(y_k, |y_k|/2) \quad \text{and} \quad i \in I. \tag{3.15}$$

On one hand, the fact $|y_k| \to \infty$ implies

$$\lim_{k \to \infty} \frac{1}{2\pi} \int_{B(0, |y_k|/4)} h_i^k(\delta_k \cdot) e^{v_i^k} \ge \sigma_{iv}, \quad i = 1, 2, .., l. \tag{3.16}$$

On the other hand, we can set

$$w_i^k(y) = v_i^k(y_k + r_k y) - v_{i_0}^k(y_k), \quad \text{for } |y| < R_k, \quad i \in I,$$

where $r_k = e^{-\frac{1}{2}v_{i_0}^k(y_k)}$ and $R_k = r_k|y_k|/2 \to \infty$ as $k \to \infty$ by (3.14). By (3.15), $w_i^k(y)$ is bounded from above by $2\log 2$, and $w_{i_0}^k(0) = 0$. From the standard estimate of elliptic equations, there is $J \subseteq I$ such that $w_i^k(y)$ converges in $C^2_{loc}(\mathbb{R}^2)$ for $i \in J$ and $w_i^k(x) \to -\infty$ for $i \notin J$ in any compact set of $\mathbb{R}^2$. Clearly, $i_0 \in J$. In particular,

$$\int_{|y| < R_k} e^{w_{i_0}^k} \ge \delta > 0 \text{ for some } \delta > 0.$$

Thus, together with (3.16) we have

$$\sigma_{i_0} = \int_{B_1} h_{i_0}^k e^{u_{i_0}^k} \geq \sigma_{i_0 v} + \delta > \sigma_{i_0 v},$$

a contradiction to step three. Hence, step four is established.

Step five: Estimate of the decay of v_i^k.

First we choose $R >> 1$ such that

$$\frac{1}{2\pi}\int_{B_{R/2}} \sum_{j=1}^{l} a_{ij} h_j^k(\delta_k y) e^{v_j^k(y)} dy > 2 + \delta, \quad i \in I. \tag{3.17}$$

for some $\delta > 0$. Note that (3.17) obviously holds for $1 \leq i \leq l$ because of of the convergence from $(v_1^k, ..v_l^k)$ to $(v_1, .., v_l)$. Then by step one, it also holds for $i > l$. In this step we study the behavior of v_i^k for $|y| > R$ $(i \in I)$. For each $R < |y| < \delta_k^{-1}/2$, let

$$\hat{v}_i^k(z) = v_i^k(|y|z) + 2\log|y|, \quad \frac{1}{2} < |z| < 2.$$

By step four $\hat{v}_i^k \leq C_1$ for some C_1 over $B_2 \setminus B_{1/2}$. Consider the equation for $\hat{v}_i^k$ in $B_2 \setminus B_{1/2}$:

$$-\Delta \hat{v}_i^k(z) = \sum_{j=1}^{n} a_{ij} h_j^k(\delta_k |y| z) e^{\hat{v}_j^k} \quad B_2 \setminus B_{\frac{1}{2}}.$$

Let f_i^k satisfy

$$\begin{cases} -\Delta f_i^k(z) = \sum_{j=1}^n a_{ij} h_j^k(\delta_k |y| z) e^{\hat{v}_j^k} \quad B_2 \setminus B_{1/2}, \\ \\ f_i^k(z) = 0, \quad \text{on} \quad \partial B_{1/2} \cup \partial B_2. \end{cases}$$

Clearly $f_i^k \geq 0$ in $B_2 \setminus B_{1/2}$. As a result of the upper bound of $\hat{v}_i^k$, we have

$$0 \leq f_i^k(z) \leq C_2, \quad z \in B_2 \setminus B_{1/2}. \tag{3.18}$$

Obviously, $\hat{v}_i^k - C_1 - f_i^k$ is a non-positive harmonic function. Hence the Harnack inequality holds:

$$-\min_{\partial B_1}\left(\hat{v}_i^k - C_1 - f_i^k\right) \leq C\left(-\max_{\partial B_1}\left(\hat{v}_i^k - C_1 - f_i^k\right)\right).$$

Equivalently,

$$\max_{\partial B_1} \hat{v}_i^k \leq \frac{1}{C}\min_{\partial B_1} \hat{v}_i^k + C_2.$$

Going back to v_i^k, we have

$$\max_{\partial B_r} v_i^k(y) \leq \frac{1}{C}\min_{\partial B_r} v_i^k + (-2 + \frac{2}{C})\log r + C_2, \tag{3.19}$$

for $R < r \leq \delta_k^{-1}$ and $i = 1, .., n$.

Let $\bar{v}_i^k(r)=\frac{1}{2\pi r}\int_{\partial B_r} v_i^k$ be the average of v_i^k on ∂B_r.

$$(\bar{v}_i^k)'(r)=\frac{1}{2\pi r}\int_{B_r}\Delta v_i^k.$$

By (3.17) we have

$$(\bar{v}_i^k)'(r)<-\frac{2+\delta}{r},\quad r>R$$

for some $\delta>0$. So for $r>R$,

$$\bar{v}_i^k(r)\le -(2+\delta)\log r+\bar{v}_i^k(R).$$

By (3.19), we have, for $i\in I$

$$\begin{aligned} v_i^k(y) &\le \frac{1}{C}\bar{v}_i^k(r)+(-2+\frac{2}{C})\log|y|+C_2,\\ &\le -(2+\frac{\delta}{C})\log|y|+C_2+\bar{v}_i^k(R). \end{aligned}\tag{3.20}$$

Note that for $i>l$, $\bar{v}_i^k(R)\to-\infty$. Thus

$$v_i^k(y)\le -R_k-(2+\frac{\delta}{C})\log|y|,\quad \forall i>l\tag{3.21}$$

for some $R_k\to\infty$ as $k\to\infty$.

Step Six: The proof of (3.10) and (3.11)

Let y_1,y_2 be two points on the same circle centered at 0: $|y_1|=|y_2|$. Then the Green's representation formula gives

$$\begin{aligned} & v_i^k(y_1)-v_i^k(y_2)\\ =& \int_{B(0,\delta_k^{-1})}(G(y_1,\eta)-G(y_2,\eta))(\sum_{j=1}^n a_{ij}h_j^k(\delta_k\eta)e^{v_j^k})d\eta\\ & +\hat{h}_i^k(y_1)-\hat{h}_i^k(y_2) \end{aligned}\tag{3.22}$$

where $\hat{h}^k=(\hat{h}_1^k,..,\hat{h}_n^k)$ is harmonic defined by $\hat{h}_i^k=v_i^k$ on $\partial B(0,\delta_k^{-1})$. Clearly

$$|\hat{h}_i^k(y)-\hat{h}_i^k(y')|=O(1),\quad \forall y,y'\in B(0,\delta_k^{-1})$$

because v_i^k has bounded oscillation on $\partial B(0,\delta_k^{-1})$. Next we claim

$$|v_i^k(y_1)-v_i^k(y_2)|\le C,\quad \forall r\in(0,\delta_k^{-1}),\quad \forall |y_1|=|y_2|=r,\quad i\in I.\tag{3.23}$$

We only need to verify (3.23) for $r<\delta_k^{-1}/2$, as the case for $r>\frac{1}{2}\delta_k^{-1}$ is obvious. To evaluate the expression for $v_i^k(y_1)-v_i^k(y_2)$, we first have

$$G(y_1,\eta)-G(y_2,\eta)=\frac{1}{2\pi}\log\frac{|y_2-\eta|}{|y_1-\eta|}+\frac{1}{2\pi}\log\frac{|1-\delta_k^2\bar{y}_1\eta|}{|1-\delta_k^2\bar{y}_2\eta|}$$

where $\bar{y}_1$ is the conjugate of y_1 when it is considered as a complex number. To prove (3.23) we decompose $B(0,\delta_k^{-1})$ into the following four sets:

$$E_1:=\{\eta;\ |\eta|\le r/2\},\quad E_2:=\{\eta;\ |\eta-y_1|<|\eta-y_2|,\frac{r}{2}\le|\eta|\le 2r\},$$

$$E_3 := \{\eta;\ |\eta - y_2| \le |\eta - y_1|,\ \frac{r}{2} \le |\eta| \le 2r\}, \quad E_4 = B(0,\delta_k^{-1}) \setminus (\cup_{i=1}^3 E_i).$$

By (3.20)

$$e^{v_j^k(y)} \le (1+|y|)^{-2-\delta'}, \quad j \in I, \quad |y| < \delta_k^{-1}$$

where $\delta' > 0$. Applying this decaying estimate, we can prove the integral of the right hand side of (3.22) over each E_j is bounded. We omit the proof because it is a standard computation. Thus (3.23) is proved. Once (3.23) is established, we have

$$v_i^k(y) = \bar{v}_i^k(r) + O(1), \quad i \in I, \quad |y| = r. \tag{3.24}$$

For $\bar{v}_i^k(r)$ we have

$$(\bar{v}_i^k)'(r) = -\frac{1}{2\pi r}\int_{B_r}\sum_{j=1}^n a_{ij}h_j^k(\delta_k y)e^{v_j^k}dy. \tag{3.25}$$

For $r > R$, the decay rates of v_i^k in (3.20) and (3.21) imply

$$\begin{aligned}\frac{1}{2\pi r}\int_{B_r}\sum_{j=1}^n a_{ij}h_i^k(\delta_k y)e^{v_j^k} &= \frac{1}{2\pi r}\left(\int_{B(0,\delta_k^{-1})} - \int_{B(0,\delta_k^{-1})\setminus B_r}\right) \\ &= \frac{m_i^k}{r} + O(r^{-1-\delta}).\end{aligned} \tag{3.26}$$

Using (3.26) in (3.25) we have

$$\bar{v}_i^k(r) = v_i^k(0) - m_i^k\log(1+r) + O(1), \quad 0 \le r < \delta_k^{-1}. \tag{3.27}$$

Clearly (3.10) and (3.11) follow from (3.24) and (3.27). This completes the proof of Proposition 3.1. □

Remark 3.1. *It is easy to see from* (3.6) *that the sequence* $u_i^k(0) - M_k$ *is bounded for* $1 \le i \le l$ *and tends to* $-\infty$ *for* $l+1 \le i \le n$.

4. The strong interaction between bubbles

In this section, we suppose u^k has two blow-up points p_1 and p_2. By Proposition 3.1, at each blow-up point p_i, u^k after scaling will converges to an entire solution of a subsystem (3.6). The following question naturally arises:

Are these two entire solutions the same?

The following theorem will answer this question affirmatively.

Proposition 4.1. *Let Ω_0 be an open and bounded set with smooth boundary, $p_1, p_2 \in \Omega_0$ be two distinct points. Suppose $u^k = (u_1^k, .., u_n^k)$ satisfies (3.1), (3.7) and (3.8) on Ω_0 and and h^k satisfies (3.2) over Ω_0 as well. Let p_1, p_2 be the only two blow-up points of u^k on Ω:*

$$\exists x_{tk} \to p_t, i_t \in I, \quad \text{such that} \ \ u_{i_t}^k(x_{tk}) \to \infty, \quad t = 1, 2.$$

$$\max_K u_i^k \le C(K), \quad \forall K \subset\subset \Omega_0 \setminus \{p_1, p_2\}, \quad i \in I.$$

Then for $\delta < \frac{1}{2}|p_1 - p_2|$

$$\lim_{k\to\infty}\int_{B(p_1,\delta)} h_i^k e^{u_i^k}dx = \lim_{k\to\infty}\int_{B(p_2,\delta)} h_i^k e^{u_i^k}dx, \quad i \in I.$$

Proof of Proposition 4.1: Let

$$\sigma_i^k = \frac{1}{2\pi}\int_{B(p_1,\delta)} h_i^k e^{u_i^k}dx, \quad \bar\sigma_i^k = \frac{1}{2\pi}\int_{B(p_2,\delta)} h_i^k e^{u_i^k}dx,$$

for $\delta > 0$ small and $i \in I$. Also we let

$$m_i^k = \sum_{j=1}^n a_{ij}\sigma_j^k, \quad \bar m_i^k = \sum_{j=1}^n a_{ij}\bar\sigma_j^k, \quad i \in I.$$

We use $\sigma_i, m_i, \bar\sigma_i$ and $\bar m_i$ to denote the limit of $\sigma_i^k, m_i^k, \bar\sigma_i^k$ and $\bar m_i^k$, respectively. Let

$$M_k = \max_{i\in I} u_i^k(x), \quad x \in B(p_1,\delta),$$

and M_k is attained by some component of u^k at p_{1k} which tends to p_1. $\bar M_k$ and p_{2k} can be defined similarly. By comparing the value of u_i^k over $\Omega_0 \setminus (B(p_1,\delta)\cup B(p_2,\delta))$, using Proposition 3.1 we have

$$\frac{m_i^k - 2}{2}M_k + (M_k - u_i^k(p_{1k})) = \frac{\bar m_i^k - 2}{2}\bar M_k + (\bar M_k - u_i^k(p_{2k})) + O(1). \tag{4.1}$$

Here we remind the reader that, for example around p_1, if the first l components of u^k converge to a system of l equations after scaling, then $M_k - u_i^k(p_{1k})$ are uniformly bounded for $1 \le i \le l$. In this case, $M_k - u_i^k(p_{1k})$ can be combined with the $O(1)$ term. For $i > l$, $M_k - u_i^k(p_{ik})$ tends to $+\infty$. The right hand side of (4.1) can also be understood this way. For each $i \in I$, if

$$M_k - u_i^k(p_{1k}) > \bar M_k - u_i^k(p_{2k})$$

we let

$$l_i^k = (M_k - u_i^k(p_{1k})) - (\bar M_k - u_i^k(p_{2k})), \quad \bar l_i^k = 0.$$

On the other hand if

$$M_k - u_i^k(p_{1k}) \le \bar M_k - u_i^k(p_{2k})$$

we let

$$l_i^k = 0, \quad \bar l_i^k = (\bar M_k - u_i^k(p_{2k})) - (M_k - u_i^k(p_{1k})).$$

Set

$$I_1 := \{i \in I;\ \lim_{k\to\infty}\frac{l_i^k}{\bar M_k} > 0.\}, \quad I_2 := \{i \in I;\ \lim_{k\to\infty}\frac{\bar l_i^k}{\bar M_k} > 0.\}$$

and I_3 be the compliment of $I_1 \cup I_2$. From this definition we see easily that $I_1 \cap I_2 = \emptyset$.

We claim that I_1 is empty. We prove this by contradiction. Suppose I_1 is not empty, then we consider the following two cases: I_2 is not empty or I_2 is empty.

Case one: $I_2 \neq \emptyset$

Let

$$\lambda = \lim_{k\to\infty} \frac{M_k}{\bar{M}_k}, \quad \delta_i = \lim_{k\to\infty} \frac{l_i^k}{\bar{M}_k}, \quad \bar{\delta}_i = \lim_{k\to\infty} \frac{\bar{l}_i^k}{\bar{M}_k}.$$

We claim that all these limits exist. Indeed, using the definitions of l_i^k and $\bar{l}_i^k$, (4.1) can be written as

$$\frac{m_i^k - 2}{2} \cdot \frac{M_k}{\bar{M}_k} + \frac{l_i^k}{\bar{M}_k} = \frac{\bar{m}_i^k - 2}{2} + \frac{\bar{l}_i^k}{\bar{M}_k} + \circ(1).$$

Take $i \in I_1$, the RHS tends to $(\bar{m}_i - 2)/2$, which implies that along a subsequence, the two terms on the LHS are $\frac{m_i-2}{2}\lambda$ and δ_i. On the other hand, take $j \in I_2$, the LHS tends to $\frac{m_j-2}{2}\lambda$, then the RHS has to tend to $\frac{\bar{m}_j-2}{2} + \bar{\delta}_j$. Now (4.1) can be written as

$$\lambda\frac{m_i - 2}{2} + \delta_i = \frac{\bar{m}_i - 2}{2} + \bar{\delta}_i, \quad i \in I. \tag{4.2}$$

From the definition of δ_i and $\bar{\delta}_i$, we observe that for each $i \in I_1$, $\bar{\delta}_i = 0$ and for $i \in I_2$, $\delta_i = 0$. By (3.10) and (3.11) of Proposition 3.1, we have

$$\sigma_i = 0,\ i \in I_1; \qquad \bar{\sigma}_i = 0,\ i \in I_2. \tag{4.3}$$

Since $\delta_i = 0$ for $i \notin I_1$ and $\sigma_i = 0$ for $i \in I_1$, we have $\sigma_i\delta_i = 0$ for all $i \in I$. Similarly, $\overline{\sigma}_i\bar{\delta}_i = 0$ for all $i \in I$.

Without loss of generality we assume

$$I_1 = \{1 \le i \le i_0\}.$$

For each $i \in I_2$, the fact $\bar{\sigma}_i = 0$ yields

$$0 = \bar{\sigma}_i = \sum_{j\in I} a^{ij}\bar{m}_j = \sum_{j\in I_2} a^{ij}\bar{m}_j + \sum_{j\notin I_2} a^{ij}\bar{m}_j. \tag{4.4}$$

We observe that the last term is positive, because $\bar{m}_j > 2$ and there exists $a^{ij} > 0$ for some $i \in I_2$ and $j \notin I_2$. Multiplying $\bar{\delta}_i$ to the last term and taking the summation on i for all i in I_2, we have

$$\sum_{i\in I_2}(\sum_{j\notin I_2} a^{ij}\bar{m}_j)\bar{\delta}_i > 0. \tag{4.5}$$

Combining (4.4) and (4.5), we have

$$\sum_{i,j\in I_2} a^{ij}\bar{m}_i\bar{\delta}_j < 0.$$

Trivially, there exists $\tilde{i} \in I_2$ such that

$$\sum_{j\in I_2} a^{\tilde{i}j}\bar{\delta}_j < 0. \tag{4.6}$$

Multiplying $a^{\tilde{i}j}$ to both sides of (4.2) (with i replaced by j) and taking the summation on j, it leads to

$$\sum_{j\in I} a^{\tilde{i}j}(\frac{m_j-2}{2})\lambda + \sum_{j\in I} a^{\tilde{i}j}\delta_j = \sum_{j\in I} a^{\tilde{i}j}(\frac{\bar{m}_j-2}{2}) + \sum_{j\in I} a^{\tilde{i}j}\bar{\delta}_j. \tag{4.7}$$

Using the definition of $\sigma_{\tilde{i}}$ as well as $\delta_i = 0$ for $i \notin I_1$, we can write the LHS of (4.7) as

$$\frac{1}{2}\lambda\sigma_{\tilde{i}} - \lambda\sum_{j\in I} a^{\tilde{i}j} + \sum_{j\in I_1} a^{\tilde{i}j}\delta_j.$$

Since the first term and the third term are both nonnegative, the LHS is no less than the second term. Similarly the RHS can be written as

$$\frac{1}{2}\bar{\sigma}_{\tilde{i}} - \sum_{j\in I} a^{\tilde{i}j} + \sum_{j\in I_2} a^{\tilde{i}j}\bar{\delta}_j.$$

Note that we have used $\bar{\delta}_i = 0$ for $i \notin I_2$. The first term of the above is 0 (because $\tilde{i} \in I_2$) and the last term is negative (because of (4.6)). Therefore the RHS is strictly less than the second term. Putting the estimates on both sides together we have

$$-\lambda\sum_{j\in I} a^{\tilde{i}j} < -\sum_{j\in I} a^{\tilde{i}j}.$$

Since $\sum_{j\in I} a^{\tilde{i}j} \geq 0$ ((**H2**)) we conclude $\lambda > 1$. On the other hand, by exchanging I_1 and I_2 in the above argument, we obtain $\lambda < 1$. Thus we have ruled out the first case.

Case two: $I_2 = \emptyset$

One immediately has $\bar{\delta}_i = 0$ for all $i \in I$. Hence, the limits $\lambda = \lim_{k\to\infty} M_k/\bar{M}_k$ and $\delta_k = \lim_{k\to\infty} l_i^k/\bar{M}_k$ both exist and (4.2) holds with $\bar{\delta}_k = 0$. Here we recall that both $\sigma = (\sigma_1, .., \sigma_n)$ and $\bar{\sigma} = (\bar{\sigma}_1, .., \bar{\sigma}_n)$ satisfy

$$4\sum_{i\in I}\sigma_i = \sum_{i,j\in I} a_{ij}\sigma_i\sigma_j$$

which can be written as

$$\sum_{i,j\in I} a^{ij}(\frac{m_i-2}{2})(\frac{m_j-2}{2}) = \sum_{i,j\in I} a^{ij}. \tag{4.8}$$

We further remark that

$$\sum_{i,j\in I} a^{ij} > 0 \tag{4.9}$$

because for each i, $\sum_{j\in I} a^{ij} \geq 0$ and A^{-1} is non-singular. To prove our result, we need another fact:

(4.10) All the eigenvalues of $\mathbf{F}$ are nonpositive, where

$$\mathbf{F} = (a^{ij})_{i_0\times i_0} \quad i,j\in I_1=\{1,..,i_0\}.$$

Indeed, let μ be the largest eigenvalue of $\mathbf{F}$ and $\eta=(\eta_1,..,\eta_{i_0})$ be an eigenvector corresponding to μ. Here η is the vector that attains

$$\max_{\mathbf{v}\in\mathbb{R}^n} \mathbf{v}^{\mathbf{T}}\,\mathbf{F}\,\mathbf{v}, \quad \mathbf{v}^{\mathbf{T}}\mathbf{v}=1.$$

Since $a^{ij}\geq 0$ for all $i\neq j$, we can choose $\eta_i\geq 0$ for all $i\in I_1$. For each $i\in I_1$,

$$0=\sigma_i=\sum_{j\in I_1} a^{ij}m_j+\sum_{j\notin I_1} a^{ij}m_j.$$

Plainly by (**H2**)

$$\sum_{j\in I_1} a^{ij}m_j\leq 0, \quad i\in I_1.$$

Multiplying η_i on both sides and taking the summation on i, then we have

$$0\geq \sum_{i,j\in I_1} a^{ij}\eta_i m_j=\sum_{j\in I_1}\mu\eta_j m_j.$$

Since each $\eta_i\geq 0$ (one of them is strictly positive) and $m_i>0$ for $i\in I_1$, we have $\mu\leq 0$, and it proves (4.10). Now we go back to our proof to rule out case two.

Since $\bar{\delta}_i=0$ in (4.2), the Pohozaev identity (4.8) for $\bar{\sigma}$ can be written as

$$\sum_{i,j\in I} a^{ij}(\frac{m_i-2}{2}\lambda+\delta_i)(\frac{m_j-2}{2}\lambda+\delta_j)=\sum_{ij} a^{ij}.$$

Expanding the LHS of the above and using (4.8) again for σ_i, we obtain

$$\lambda^2\sum_{i,j\in I} a^{ij}+2\lambda\sum_{i,j\in I} a^{ij}(\frac{m_i-2}{2})\delta_j+\sum_{i,j\in I_1} a^{ij}\delta_i\delta_j=\sum_{i,j} a^{ij}. \tag{4.11}$$

The third term of LHS is nonpositive by (4.10). The second term of the LHS can be written as

$$\lambda\sum_{j\in I}(\sigma_j-2\sum_{i\in I} a^{ij})\delta_j=-\lambda\sum_{j\in I}(\sum_{i\in I} a^{ij})\delta_j\leq 0$$

because $\sigma_j\delta_j=0$ for all $j\in I$. Thus we conclude from (4.11) that $\lambda\geq 1$. On the other hand from $\sigma_i=0$ ($i\in I_1$), argued as (4.6) we obtain an index $\tilde{i}\in I_1$ such that

$$\sum_{j\in I_1} a^{\tilde{i}j}\delta_j<0. \tag{4.12}$$

Then as we did for (4.7), we obtain

$$\lambda\sum_{j\in I} a^{\tilde{i}j}(\frac{m_j-2}{2})+\sum_{j\in I} a^{\tilde{i}j}\delta_j=\sum_{j\in I} a^{\tilde{i}j}(\frac{\bar{m}_j-2}{2}).$$

Following the same calculation as before we obtain

$$0 > \sum_{j\in I_1} a^{\tilde{i}j}\delta_j = \frac{\bar{\sigma}_{\tilde{i}}}{2} + (\lambda-1)\sum_{j\in I} a^{\tilde{i}j},$$

which forces λ to be less than 1. Thus, we have obtained a contradiction to $\lambda \ge 1$. Case two is also ruled out. Thus we have proved that I_1 has to be empty. Using exactly the same argument we also have $I_2 = \emptyset$.

Since $I_1 = I_2 = \emptyset$, (4.2) becomes

$$\lambda \frac{m_i - 2}{2} = \frac{\bar{m}_i - 2}{2}, \quad i \in I.$$

Using (4.8) for both $(m_1, .., m_n)$ and $(\bar{m}_1, .., \bar{m}_n)$ we have $\lambda = 1$. Consequently $\sigma_i = \bar{\sigma}_i$ for all $i \in I$. Proposition 4.1 is established. □

5. Proof of Theorem 1.1 and Theorem 1.2

Let $u = (u_1, .., u_n)$ be a solution of (1.1). Define

$$v_i = u_i - \log \int_M h_i e^{u_i} dV_g. \tag{5.1}$$

Clearly $v = (v_1, .., v_n)$ satisfies

$$\int_M h_i e^{v_i} dV_g = 1 \tag{5.2}$$

and

$$\Delta_g v_i + \sum_{j=1}^n \rho_j a_{ij}(h_j e^{v_j} - 1) = 0, \quad i \in I. \tag{5.3}$$

To prove the a priori bound for u, we only need to establish

$$|v_i(x)| \le C, \quad \forall x \in M, \quad i \in I. \tag{5.4}$$

Indeed, once we have (5.4), for u we have

$$\log \int_M h_i e^{u_i} - C \le u_i(x) \le \log \int_M h_i e^{u_i} + C, \quad \forall x \in M. \tag{5.5}$$

Since $u \in \mathring{H}^1(M)$, there exists x_0 such that $u(x_0) = 0$. Using this in (5.5) we have

$$-C \le \log \int_M h_i e^{u_i} \le C. \tag{5.6}$$

With (5.1) and (5.6) we see that a bound for u can be obtained from the bound for v. To prove (5.4) we only need to prove

$$v_i \le C, \quad i \in I. \tag{5.7}$$

because a lower bound for v_i can be obtained easily from the upper bound in (5.7) by standard elliptic estimate. So we only need to establish (5.7).

We prove (5.7) by contradiction. Suppose there are solutions v^k to (5.3) such that $\max_{M, i\in I} v_i^k(x) \to +\infty$. We consider the following two cases.

Case one: $\rho_i^k \to \rho_i > 0$ as $k \to +\infty$ for all $i \in I$.

The equation for v^k is

$$\Delta_g v_i^k + \sum_{j=1}^{n} \rho_j^k a_{ij}(h_j e^{v_j^k} - 1) = 0, \quad i \in I. \tag{5.8}$$

In [33] the authors prove a Brezis-Merle type lemma (Lemma 4.1) which guarantees that there exists a positive constant $\epsilon_0 > 0$ such that if

$$\int_{B(p,r_0)} e^{v_i^k} dx \le \epsilon_0 \quad \text{for all} \quad i \in I,$$

then

$$v_i^k(x) \le C, \quad x \in \overline{B}\left(p, \frac{r_0}{2}\right). \tag{5.9}$$

Thus, v^k blows up only at a finite set $\{p_1, \cdots, p_N\}$. Since $v_i^k(x)$ is uniformly bounded from above in any compact set of $M \setminus \{p_1, \cdots, p_N\}$, by (5.8), v_i^k converges to $\sum_{l=1}^N m_i(p_l) G(x, p_l)$ in $C^\infty_{loc}(M \setminus \{p_1, \cdots, p_N\})$, where

$$\begin{cases} m_i(p_l) = \sum_{j \in I} a_{ij} \sigma_j(p_l), \\ \sigma_j(p_l) = \lim_{k\to\infty} \frac{1}{2\pi} \int_{B(p_l,\delta_0)} \rho_j^k h_j e^{v_j^k} dV_g \end{cases} \tag{5.10}$$

for some $\delta_0 > 0$ such that $B(p_l, 2\delta_0) \cap B(p_{l'}, 2\delta_0) = \emptyset$, $l \neq l'$. Here, $G(x,p)$ is the Green function:

$$\begin{cases} -\Delta_{g,x} G(x,p) = \delta_p - 1, \\ \int_M G(x,p) dV_g(x) = 0. \end{cases}$$

To apply Proposition 4.1, we rewrite (5.8) in local coordinates. For $p \in M$, let $y = (y^1, y^2)$ be the isothermal coordinates near p such that $y_p(p) = (0,0)$ and y_p depends smoothly on p. In this coordinate, ds^2 has the form

$$e^{\phi(y_p)} \left[(dy^1)^2 + (dy^2)^2 \right],$$

where $\nabla \phi(0) = 0$, $\phi(0) = 0$. Also near p we have

$$\Delta_{y_p} \phi = -2K e^{\phi}, \quad \text{where } K \text{ is the Gauss curvature.}$$

When there is no ambiguity, we write $y = y_p$ for simplicity. In this local coordinate, (5.8) is of the form:

$$-\Delta v_i^k = e^{\phi} \sum_{j=1}^{n} a_{ij} \rho_j^k \left(h_j e^{v_j^k} - 1 \right) \quad \text{in } B(0,\delta_0), \quad i \in I. \tag{5.11}$$

Let f_i^k be defined as

$$-\Delta f_i^k = -e^{\phi} \sum_{j=1}^{n} \rho_j^k a_{ij} \quad \text{in } B(0,\delta_0), \quad i \in I,$$

and $f_i^k(0) = |\nabla f_i^k(0)| = 0$. Let $\tilde{v}_i^k = v_i^k - f_i^k$ and

$$H_i^k = e^{\phi} \rho_i^k e^{f_i^k} h_i.$$

Then (5.11) becomes

$$-\Delta \tilde{v}_i^k = \sum_{j=1}^n a_{ij} H_j^k e^{\tilde{v}_j^k} \quad \text{in } B(0,\delta_0). \tag{5.12}$$

Here, we observe that $\int_{B(0,\delta_0)} H_i^k e^{\tilde{u}_i^k} dx = \int_{B(0,\delta_0)} \rho_i^k h_i e^{v_i^k} dV_g$.

Since v_i^k converges in $M \setminus \bigcup_{j=1}^N B(p_j, 2\delta_0)$, we have

$$|\tilde{v}_i^k(x) - \tilde{v}_i^k(y)| \le C, \quad \forall x,y \in M \setminus \bigcup_{j=1}^N B(p_j, 2\delta_0), \quad i \in I. \tag{5.13}$$

By (3.10) of Proposition 3.1, we also have

$$\int_{M\setminus \bigcup_{j=1}^N B(p_j,\delta_0)} h_i e^{v_i^k} dV_g \to 0, \quad i \in I, \tag{5.14}$$

and by Proposition 4.1,

$$\lim_{k\to\infty} \int_{B(p_l,\delta_0)} \rho_i^k h_i e^{v_i^k} dV_g = \lim_{k\to\infty} \int_{B(p_m,\delta_0)} \rho_i^k h_i e^{v_i^k} dV_g \tag{5.15}$$

for $i \in I$ and for any pair of integers l, m between 1 and N. (5.14) combined with (5.15) yields for $i \in I$ and $j \in \{1, 2, \cdots, N\}$,

$$\sigma_i = \lim_{k\to\infty} \frac{1}{2\pi} \int_{B(0,\delta_0)} H_i^k e^{\tilde{v}_i^k} dx = \lim_{k\to\infty} \frac{1}{2\pi} \int_{B(p_j,\delta_0)} \rho_i^k h_i e^{v_i^k} dV_g = \frac{\rho_i}{2\pi N}.$$

On the other hand, $(\sigma_1, \cdots, \sigma_n)$ satisfies the Pohozaev identity:

$$4 \sum_{i\in I} \sigma_i = \sum_{i,j\in I} a_{ij} \sigma_i \sigma_j. \tag{5.16}$$

Consequently,

$$8\pi N \sum_{i=1}^n \rho_i = \sum_{i,j=1}^n a_{ij} \rho_i \rho_j.$$

Thus, a contradiction to the assumption of the theorem.

Case two: $\lim_{k\to\infty} \rho_i^k = \rho_i > 0$, $i = 1, .., l$, $\lim_{\to\infty} \rho_i^k = 0$ for $i > l$.

Let $M_k = \max\{v_1^k, .., v_l^k\}$ and $\bar{M}_k = \{v_{l+1}^k, .., v_n^k\}$. We first show

$$\bar{M}_k - M_k \le C \tag{5.17}$$

by contradiction. Suppose $\bar{M}_k - M_k \to \infty$, let

$$V_i^k(y) = v_i^k(e^{-\frac{\bar{M}_k}{2}} y + p_k) - \bar{M}_k$$

where p_k is where $\bar{M}_k$ is attained: $v_{i_0}^k(p_k) = \bar{M}_k$. Clearly $i_0 > l$. Thanks to the fact that $V_i^k \to -\infty$ for $i \le l$ and $\rho_i^k \to 0$ for $i > l$, $V_{i_0}^k$ converges in $C^2_{loc}(\mathbb{R}^2)$ to

$$\begin{cases} -\Delta V_{i_0} = 0, & \mathbb{R}^2, \\ V_{i_0}(0) = 0, & V_{i_0} \le 0. \end{cases}$$

Clearly $V_{i_0} \equiv 0$, $\int_{B_R} e^{V_{i_0}}$ can be arbitrarily large if R is large, this is a contradiction to (5.2). (5.17) is proved.

We use the same notations as in case one. Let $p_1, .., p_N$ be blowup points for v_i^k. Then around each blowup point, say p_1, the equation for v^k can be written in local coordinates as (5.12) with $\tilde{v}_i^k$ and H_i^k defined the same as in case one. Without loss of generality we assume $\rho_i^k > 0$ for all k and $l+1 \le i \le L$ and $\rho_i^k = 0$ for all k and all $L+1 \le i \le n$. Then we observe from the definition of H_i^k that $H_i^k \to 0$ for $l+1 \le i \le L$ and $H_i^k = 0$ for $i > L$. To reduce case two to case one, we need to adjust the terms involved with these vanishing H_i^ks. To do this we set $\hat{f}_i^k$ as

$$\begin{cases} -\Delta \hat{f}_i^k = \sum_{j=L+1}^n a_{ij} e^{\tilde{v}_j^k - M_k}, \quad B(0,\delta), \\ \\ \hat{f}_i^k = 0 \quad \text{on} \quad \partial B(0,\delta). \end{cases}$$

Since $\max v_i^k - M_k$ is uniformly bounded for all i, we have

$$\|\hat{f}_i^k\|_{C^1} \le C$$

for some C independent of k. Now we define

$$\hat{v}_i^k = \begin{cases} \tilde{v}_i^k + \hat{f}_i^k, \quad i = 1, .., l, \\ \tilde{v}_i^k + \log \rho_i^k + \hat{f}_i^k, \quad l+1 \le i \le L, \\ \tilde{v}_i^k - M_k + \hat{f}_i^k, \quad L+1 \le i \le n. \end{cases}$$

and

$$\hat{H}_i^k = \begin{cases} H_i^k e^{-\hat{f}_i^k}, \quad 1 \le i \le l, \\ \\ \frac{H_i^k}{\rho_i^k} e^{-\hat{f}_i^k} = e^{\phi + f_i^k - \hat{f}_i^k} h_i, \quad l+1 \le i \le L, \\ \\ e^{-\hat{f}_i^k}, \quad L+1 \le i \le n. \end{cases}$$

Easy to see there exists $c > 0$ independent of k such that

$$\frac{1}{c} \le \hat{H}_i^k \le c, \quad |\nabla \hat{H}_i^k| \le c, \quad B(0,\delta).$$

On the other hand $\hat{v}_i^k$ satisfies

$$-\Delta \hat{v}_i^k = \sum_{j \in I} a_{ij} \hat{H}_j^k e^{\hat{v}_j^k}, \quad B(0,\delta), \quad i \in I.$$

Easy to observe that $\max \hat{v}_i^k - M_k \to -\infty$ for $i = l+1, ..., n$. Therefore case two is reduced to case one, which gives

$$\sigma_i(p_t) = \sigma_i(p_m) \quad \forall t, m \in \{1, .., N\}, \quad \forall 1 \le i \le l.$$

Note that $\sigma_i(p_t) = 0$ for all $i > l$ and all t because $\rho_i^k \to 0$ for $i > l$. Then as in case one we obtain a contradiction. Theorem 1.1 is established. $\Box$

Proof of Theorem 1.2:

Theorem 1.2 will be discussed in two cases.

Case 1. One of a_{ii} is positive.

We may suppose $a_{11} > 0$. Thanks to Theorem 1.1 the Leray-Schauder degree of (1.1) for $\rho \in \mathcal{O}_{N-1}$ is equal to the degree for the following specific system corresponding to $(\rho_1, 0, .., 0)$:

$$\Delta_g u_1 + \rho_1 a_{11}\left(\frac{h_1 e^{u_1}}{\int_M h_1 e^{u_1} dV_g} - 1\right) = 0, \tag{5.18}$$

$$\Delta_g u_j + \rho_1 a_{j1}\left(\frac{h_1 e^{u_1}}{\int_M h_1 e^{u_1} dV_g} - 1\right) = 0 \quad \text{for } j \geq 2.$$

where ρ_1 satisfies

$$8\pi(N-1) < a_{11}\rho_1 < 8\pi N. \tag{5.19}$$

Easy to see $(\rho_1, 0, .., 0) \in \mathcal{O}_{N-1}$. Using Theorem 1.2 of [13], we obtain the degree counting formulas (1.6) in this case.

Case 2. $a_{ii} = 0$ for all $i \in I$.

By Lemma 2.1, $a_{12} > 0$. The degree counting formula of (1.1) for $\rho \in \mathcal{O}_N$ can be computed by the degree of the following specific system

$$\begin{cases} \Delta_g u_1 + a_{12}\rho_2\left(\frac{h_2 e^{u_2}}{\int_M h_2 e^{u_2} dV_g} - 1\right) = 0, \\ \Delta_g u_2 + a_{12}\rho_1\left(\frac{h_1 e^{u_1}}{\int_M h_1 e^{u_1} dV_g} - 1\right) = 0, \\ \Delta_g u_i + \rho_1 a_{i1}\left(\frac{h_1 e^{u_1}}{\int_M h_1 e^{u_1} dV_g} - 1\right) \\ \qquad +\rho_2 a_{i2}\left(\frac{h_2 e^{u_2}}{\int_M h_2 e^{u_2} dV_g} - 1\right) = 0, \quad i \geq 3. \end{cases} \tag{5.20}$$

where ρ_1, ρ_2 satisfy

$$8\pi(N-1)(\rho_1 + \rho_2) < 2a_{12}\rho_1\rho_2 < 8\pi N(\rho_1 + \rho_2). \tag{5.21}$$

Easy to see $(\rho_1, \rho_2, 0, .., 0) \in \mathcal{O}_{N-1}$. Now we consider the special case of (5.20): $\rho_1 = \rho_2$ and $h_1 = h_2 = h$. In this case, the maximum principle implies $u_1 = u_2 + c$ for some constant c. Since u_1, u_2 are both in $\mathring{H}^1(M)$, $c = 0$. Then the first two equations of (5.20) turn out to be:

$$\Delta_g u + a_{12}\rho\left(\frac{h e^{u}}{\int_M h e^{u} dV_g} - 1\right) = 0, \tag{5.22}$$

where ρ satisfies

$$8\pi(N-1) < a_{12}\rho < 8\pi N.$$

Hence, again the Leray-Schauder degree for equation (1.1) can be reduced to the Leray-Schauder degree for the single equation (5.22). By applying Theorem 1.2 in [13], the degree counting formulas (1.6) is also obtained in this case. This completes the proof of Theorem 1.2. $\square$

6. Proof of Theorem 1.4

For equation (1.3), we have to show u^k never blows up near the boundary $\partial\Omega$. This fact is standard, we include the argument for the convenience of the reader (see [34]). Since Ω is a bounded set with smooth boundary, there is a uniform constant r_0 such that for any point on $\partial\Omega$, there is a ball of radius r_0 tangent to $\partial\Omega$ at this point from the outside. Let $x_0 \in \partial\Omega$ and $B(x_1, \lambda)$ be a ball tangent to $\partial\Omega$ at x_0 from the outside. $\lambda \leq r_0$ will be determined later. Let

$$H_i = \frac{\rho_i h_i}{\int_\Omega h_i e^{u_i} dx}, \quad \rho_i > 0, \ i \in I$$

then the equation for u_i becomes

$$-\Delta u_i = \sum_{j=1}^n a_{ij} H_j e^{u_j}, \quad \Omega, \quad i \in I.$$

For H_i we obviously have

$$|\nabla \log H_i(x)| \leq C, \quad \forall x \in \Omega. \tag{6.1}$$

Let $y = x - x_1$ and

$$u_i^\lambda(y) = u_i(x_1 + \lambda^2 \frac{y}{|y|^2}), \quad i \in I.$$

Then

$$-\Delta u_i^\lambda(y) = \sum_{j=1}^n a_{ij} \left(\frac{\lambda^4}{|y|^4} H_j(x_1 + \lambda^2 \frac{y}{|y|^2}) \right) e^{u_j^\lambda(y)} \quad \text{in} \quad \Omega^\lambda$$

where Ω^λ is the image of Ω under the Kelvin transformation. Moreover, we have

$$u_i^\lambda = 0 \quad \text{on} \quad \partial\Omega^\lambda.$$

Let

$$\bar{H}_i(y) = \frac{\lambda^4}{|y|^4} H_i(x_1 + \lambda^2 \frac{y}{|y|^2}).$$

Using (6.1), we see by direct computation that in a small neighborhood of x_0, $\bar{H}_i$ is strictly decreasing in the outer normal direction to $\partial\Omega^\lambda$, as long as λ is small. The smallness of the neighborhood of x_0 and λ can both be represented by ϵ_0 which depends on the usual constants. Thus we have the monotonicity of $\bar{H}_i$ in a neighborhood of the whole $\partial\Omega^\lambda$. Using the standard moving plane argument we see u_i^λ is increasing along the inner normal of $\partial\Omega^\lambda$ in a small neighborhood of $\partial\Omega^\lambda$, which implies that for any sequence of function u^k of (1.3), no blowup point for u^k exists in a fixed neighborhood of $\partial\Omega$. Then the remaining part of the proof of Theorem 1.4 is the same as Theorem 1.1 and Theorem 1.2. So, the details are omitted here. □

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