ON MAXIMAL PRIMITIVE QUOTIENTS OF INFINITESIMAL CHEREDNIK ALGEBRAS OF $\mathfrak{gl}_n$

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Abstract. We prove analogues of some of Kostant's theorems for infinitesimal Cherednik algebras of $\mathfrak{gl}_n$. As a consequence, it follows that in positive characteristic the Azumaya and smooth loci of the center of these algebras coincide.

Infinitesimal Cherednik algebras (more generally, infinitesimal Hecke algebras) were introduced by Etingof, Gan and Ginzburg [EGG]. Here we will be concerned with infinitesimal Cherednik algebras of $\mathfrak{gl}_n$. Let us recall the definition. Let $\mathfrak{h} = \mathbb{C}^n$ denote the standard representation of $\mathfrak{g} = \mathfrak{gl}_n$. Denote by y_i the standard basis elements of $\mathfrak{h}$, and by x_i the dual basis of $\mathfrak{h}^*$. For the given deformation parameter $b = b_0+b_1\tau+\cdots+b_m\tau^m \in \mathbb{C}[\tau], b_m \neq 0, m \geq 0$, one defines the infinitesimal Cherednik algebra of $\mathfrak{gl}_n$ with parameter b, to be denoted by H_b, as the quotient of the semi-direct product $\mathfrak{U}\mathfrak{g} \ltimes T(\mathfrak{h}\oplus\mathfrak{h}^*)$ by the relations

$$[x, x'] = 0, \qquad [y, y'] = 0, \qquad [y, x] = b_0r_0(x, y)+b_1r_1(x, y)+\cdots+b_mr_m(x, y),$$

where $x, x' \in \mathfrak{h}, y, y' \in \mathfrak{h}^*$, and $r_i(x, y) \in \mathfrak{U}\mathfrak{g}$ are the symmetrizations of the following functions on $\mathfrak{g}$ (thought of as elements in $\operatorname{Sym}\mathfrak{g}$ in the standard way):

$$(x,(1-tA)^{-1}y)\det(1-tA)^{-1} = r_0(x, y)(A)+r_1(x, y)(A)t+r_2(x, y)(A)t^2+\cdots$$

The algebras H_b have the following PBW property. If we introduce the filtration on H_b by setting $\deg x = \deg y = 1, x \in \mathfrak{h}^*, y \in \mathfrak{h}, \deg g = 0, g \in \mathfrak{g}$, then the natural map : $\mathfrak{U}\mathfrak{g} \ltimes \operatorname{Sym}(\mathfrak{h}\oplus\mathfrak{h}^*) \to \operatorname{gr} H_b$ is an isomorphism.

Besides the action of $G = GL_n(\mathbb{C})$ on H_b, we also have the action of $\mathfrak{h}$ and $\mathfrak{h}^*$ defined as follows. For any $v \in \mathfrak{h}$, the adjoint action $\operatorname{ad}(v)$ is locally nilpotent on H_b. Thus $exp(\operatorname{ad}(v))$ gives an automorphism of H_b, and in this way $\mathfrak{h}$ acts on H_b. The action of $\mathfrak{h}^*$ on H_b is defined similarly. Combining these actions with the G-action, we get the actions of $G \ltimes \mathfrak{h}, G \ltimes \mathfrak{h}^*$ on H_b.

The enveloping algebra $\mathfrak{U}(\mathfrak{sl}_{n+1})$ is an example of H_b (for $m = 1$). In fact, the algebras H_b have many properties similar to the enveloping algebras of simple Lie algebras.

Let $Q_1, ..., Q_n \in k[\mathfrak{g}]^G$ be defined as follows:

$$\det(t\operatorname{Id} - X) = \sum_{j=0}^{n}(-1)^j t^{n-j}Q_j(X).$$

Also let $\alpha_1, \cdots \alpha_n$ be the corresponding elements of $\mathbf{Z}(\mathfrak{U}\mathfrak{g})$ under the standard identification of $\mathbb{C}[\mathfrak{g}]^G$ and $\mathbf{Z}(\mathfrak{U}\mathfrak{g})$. It was shown in [T1] that the following elements generate the center of H_b

$$t_i = \sum_j [\alpha_i, x_j] y_j - c_i = \sum x_j [\alpha_i, y_j] - c_i \in \mathbf{Z}(H_b),$$

where $c_i \in \mathbf{Z}(\mathfrak{U}\mathfrak{g})$ are certain elements. Namely, if we consider the following element of $\mathbb{C}[\mathfrak{g}][t, \tau]$ given by

$$c' = b_m \frac{\det(t - A)}{(t\tau - 1)\det(1 - \tau A)}$$

then the top symbol of c_i considered as an element of $\mathbb{C}[\mathfrak{g}]$ is the coefficient of $t^{n-i}\tau^m$ in c'. We have that $\mathbf{Z}(H_b) = \mathbb{C}[t_1, \cdots, t_n]$. For a character $\chi : \mathbb{C}[t_1, \cdots, t_n] \to \mathbb{C}$, denote by $U_{b,\chi}$ the quotient $H_b / \ker(\chi) H_b$.

From now on we will assume that $m \geq 1$. Let us introduce a new filtration on H_b by setting $\deg x_i = m, \deg y_i = 1, \deg g = 1, g \in \mathfrak{g}$. Then, $\operatorname{gr} H_b = \operatorname{Sym}(\mathfrak{g} \oplus \mathfrak{h} \oplus \mathfrak{h}^*)$ is a Poisson algebra (and the Poisson bracket only depends on m). We will denote it by A_m. Denote $B_m = \operatorname{gr} H_b / (\operatorname{gr} t_i)$. Again, B_m is a Poisson algebra $m \geq 0$. As remarked earlier, $\operatorname{Spec} B_1$ is the nilpotent cone of $\mathfrak{sl}_{n+1}(\mathbb{C})$. The main result of this paper is the following analogue of some of Kostant's theorems for semi-simple Lie algebras [K].

Theorem 0.1. *The algebra H_b is a free module over its center. B_m is a an integral domain which is a normal, complete intersection ring. Moreover, the smooth locus of $\operatorname{Spec} B_m$ under the Poisson bracket is symplectic.*

Proof. We will partially follow [BL]. Denote by f_y (respectively f_x) the element $\det\{\alpha_i, y_j\} \in B_m$. Then the localization $(B_m)_{f_y}$ is isomorphic to the polynomial algebra $\operatorname{Sym}(\mathfrak{g} \oplus \mathfrak{h})$. We will use the notation $D(f) = \operatorname{Spec}(B_m)_f \subset \operatorname{Spec} B_m, f \in B_m$. Let us set $U = D(f_x) \cup D(f_y)$. To show that $X = \operatorname{Spec} B_m$ is an irreducible, reduced and normal variety, it is enough to show that it is Cohen-Macaulay, U is connected, and $\dim(X \backslash U) \leq \dim X - 2$ [BL].

We have an action of the affine group $G \ltimes \mathfrak{h}$ on $\operatorname{Sym}(\mathfrak{g} \oplus \mathfrak{h})$. Then f_y is a semi-invariant of this action, i.e., $(g, v) f_y = \det(g) f_y, g \in G, v \in \mathfrak{h}$. As explained in [R1], the set $D(f_y) \subset \operatorname{Spec} \operatorname{Sym}(\mathfrak{g} \oplus \mathfrak{h})$ is the dense orbit under the action of $G \ltimes \mathfrak{h}$ on $\operatorname{Spec} \operatorname{Sym}(\mathfrak{g} \oplus \mathfrak{h})$. In fact, this set consists of pairs (A, v) with $A \in \mathfrak{g}, v \in \mathfrak{h}$, such that $v, Av, \cdots A^{n-1}v$ are linearly independent. We have a similar statement about $D(f_x)$, and the action of $G \ltimes \mathfrak{h}^*$ on $\operatorname{Sym}(\mathfrak{g} \oplus \mathfrak{h}^*)$.

Recall that the algebra $\mathbb{C}[\alpha_1, \cdots, \alpha_n]$ is finite over $\mathbb{C}[c_1, \cdots, c_n]$. In particular, $\mathbb{C}[c_1, \cdots, c_n]$ is isomorphic to the polynomial algebra in n variables. Therefore $\mathbb{C}[\alpha_1, \cdots, \alpha_n]$ is a finitely generated free module over $\mathbb{C}[c_1, \cdots, c_n]$.

Let us introduce a filtration on A_m, where $\deg g = 1, g \in \mathfrak{g} \deg x_i, \deg y_j = 0$. Since $\operatorname{Sym} \mathfrak{g}$ is a free $\mathbb{C}[\alpha_1, \cdots, \alpha_n]$-module (by Kostant's theorem for $\mathfrak{g}$ [BL]), we conclude that $\operatorname{Sym} \mathfrak{g}$ is a free $\mathbb{C}[c_1, \cdots, c_n]$-module. This implies that $(t_1, \cdots, t_n)$ is a regular sequence (since $c_j = \operatorname{gr} t_j$) and $\operatorname{gr} A_m$ is a free module over $\mathbb{C}[\operatorname{gr} t_1, \cdots, \operatorname{gr} t_n]$. Therefore A_m is a free module over $\mathbb{C}[t_1, \cdots, t_n]$. In particular, H_b is a free $\mathbf{Z}(H_b)$-module. Also, we obtain that B_n is a complete intersection ring.

The latter filtration on A_m induces the corresponding filtration on its quotient B_m. Then the degeneration of $X \setminus U$ under this filtration will be given by equations $c_i = 0, i = 1, \cdots n, f_x = 0, f_y = 0$. Therefore, what we get is nothing but $Y = \mathfrak{h} \times \mathfrak{h}^* \times N \cap (f_x = 0 = f_y)$, where N denotes the nilpotent cone of $\mathfrak{g}$. We need to prove that $\dim Y \leq= \dim X = \dim N + 2\dim \mathfrak{h} - 2$. Consider the projection map $p : Z \to N$. Let $U \subset N$ denote the open subset of regular nilpotent matrices. Then clearly $\dim p^{-1}(U) \leq \dim N + 2\dim \mathfrak{h} - 2$, and $p^{-1}(N \setminus U) = (N \setminus U) \times \mathfrak{h} \times \mathfrak{h}^*$, whose dimension is $\dim N + 2\dim \mathfrak{h} - 2$.

Finally, it is obvious that $D(f_x) \cap D(f_y)$ is nonempty. It is also clear that $D(x_f) \cup D(y_f)$ is in the orbit of any element of $D(x_f) \cap D(y_f)$ under the actions of $G \ltimes \mathfrak{h}, G \ltimes \mathfrak{h}^*$. Therefore $D(x_f) \cup D(y_f)$ lies in a single symplectic leaf. □

As a consequence we get that $\operatorname{gr} U_{b,\chi} = B_m$ is a domain, so $U_{b,\chi}$ is also a domain.

In analogy with semi-simple Lie algebras, one defines an analogue of the category $\mathcal{O}$, and Verma modules for H_b [T1]. Let us recall their definition. Denote by n_+ (respectively n_-) the Lie subalgebra of $\mathfrak{g}$ consisting of upper (lower) triangular matrices. Then we have a triangular decomposition $H_b = H_- \otimes \mathfrak{U}(C) \otimes H_+$, where H_+ (respectively H_-) denotes the subalgebra of H_b generated by n_+ $(n_-), \mathfrak{h}(\mathfrak{h}^*)$, and $C \subset \mathfrak{g}$ is the Cartan subalgebra of all diagonal matrices. For a weight $\lambda \in C^*$, the corresponding Verma module $M(\lambda)$ is defined as $H_b \otimes_{\mathfrak{U}(C) \otimes H_+} \mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ is the 1-dimensional representation of $\mathfrak{U}(C) \otimes H_+$ on which C acts by λ and $n_+, \mathfrak{h}$ act like 0.

We have the following analogue of a theorem of Duflo [D].

Corollary 0.1. *The annihilator of a Verma module $M(\lambda)$ is generated by $\operatorname{Ann}(M(\lambda)) \cap \mathbf{Z}(H_b)$.*

Proof. Directly following [J], using the fact that Verma mudules have finite length [T1], it follows that $H_b/\operatorname{Ann}(M(\lambda))$ has the same Gelfand-Kirillov dimension as $H_b/\operatorname{Ann}(M(\lambda)) \cap \mathbf{Z}(H_b)$. But since the latter is a domain, we get that $\operatorname{Ann}(M(\lambda)) = (\operatorname{Ann}(M(\lambda)) \cap \mathbf{Z}(H_b))H_b$. □

Let us discuss the case of a field $\mathbf{k} = \bar{\mathbf{k}}$ of positive characteristic. We will assume that $char(\mathbf{k}) \gg 0$, then the definition of H_b over $\mathbf{k}$ makes sense. One checks easily that $\mathfrak{h}^p, \mathfrak{h}^{*p}, g^p - g^{[p]} \in \mathbf{Z}(H_b), g \in \mathfrak{g}$, where $g^{[p]} \in \mathfrak{g}$ denotes the p-th power of g as a matrix. We will denote by $\mathbf{Z}_0(H_b)$ the algebra generated by the above elements. We have following result which we conjectured in [T1].

Corollary 0.2. *The smooth and Azumaya loci of $\mathbf{Z}(H_b)$ coincide, and $\mathbf{Z}(H_b)$ is generated by $t_1, \cdots, t_n$ over $\mathbf{Z}_0(H_b)$. The PI-degree of H_b is $\frac{1}{2}(n^2 + n)$.*

The corollary follows from the following trivial proposition and [T2].

Proposition 0.1. *Let S be a Poisson algebra over* $\mathbf{k}$*, and let* $(f_1, \cdots, f_n)$ *be a regular sequence of Poisson central elements. Let* $S/(f_1, \cdots, f_n)$ *be a normal domain such that its smooth locus is symplectic. Then the Poisson center of S is generated as an algebra by* $S^p, f_1, \cdots, f_n$.

Proof. It follows immediately that the Poisson center of S lies in $S^p + I$ [T1]. Let $f \in I^k$ be in the Poisson center of S. But I^k/I^{k+1} is a free Poisson S/I-module, so $f \in S^p[f_1, \cdots, f_n] + I^{k+1}$. Continuing by induction on k, we are done. □

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