EXAMPLES OF MINIMAL DIFFEOMORPHISMS ON $\mathbb{T}^2$ SEMICONJUGATED TO AN ERGODIC TRANSLATION

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Abstract. We prove that for every $\epsilon > 0$ there exists a minimal diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ of class $C^{3-\epsilon}$ and semiconjugate to an ergodic traslation, and have the following properties: zero entropy, sensitivity with respect to initial conditions, uncountable number of Li-Yorke pairs. These examples are obtained through the holonomy of the unstable foliation of Mañé's example of derived from Anosov diffeomorphism on $\mathbb{T}^3$.

1. Introduction.

The classical result of Denjoy ([D]) can be stated as follows: if $f : S^1 \to S^1$ is a C^2 diffeomorphism semiconjugated to an ergodic rotation then it is indeed conjugated to it. One may ask to what extent Denjoy´s theory on S^1 can be extended to higher dimensional tori. In particular one may ask: Does there exist r so that if $f : \mathbb{T}^2 \to \mathbb{T}^2$ is a C^r diffeomorphism semiconjugated to an ergodic translation then f is conjugated to it? This seems to be a very difficult question. Nevertheless, KAM theory provides a particular result when f is close to an ergodic translation of diophantine type. There are also some indications that if the above question has a positive answer, then $r = 3$ (see for instance [McS] for $C^{3-\epsilon}$ examples with wandering domains and [NS]).

One may ask if there are extra restrictions on the differentiability class of a Denjoy type map $f : \mathbb{T}^2 \to \mathbb{T}^2$ (i.e. semiconjugataed -but not conjugated- to an ergodic traslation) if we also assume that f is minimal (i.e. every orbit is dense). In this paper we give examples of this type of class $C^{3-\epsilon}$ for any $\epsilon > 0$. If we denote by h the semiconjugacy then, if f is minimal, the fibers $h^{-1}(x)$ have empty interior and we may ask how they look-like. We show that the fibers are points or arcs. The first (topological) example of this kind was given by M. Rees in [R1] in order to construct a non-distal (but point distal) homeomorphism of $\mathbb{T}^2$. Recall that f is *non-distal* if there exist $x \neq y$ such that $\inf_{n\in\mathbb{Z}}\{dist(f^n(x), f^n(y))\} = 0$ and f is *point distal* if there exists x such that for any $y \neq x, \inf_{n\in\mathbb{Z}}\{dist(f^n(x), f^n(y))\} > 0$.

The dynamics on the nontrivial fibers by the action of the map f has a chaotic flavour: it compress them to an arc of lenght arbitrarily small and then stretches to an arc of fixed length and then compresses them and so on. This will imply interesting properties: *sensitivity with respect to initial conditions* (that is, there exists some $\epsilon > 0$ so that for any $x \in \mathbb{T}^2$ and any nieghborhood $U(x)$ there exists $y \in U$ and $n > 0$ such that $dist(f^n(x), f^n(y)) > \epsilon$) and the existence of *Li-Yorke pairs* (i.e, pair of points $x \neq y$ such that $\liminf_n dist(f^n(x), f^n(y)) = 0$ and $\limsup_n dist(f^n(x), f^n(y)) > 0$).

It is also interesting to ask about the ergodic properties of these Denjoy type maps. Unfortunately, our examples are simple from this point of view: they have just one invariant measure, i.e., are *uniquely ergodic*. Neverhteless we post the question: does there exist minimal diffeomorphism semiconjugated to an ergodic translation not uniquely ergodic? For homeomorphisms the answer is positive (see [R2]).

Our result is the following:

Main Theorem. *For all $r \in [1,3)$ there exists a diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ of class C^r such that:*

- *f is minimal.*
- *f is isotopic and semiconjugated (but not conjugated) to an ergodic traslation. If we denote by h the semiconjugacy, then $h^{-1}(x)$ is either a point or an arc. Moreover, there are uncountable points x such that $h^{-1}(x)$ is a nontrivial arc.*
- *f preserves a minimal and invariant foliation with one dimensional C^1 leaves. The fibers $h^{-1}(x)$ are contained in the leaves of this foliation.*
- *f has zero entropy.*
- *f has sensitivity with respect to initial conditions*
- *There is an uncountable number of Li-Yorke pairs.*
- *f is point-distal and non-distal.*
- *f is uniquely ergodic.*

The proof of our theorem is inspired by [McS]. There, the examples are constructed through the holonmy map from a cross section to itself of the unstable foliation of a derived from Anosov diffeomorphism obtained through a Hopf´s bifurcation. In this paper we use instead Mañé's example of derived from Anosov diffeormphism ([M]) where we prove that the unstable foliation is minimal. However, there is a main difference with [McS]: while there the starting linear Anosov map is fixed and for any ϵ a modification is taken so that the unstable foliation is $C^{3-\epsilon}$ we have to do it the other way around, that is, given $\epsilon > 0$ we have to find the linear Anosov map to begin with so that a modification can be done such that the unstable foliation of the resulting diffeomorphism is $C^{3-\epsilon}$. This modification also includes the existence of periodic points of different unstable indices and the existence of transversal homoclinic points associated to them.

The paper is organized as follows: in Section 2 we give our construction of Mañé's derived from Anosov diffeomorphism and we prove the minimality of the unstable foliation (see Section 2.1) and the minimality of the central foliation through the semiconjugacy with the linear Anosov map (see Section 2.2); in Section 3 we give the topological version of our main result and in Section 4 we prove the differentiability of the unstable foliation through the C^r Section Theorem ([HPS]).

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2. On Mañe's Derived from Anosov diffeomorphism

In [M] R.Mañé construct an example on $\mathbb{T}^3$ which is robustly-transitive but not Anosov. This is known as Mañe's Derive from Anosov diffeomorphisms due to the construction: it begins with an Anosov linear map on $\mathbb{T}^3$ with partially hyperbolic structure $E^s \oplus E^c \oplus E^u$ and modifis it in a neighborhood of the fixed point in order to change the unstable index of it (and preserving the partially hyperbolic structure). See Figure 1.

Let us be more precise. Let $\mathbb{T}^3 = \mathbb{R}^3/_{\mathbb{Z}^3}$ be the three dimensional torus and denote by $\pi : \mathbb{R}^3 \to \mathbb{T}^3$ the canonical projection, and set $p = \pi(0)$.

Consider $B \in SL(3,\mathbb{Z})$ with eigenvalues $0 < \lambda_s < \lambda_c < 1 < \lambda_u$ and denote also by B the induced Linear Anosov system on $\mathbb{T}^3$ with hyperbolic structure $T\mathbb{T}^3 = E^s \oplus E^c \oplus E^u$ (corresponding to the eigenspaces associated to λ_s, λ_c and λ_u). For the sake of simplicity to do our calculations we will define an Euclidean metric on $\mathbb{R}^3$ so that E^s_B, E^c_B and E^u_B are mutally orthogonal.

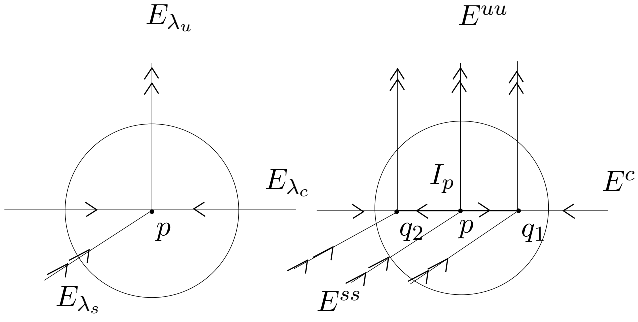

FIGURE 1. Modfication.

Let ρ be small and consider $B(p,\rho)$ the ball centered at p. Let $Z : \mathbb{R} \to \mathbb{R}$ be a C^∞ bump function such that $Z(0) = 1$, $sop[Z] \subset (-\frac{\rho}{2}, \frac{\rho}{2})$ and $|Z'(z)| < \frac{4}{\rho}$. (See Figure 2)

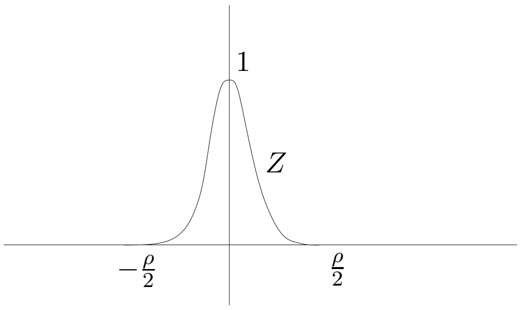

FIGURE 2. The bump function Z.

For our construction of the Mañe's Derived from Anosov[1] we need an auxiliary function as in the next lemma.

Lemma 2.0.1. *For all $k > 0$ arbitrarily small there exist a function $\beta_k : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ such that:*

(1) *β_k is C^∞, decreasing and such that $-k \leq \beta_k'(t)t \leq 0$.*
(2) *$sop[\beta_k] \subset [0,k]$.*
(3) *$\lambda_s + \beta_k(0) < 1 < \lambda_c + \beta_k(0) < 1 + k$.*

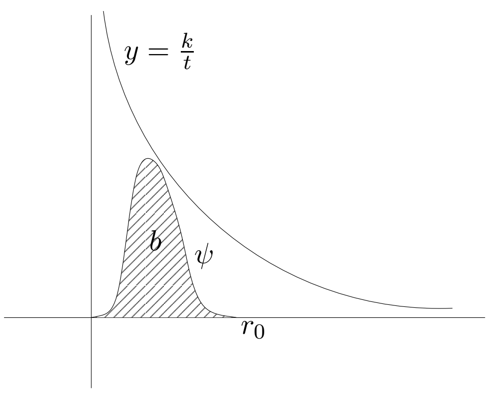

FIGURE 3. The function ψ.

[1]Our contruction is slighty different because we need to keep control of the relation between E^s and E^c to obtain higher differentiability of the unstable foliation

Proof. We may assume that $0 < k < \lambda_c - \lambda_s$ and take b such that $1 - \lambda_c < b < 1 - \lambda_c + k$. Let $r_0 < k$. Since $\int_0^{r_0} \frac{k}{t} dt$ is divergent we may find a C^∞ non negative function ψ with support in $[0, r_0]$ such that $\int_0^{r_0} \psi(t)dt = b$ and $\psi(t) \leq \frac{k}{t}$ (in other words the graph of ψ is below the graph of $h(t) = \frac{k}{t}$.

Define

$$\beta_k(t) = b - \int_0^t \psi(s)ds.$$

This function satisfies the lemma. □

Finally, define $g_{B,k} : \mathbb{T}^3 \to \mathbb{T}^3$ defined by:

$$g_{B,k}(\xi) = B(\xi) \quad \text{for} \quad \xi \notin B(p, \rho) \tag{1}$$

and for $\xi \in B(p, \rho)$ in local coordinates with respect to $E^s B \oplus E^c_B \oplus E^u_B, \xi = (x, y, z)$

$$g_{B,k}(\xi) = (\lambda_s x, \lambda_c y, \lambda_u z) + Z(z)\beta_k(r)(x, y, 0) \tag{2}$$

where $r = x^2 + y^2$.

Proposition 2.0.1. *If k is sufficiently small, then $g_{B,k} : \mathbb{T}^3 \to \mathbb{T}^3$ defined above is a diffeomorphism with partially hyperbolic structure $T\mathbb{T}^3 = E^s_{g_{B,k}} \oplus E^c_{g_{B,k}} \oplus E^u_{g_{B,k}}$ where $E^s_{g_{B,k}}$ is uniformly contracting and $E^u_{g_{B,k}}$ is uniformly expanding. Moreover, given cones C^s, C^c and C^u around E^s_B, E^c_B and E^u_B respectively we have that $E^s_{g_{B,k}} \in C^s, E^c_{g_{B,k}} \in C^c$ and $E^u_{g_{B,k}} \in C^u$. Furthermore, the same is true for any g in any sufficiently small C^1 neighborhood $\mathcal{U}$ of $g_{B,k}$.*

Proof. First of all, the C^0 distance between $g_{B,k}$ and B is smaller than $\sqrt{k}$ and hence (assuming k small) we conclude that $g_{B,k}$ is a differentiable homeomorphism. To avoid notation, set $g = g_{B,k}$ for the time being.

For $\xi \notin B(p, \rho)$ we have $dg_\xi = B$. For $\xi \in B(p, \rho)$ we have (with respect to the decomposition $E^s \oplus E^c \oplus E^u$)

$$dg_\xi = \begin{pmatrix} \lambda_s + Z(z)(\beta(r) + \beta'(r)2x^2) & Z(z)\beta'(r)2xy & Z'(z)\beta(r)x \\ Z(z)\beta'(r)2xy & \lambda_c + Z(z)(\beta(r) + \beta'(r)2y^2) & Z'(z)\beta(r)y \\ 0 & 0 & \lambda_u \end{pmatrix} \tag{3}$$

We may write $dg_\xi = A_\xi + M_\xi$ (and agreeing that Z and β are identically zero outside $B(p, \rho)$) where

$$A_\xi = \begin{pmatrix} \lambda_s + Z(z)\beta(r) & 0 & 0 \\ 0 & \lambda_c + Z(z)\beta(r) & 0 \\ 0 & 0 & \lambda_u \end{pmatrix} \tag{4}$$

and

$$M_\xi = \begin{pmatrix} Z(z)\beta'(r)2x^2 & Z(z)\beta'(r)2xy & Z'(z)\beta(r)x \\ Z(z)\beta'(r)2xy & Z(z)\beta'(r)2y^2 & Z'(z)\beta(r)y \\ 0 & 0 & 0 \end{pmatrix} \tag{5}$$

Since $|\beta'(r)r| \leq k$ it is straighborward to check that $\|M_\xi\| \leq \max\{2k, 8\beta(0)\sqrt{k}/\rho\}$. Therefore, choosing k arbitrarily small we get that $\|M_\xi\|$ is also arbitrarily small. Since the co-norm $(= \|A_\xi^{-1}\|^{-1})$ of A_ξ is bounded away from zero we have that dg_ξ is an isomorphism and hence g a diffeomorphism. On the other hand, $A_\xi(E^j_B) = E^j_B,\ j = s, c, u$ and

- $\lambda_s \leq \|A_{\xi/E^s_B}\| \leq \lambda_s + \beta_k(0) < 1$
- $\lambda_c \leq \|A_{\xi/E^c_B}\| \leq \lambda_c + \beta_k(0) < 1 + k$

- $\frac{\|A_{\xi/E^s_B}\|}{\|A_{\xi/E^c_B}\|} \leq \frac{\lambda_s}{\lambda_c} < 1$.
- $\|A^-1_{\xi/E^u_B}\| \leq \lambda_u^{-1}$.

From this it is easy to conclude the proof of the proposition, taking k sufficiently small (and so $\|M_\xi\|$ is sufficiently small) and taking $\mathcal{U}$ sufficiently small.

□

For $g_{B,k} : \mathbb{T}^3 \to \mathbb{T}^3$ with k small and $g \in \mathcal{U}(g_{B,k})$ so that the above proposition applies we set:

- $\lambda_s(g)(\xi) = \|dg_{\xi/E^s_g}\|$, and $\lambda_s(g) = max_{\xi \in \mathbb{T}^3}\{\lambda_s(g)(\xi)\}$.
- $\lambda_c(g)(\xi) = \|dg_{\xi/E^c_g}\|$, and $\lambda_c(g) = max_{\xi \in \mathbb{T}^3}\{\lambda_c(g)(\xi)\}$.
- $\lambda_u(g)(\xi) = \|dg_{\xi/E^u_g}\|$, and $\lambda_u(g) = min_{\xi \in \mathbb{T}^3}\{\lambda_u(g)(\xi)\}$.

Notice that, given $\epsilon > 0$ small, the following conditions hold for $g \in \mathcal{U}(g_{B,k})$ with k and $\mathcal{U}$ sufficiently small:

(1) $0 < \lambda_s(g)(\xi) < \lambda_c(g)(\xi) < \lambda_u(g)(\xi)$ for all $\xi \in \mathbb{T}^3$.
(2) $\lambda_s(g) < \lambda_s + \beta(0) + \epsilon < 1$.
(3) $\lambda_c(g) < \lambda_c + \beta(0) + \epsilon$.
(4) $\lambda_u(g) > \lambda_u - \epsilon > 1$.

Once we know that $g \in \mathcal{U}(g_{b,k})$ is partially hyperbolic, by well known results (see [HPS]) we get that the bundles E^s_g and E^u_g uniquely integrate to foliations $\mathcal{F}^s_g$ and $\mathcal{F}^u_g$ called the (strong) stable and unstable foliations respectively.

Moreover, since E^s_g and E^u_g are contained in tiny cones around E^s_B and E^u_B we conclude that $\mathcal{F}^s_g$ and $\mathcal{F}^u_g$ are quasi isometric and hence, by [B] (see also [BBI]), we also conclude that E^c_g is uniquely integrable. We denote by $\mathcal{F}^c_g$ this central foliation. And therefore, the bundles $E^s_g \oplus E^c_g$ and $E^c_g \oplus E^u_g$ are uniquely integrable and leads to the central stable and central unstable foliations. We also remark that in the particular case $g = g_{B,k}$ it holds that $E^s_g \oplus E^c_g = E^s_B \oplus E^c_B$ and so the central stable foliation of $g_{B,k}$ coincides with the two dimensional stable foliation of B.

In the following subsections we are going to study properties of the invariant foliations and also consequences of the semiconjugacy with the linear Anosov map. These results are fundamental for our purposes.

Theorem 2.0.1. *For all k sufficiently small and $\mathcal{U}(g_{B,k})$ sufficiently small as well, the central and unstable foliations $\mathcal{F}^c_g, \mathcal{F}^u_g$ of $g \in \mathcal{U}(g_{B,k})$ are minimal, i.e., all leaves are dense.*

The minimality of $\mathcal{F}^u_g$ can be obtained from [PS], and the minimality of $\mathcal{F}^c_g$ is kind of folklore result. We are going to give a complete proof of the theorem in Sections 2.1 (see Theorem 2.1.1) and 2.2 (see Corollary 2.2.3). Neverhteless, let us state and proof the following

Corollary 2.0.1. *Let k and $\mathcal{U}(g_{B,k})$ be as in the above theorem. Then, there exists $g \in \mathcal{U}(g_{B,k})$ of class C^∞ having p as a fixed point with unstable index 2 and such that g has a transversal homoclinic point associated to p.*

Proof. Notice that for $g_{B,k}$ the fixed point $p = \pi(0)$ has unstable index equals to 2 since $dg_{B,k/E^c} = \lambda_c + \beta(0) > 1$. On the other hand, since $\mathcal{F}^u_{g_{B,k}}(p)$ is dense (and hence accumulates on $\mathcal{F}^s_{g_{B,k}}(p)$) by the Hayashi's connecting lemma (see [H])) we can perturb $g_{B,k}$ (with support disjoint from a ball at p) and find g as in the conditions of the statement. □

Remark 2.1. Indeed, a stronger results holds: using the notion of blenders, the minimality of $\mathcal{F}_g^u$ and $\mathcal{F}_g^c$, we may find $g \in \mathcal{U}(g_{B,k})$ of class C^∞ and such that $\mathcal{F}_g^s(p)$ is dense in $\mathbb{T}^3$ (see [DR] and also [BDV]). However, for the time being we won´t make use of this fact.

2.1. **Minimality of the unstable foliation.** In this subsection we will prove that $\mathcal{F}_g^u$ is minimal for $g \in \mathcal{U}(g_{B,k})$ for k and $\mathcal{U}$ small enough. The proof is based on the ideas and methods in [PS]:

Theorem 2.1.1. *For all k sufficiently small and $\mathcal{U}(g_{B,k})$ sufficiently small as well, the unstable foliation $\mathcal{F}_g^u$ of $g \in \mathcal{U}(g_{B,k})$ is minimal, i.e., all leaves are dense.*

Proof. Recall that $0 < \lambda_s < \lambda_c < 1 < \lambda_u$ are the eigenvalues of B. Choose $\sigma, 1-(\lambda_c-\lambda_s) < \sigma < 1$. We may assume that ρ (the radius of the ball centered at p where the modification of B is performed) is small so that any arc I^s in $\mathcal{F}_B^s$ of length one has a subarc I_1^s of length at least $1/3$ with empty intersection with $B(p, 2\rho)$.

Let n_0 so that

$$\sigma^{-n_0} > 3. \tag{6}$$

Let $\epsilon, 0 < \epsilon < \rho$ be such that $1 - (\lambda_c - \lambda_s) + \epsilon < \sigma$ and

$$\lambda := \lambda_c(1+\epsilon)^{n_0} < 1. \tag{7}$$

Let us denote by $D_g^{cs}(x, \epsilon)$ a disc centered at x and radius ϵ in the central stable leaf through x, $\mathcal{F}_g^{cs}(x)$.

Now, we may assume that k and $\mathcal{U}$ are so small so that the following holds for $g \in \mathcal{U}(g_{B,k})$:

- (i) $\lambda_s(g) < \sigma$.
- (ii) $\lambda_c(g) < 1 + \epsilon$.
- (iii) $\|dg_{/E_g^{cs}(\xi)}\| \leq \lambda_c(1+\epsilon)$ if $\xi \notin B(p, \rho)$.
- (iv) Any arc I^s of $\mathcal{F}_g^s$ of length at least one has a subarc I_1^s of length at least $1/3$ with empty intersection with $B(p, 2\rho)$.
- (v) Any leaf of $\mathcal{F}_g^u$ has nonempty intersection with $D_g^{cs}(x, \epsilon)$ for any x (since $\mathcal{F}_B^u$ is minimal and for k and $\mathcal{U}$ small the bundles E_B^u and E_g^u are close).

Given $x \in \mathbb{T}^3$ let $I^s(x)$ an arc of length one such that $x \in I^s(x) \subset \mathcal{F}_g^s(x)$. We know that there exists a subarc I_1^s of length at least $1/3$ such that $I_1^s \cap B(p, 2\rho) = \emptyset$. Now, by (6), we conclude that $g^{-n_o}(I_1^s) \subset \mathcal{F}_g^s(g^{-n_0}(x)$ is an arc of length at least one. Therefore, there exists a subarc $I_2^s \subset g^{-n_o}(I_1^s)$ of length at least $1/3$ such that $I_2^s \cap B(p, 2\rho) = \emptyset$. Arguing by induction, we conclude that for each $j \geq 1$ there exists $I_{j+1}^s \subset g^{-n_0}(I_j^s)$ such that $I_{j+1}^s \cap B(p, 2\rho) = \emptyset$.

Define

$$z_x = \bigcap_{j \geq 1} g^{jn_0}(I_{j+1}).$$

Notice that

$$z_x \in I^s(x) \quad \text{and} \quad g^{-jn_0}(z_x) \notin B(p, 2\rho) \ \forall \ j \geq 0. \tag{8}$$

In other words, in any arc of length one on any leaf of $\mathcal{F}_g^s$ there exists a point whose g^{n_0}-backward orbit never meets $B(p, 2\rho)$. Let $z = z_x$ be such a point and let $j \geq 1$. Then we have that

$$D_g^{cs}(g^{-jn_0}(z), \epsilon) \cap B(p, \rho) = \emptyset$$

and so, for any $y \in D_g^{cs}(g^{-jn_0}(z), \epsilon)$ we have that, by (7), (ii) and (iii)

$$\|dg_y^{n_0}\| \leq \lambda_c(1+\epsilon)^{n_0} = \lambda < 1$$

and therefore

$$g^{n_0}(D_g^{cs}(g^{-jn_0}(z),\epsilon)) \subset D_g^{cs}(g^{-(j-1)n_0}(z),\lambda\epsilon) \tag{9}$$

and so, for any $1 \le m \le j$ we have

$$g^{mn_0}(D_g^{cs}(g^{-jn_0}(z),\epsilon)) \subset D_g^{cs}(g^{-(j-m)n_0}(z),\lambda^m\epsilon).$$

Now, we are ready to conclude the proof of the minimality of $\mathcal{F}_g^u$ (for the argument see Figure 4). Let $\xi \in \mathbb{T}^3$ and U some open set in $\mathbb{T}^3$. We want to prove that

$$\mathcal{F}_g^u(\xi) \cap U \neq \emptyset.$$

Let $y \in U$, and consider an arc $J_y \subset F_g^s(y), J_y \subset U$. There exists m_0 so that $g^{-m_0}(J_y)$ has length grater than one. Let $z \in g^{-m_0}(J_y)$ the point constructed above, and let μ be such that

$$g^{m_0}(D_g^{cs}(z,\mu)) \subset U. \tag{10}$$

Let m_1 be such that $\lambda^{m_1}\epsilon < \mu$. From (9) we conclude that

$$g^{m_1 n_0}(D_g^{cs}(g^{-m_1 n_0}(z),\epsilon) \subset D_g^{cs}(z,\mu).$$

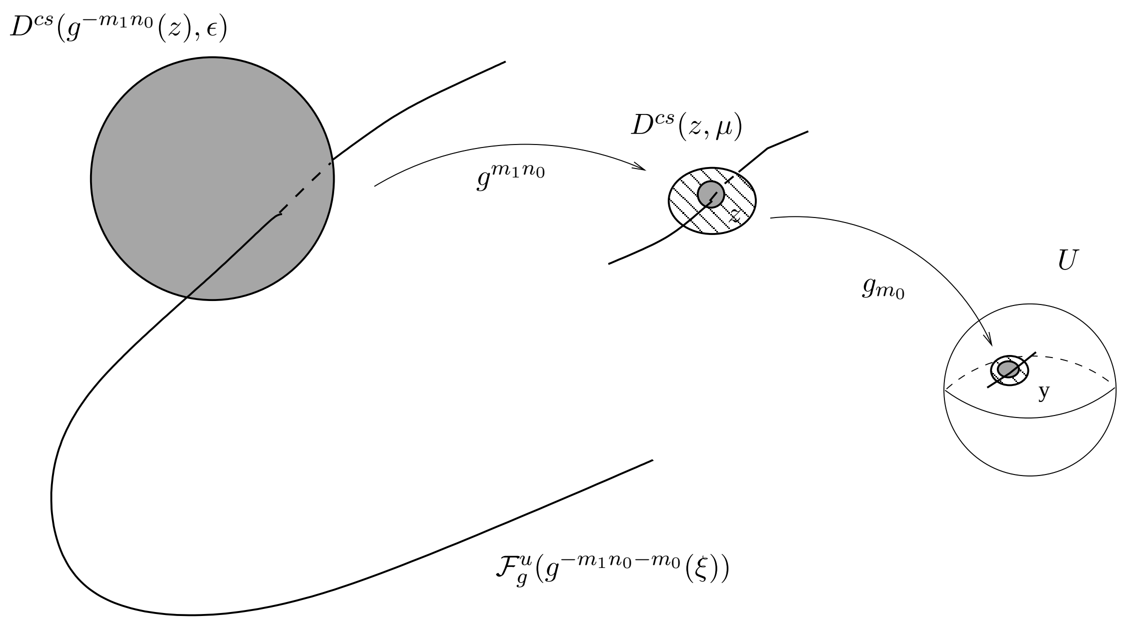

FIGURE 4.

On the other hand, from (v) we know that $\mathcal{F}_g^u(g^{-m_1 n_0 - m_0}(\xi)) \cap D_g^{cs}(g^{-m_1 n_0}(z),\epsilon) \neq \emptyset$. Using (10), iterating $m_1 n_0 + m_0$ times we conclude that

$$\mathcal{F}_g^u(\xi) \cap U \neq \emptyset$$

as we wished. This completes the proof of the minimality of $\mathcal{F}_g^u$ for $g \in \mathcal{U}(g_{B,k})$ with k and $\mathcal{U}$ small enough. $\square$

2.2. **Semiconjugacy with the linear Anosov System.** In this subsection we establish a well known result about the semiconjugacy of any map isotopic to an Anosov map on the torus (see for instance [S]) and also we derive some consequence of it. Indeed, we establish it in the universal cover $\mathbb{R}^3$.

Theorem 2.2.1. *Let $B : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear hyperbolic isomorphism. Then, there exists $C > 0$ such that if $G : \mathbb{R}^3 \to \mathbb{R}^3$ is homoemorphism such that* $\sup\{\|G(x) - Bx\| : x \in \mathbb{R}^3\} = K < \infty$ *then there exists $H : \mathbb{R}^3 \to \mathbb{R}^3$ continuous and onto such that:*

(1) $B \circ H = H \circ G$.
(2) $\|H(x) - x\| \leq CK$ *for all* $x \in \mathbb{R}^3$.
(3) $H(x)$ *is characterized as the unique point* y *such that*

$$\|B^n(y) - G^n(x)\| \leq CK \ \forall \ n \in \mathbb{Z}.$$

(4) $H(x) = H(y)$ *if and only if* $\|G^n(x) - G^n(y)\| \leq 2CK \forall n \in \mathbb{Z}$ *and if and only if* $\sup_{n\in\mathbb{Z}}\{\|G^n(x) - G^n(y)\|\} < \infty$.
(5) *If* $B \in SL(3, \mathbb{Z})$ *and* G *is the lift of* $g : \mathbb{T}^3 \to \mathbb{T}^3$ *then* H *induces* $h : \mathbb{T}^3 \to \mathbb{T}^3$ *continuos and onto such that* $B \circ h = h \circ g$ *and* $dist_{C^0}(h, id) \leq C dist_{C^0}(B, g)$.

We will prove some consequence of the above theorem to our $B \in SL(3, \mathbb{Z})$ and our construction of Mañe's derived from Anosov diffeomorphism $g_{B,k} : \mathbb{T}^3 \to \mathbb{T}^3$ and any $g \in \mathcal{U}(g_{B,k})$. Let $G : \mathbb{R}^3 \to \mathbb{R}^3$ the lift of g such that $\sup\{\|G(x) - Bx\| : x \in \mathbb{R}^3\} = dist_{C^0}(B, g)$ (that we may assume that is less than $\sqrt{k}$). Denote by $\tilde{\mathcal{F}}^j, j = s, c, u, cs, cu$ the lift of the stable, central, ustable, central stable and central unstable foliation respectively.

Theorem 2.2.2. *With the above notations we have:*

(1) $H(\tilde{\mathcal{F}}^{cu}_G(x)) = \tilde{\mathcal{F}}^{cu}_B(H(x))$ *and* $H(\tilde{\mathcal{F}}^{cs}_G(x)) = \tilde{\mathcal{F}}^{cs}_B(H(x))$
(2) $H(\tilde{\mathcal{F}}^{c}_G(x)) = \tilde{\mathcal{F}}^{c}_B(H(x))$.
(3) $H(\tilde{\mathcal{F}}^{u}_G(x)) = \tilde{\mathcal{F}}^{u}_B(H(x)) = H(x) + E^u_B$ *and* $H : \tilde{\mathcal{F}}^u_G(x) \to \tilde{\mathcal{F}}^u_B(H(x))$ *is a homeomorphism.*
(4) *For any* $x, y \in \mathbb{R}^3$ *hold*

$$\sharp\{\tilde{\mathcal{F}}^{cs}_G(x) \cap \tilde{\mathcal{F}}^{u}_G(y)\} = 1 \quad \textit{and} \quad \sharp\{\tilde{\mathcal{F}}^{cu}_G(x) \cap \tilde{\mathcal{F}}^{s}_G(y)\} = 1.$$

Proof. For the first item we only prove that $H(\tilde{\mathcal{F}}^{cu}_G(x)) = \tilde{\mathcal{F}}^{cu}_B(H(x))$, the other one is similar. Let us prove first that $H(\tilde{\mathcal{F}}^{cu}_G(x)) \subset \mathcal{F}^{cu}_B(H(x)) = H(x) + E^{cu}_B$. Arguing by contradiction, assume that there exists $y \in \tilde{\mathcal{F}}^{cu}_G(x)$ such that $H(y) \notin \tilde{\mathcal{F}}^{cu}_B(H(x))$ let $z = \tilde{\mathcal{F}}^{s}_B(H(y)) \cap \tilde{\mathcal{F}}^{cu}_B(H(x))$. By backward iteration we have that

$$\begin{aligned} \|B^{-n}(H(y)) - B^{-n}(H(x))\| &\geq \|B^{-n}(H(y)) - B^{-n}(H(z))\| - \|B^{-n}(H(z)) - B^{-n}(H(x))\| \\ &\geq \lambda_s^{-n}\|H(y) - H(z)\| - \lambda_c^{-n}\|H(z) - H(x)\| \end{aligned}$$

On the other hand, since $y \in \tilde{\mathcal{F}}^{cu}_G(x)$ and (for k and $\mathcal{U}$ small) $\|dG^{-1}_{/E^{cu}_G}\| \leq (\lambda_c - \epsilon)^{-1}$ we have

$$\begin{aligned} \|B^{-n}(H(x)) - B^{-n}(H(y))\| &\leq \|B^{-n}(H(x)) - G^{-n}(x)\| \\ &\quad + \|G^{-n}(x) - G^{-n}(y)\| \\ &\quad + \|B^{-n}(H(y)) - G^{-n}(y)\| \\ &\leq 2C\sqrt{k} + (\lambda_c - \epsilon)^{-n} dist_{\tilde{\mathcal{F}}^{cu}_G(x)}(x, y). \end{aligned}$$

For n large enough we arrive to a contradiction with the previous equation.

Now, since $\|H - Id\| \leq C\sqrt{k}$ we have:

- $\tilde{\mathcal{F}}^{cu}_G(x) \subset \{z : dist_{\mathbb{R}^3}(z, H(x) + E^{cu}_B) \leq C\sqrt{k}\}$ (that is, roughly speaking, $\tilde{\mathcal{F}}^{cu}_G$ is a surface in a sandwich of size $C\sqrt{k}$ with central slice the plane $H(x) + E^{cu}_B$. See Figure 5.
- $\tilde{\mathcal{F}}^{cu}_G(x)$ is transversal to E^s_B.
- $\tilde{\mathcal{F}}^{cu}_G(x)$ is a complete manifold

and it is not difficult to see that $\tilde{\mathcal{F}}^{cu}_G(x)$ is a graph of a map $E^{cu}_B \to E^s_B$. Then, since $\|H - Id\| \leq C\rho$ is follows that $H : \tilde{\mathcal{F}}^{cu}_G(x) \to \tilde{\mathcal{F}}^{cu}_B(H(x))$ is onto.

Let us prove the second item. From the first one it follows that:

$$H(\tilde{\mathcal{F}}^{c}_G(x)) = H(\tilde{\mathcal{F}}^{cs}_G(x) \cap \tilde{\mathcal{F}}^{cu}_G(x)) \subset \tilde{\mathcal{F}}^{cs}_B(H(x)) \cap \tilde{\mathcal{F}}^{cu}_B(H(x)) = \tilde{\mathcal{F}}^{c}_B(H(x)).$$

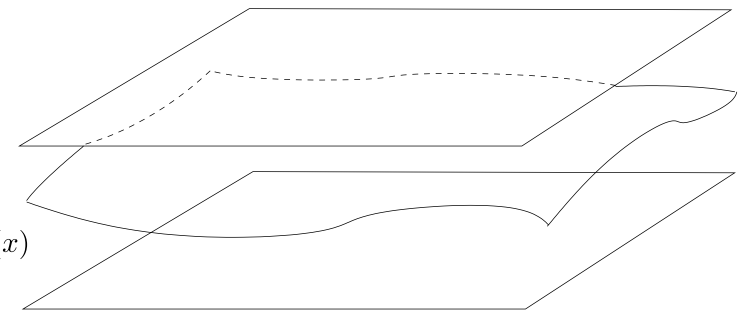

FIGURE 5. The $\mathcal{F}_G^{cu}$ leaf

Since $\|H-Id\|\le C\sqrt{k}$ we have that $\tilde{\mathcal{F}}_G^c(x)$ ia in a cylinder of radius $C\sqrt{k}$ with axe $H(x)+E_B^c=\tilde{\mathcal{F}}_B^c(H(x))$. Since E_G^c is in tiny cone around E_B^c we may assume that E_B^c is always transversal to $E_B^s\oplus E_B^u$ and moreover $\tilde{\mathcal{F}}_G^c(x)$ is the graph of a map $E_B^c\to E_B^s\oplus E_B^u$. Using again that $\|H-Id\|\le C\sqrt{k}$ we conclude that $H:\tilde{\mathcal{F}}_G^c(x)\to\tilde{\mathcal{F}}_B^c(H(x))$ is onto.

For the third item item observe also that $H(\tilde{\mathcal{F}}_G^u(x))\subset\tilde{\mathcal{F}}_B^u(H(x))$ since for $y\in\tilde{\mathcal{F}}_G^u(x)$ we have that $\|G^n(y)-G^n(x)\|\to_{n\to-\infty}0$ and hence the distance between $H(G^n(y))=B^n(H(y))$ and $H(G^n(x))=B^n(H(x))$ is bounded for $n\le 0$ which implies that $H(y)\in H(x)+E_B^u$. By similar arguments as in the previous item we have that $H:\tilde{\mathcal{F}}_G^u(x)\to\tilde{\mathcal{F}}_B^u(H(x))$ is onto. On the other hand, $H_{/\tilde{\mathcal{F}}_G^u(x)}$ is injective: otherwise, if for some $z,y\in\tilde{\mathcal{F}}_G^u(x)$ we have that $H(z)=H(y)$ by forward iteration we have that $\|G^n(y)-G^n(z)\|$ goes to infinity (recall that $\tilde{\mathcal{F}}_G^u$ is quasi isometric) and so $\|H(G^n(y))-H(G^n(z))\|$ also goes to infinity by forward iteration, this is imposible since $H(G^n(y))=B^n(H(y))=B^n(H(z))=H(G^n(z))$.

For the fourth and last item observe that

$$\sharp\{\tilde{\mathcal{F}}_G^{cs}(x)\cap\tilde{\mathcal{F}}_G^u(y)\}\le 1.$$

Otherwise, let $z,w\in\tilde{\mathcal{F}}_G^{cs}(x)\cap\tilde{\mathcal{F}}_G^u(y)$ and iterating forward we have (since $\tilde{\mathcal{F}}^u$ is quasi isometric) that $\|G^n(z)-G^n(w)\|\sim dist_{\tilde{\mathcal{F}}^u}(G^n(z),G^n(w))$ which grows with exponential rate $\sim\lambda_u$. On the other hand, since $z,w\in\tilde{\mathcal{F}}^{cs}$ the distance can grow at most with rate $\lambda_c(g)<1+\epsilon<\lambda_u$ and we get a contradiction.

To see the intersection is nonempty just recall that $\tilde{\mathcal{F}}^{cs}(x)$ is a graph of a (bounded) map $E_B^{cs}\to E_B^u$ and $\tilde{\mathcal{F}}^u(y)$ is a graph of a (bounded) map $E_B^u\to E_B^{cs}$.

The second part of this item is very similar to what we already done. Nevertheless (for the very last argument) it worth to mention that it is not true in general that $H(\tilde{\mathcal{F}}_G^s(x))=\tilde{\mathcal{F}}_B^s(H(x))$, and so we can not be sure that $\tilde{\mathcal{F}}_G^s(x)$ is at a bounded distance of $H(x)+E_B^s$ but still it is not difficult to see (since E_G^s is in a tiny cone around E_B^s) that $\tilde{\mathcal{F}}_G^s(x)$ is the graph of a map $E_B^s\to E_B^{cu}$.

□

Corollary 2.2.1. *With the above notations, assume that $H(x)=H(y)$. Then x,y belongs to the same central leaf $\tilde{\mathcal{F}}_G^c(x)=\tilde{\mathcal{F}}_G^c(y)$. Moreover, if we denote by $[x,y]^c$ the central arc in $\tilde{\mathcal{F}}_G^c(x)$ with ends x and y then $H([x,y]^c)=H(x)=H(y)$ and the diameter of $[x,y]^c$ is bounded by $2C\sqrt{k}$. In particular, for any z, $H^{-1}(z)$ is either a point or an arc.*

Proof. Let x,y be such that $H(x)=H(y)$. We claim that $y\in\tilde{\mathcal{F}}_G^{cs}(x)$. Otherwise, from the last theorem we may consider $z=\tilde{\mathcal{F}}_G^{cs}(x)\cap\tilde{\mathcal{F}}_G^u(y)$. By similar arguments as before, since by forward iteration the distance between $G^n(z)$ and $G^n(y)$ grows with a rate much higher than the one between $G^n(z)$ and $G^n(x)$ could do, we conclude that

$$\|G^n(x) - G^n(y)\| \to_{n\to\infty} \infty.$$

This is impossible due to $H(G^n(y)) = H(G^n(x))$ and so $G^n(z)$ and $G^n(y)$ are at bounded distance for every n.

In a similar way we prove that $y \in \tilde{\mathcal{F}}^{cu}_G(x)$. Therefore

$$y \in \tilde{\mathcal{F}}^{cs}_G(x) \cap \tilde{\mathcal{F}}^{cu}_G(x) = \tilde{\mathcal{F}}^{c}_G(x).$$

Now, recall that $\tilde{\mathcal{F}}^c(z)$ is the graph of a map $H(z) + E^c_B \to H(z) + E^s_B \oplus E^u_B$ and bounded by $C\sqrt{k}$ (in particular $\tilde{\mathcal{F}}^c(z)$ is quasi isometric) for any z. We shall denote by $\Pi^{su} : \mathbb{R}^3 \to E^c_B$ the projection along $E^s_B \oplus E^u_B$.

Now, if $w \in [x, y]^c$ it follows that for any n that $\Pi^{su}(G^n(x)) < \Pi^{su}(G^n(w)) < \Pi^{su}(G^n(y))$. From this it follows that $\sup_{n\in\mathbb{Z}}\{\|G^n(x) - G^n(y)\|\} < \infty$ and so $H(x) = H(w)$. Finally, if $H(w) = H(z)$ then $\|z - w\| \leq 2C\sqrt{k}$.

□

Let us set the following notation: for $x \in \mathbb{R}^3$ let $[x] = \{y \in \mathbb{R}^3 : H(y) = H(x)\} = H^{-1}(H(x))$. In other words $[x]$ is the equivalent class or the equivalence relation $x \sim y$ if and only if $H(x) = H(y)$. From the above lemma we have that $[x]$ is a point or an arc contained in the central leaf $\mathcal{F}^c_G(x)$. In particular from the fact $H : \tilde{F}^u_G(x) \to \tilde{F}^u_B(H(x))$ is a homeomorphism, we have (see Figure 6):

Corollary 2.2.2. *Let $x \in \mathbb{R}^3$ and let $z \in \mathcal{F}^u_G(x)$. Then*

$$[z] = \left(\bigcup_{y\in[x]} \tilde{\mathcal{F}}^u_G(y)\right) \bigcap \tilde{\mathcal{F}}^c_G(z). \tag{11}$$

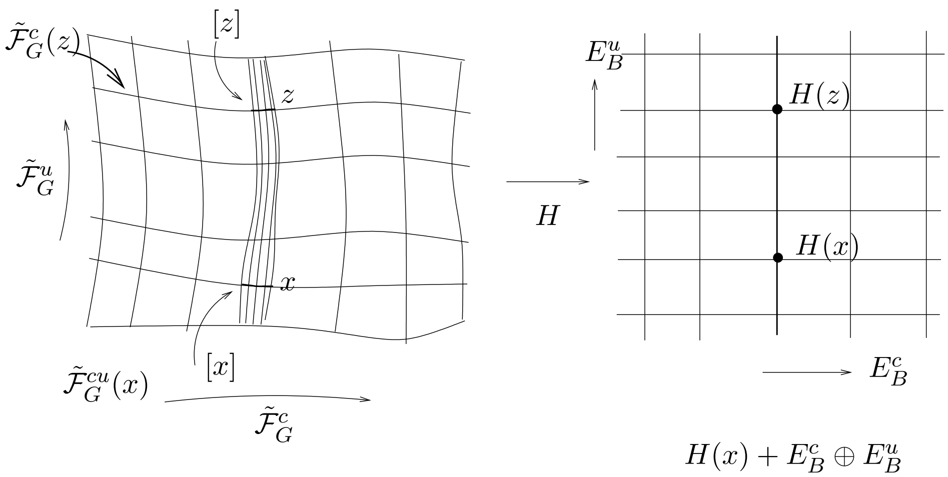

FIGURE 6.

Now, going back to the induced linear anosov diffeomorphism on the 3-torus by $B \in SL(3, Z)$ and the Mañé's DA $g \in \mathcal{U}(g_{B,k})$ and applying the previous results we get the following

Theorem 2.2.3. *There exists $h : \mathbb{T}^3 \to \mathbb{T}^3$ continuous and onto such that*

(1) $B \circ h = h \circ g$
(2) $dist_{C^0}(h, id) \leq C\sqrt{k}$.

(3) *$h(\mathcal{F}_g^j(x)) = \mathcal{F}_B^j(h(x))$ where $j = cs, cu, c, u$ and $h : \mathcal{F}_g^u(x) \to \mathcal{F}_B^u(h(x))$ is a homeomorphism.*
(4) *If $h(x) = h(y)$ then $y \in \mathcal{F}_g^c(x)$.*
(5) *$h^{-1}(z)$ is either a point or an arc contained in a central leaf (with diameter less than $2C\sqrt{k}$).*
(6) *If we set $[x] = h^{-1}(h(x)) = \{y \in \mathbb{T}^3 : h(x) = h(y)\}$ then, for $z \in \mathcal{F}_g^u(x)$ we have*

$$[z] = \left(\bigcup_{y\in[x]} \mathcal{F}_g^u(y)\right) \bigcap \mathcal{F}_g^c(z).$$

Corollary 2.2.3. *Let $g \in \mathcal{U}(g_{B,k})$ as above. Then, $\mathcal{F}_g^c$ is minimal, i.e., every leaf is dense in $\mathbb{T}^3$.*

Proof. Let $x \in \mathbb{T}^3$ and let $U \subset \mathbb{T}^3$ be an open set. We want to prove that $\mathcal{F}_g^c(x) \cap U \neq \emptyset$. Consider $S \subset U$ a small two dimensional disk transverse to E_g^c. We know that $h_{/S}$ is injective and hence $h(S)$ is a two dimensional topological manifold transverse to E_B^c. Since $\mathcal{F}_B^c$ is minimal, we get that $\mathcal{F}_B^c(h(x)) \cap h(S) \neq \emptyset$, that is, there exists $y \in S$ such that $h(y) \in \mathcal{F}_B^c(h(x)) = h(\mathcal{F}_g^c(x))$. Therefore $y \in \mathcal{F}_g^c(x)$ and so $\mathcal{F}_g^c(x) \cap U \neq \emptyset$. □

2.3. **Further analysis on the semiconjugacy.** In this section we give a more detailed consequences of the semiconjugacy with the linear Anosov diffeomorphism B and on the equivalent classes $[x] = h^{-1}(h(x)) = \{y : h(y) = h(x)\}$. Let us begin with the following

Lemma 2.3.1. *For $g \in \mathcal{U}(g_{B,k})$ as above the following hold:*

(1) *If*

$$\liminf_{n\to-\infty} \frac{1}{n} \log \|dg^n_{/E_g^c(x)}\| > 0$$

then $[x] = h^{-1}(h(x)) \supsetneq \{x\}$.
(2) *The set $\mathcal{A} = \{z \in \mathbb{T}^3 : h^{-1}(z)$ is a point $\}$ has full Lebesgue measure.*

Proof. For the proof of the first item, let γ,

$$\liminf_{n\to-\infty} \frac{1}{n} \log \|dg^n_{/E_g^c(x)}\| > \gamma > 0.$$

Then, for n large enough we have

$$\|Dg^{-n}_{E_g^c(x)}\| \leq e^{-\gamma}n$$

and therefore, by standard arguments, there exists a central arc I^c containing x such that the length of $g^{-n}(I^c)$ is uniformly bounded for $n \geq 0$ (indeed, $I^c \subset W^u(x)$). We claim that $g^n(I^c)$ has bounded length for $n \geq 0$. We will denote by $\ell(I)$ the length of I.

We may assume that ρ is small (recall that the support of the modification of B is in $B(p,\rho)$) so that if J^c is a central arc such that $4\rho \leq \ell(J^c) \leq 6\rho$ then $J^c \cap B(p,\rho)$ has at most one connected component of length at most 2ρ. Recall also that $\lambda_c(g) < 1+\epsilon$ where ϵ is small (taking k small) (for instance, $\epsilon < 1-\lambda_c$ and $\epsilon < 1/2$.)

To prove the claim we may assume that $\ell(I^c) < 2\rho$ and arguing by contradiction, consider the case where the length of $g^n(I^c)$ is unbounded for $n \geq 0$. Let n_0 be the first time such that $\ell(g^n(I^c) \geq 6\rho$. Since $4\rho\lambda_c(g) < 4\rho(1+\epsilon) < 6\rho$ it follows that

$$4\rho \leq \ell(g^{n_0-1}(I^c)) < 6\rho.$$

Set $J^c = g^{n_0-1}(I^c)$. By the above condition on J^c and recalling that $\|dg_\xi\| = \|B\| = \lambda_c$ if $\xi \notin B(p,\rho)$ we get

$$6\rho \leq \ell(g^{n_0}(I^c)) = \ell(g(J^c)) \leq (1+\epsilon)\frac{\ell(J^c)}{2} + \lambda_c\frac{\ell(J^c)}{2} < \ell(J^c) < 6\rho,$$

a contradiction. Now since, $\ell(g^n(I^c))$ is bounded for all $n \in \mathbb{Z}$ we conclude that $h(I^c) = h(x)$ (this can be seen by lifting to $\mathbb{R}^3$ where inmediately follows that $\|G^n(x) - G^n(y)\|$ is bounded for all $n \in \mathbb{Z}$ and $y \in I^c$.)

For the proof of the second item, we may assume that $\lambda_c(1+\epsilon)\lambda_c(g) < 1$ and also that $m(B(p,4\rho)) < \frac{1}{2}$, where m is the lebesgue measure in $\mathbb{T}^3$. Since $B : \mathbb{T}^3 \to \mathbb{T}^3$ preserves measure and it is ergodic there is a full measure set $\mathcal{R}$ such that if $y \in \mathcal{R}$ we have

$$\lim_{n\to\infty} \frac{\sharp\{j : 0 \leq j \leq n, B^{-j}(y) \in B(p,4\rho)\}}{n} = m(B(p,4\rho)) < \frac{1}{2}.$$

We will show that $\mathcal{R} \subset \mathcal{A}$. Arguing by contradiction, let $y \in \mathcal{R}$ such that $h^{-1}(y)$ is a nontrivial center arc I^c. Recall that $\ell(g^n(I^c))$ is bounded (by $2C\sqrt{k} < \rho$) for all $n \in \mathbb{Z}$. Thus, whenever we have that $B^{-j}(y) \notin B(p,4\rho)$ then $g^{-j}(I^c) \cap B(p,\rho) = \emptyset$. Since $dg_{/E^c_g(\xi)} \leq \lambda_c(1+\epsilon)$ for $\xi \notin B(p,\rho)$, if $J^c \cap B(p,\rho) = \emptyset$ then $\ell(g^{-1}(J^c)) \geq (\lambda_c(1+\epsilon))^{-1}\ell(J_c)$. And in any case $\ell(g^{-1}(J^c)) \geq \lambda_c(g)^{-1}\ell(J_c)$

Now, for n large enough we have:

$$\begin{aligned}\ell(g^{-n}(I^c)) &\geq \left(\prod_{j:B^{-j}\notin B(p,4\rho)} (\lambda_c(1+\epsilon))^{-1} \prod_{j:B^{-j}\in B(p,4\rho)} \lambda_c(g)^{-1}\right)\ell(I^c) \\ &\geq (\lambda_c(1+\epsilon)\lambda_c(g))^{-\frac{n}{2}}\ell(I^c) \to_{n\to\infty} \infty,\end{aligned}$$

a contradiction. □

The following lemma says that in any unstable leaf there is a point whose forward orbit never meets $B(p,2\rho)$ and is similar to what we have done in Section 2.1. Notice also that $\mathcal{F}^u_g$ is orientable and choose an orientation. For $x \in \mathbb{T}^3$ denote by $\mathcal{F}^{u,+}_g(x,t)$ an arc of length t in $\mathcal{F}^u_g(x)$ starting at x in the chosen orientation.

Lemma 2.3.2. *Assume that $\lambda > 3$. Then for ρ, k and $\mathcal{U}$ small the following holds for $g \in \mathcal{U}(g_{B,k})$: for any $x \in \mathbb{T}^3$ there exists a point $z_x \in \mathcal{F}^{u,+}_g(x,1)$ such that $g^n(z) \cap B(p,2\rho) = \emptyset$ for any $n \geq 0$.*

Proof. We may assume that ρ is so small that any segment I^u in $\mathcal{F}^u_B$ of length one has a subsegment I^u_1 of length 1/3 such that $I^u_1 \cap B(p,2\rho) = \emptyset$. Now, if k is small and $\mathcal{U}(g_{B,k})$ as well we may assume that the same property holds for $g \in \mathcal{U}$, that is, any arc I^u in $\mathcal{F}^u_g$ of length one has a subarc I^u_1 of length 1/3 such that $I^u_1 \cap B(p,2\rho) = \emptyset$. Moreover, we may assume that $\lambda^u(g) > 3$. Now, $g(I^u_1)$ has length at leat one and so it has a subarc I^u_2 such that $I^u_2 \cap B(p,2\rho) = \emptyset$. By induction, for any n we have that $g(I^u_n)$ contains I^u_{n+1} such that $I^u_{n+1} \cap B(p,2\rho) = \emptyset$. Therefore,

$$z_x \in \bigcap_{n\geq 0} g^{-n}(I^u_{n+1})$$

satisfies the lemma. □

Corollary 2.3.1. *Let $g \in \mathcal{U}$ as above and let $x \in \mathbb{T}^3$ be such that $[x] \supsetneq \{x\}$. Then, given ay $\eta > 0$ there is point $y \in \mathcal{F}^{u,+}_g(x)$ (the positive side of $\mathcal{F}^u_g$ in the chosen orientation) such that the length $\ell([y]) < \eta$.*

Proof. Recall that for $g \in \mathcal{U}$ we have $\|dg_{/E^c_g(\xi)}\| < \lambda_c(1+\epsilon) < 1$ if $\xi \notin B(p,\rho)$. Also, if k is small then $2C\sqrt{k} < \rho$. Let η be given and let n_0 be such that

$$(\lambda_c(1+\epsilon))^{n_0}\, 2C\sqrt{k} < \eta.$$

Consider x such that $[x] \supsetneq \{x\}$. From the above lemma, consider $z \in \mathcal{F}^u_g(g^{-n_0}(x), 1)$ such that $g^n(z) \notin B(p, 2\rho)$ for any $n \geq 0$. Notice that, since $[g^{-n_0}(x)]$ is not trivial, the same is true for z. On the other hand $[z]$ is a central segment of length at most $2C\sqrt{k}$. Therfore, $g^n([z]) \cap B(p,\rho) = \emptyset$ for $n \geq 0$. Therefore,

$$\ell(g^n[z]) \leq (\lambda_c(1+\epsilon))^n 2C\sqrt{k}.$$

Finally, setting $y = g^{n_0}(z) \in \mathcal{F}^{u,+}_g(x)$ we have

$$\ell([y]) = \ell(g^{n_0}[z]) \leq (\lambda_c(1+\epsilon))^{n_0} 2C\sqrt{k} < \eta.$$

□

The next result is fundamental for our purpose on the behavior of the holonomy map along the unstable foliation. The main tool is the existence of a transversal homoclinic point (recall Corollary 2.0.1).

Lemma 2.3.3. *Let $g \in \mathcal{U}(g_{B,k})$ having a transversal homoclinic point associated to the fixed point p of unstable index two. There exists ϵ_0 and $z_p \in \mathcal{F}^{u,+}_g(p)$ such that*

$$\limsup_{n\to\infty} \ell(g^n[z_p]) > \epsilon_0.$$

Proof. Recall that $[p]$ si the central segment between q_1, q_2. Let $\epsilon_0 < \min\{\ell[q_1,p]^c, \ell[p,q_2]^c\}$. Notice also that

$$W^u(p) = \bigcup_{y\in(q_1,q_2)^c} \mathcal{F}^u_g(y).$$

Let z be a homoclinic point associated to p, that is $z \in \mathcal{F}^s_g(p) \cap W^u(p)$. We know that

$$[z] = \left(\bigcup_{y\in[p]} \mathcal{F}^u_g(y)\right) \bigcap \mathcal{F}^c_g(z).$$

We may assume that the orientation in $\mathcal{F}^u_g$ is such that $z_p = [z] \cap \mathcal{F}^u_g(p) \in \mathcal{F}^{u.+}_g(p)$. Since $[z] = [z_p], z \in \mathcal{F}^s(p)$ and $[z] \subset \mathcal{F}^c_g(z) \subset \mathcal{F}^{cs}_g(p)$ by forward iteration $g^n([z]$ must approach at leat to $[q_1,p]$ or $[p,q_1]$ (see also Figure 7), and the lemma follows. □

Indeed, a more extensive result holds:

Proposition 2.3.1. *Let $g \in \mathcal{U}(g_{B,k})$ having a transversal homoclinic point associated to the fixed point $= p$ of unstable index two. Then there exists an uncountable set Λ_0 such that:*

1. *If $x, y \in \Lambda_0, x \neq y$ then $\mathcal{F}^{cu}_g(x) \neq \mathcal{F}^{cu}_g(y)$.*
2. *For any $x \in \Lambda_0, [x]$ is nontrival.*
3. *There exists ϵ_0 such that for any $x \in \Lambda_0$ and any $t > 0$ there exists $z_x \in \mathcal{F}^{u,+}_g(x) \backslash \mathcal{F}^u_g(x,t)$ such that $\ell([z_x]) > \epsilon_0$.*

Proof. From the existence of a transversal homoclinic point associated to p of index two we conclude the existence of a non trivial hyperbolic compact invariant set Λ (of unstable index 2) and with local product structure. In particular, from Lemma 2.3.1 we get that for any $x \in \Lambda, [x]$ is nontrival.

Notice that for $x \in \Lambda, W^u(x)$ is two dimensional and contained in $\mathcal{F}^{cu}_g(x)$ and there exists some $\delta > 0$ such that $W^u_\delta(x)$ has uniform size. We will denote by $W^{u,+}_\delta(x)$ the component of

$W^u_\delta(x) \setminus \mathcal{F}^c_g(x)$ which is in the positive direction of $\mathcal{F}^u_g(x)$. Moreover, there is an uncountable number of disjoints unstable manifolds W^u. Furhtermore, there is some L such that (setting $\mathcal{F}^s_g(p,L) = W^s_L(p)$) we have that

$$\mathcal{F}^s_g(p,L) \cap W^u_\delta(x) \neq \emptyset \ \ \forall \ x \in \Lambda.$$

Indeed, it is not difficult to see that if x is not in a central stable periodic leaf, then

$$\mathcal{F}^s_g(p,L) \cap W^{u,+}_\delta(x) \neq \emptyset.$$

Let us see a consequence of the above fact. Let $z \in \mathcal{F}^s_g(p,L) \cap W^{u,+}_\delta(x)$ and let $\epsilon_0 < \ell([p])/2$. Since $g^n(z) \to_n p$ we conclude that $\ell(g^n[z]) = \ell([g^n(z)]) > \epsilon_0$ for n large enough (see Figure 7). Indeed, $[z]$ is a central arc of uniforme size and therefore, and since exists m_0 such that $g^{m_0}(z) \in W^s_{loc}(p)$ we have that $g^{m_0}([z])$ is central arc of uniform size in $\mathcal{F}^{cs}_{loc}(p)$ Now, by forward iteration, we have that $\ell(g^n([z])) > \epsilon_0$ for all $n \geq m_1$ for some m_1 (which is independent of x).

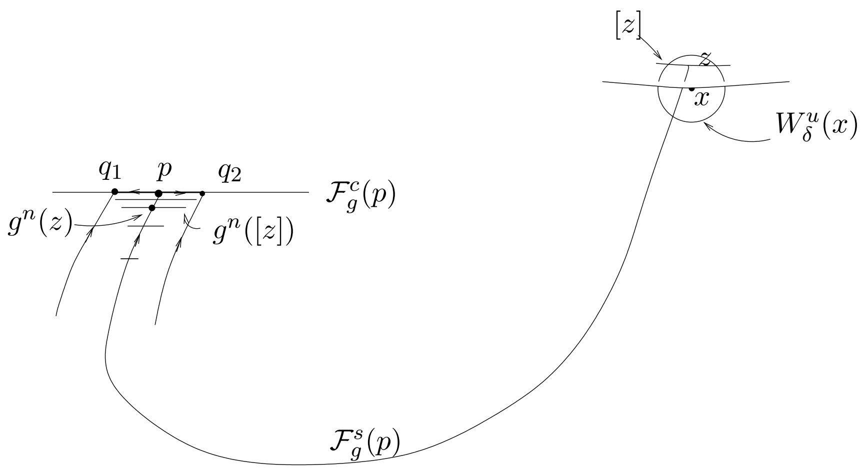

FIGURE 7.

Now choose an uncountable set $\Lambda_0 \subset \Lambda$ such that for $x \neq y \in \Lambda_0$ we have $\mathcal{F}^{cu}_g(x) \neq \mathcal{F}^{cu}_g(y)$ and that for any $x \in \Lambda_0$ then x is not in a periodic central stable leaf. It remains to prove (3). Let $x \in \Lambda_0$ and let $t > 0$. Choose n_1 bigger than m_1 such that $g^{-n_1}(\mathcal{F}^{u,+}_g(x,t)) \subset W^u_\eta(g^{-n_1}(x))$ where η is such that $W^u_\eta(g^{-n_1}(x)) \cap \mathcal{F}^s_g(x,L) = \emptyset$. Let $w \in \mathcal{F}^s_g(p,L) \cap W^{u,+}_\delta(g^{-n_1}(x)))$. It follows that $[w] \cap \mathcal{F}^{u,+}_g(g^{-n_1}(x))) \neq \emptyset$ and set y the point of intersection. Notice that $y \notin g^{-n_1}(\mathcal{F}^{u,+}_g(x,t))$ and therefore $z_x = g^{n_1}(y) \in \mathcal{F}^{u,+}_g(x) \setminus \mathcal{F}^{u,+}_g(x,t)$. On the other hand

$$\ell([z_x]) = \ell([g^{n_1}(y)]) = \ell(g^{n_1}([y])) = \ell(g^{n_1}([z])) > \epsilon_0.$$

□

3. The induced holonomy map on $\mathbb{T}^2$.

Let $B \in SL(3,\mathbb{Z})$ (with eigenvalues $0 < \lambda_s < \lambda_c < 1 < \lambda_u$) and $g_{B,k}$ defined in (1) and (2) and let $g \in \mathcal{U}(g_{B,k})$ with k and $\mathcal{U}$ small, and having a transversal homoclinic point associated to the fixed point p of unstable index 2 so that all results of the last section hold.

Consider a bidimensional torus transversal to $\mathcal{F}^u_B$ and (assuming k and $\mathcal{U}$ small) also transversal to $\mathcal{F}^u_g$. For instance, we may consider $\mathbb{T}^2 = \mathbb{R}^2_{/\mathbb{Z}^2} \subset \mathbb{R}^3_{/\mathbb{Z}^3} = \mathbb{T}^3$.

The foliations $\mathcal{F}^u_B$ and $\mathcal{F}^u_g$ are orientable and choose similar orientation on both (that is, take unit vector fields $X_B = e^u$ and X_g close to X_B).

Definition 3.1. For g as above we define $f = f_g : \mathbb{T}^2 \to \mathbb{T}^2$ the holonomy map on $\mathbb{T}^2$ induced by the unstable foliation $\mathcal{F}^u_g$. In other words, $f(x)$ is the first return map of $\mathcal{F}^u_g(x)$ to $\mathbb{T}^2$ in the given orientation. Moreover, we can define $F : \mathbb{T}^3 \to \mathbb{T}^2$ as the first return to $\mathbb{T}^2$ of any $x \in \mathbb{T}^3$ along the positive orientation of $\mathcal{F}^u_g(x)$.

Remark 3.1. Notice that the induced map $f = f_g$ is a homeomorphism. Morevoer, f is of class C^r if the unstable foliation $\mathcal{F}^u_g$ is of class C^r. Furthermore, the unstable foliation $\mathcal{F}^u_g$ is of class C^r if the unstable bundle E^u_g is of class C^r.

Besides, if we consider the holonomy map $T_B : \mathbb{T}^2 \to \mathbb{T}^2$ induced by $\mathcal{F}^u_B$ we obtain that T_B is a minimal (and hence ergodic) translation. Moreover, $f = f_g$ and T_B are close as we wish if k is small.

If we apply the results on the previous section we obtain the topological version of our main result:

Theorem 3.0.1. *For $g : \mathbb{T}^3 \to \mathbb{T}^3$ as above and $f = f_g : \mathbb{T}^2 \to \mathbb{T}^2$ and $T_B : \mathbb{T}^2 \to \mathbb{T}^2$ as above we have:*

(i) *f is minimal.*
(ii) *f is isotopic and semiconjugated to the ergodic translation T_B. If we denote by $\tilde{h}$ the semiconjugacy, then $\tilde{h}^{-1}(x)$ is either a point or an arc. Moreover, there are uncountable points x such that $\tilde{h}^{-1}(x)$ is a nontrivial arc.*
(iii) *f preserves a minimal and invariant C^0 foliation with one dimensional C^1 leaves. The fibers $\tilde{h}^{-1}(x)$ are contained in the leaves of this foliation.*
(iv) *f has zero entropy.*
(v) *f is point-distal non-distal.*
(vi) *There is an uncountable number of Li-Yorke pairs.*
(vii) *f has sensitivity with respect to initial conditions.*
(viii) *f is uniquely ergodic.*

Proof. (i) follows from the minimality of the unstable foliation $\mathcal{F}^u_g$ (see section 2.1).

Let's prove (ii). Since f and T_B are C^0 close, we get that f and T_B are isotopic. Recall h to be the semiconjugacy between $g = g_{B,k}$ and $B : \mathbb{T}^3 \to \mathbb{T}^3$ given in section 2.2.

Since $dist_{C^0}(h, id) < C\sqrt{k}$ (which we may assume less than 1/4), then for every point of $h(\mathbb{T}^2)$ we can define a natural projection $P : h(\mathbb{T}^2) \to \mathbb{T}^2$ along the unstable foliation $\mathcal{F}^u_B$, that is $P(h(x))$ is the closest within $\mathcal{F}^u_B(h(x)$ in $\mathbb{T}^2$ to $h(x)$. Define

$$\tilde{h} : \mathbb{T}^2 \to \mathbb{T}^2 \qquad \tilde{h}(x) = P(h(x)).$$

Clearly, $\tilde{h}$ is continuous and close to the identity (if k is small) and hence onto (and isotopic to the identity as well).

Now, if we take $x \in \mathbb{T}^2$ and $f(x) \in \mathbb{T}^2$ we have that they are the ends of an arc $I^u \subset \mathcal{F}^u_g(x)$ and when lifted to $\mathbb{R}^3$ their coordinates have z-difference equal to one.

On the other hand $h(I^u)$ is an arc (segment) of $\mathcal{F}^u_B(h(x))$ so that, when lifted to $\mathbb{R}^3$ the ends have coordinates whose z-difference is between $1-2C\sqrt{k}$ and $1+2C\sqrt{k}$. Therefore $P(h(f(x)) = T_B(\tilde{h}(x)$ that is

$$\tilde{h} \circ f = T_B \circ \tilde{h}.$$

Notice that

- If $h^{-1}(x) = \{y\}$ then clearly holds that $\tilde{h}^{-1}(x)$ is a unique point.

- If $h^{-1}(x)$ is a non trivial central arc, then the projection (by P) on $\mathbb{T}^2$ is a non trivial arc and it is $\tilde{h}^{-1}(x)$.

Moreover, by Proposition 2.3.1, we get that there are an uncountable number of points x such that $\tilde{h}^{-1}(x)$ is a nontrivial arc. This finishes the proof of (ii).

To prove (iii), for $x \in \mathbb{T}^2$ let $\mathcal{C}(x)$ the connected component that contains x of $\mathcal{F}_g^{cu}(x) \cap \mathbb{T}^2$. It follows that $\mathcal{C}$ is a continuous foliations with C^1 dimensional leaves (recall that $\mathcal{F}_g^{cu}(x)$ is a C^1 manifold) and obviously invariant by f, the holonomy map. Furhtermore, since $h(\mathcal{F}_g^{cu})(x) = \mathcal{F}_B^{cu}(h(x))$ it follows that $\tilde{h}(\mathcal{C}(x))$ is the connected component of $\mathcal{F}_B^{cu}(\tilde{h}(x)) \cap \mathbb{T}^2$ which contains $\tilde{h}(x)$. Since this foliation by lines on $\mathbb{T}^2$ is minimal we also conclude that $\mathcal{C}$ is minimal (similar proof as in Corollary 2.2.3). Since $h^{-1}(x)$ live in a central unstable leaf, we get that $\tilde{h}^{-1}$ live in the leaves of this foliations.

The proof of (iv) is rather easy. Indeed, by Bowen's formula ([Bo]) we have

$$h_{top}(f) \leq h_{top}(T_B) + \sup_{x \in \mathbb{T}^2} h_{top}(f, \tilde{h}^{-1}(x))$$

where $h_{top}(f, K) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(\epsilon, n, f, K)$ and $N(\epsilon, n, f, K)$ is the minimum cardinality of (n, ϵ) separated set in K. Since for all x, $\tilde{h}^{-1}(x)$ is either a point or an arc (with bounded length in the future and in the past) we have the result (see also [BFSV]).

Let us prove (v). Recall that f is point distal if there exists $x \in \mathbb{T}^2$ such that for every $y \neq x$ there exists $r_y > 0$ so that $r_y \leq \inf\{dist(f^n(x), f^n(y)) : n \in \mathbb{Z}\}$ and f is non distal if there exists a pair of points z, w such that $\inf\{dist(f^n(z), f^n(w)) : n \in \mathbb{Z}\} = 0$. We will show first that f is point distal. Let $x \in \mathbb{T}^2$ be such that $\tilde{h}^{-1}(\tilde{h}(x)) = \{x\}$ and consider any $y \in \mathbb{T}^2$. Let $\alpha = dist(\tilde{h}(x), \tilde{h}(y))$. By the (uniform) continuity of $\tilde{h}$, there exists r such that if $dist(z, w) < r$ then $dist(\tilde{h}(z), \tilde{h}(w)) < \alpha$ for any $z, w \in \mathbb{T}^2$. We claim that $\inf\{dist(f^n(x), f^n(y)) : n \in \mathbb{Z}\} \geq r > 0$. Otherwise, if for some n we have $dist(f^n(x), f^n(y)) < r$ then we get (since T_B is an isometry):

$$\alpha > dist(\tilde{h}(f^n(x)), \tilde{h}(f^n(y))) = dist(T_B^n(\tilde{h}(x)), T_B(\tilde{h}(y))) = dist(\tilde{h}(x), \tilde{h}(y)) = \alpha.$$

We will prove now that f is non-distal. We will go back to $g =: \mathbb{T}^3 \to \mathbb{T}^3$ and let x such that $[x] = h^{-1}(h(x) \supsetneq \{x\}$. Let $I_x = F([x])$ the first return map to $\mathbb{T}^2$ of $[x]$ along the unstable foliation $\mathcal{F}_g^u$ if $[x] \cap \mathbb{T}^2 = \emptyset$, otherwise, set $I_x = P([x])$. From Corollary 2.3.1 we know that for any η there exists $y \in \mathcal{F}_g^{u,+}(x)$ such that $\ell([y]) < \eta$. On the other hand, since $[y] = \bigcup_{z \in [x]} \mathcal{F}_g^{u,+}(z) \bigcap \mathcal{F}_g^c(y)$ we have that there exists n_y such that $f^{n_y}(I_x) = I_y$. It follows that $\liminf_{n \to \infty}(f^n(I_x))) = 0$. Finally, if we take $z \neq w \in I_x$ we conclude that $\inf\{dist(f^n(z), f^n(w)) : n \in \mathbb{Z}\} = 0$, i.e., f is non-distal.

We prove now (vi). Consider $I_p = \tilde{h}^{-1}(p)$. As we argue above, by Corollary 2.3.1 we have that $\liminf_{n \to \infty} \ell(f^n(I_p)) = 0$. On the other hand, notice that given ϵ_0 there exists ϵ_1 such that if I is an arc contained in a central leaf $\mathcal{F}_g^c$ of length ϵ_0 then its projection on $\mathbb{T}^2$ under the first return map F (or under the projection P) has length at least ϵ_1. Now, if we apply Lemma 2.3.3 we get that $\limsup \ell(f^n(I_p)) \geq \epsilon_1$. In particular, the ends of I_p (which are $P(q_1), P(q_2)$) is a Li-Yorke pair. In a similar way, if $x \in \Lambda_0$ as in Proposition 2.3.1 and denote by $I_x = F([x])$ the first return map to $\mathbb{T}^2$ of $[x]$ along the unstable foliation $\mathcal{F}_g^u$ if $[x] \cap \mathbb{T}^2 = \emptyset$ and otherwise $I_x = P([x])$, we get from Corollary 2.3.1 and Proposition 2.3.1 that

$$\liminf_{n \to \infty} \ell(f^n(I_x)) = 0 \quad \text{and} \quad \limsup_{n \to \infty} \ell(f^n(I_x)) \geq \epsilon_1$$

and therefore the ends of I_x are a Li-Yorke pair.

For the proof of (vii), recall that f has sensitivity with respect to initial conditions if there exists some ϵ_2 such that for any $x \in \mathbb{T}^2$ and any open set U containing x there exist $y \in U$

and $n > 0$ such that $dist(f^n(x), f^n(y)) \geq \epsilon_2$. So, given ϵ_1 let ϵ_2 be such that any arc in $\mathcal{C}$ of length ϵ_1 then the endpoints are at distance at least $2\epsilon_2$. Let x and U be given. Assume first that $\tilde{h}^{-1}(\tilde{h}(x)) = \{x\}$ which is the same as $[x] = \{x\}$. Since f is minimal we have that there is m_k such that $f^{m_k}(p) \to_k x$. We claim that for k large enough $f^{m_k}(I_p) \subset U$. Indeed, it follows that $\ell(f^{m_k}) \to 0$, otherwise we conclude that $[x] \neq \{x\}$ (the equivalent classes are lower semicontinuous). Thus, choose some m so that $f^m(I_p) \subset U$. Since $\limsup \ell(f^n(I_p)) \geq \epsilon_1$ we get the result taking y as the appropriate end point of $f^m(I_p)$. Now, if $[x]$ is non trivial we can argue as before, since in U there are points z such that $[z]$ si trivial and so for some m we have that $f^m(I_p) \subset U$.

It is left to prove (viii). Consider the set

$$\tilde{\mathcal{A}} = \{x \in \mathbb{T}^2 : \tilde{h}^{-1}(x) \text{ is a point}\}.$$

Observe that $\tilde{h}^{-1}(x)$ is a point if and only if $h^{-1}(x)$ is a point. Moreover, if $h^{-1}(x)$ is just a point the same is true for any $y \in \mathcal{F}^u_B(x)$. Therefore, since

$$\mathcal{A} = \{x \in \mathbb{T}^3 : h^{-1}(x) \text{ is a point }\}$$

has full lebesgue measure on $\mathbb{T}^3$ by Lemma 2.3.1 we get that $\tilde{\mathcal{A}}$ has full lebesgue measure on $\mathbb{T}^2$.

Denote by $\mathcal{M}(f)$ the set of invariant probabilities of f. Given $\mu \in \mathcal{M}_f$ we may define a measure $\nu \in \mathcal{M}_{T_B}$ by $\nu(A) = \mu(h^{-1}(A))$. Since T_B is uniquely ergodic, $\nu = m$ (the lebesgue measure on $\mathbb{T}^2$). That is, for every borelean set D and $\mu \in \mathcal{M}_f$ we have $\mu(h^{-1}(D)) = m(D)$. And therefore, for every $\mu \in \mathcal{M}_f$, setting $\mathcal{D} = \tilde{h}^{-1}(\tilde{\mathcal{A}})$ we have

$$\mu(\mathcal{D}) = \mu(\tilde{h}^{-1}(\tilde{\mathcal{A}})) = m(\tilde{\mathcal{A}}) = 1.$$

Observe that for any Borel set A we have $A \cap \mathcal{D} = \tilde{h}^{-1}(\tilde{h}(A \cap \mathcal{D})$.

Given $\mu_1, \mu_2 \in \mathcal{M}_f$ and A any Borel set we have

$$\begin{aligned} \mu_1(A) &= \mu_1(A \cap \mathcal{D}) = \mu_1(\tilde{h}^{-1}(\tilde{h}(A \cap \mathcal{D}))) = m(\tilde{h}(A \cap \mathcal{D})) \\ &= \mu_2(\tilde{h}^{-1}(h(A \cap \mathcal{D}))) = \mu_2(A \cap \mathcal{D}) \\ &= \mu_2(A). \end{aligned}$$

Thus f is uniquely ergodic.

□

Remark 3.2. If f were of class C^2 and the leaves of the foliation $\mathcal{C}$ also were of class C^2 one is tempted to use Schwarz's argument ([Sch]) to show that non trivial fibers $\tilde{h}^{-1}$ are not possilbe. However, in our case there is extra difficulty: we don´t know *a priori* that the sum of the length of the iterates of a nontrivial fiber (if exists) does converge. In our examples, this sum does not converge!

Let us point out as well that with our method, the differentiability of th sysetm and of the foliation are like the dishes on a balance. More differentiability ensured for the system implies less ensured for the folliation.

4. On the smoothness of E^u_g.

From Theorem 3.0.1 and Remark 3.1 the only thing that is left to prove for the proof of our Main Theorem is the following: given $r \in [1, 3)$ there exists g so that the unstable bundle E^u_g is of class C^r.

In order to establish the differentiability class of E^u_g we recall a classical result from [HPS] that is very useful for these type of problems.

Theorem 4.0.2. *C^r-section theorem.*

Consider M a compact C^r-manifold and $g : M \to M$ a C^r-diffeomorphism. Let $\pi : L \to M$ be a finite-dimensional Finslered vector bundle and let D be disc subbundle, $\pi(D) = M$. Let $F : D \to D$ be a homeomorphisms such that $F(L_\xi) = L_{g(\xi)}$, and let $l_\xi = l_\xi(F, g)$ be the Lipchitz constant of $F|_{L_\xi}$ for $\xi \in M$.

Then if $l_\xi < 1$ for every $\xi \in M$ there exists a unique continue section $\sigma : M \to L$ such that $F \circ \sigma = \sigma \circ g$ (an invariant section).

Moreover, if $\pi : L \to M$ is a C^r-vector bundle (with some estructure wich is compatible with the Finslered estructure), F is C^r and setting $\tau_\xi = \tau_\xi(g) = \|(dg_\xi)^{-1}\|$ we have $l_\xi . \tau_\xi^r < 1$, then the invariant section $\sigma : M \to L$ is C^r.

Let $B \in Sl(3, \mathbb{Z})$ be a linear transformation with eigenvalues $0 < \lambda_s < \lambda_c < 1 < \lambda_u$ and invariant hyperbolic structure $E^s_B \oplus E^c_B \oplus E^u_B$ as we have been considering and the euclidean metric on $\mathbb{R}^3$ such that the above spaces are mutually orthogonal. Consider the vector space

$$\mathcal{L}(E^u_B, E^s_B \oplus E^c_B) = \{t : E^u_B \to E^s_B \oplus E^c_B, t \text{ linear}\}$$

endowed with the natural norm structure.

Consider the (trivial) vector bundle

$$L = \{(\xi, t) : \xi \in \mathbb{T}^3, t \in \mathcal{L}(E^u_B, E^s_B \oplus E^c_B)\}. \tag{12}$$

Then $\pi : L \to M$ given by $\pi(\xi, t) = \xi$ is a (finite dimensional) C^∞ Finslered vector bundle.

Now, for $g = g_{B,k} : \mathbb{T}^3 \to \mathbb{T}^3$ we define the associated vector bundle map $F = F_{B,g} : L \to L$ as follows: for $(\xi, t) \in L$,

$$F(\xi, t) = (g(\xi, s)), s \in \mathcal{L}(E^u_B, E^s_B \oplus E^c_B) \text{ such that } graph(s)) = dg_\xi(graph(t)). \tag{13}$$

Recall that $E^s_B \oplus E^c_B$ is invariant by dg_ξ for any $\xi \in \mathbb{T}^3$ and so F is well defined vector bundle homeomorphism. Neverhteless, for g close to $g_{B,k}$ the associated map $F : L \to L$ may not be well defined on the whole L. To overcome this difficulty just set

$$D = \{(\xi, t) : \xi \in \mathbb{T}^3, t \in \mathcal{L}(E^u_B, E^s_B \oplus E^c_B), \|t\| \le 1\}$$

and from the above theorem we have

Corollary 4.0.2. *Assume that for some r, B and k we have*

$$l_\xi(F, g_{B,k}) < 1, \qquad l_\xi(F, g_{B,k}) \left(\tau(g_{B,k})\right)^r < 1.$$

Then, there exists $\mathcal{U}(g_{B,k})$ such that for any $g \in \mathcal{U}(g_{B,k})$ of class C^∞ we have that the associated map $F_g : D \to D$ is well defined, $l_\xi(F) < 1$ and $l_\xi(F)\tau(g)^r < 1$. In particular there exists a unique invariant section in D and it is of class C^r.

Remark 4.1. Observe that if $\sigma : \mathbb{T}^3 \to L$ is an invariant section by F, i.e., $F \circ \sigma = \sigma \circ g$ then it holds that $graph(\sigma(\xi)) = E^u_g(\xi)$. So, in order to find the differentiability class we will apply the C^r Section Theorem to our $F : L \to L$ over g.

Remark 4.2. If we use the C^r-section theorem to calculate the differentiability of the unstable vector bundle of the Anosov system induced by B, then we will have differentiability less than C^3: Let $r = 3$, then compute $l_\xi \tau_\xi^r = \frac{\lambda_c}{\lambda_u} \frac{1}{\lambda_s^3} = \frac{\lambda_c}{\lambda_s^2} > 1$. Moreover, the last estimate shows that in order to have proximity to C^3 differentiability we must find linear Anosov systems with λ_s close to λ_c. This will be done in Section 4.1

Through the rest of this subsection and to avoid notation we set $g = g_{B,k}$. We want to estimate $l_\xi(F,g)$ and $\tau_\xi(g)$ for the graph transform F associated to $g = g_{B,k}$. Recall that the differential of g in the decomposition $E^s_B \oplus E^c_B \oplus E^u_B$ is given by:

$$dg_\xi = \begin{pmatrix} \lambda_s & 0 & 0 \\ 0 & \lambda_c & 0 \\ 0 & 0 & \lambda_u \end{pmatrix} \quad \text{for } \xi \in \mathbb{T}^3 \setminus B(p,\rho)$$

and

$$dg_\xi = \begin{pmatrix} \lambda_s + Z(z)(\beta(r)+\beta'(r)2x^2) & Z(z)\beta'(r)2xy & Z'(z)\beta(r)x \\ Z(z)\beta'(r)2xy & \lambda_c + Z(z)(\beta(r)+\beta'(r)2y^2) & Z'(z)\beta(r)y \\ 0 & 0 & \lambda_u \end{pmatrix}$$

for $\xi \in B(p,\rho)$.

Set $T_\xi = dg_{\xi/E^s_B \oplus E^c_B}$.

Lemma 4.0.4. *With the above notations we have*

$$l_\xi = l_\xi(F) \leq \frac{\|T_\xi\|}{\lambda_u}.$$

Moreover the following estimations hold:

(i) *For $\xi \notin B(p,\rho)$ we have $l_\xi \leq \frac{\lambda_c}{\lambda_u}$*
(ii) *For $\xi \in B(p,\rho)$ we have $l_\xi < \frac{\lambda_c + Z(z)\beta(r)+k}{\lambda_u}$*

In particular $l_\xi(F) < 1$ for all $\xi \in \mathbb{T}^3$ (if k is small).

Proof. If we write

$$dg_\xi = \begin{pmatrix} T_\xi & A_\xi \\ 0 & \lambda_u \end{pmatrix}$$

then it is not difficult to see that

$$F(\xi,t)(v) = \frac{1}{\lambda_u}\left(T_\xi(t(v)) + A_\xi v\right)$$

and therefore

$$\|F(\xi,t_1) - F(\xi,t_2)\| \leq \frac{\|T_\xi\|}{\lambda_u}\|t_1 - t_2\|$$

which implies $l_\xi \leq \frac{\|T_\xi\|}{\lambda_u}$. Since for $\xi \notin B(p,\rho)$ it holds that $\|T_\xi\| = \lambda_c$ we obtain (i).

In order to prove (ii), set $T_\xi = D + S_\xi$ where $D = \begin{pmatrix} \lambda_s & 0 \\ 0 & \lambda_c \end{pmatrix}$ and

$$S_\xi = \begin{pmatrix} Z(z)(\beta(r)+\beta'(r)2x^2) & Z(z)\beta'(r)2xy \\ Z(z)\beta'(r)2xy & Z(z)(\beta(r)+\beta'(r)2y^2) \end{pmatrix}$$

Observe that S_ξ is selfadjoint and has eigenvectors (when $\xi \neq p$) (x,y),$(-y,x)$ and eigenvalues

$$\lambda_1 = Z(z)(\beta(r) + 2\beta'(r)r) \qquad \lambda_2 = Z(z)\beta(r). \tag{14}$$

When, $\xi = p$ then $S_\xi = Z(0)\beta(0)Id$. From the definition of g (recall Lemma 2.0.1 and (2)) we have $-k < \lambda_1 < \lambda_2 < \beta(0)$ and $\lambda_2 > 0, \lambda_2 - \lambda_1 < k$. Then, $\|S_\xi\| \leq \max\{|\lambda_1|, |\lambda_2|\} \leq \lambda_2 + k = Z(z)\beta(r) + k$ and so $\|T_\xi\| \leq \lambda_c + \lambda_2 + k$.

□

Lemma 4.0.5. *Let $\lambda_1 = \lambda_{1,g} : \mathbb{T}^3 \to \mathbb{R}$ be the function defined by: $\lambda_{1,g}(\xi) = 0$ if $\xi \notin B(p,\rho)$ and $\lambda_{1,g}(\xi) = Z(z)(\beta(r) + 2\beta'(r)r)$ for $\xi \in B(p,\rho)$. Then, we have*

$$\|(dg_\xi)^{-1}\| = \tau_\xi = \tau_\xi(g) \leq \frac{1}{\lambda_s + \lambda_{1,g}(\xi)}$$

Proof. Write

$$dg_\xi = \begin{pmatrix} T_\xi & A_\xi \\ 0 & \lambda_u \end{pmatrix}.$$

Then

$$(dg_\xi)^{-1} = \begin{pmatrix} T_\xi^{-1} & -\lambda_u^{-1} T_\xi^{-1} A_\xi \\ 0 & \lambda_u^{-1} \end{pmatrix}.$$

Since $\|A_\xi\|$ is small, $\lambda_u^{-1} < 1$ and $\|T_\xi^{-1}\| \geq 1$ it follows

$$\tau_\xi \leq \|T_\xi^{-1}\|.$$

So we want to estimate $\|(T_\xi)^{-1}\|$. If $\xi \notin B(p,\rho)$ then

$$\|T_\xi^{-1}\| = \frac{1}{\lambda_s} = \frac{1}{\lambda_s + \lambda_1(\xi)}.$$

If $\xi = p$ then

$$T_p = \begin{pmatrix} \lambda_s + Z(0)\beta(0) & 0 \\ 0 & \lambda_c + Z(0)\beta(0) \end{pmatrix}$$

and so

$$\|T_p^{-1}\| = \frac{1}{\lambda_s + Z(0)\beta(0)} = \frac{1}{\lambda_s + \lambda_1(p)}.$$

For $\xi \in B(p,\rho), \xi \neq p$ write

$$T_\xi = C_\xi + \widetilde{S}_\xi$$

where

$$C_\xi = \begin{pmatrix} \lambda_s - \lambda_c & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \widetilde{S}_\xi = \begin{pmatrix} Z(z)(\beta(r) + \beta'(r)2x^2) + \lambda_c & Z(z)\beta'(r)2xy \\ Z(z)\beta'(r)2xy & Z(z)(\beta(r) + \beta'(r)2y^2) + \lambda_c \end{pmatrix}.$$

The selfadjoint $\widetilde{S}_\xi$ map has eigenvectors $(x,y), (-y,x)$ associated to eigenvalues $\lambda_1 + \lambda_c$ and $\lambda_2 + \lambda_c$ where λ_1, λ_2 are as in (14).

Let $\mathcal{E}$ the elipse with axis in the (x,y) direction and $(-y,x)$ direction, with vértices of norm $\frac{1}{\lambda_2+\lambda_c}$ and $\frac{1}{\lambda_1+\lambda_c}$ respectively. We have $S_\xi(\mathcal{E}) = S^1$ (the unit circle). Thus

$$T_\xi(\mathcal{E}) \subset \left\{ v : 1 - \frac{\lambda_c - \lambda_s}{\lambda_c + \lambda_1} \leq \|v\| \leq 1 + \frac{\lambda_c - \lambda_s}{\lambda_c + \lambda_1} \right\}$$

. Setting $R = 1 - \frac{\lambda_c - \lambda_s}{\lambda_c + \lambda_1} = \frac{\lambda_s + \lambda_1}{\lambda_c + \lambda_1}$, we have that

$$T_\xi^{-1}(\{v : \|v\| = R\}) \subset int(\mathcal{E}) \subset \left\{ v : \|v\| \leq \frac{1}{\lambda_1 + \lambda_c} \right\}$$

. Then

$$\|(T_\xi)^{-1}\| \leq \frac{1}{R} \frac{1}{\lambda_1 + \lambda_c} = \frac{1}{\lambda_s + \lambda_1}$$

. ∎

4.1. **A special family of linear Anosov diffeomorphism on $\mathbb{T}^3$.** In order to construct elements with E^u bundle of class C^r with r close to 3 we have seen that we need $B \in Sl(3,\mathbb{Z})$ with eigenvalues λ_s and λ_c arbitrary close. For this we will find a special family of matrices in $SL(3,\mathbb{Z})$.

Let us begin with the following family $\mathcal{J} = \{M_a\}_{a\in\mathbb{N}\setminus\{0,1,2\}}$ of matrices in $SL(3,\mathbb{Z})$ (inspired form the one in [McS]):

$$M_a = \begin{pmatrix} 0 & -1 & 0 \\ 1 & a^2-1 & a \\ 0 & a^3+a & 1 \end{pmatrix} \tag{15}$$

Lemma 4.1.1. *For every $a \in \mathbb{N}\setminus\{0,1,2\}, M_a$ has eigenvalues $\alpha_a, \beta_a, \gamma_a$ such that*

$$\alpha_a < \frac{-a^2}{3} < -1 < \beta_a < 0 < a^2 < \gamma_a.$$

Furhtermore, we have

$$-\frac{2a^2}{3} < \alpha_a < -\frac{a^2}{3} \quad \textit{and} \quad a^2 < \gamma_a < 2a^2. \tag{16}$$

Proof. The characteristic polinomial of M_a is given by $P_a(\lambda) = -\lambda^3 + a^2\lambda^2 + a^4\lambda + 1$. The derivative of P_a is $P_a'(\lambda) = -3\lambda^2 + 2a^2\lambda + a^4$ and has one negative root $\lambda = \frac{-a^2}{3}$ and a positive one $\lambda = a^2$. On the negative root of P_a' the polynomial P_a has relative minimum, and on the positive root where there is a relative maximum of P_a. The value of P_a on such roots are:

$$P_a\left(\frac{-a^2}{3}\right) = \frac{-5a^6}{27} + 1 < 0 \quad \text{and} \quad P_a(a^2) = a^6 + 1 > 0$$

Thus, $P_a(\lambda)$ is as in Figure 8 and the eigenvalues of M_a (i.e. the roots of $P_a(\lambda)$) satisfies

$$\alpha_a < \frac{-a^2}{3} < \beta_a < 0 < a^2 < \gamma_a.$$

For the proof of the other inequalities in (16) just do some computations:

$$P_a\left(-\frac{2a^2}{3}\right) = \frac{2}{3^3}a^6 + 1 > 0 \quad \text{and} \quad P_a(2a^2) = -2a^6 + 1 < 0.$$

□

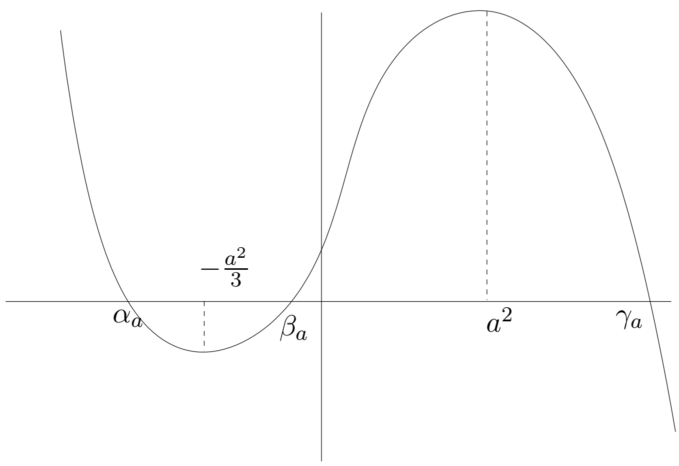

FIGURE 8. The graph of $P_a(\lambda)$

We are ready to define our special family of linear Anosov maps:

$$\mathcal{I} = \left\{B_a = \left(M_a^2\right)^{-1} : M_a \in \mathcal{J}, a \in \mathbb{N}\setminus\{0,1,2\}\right\} \tag{17}$$

Notice that $B_a \in SL(3,\mathbb{Z})$ and the eigenvalues of B_a are the inverse of the square of the eigenvalues of M_a and we have:

$$\frac{1}{4a^4} < \frac{1}{\gamma_a^2} < \frac{1}{a^4} < \frac{9}{4a^4} < \frac{1}{\alpha_a^2} < \frac{9}{a^4} < 1 < \frac{1}{\beta_a^2}.$$

We summarize this in the following

Corollary 4.1.1. *For $B_a \in \mathcal{I}$ the following holds:*

(i) *$B_a \in Sl(3,\mathbb{Z})$ and has eigenvalues $0 < \lambda_s(a) < \lambda_c(a) < 1 < \lambda_u(a)$.*
(ii) *For every $a \in \mathbb{N} \setminus \{0,1,2\}$ we may write*

$$\lambda_s(a) = K_a \frac{1}{a^4} \quad \text{and} \quad \lambda_c(a) = K'_a \frac{1}{a^4} \tag{18}$$

where $\frac{1}{10} < K_a < K'_a < 10$. In particular $\lambda_u(a) = \frac{a^8}{K_a K'_a}$.

With the next result we will conclude the proof of our Main Theorem:

Proposition 4.1.1. *For each $r \in [1,3)$ there exists $B_a \in \mathcal{I}$ such that for $g_a = g_{B_a,k}$ as defined in* (1) *and* (2) *with k sufficiently small the following holds: for the map $F = F_{B_a,g_a} : L \to L$ as defined in* (12) *and* (13) *and $l_\xi(F), \tau_\xi(g_a)$ as defined in Theorem 4.0.2 we have:*

$$l_\xi(F)(\tau_\xi(g_a))^r < 1 \quad \text{for all } \xi \in \mathbb{T}^3.$$

Proof. For the sake of simplicity, for $\xi \in \mathbb{T}^3$ set $l_{\xi,a} = l_\xi(F_{B_a,g_a})$ and $\tau_{\xi,a} = \tau_\xi(g_a)$.

Fix $r, 1 \le r < 3$. It is enough to prove the proposition to show that

$$\lim_{a\to\infty} l_{\xi,a}\tau_{\xi,a}^r = 0$$

uniformly on $\xi \in \mathbb{T}^3$. To do so, from Lemmas 4.0.4 and 4.0.5, we have for $\xi \notin B(p,\rho)$:

$$l_{\xi,a}\tau_{\xi,a}^r = \frac{\lambda_c(a)}{\lambda_u(a)} \frac{1}{\lambda_s(a)^r} = \frac{\lambda_c(a)^2}{\lambda_s(a)^{r-1}} = \frac{K'_a a^{4(r-1)}}{K_a a^8} \le 100 \frac{a^{4(r-1)}}{a^8} \tag{19}$$

and for $\xi \in B(p,\rho)$:

$$\begin{aligned} l_{\xi,a}\tau_{\xi,a}^r &= \frac{\lambda_c(a) + Z(z)\beta(r) + k}{\lambda_u(a)} \left[\frac{1}{\lambda_s(a) + \lambda_{1,g_a}(\xi)}\right]^r \\ &= \frac{1}{\lambda_u(a)} \left[\frac{\lambda_c(a) + \lambda_{1,g_a}(\xi)}{(\lambda_s(a) + \lambda_{1,g_a}(\xi))^r} + \frac{k + Z(z)\beta(r) - \lambda_{1,g_a}(\xi)}{(\lambda_s(a) + \lambda_{1,g_a}(\xi))^r}\right] \end{aligned}$$

Since $Z(z)\beta(r) - \lambda_{1,g_a}(\xi) \le 2k$ we have

$$\begin{aligned} l_{\xi,a}\tau_{\xi,a}^r &\le \frac{1}{\lambda_u(a)} \left[\frac{\lambda_c(a) + \lambda_{1,g_a}(\xi)}{(\lambda_s(a) + \lambda_{1,g_a}(\xi))^r} + \frac{3k}{(\lambda_s(a) + \lambda_{1,g_a}(\xi))^r}\right] \\ &\le \frac{1}{\lambda_u(a)} \left[\frac{\lambda_s(a) + \lambda_{1,g_a}(\xi) + (\lambda_c(a) - \lambda_s(a) + 3k)}{(\lambda_s(a) + \lambda_{1,g_a}(\xi))^r}\right] \end{aligned}$$

We may assume, for fixed a that $3k < \lambda_s(a) < 10\frac{1}{a^4}$. From the fact that $0 < \lambda_c(a) - \lambda_s(a) < 10\frac{1}{a^4}$ and also that $\lambda_{1,g_a}(\xi) \geq -k$ we have

$$\begin{aligned}
l_{\xi,a}\tau^r_{\xi,a} &\leq \frac{1}{\lambda_u(a)}\left[\frac{\lambda_s(a)+\lambda_{1,g_a}(\xi)+20\frac{1}{a^4}}{(\lambda_s(a)+\lambda_{1,g_a}(\xi))^r}\right] \\
&\leq \frac{1}{\lambda_u(a)}\left[\frac{1}{(\lambda_s(a)+\lambda_{1,g_a}(\xi))^{r-1}}+\frac{20}{a^4(\lambda_s(a)+\lambda_{1,g_a}(\xi))^r}\right] \\
&\leq \frac{1}{\lambda_u(a)}\left[\frac{1}{(\lambda_s(a)-k)^{r-1}}+\frac{20}{a^4(\lambda_s(a)-k)^r}\right] \\
&\leq \frac{100}{a^8}\left[\frac{2}{(\lambda_s(a))^{r-1}}+\frac{40}{a^4(\lambda_s(a))^r}\right] \\
&\leq \frac{100}{a^8}\left[8a^{4(r-1)}+40a^{4(r-1)}\right] \\
&\leq 10^4\frac{a^{4(r-1)}}{a^8}
\end{aligned}$$

From this and (19) and taking into account that $1 \leq r < 3$ we have for $a \in \mathbb{N}$ large enough that

$$l_{\xi,a}\tau^r_{\xi,a} < 1$$

for any $\xi \in \mathbb{T}^2$. This completes the proof of the proposition.

□

We can conclude the proof of our Main Theorem: let $r, 1 \leq r < 3$ and choose $B_a \in \mathcal{I}$ and $g_{B_a,k}$ from the above Proposition. From Corollary 4.0.2 we find $\mathcal{U}(g_{B_a,k})$ and we choose $g \in \mathcal{U}(g_{B_a,k})$ of class C^∞ and having a homoclinic intersection associated to the fixed point p of unstable index two. From Theorem 4.0.2, Corollary 4.0.2 and remark 4.1 the unstable foliation $\mathcal{F}^u_g$ is of class C^r and so, by remark 3.1 the induced map $f = f_g : \mathbb{T}^2 \to \mathbb{T}^2$ is of class C^r. Finally, Theorem 3.0.1 implies our Main Theorem.

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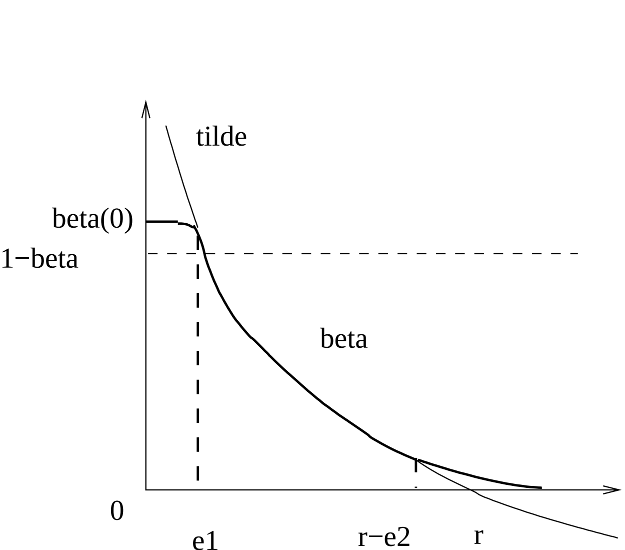
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beta(0)
1−beta
beta
0
e1
r−e2
r