

LOSS OF DERIVATIVES FOR SYSTEMS OF COMPLEX VECTOR FIELDS AND SUMS OF SQUARES

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ABSTRACT. We discuss, both for systems of complex vector fields and for sums of squares, the phenomenon discovered by Kohn of hypoellipticity with loss of derivatives.

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1. ESTIMATES FOR VECTOR FIELDS AND SUMS OF SQUARES IN \mathbb{R}^3

A system of real vector fields $\{X_j\}$ in $T\mathbb{R}^n$ is said to satisfy the bracket finite type condition if

$$(1.1) \quad \text{commutators of order } \leq h - 1 \text{ of the } X_j\text{'s span the whole } T\mathbb{R}^n.$$

Explicitly: $\text{Span}\{X_j, [X_{j_1}, X_{j_2}], \dots, [X_{j_1}, [X_{j_2}, \dots, [X_{j_{h-1}}, X_{j_h}]] \dots\} = T\mathbb{R}^n$. This system enjoys δ -subelliptic estimates for $\delta = \frac{1}{h}$ and therefore it is hypoelliptic according to Hörmander [6]. (See also [5] and [10] for elliptic regularization which yields regularity from estimates.) This remains true for systems of complex vector fields $\{L_j\}$ stable under conjugation (both in $\mathbb{C} \otimes T\mathbb{R}^n$ or $\mathbb{C} \otimes T\mathbb{C}^n$) once one applies Hörmander's result to $\{\text{Re } L_j, \text{Im } L_j\}$. Stability under conjugation can be artificially achieved by adding $\{\epsilon \bar{L}_j\}$ in order to apply Hörmander's theorem $\|u\|_\delta^2 \leq \sum_j (c_\epsilon \|L_j u\|^2 + \epsilon \|\bar{L}_j u\|^2) + c_\epsilon \|u\|^2$, $u \in C_c^\infty$. (Precision about ϵ and c_ϵ is not in the statement but transparent from the proof.) On the other hand, by integration by parts $\|\bar{L}_j u\|^2 \lesssim \|L_j u\|^2 + |([L_j, \bar{L}_j]u, u)| + \|u\|^2 \lesssim \|L_j u\|^2 + \|u\|_{\frac{1}{2}}^2 + \|u\|^2$. Thus if the type is $h = 2$, and hence $\delta = \frac{1}{2}$, the $\frac{1}{2}$ -norm is absorbed in the left: $\{\epsilon \bar{L}_j\}$ can be taken back and one has $\frac{1}{2}$ -subelliptic estimates for $\{L_j\}$. The restraint $h = 2$ is substantial and in fact Kohn discovered in [9] a pair of vector fields $\{L_1, L_2\}$ in \mathbb{R}^3 of finite type $k + 1$ (any fixed k) which are not subelliptic but, nonetheless, are hypoelliptic. Precisely, in the terminology of [9], they lose $\frac{k-1}{2}$ derivatives and the related sum of squares $\bar{L}_1 L_1 + \bar{L}_2 L_2$ loses $k - 1$ derivatives. The vector fields in question are $L_1 = \partial_{\bar{z}} + iz\partial_t$ and $L_2 = \bar{z}^k(\partial_z - i\bar{z}\partial_t)$ in $\mathbb{C} \times \mathbb{R}$. Writing $t = \text{Im } w$, they are identified to \bar{L} and $\bar{z}^k L$ for the CR vector field \bar{L} tangential to the strictly pseudoconvex hypersurface $\text{Re } w = |z|^2$ of \mathbb{C}^2 . Consider a more general hypersurface $M \subset \mathbb{C}^2$ defined by $\text{Re } w = g(z)$ for g real, and use the notations $g_1 = \partial_z g$, $g_{1\bar{1}} = \partial_z \partial_{\bar{z}} g$ and $g_{1\bar{1}\bar{1}} = \partial_z \partial_{\bar{z}} \partial_{\bar{z}} g$. Suppose that M is pseudoconvex, that is, $g_{1\bar{1}} \geq 0$ and denote by $2m$ the vanishing order of g at 0, that

is, $g = 0^{2m}$. Going further in the analysis of loss of derivatives, Bove, Derridj, Kohn and Tartakoff have considered the case where

$$(1.2) \quad g_1 = \bar{z}|z|^{2(m-1)}h(z) \text{ and } g_{1\bar{1}} = |z|^{2(m-1)}f(z) \text{ for } f > 0.$$

If $L = \partial_z - ig_1\partial_t$ is the $(1, 0)$ vector field tangential to $Re w = g$ for g satisfying (1.2), they have proved loss of $\frac{k-1}{m}$ derivatives for the operator $L\bar{L} + \bar{L}|z|^{2k}L$.

We consider here a general pseudoconvex hypersurface $M \subset \mathbb{C}^2$; ζ and ζ' will denote cut-off functions in a neighborhood of 0 such that $\zeta'|_{\text{supp}\zeta} \equiv 1$.

Theorem 1.1. *Let $\{L, \bar{L}\}$ (or better $\{\text{Re } L, \text{Im } L\}$) have type $2m$; then the system $\{\bar{L}, \bar{z}^k L\}$ loses $l := \frac{k-1}{2m}$ derivatives. More precisely*

$$(1.3) \quad \begin{aligned} \|\zeta u\|_s^2 &\lesssim \|\zeta' \bar{L} u\|_{s-\frac{1}{2m}}^2 + \|\zeta' \bar{z}^k \bar{L} u\|_{s+l}^2 \\ &+ \|\zeta' \bar{z}^k L u\|_{s+l}^2 + \|u\|_0^2. \end{aligned}$$

The estimate (1.3) says that the responsible of the loss l is $\bar{z}^k L$ (plus the extra vector field $\bar{z}^k \bar{L}$) and not \bar{L} . The proof of this here, as well as the two theorems below, follows in Section 4. What underlies the whole technicality is the basic notion of subelliptic multiplier; also the stability of multipliers under radicals is crucial (hidden in the interpolation Lemma 3.2 below). We point out that though the coefficient of the vector field \bar{L} gains much in generality ($+ig_{1\bar{1}}$ instead of $+iz$ or $+iz|z|^{2(m-1)}$ as in [9] and [1] respectively), instead, the perturbation \bar{z}^k of L remains the same. This is substantial; only an antiholomorphic perturbation is allowed. We introduce a new notation for the perturbed Kohn-Laplacian

$$(1.4) \quad \square^k = L\bar{L} + \bar{L}|z|^{2k}L \quad \text{for } L = \partial_z - ig_1\partial_t.$$

Theorem 1.2. *Let $\{L, \bar{L}\}$ have type $2m$ and assume moreover, that*

$$(1.5) \quad |g_1| \lesssim |z|g_{1\bar{1}} \quad \text{and} \quad |g_{1\bar{1}\bar{1}}| \lesssim |z|^{-1}g_{1\bar{1}}.$$

Then \square^k loses $l = \frac{k-1}{m}$ derivatives, that is

$$(1.6) \quad \|\zeta u\|_s^2 \lesssim \|\zeta' \square^k u\|_{s+2l}^2 + \|u\|_0^2.$$

Differently from vector fields, loss for sums of squares requires the additional assumption (1.5); whether finite type suffices is an open question.

Example 1.3. Consider the boundary defined by $Re w = g$ with $g(z) = 0^{2m}$ and assume

$$(1.7) \quad g_{1\bar{1}} \gtrsim |z|^{2(m-1)}.$$

This boundary is pseudoconvex, has bracket finite type $2m$ and (1.5) is satisfied. Thus Theorem 1.2 applies and we have (1.6). This is more general than [1] where it is assumed (1.2). Thus, for example, for the domain graphed by g with

$$g = |z|^{2(m-1)}x^2h(z) \quad \text{for } h > 0 \text{ and } h_{1\bar{1}} > 0,$$

we have (1.7) though the second of (1.2) is never true, not even for $h \equiv 1$. For general h , neither of (1.2) is fulfilled.

There is a result for sum of squares which stays close to Theorem 1.1 and in particular only assumes finite type without the additional hypothesis (1.5). This requires to modify the Kohn-Laplacian as

$$\tilde{\square}^k = \Lambda_{\partial_t}^{-2l} L \bar{L} + L |z|^{2k} \bar{L} + \bar{L} |z|^{2k} L,$$

where $\Lambda_{\partial_t}^{-2l}$ is the standard pseudodifferential operator of order $-2l$ in t .

Theorem 1.4. *Let $\{L, \bar{L}\}$ have type $2m$; then*

$$(1.8) \quad \|\zeta u\|_s^2 \lesssim \|\zeta' \tilde{\square}^k u\|_{s+2l}^2 + \|u\|_0^2.$$

Some references to current literature are in order. Hypoellipticity in presence of infinite degeneracy has been intensively discussed in recent years. The ultimate level to which the problem is ruled by estimates, are superlogarithmic estimates (Kusuoka and Strooke [11], Morimoto [12] and Kohn [8]). Related work is also by Bell and Mohammed [2] and Christ [3]. Beyond the level of estimates are the results by Kohn [7] which develop, in a geometric framework, an early result by Fedi [4]: the point here is that the degeneracy is confined to a real curve transversal to the system. This explains also why if the set of degeneracy is big, superlogarithmicity becomes in certain cases necessary ([12] and [3]). In all these results, however, there is somewhat a gain of derivatives (such as sublogarithmic). The simplest example of hypoellipticity without gain (nor loss) is $\square_b + \lambda \text{id}$, $\lambda > 0$ where \square_b is the Kohn-Laplacian of $\text{Re } w = |z|^2$ (cf. Stein [14] where the bigger issue of the analytic-hypoellipticity is also addressed). As for loss of derivatives, the phenomenon has been discovered by Kohn in [9] and further developed by Bove, Derridj, Kohn and Tartakoff in [1]. Additional contribution is, among others, by Parenti and Parmeggiani [13] and Tartakoff [15].

2. SUMS OF SQUARES IN \mathbb{R}^{2n+1} FOR $n > 1$

We restate in higher dimension the results of Section 1; we can better appreciate the different role which is played by the finite type with respect to (1.5). The content of this section is a direct consequence of the results of Section 1 (plus ellipticity and maximal hypoellipticity related to microlocalization) and therefore it does not need a specific proof. In $\mathbb{C}^n \times \mathbb{R}_t$ we start, as in Section 1, from $L_1 = \partial_{z_1} - ig_1(z_1)\partial_t$ and complete L_1 to a system of smooth complex vector fields in a neighborhood of 0

$$L_j = \partial_{z_j} - ig_j(z)\partial_t, \quad j = 1, \dots, n \quad \text{for } g_j|_0 = 0.$$

For a system of vector fields, we denote by $\mathcal{L}ie_{2m}$ the span of commutators of order $\leq 2m - 1$ belonging to the system. We have $\|u^0\|_1^2 \lesssim \sum_{j=1}^n \|\bar{L}_j u^0\|_0^2 + \|u\|_0^2$ and, if for some

index j , say $j = 1$, $\partial_t \in \mathcal{L}ie_{2m_1}\{L_1, \bar{L}_1\}$, then $\|u^-\|_{\frac{1}{2m_1}}^2 \lesssim \sum_{j=1}^n \|\bar{L}_j u\|_0^2 + \|u\|_0^2$ (cf. the end of Section 3). Summarizing up, if we only have (1.3) for u^+ , we get, for the full u and with l replaced by $l_1 = \frac{k_1}{2m_1}$:

$$(2.1) \quad \|\zeta u\|_s^2 \lesssim \left(\|\zeta' \bar{L}_1 u\|_{s-\frac{1}{2m_1}}^2 + \|\zeta' z_1^{k_1} \bar{L}_1 u\|_{s+l_1}^2 + \|\zeta' z_1^{k_1} L_1 u\|_{s+l_1}^2 \right) + \sum_{j=2}^n \|\bar{L}_j u\|_{s-\frac{1}{2m_1}}^2 + \|u\|_0^2.$$

We assume that each coefficient satisfy $g_j = \partial_{z_j} g$ for a real function $g = g(z)$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and denote by \mathbb{L} the bundle spanned by the L_j 's. We note that this defines a CR structure because, on account of $g_{i\bar{j}} = g_{j\bar{i}}$,

$$\mathbb{L} \text{ is involutive.}$$

Also, this structure is of hypersurface type in the sense that

$$T(\mathbb{C}_z^n \times \mathbb{R}_t) = \mathbb{L} \oplus \bar{\mathbb{L}} \oplus \mathbb{R}\partial_t.$$

Note that, in fact, the L_j 's commute; therefore, the Levi form is defined directly by $[L_i, \bar{L}_j] = g_{ij}\partial_t$, without passing to the quotient modulo $\mathbb{L} \oplus \bar{\mathbb{L}}$. We also assume that the Levi form $(g_{i\bar{j}})$ is positive semidefinite; in particular $g_{j\bar{j}} \geq 0$ for any j . (Geometrically, this means that the hypersurface $\text{Im } w = g$ graphed by g , is pseudoconvex.) We choose $\kappa = (k_1, \dots, k_n)$ and define the perturbed Kohn-Laplacian

$$\square^\kappa = \sum_{j=1}^n L_j \bar{L}_j + \bar{L}_j |z_j|^{2k_j} L_j.$$

Theorem 2.1. *Assume that for any j , $\partial_t \in \mathcal{L}ie_{2m_j}\{L_j, \bar{L}_j\}$, and that*

$$(2.2) \quad |g_j| \lesssim |z_j| g_{j\bar{j}} \text{ and } |g_{j\bar{j}}| \lesssim |z_j|^{-1} g_{j\bar{j}} \text{ for any } j = 1, \dots, n.$$

Define $l_j := \frac{k_j-1}{2m_j}$ and put $l = \max_j \frac{k_j-1}{2m_j}$; then

$$(2.3) \quad \|\zeta u\|_s^2 \lesssim \|\zeta' \square^\kappa u\|_{s+2l}^2 + \|u\|_0^2.$$

The proof of Theorem 2.1 and Theorem 2.2 below, are just a variation of those of the twin Theorems 1.2 and 1.4. We define now

$$\tilde{\square}^\kappa = \sum_{j=1}^n \left(\Lambda_{\partial_t}^{-2l_j} L_j \bar{L}_j + \sum_{j=1}^n L_j |z_j|^{2k_j} \bar{L}_j + \bar{L}_j |z_j|^{2k_j} L_j \right).$$

Theorem 2.2. *Assume that for any j , $\partial_t \in \mathcal{L}ie_{2m_j}\{L_j, \bar{L}_j\}$; then*

$$(2.4) \quad \|\zeta u\|_s^2 \lesssim \|\zeta' \tilde{\square}^\kappa u\|_{s+2l}^2 + \|u\|_0^2.$$

3. PRELIMINARIES

We identify $\mathbb{C} \times \mathbb{R}$ to \mathbb{R}^3 with coordinates (z, \bar{z}, t) or $(\operatorname{Re} z, \operatorname{Im} z, t)$. We denote by $\xi = (\xi_z, \xi_{\bar{z}}, \xi_t)$ the variables dual to (z, \bar{z}, t) , by Λ_ξ^s the standard symbol $(1 + |\xi|^2)^{\frac{s}{2}}$, and by Λ_∂^s the pseudodifferential operator with symbol Λ_ξ^s ; this is defined by $\Lambda_\partial^s(u) = \mathcal{F}^{-1}(\Lambda_\xi^s \mathcal{F}(u))$ where \mathcal{F} is the Fourier transform. We consider the full (resp. totally real) s -Sobolev norm $\|u\|_s := \|\Lambda_\partial^s u\|_0$ (resp. $\|u\|_{\mathbb{R}, s} := \|\Lambda_{\partial_t}^s u\|_0$). In \mathbb{R}_ξ^3 , we consider a conical partition of the unity $1 = \psi^+ + \psi^- + \psi^0$ where ψ^\pm have support in a neighborhood of the axes $\pm \xi_t$ and ψ^0 in a neighborhood of the plane $\xi_t = 0$, and introduce a decomposition of the identity $\operatorname{id} = \Psi^+ + \Psi^- + \Psi^0$ by means of Ψ^\pm , the pseudodifferential operators with symbols ψ^\pm ; we accordingly write $u = u^+ + u^- + u^0$. Since $|\xi_z| + |\xi_{\bar{z}}| \lesssim \xi_t$ over $\operatorname{supp} \psi^+$, then $\|u^+\|_{\mathbb{R}, s} = \|u^+\|_s$.

We carry on the discussion by describing the properties of commutation of the vector fields L and \bar{L} for $L = \partial_z - ig_1(z)\partial_t$. The crucial equality is

$$(3.1) \quad \|Lu\|^2 = ([L, \bar{L}]u, u) + \|\bar{L}u\|^2, \quad u \in C_c^\infty,$$

which is readily verified by integration by parts. Note here that errors coming from derivatives of coefficients do not occur since g_1 does not depend on t . Recall that $[L, \bar{L}] = g_{1\bar{1}}\partial_t$; this implies

$$(3.2) \quad |(g_{1\bar{1}}\partial_t u, u)| \lesssim s.c. \|\partial_t u\|^2 + l.c. \|u\|^2.$$

We have

$$(3.3) \quad \begin{aligned} \|u^0\|_1^2 &\lesssim \|\bar{L}u^0\|^2 + \|Lu^0\|^2 + \|u\|^2 \\ &\leq 2\|\bar{L}u^0\|^2 + sc\|\partial_t u^0\|^2 + lc\|u\|^2. \end{aligned}$$

To check (3.3), we point our attention to the estimate for operator's symbols $(1 + |\xi|^2)|\alpha|^2 \lesssim |\alpha|^2 + |\sigma(\bar{L})\alpha|^2 + |\sigma(L)\alpha|^2$ (α complex) over $U \times \operatorname{supp} \psi^0$ for a neighborhood U of 0; in addition to the fact that $[L, \Psi^0]$ is of order 0, this yields the first inequality of (3.3). The second follows from (3.1) combined with (3.2). As for u^- , since $g_{11}\sigma(\partial_t) < 0$ over $\operatorname{supp} \psi^-$, then

$$(g_{11}\partial_t u^-, u^-) = - |(g_{11}\Lambda_{\partial_t} u^-, u^-)|.$$

Thus (3.1) implies $\|Lu^-\| \leq \|\bar{L}u^-\|$ (the second inequality in (3.4) below). Suppose now that $\{L, \bar{L}\}$ have type $2m$; this yields the first inequality below which, combined with the former, yields

$$(3.4) \quad \begin{aligned} \|u^-\|_{\frac{1}{2m}}^2 &\lesssim \|Lu^-\|_0^2 + \|\bar{L}u^-\|_0^2 + \|u\|_0^2 \\ &\lesssim \|\bar{L}u^-\|_0^2 + \|u\|_0^2. \end{aligned}$$

In conclusion, only estimating u^+ is relevant. For this purpose, we have a useful statement

Lemma 3.1. *Let $|[L, \bar{L}]|^{\frac{1}{2}}$ be the operator with symbol $|g_{11}|^{\frac{1}{2}}\Lambda_{\xi_t}^{\frac{1}{2}}$; then*

$$(3.5) \quad \| |[L, \bar{L}]|^{\frac{1}{2}} u^+ \|^2 \leq \|Lu^+\|^2 + \|\bar{L}u^+\|^2.$$

Proof. From (3.1) we get

$$|([L, \bar{L}]u, u)| \leq \|Lu\|^2 + \|\bar{L}u\|^2.$$

The conclusion then follows from

$$[L, \bar{L}] = |[L, \bar{L}]| \quad \text{over } \text{supp } \psi^+.$$

□

We pass to a result about interpolation which plays a central role in our discussion.

Lemma 3.2. *Let $f = f(z)$ be smooth and satisfy $f(0) = 0$. Then for any ρ, r, n_1 and n_2 with $0 < n_1 \leq r, n_2 > 0$*

$$(3.6) \quad \|f^r u\|_0^2 \lesssim sc \|f^{r-n_1} u\|_{\mathbb{R}, -n_1\rho}^2 + lc \|f^{r+n_2} u\|_{\mathbb{R}, n_2\rho}^2.$$

Proof. Set $A := \Lambda_{\partial_t}^\rho f$; interpolation for the pseudodifferential operator A yields

$$\begin{aligned} \|f^r u\|_0^2 &= \|(\Lambda^\rho f)^r u\|_{\mathbb{R}, -\rho r}^2 \\ &= (\Lambda^{\rho(r-n_1)} f^{r-n_1}, \Lambda^{\rho(r+n_1)} f^{r+n_1})_{-\rho r} \\ &= (\Lambda^{-\rho n_1} f^{r-n_1}, \Lambda^{\rho n_1} f^{r+n_1})_0 \lesssim sc \|f^{r-n_1}\|_{\mathbb{R}, -n_1\rho}^2 + lc \|f^{r+n_1} u\|_{\mathbb{R}, n_1\rho}^2. \end{aligned}$$

This proves the lemma for $n_2 = n_1$; the general conclusion is obtained by iteration. □

We have now a result about factors in a scalar product.

Lemma 3.3. *Let $h = h(z)$ satisfy $|h| \leq |h_1||h_2|$ and take $f = f(z, t)$ and $g = g(z, t)$. Then*

$$(3.7) \quad |(f, hg)|_{\mathbb{R}, s} \lesssim \|fh_1\|_{\mathbb{R}, s}^2 + \|gh_2\|_{\mathbb{R}, s}^2.$$

Proof. We use the notation \mathcal{F}_t for the partial Fourier transform with respect to t and $d\lambda$ for the element of volume in $\mathbb{C}_z \simeq \mathbb{R}_{\text{Re } z, \text{Im } z}^2$. The lemma follows from the following sequence of inequalities in which the crucial fact is that h, h_1 and h_2 are constant in the

integration in ξ_t :

$$\begin{aligned}
|(f, hg)_{\mathbb{R}, s}| &= \left| \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}_{\xi_t}^1} \Lambda_{\xi_t}^{2s} \mathcal{F}_t(f) h \mathcal{F}_t(g) d\xi_t \right) d\lambda \right| \\
&\leq \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}_{\xi_t}^1} \Lambda_{\xi_t}^{2s} |\mathcal{F}_t(f) h_1 h_2 \mathcal{F}_t(g)| d\xi_t \right) d\lambda \\
&\leq \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}_{\xi_t}^1} \Lambda_{\xi_t}^{2s} |\mathcal{F}_t(f) h_1|^2 d\xi_t \right) d\lambda + \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}_{\xi_t}^1} \Lambda_{\xi_t}^{2s} |\mathcal{F}_t(g) h_2|^2 d\xi_t \right) d\lambda \\
&\stackrel{\text{Plancherel}}{=} \|fh_1\|_{\mathbb{R}, s}^2 + \|gh_2\|_{\mathbb{R}, s}^2.
\end{aligned}$$

□

We say a few words for the case of higher dimension. In $\mathbb{C}_{z_1, \dots, z_n}^n \times \mathbb{R}_t$, we consider a full system $L_j = \partial_{z_j} - ig_j \partial_t$, $j = 1, \dots, n$ with $g_j|_0 = 0$. The same argument used in proving (3.3) yields

$$(3.8) \quad \|u^0\|_1^2 \lesssim \sum_{j=1}^n \|\bar{L}_j u^0\|^2 + \|u\|^2.$$

Similarly as above, we have $\|L_j u^-\|^2 \leq \|\bar{L}_j u^-\|^2 + \|u\|^2$ for any j . Then, if at least one index j , say $j = 1$, the pair $\{L_1, \bar{L}_1\}$ has type $m = m_1$, we get, in the same way as in (3.4)

$$\|u^-\|_{\frac{1}{2m}}^2 \lesssim \sum_{j=1}^n \|\bar{L}_j u^-\|^2 + \|u\|^2.$$

Again, only estimating u^+ is therefore relevant.

4. PROOF OF THEOREM 1.1, THEOREM 1.2 AND THEOREM 1.4

Terminology. In an estimate we call “good” a term in the right side (upper bound). We call “absorbable” a term that we encounter in the course of the estimate and which comes as a fraction (small constant or sc) of a former term. If cut-off are involved in the estimate, and in the right side the cut-off can be expanded, say passing from ζ to ζ' , we call “neglectable” a term which comes with lower Sobolev index and possibly with a bigger cut-off. Neglectable is meant with respect to the initial (left-hand side) term of the estimate, to further terms that one encounters and even to extra terms provided that they can be estimated by “good”. These latter are sometimes artificially added to expand the range of “neglectability”.

Proof of Theorem 1.1. According to (3.3) and (3.4), it suffices to prove (1.3) for $u = u^+$; so, throughout the proof we write u but mean u^+ . Also, we use the equivalence, over u^+ ,

between the totally real $\|\cdot\|_{\mathbb{R},s}$ – with the full $\|\cdot\|_s$ -Sobolev norm; the specification of the norm will be omitted. Moreover, we can use a cut-off $\zeta = \zeta(t)$ in t only. In fact, for a cut-off $\zeta = \zeta(z)$ we have $[L, \zeta(z)] = \dot{\zeta}$ and $\dot{\zeta} \equiv 0$ at $z = 0$. On the other hand, $z^k L \sim L$ outside $z = 0$ which yields (4.1) below (so that we have gain, instead of loss). Recall in fact that we are assuming that M has type $2m$. It is classical that the tangential vector fields L and \bar{L} satisfy $\frac{1}{2m}$ -subelliptic estimates, that is, the first inequality in the estimate below. In combination with (3.1) which implies the second inequality below, we get

$$(4.1) \quad \begin{aligned} \|\zeta u\|_s^2 &\lesssim \|\zeta \bar{L}u\|_{s-\frac{1}{2m}}^2 + \|\zeta Lu\|_{s-\frac{1}{2m}}^2 + \|\zeta' u\|_{s-\frac{1}{2m}}^2 \\ &\lesssim \|\zeta \bar{L}u\|_{s-\frac{1}{2m}}^2 + \|\zeta |[L, \bar{L}]\|_{\frac{1}{2}} u\|_{s-\frac{1}{2m}}^2 + \|\zeta' u\|_{s-\frac{1}{2m}}^2. \end{aligned}$$

Remark that $\|\zeta' u\|_{s-\frac{1}{m}}^2$ (for a new ζ') takes care of the error $\|\zeta' \bar{L}u\|_{s-\frac{1}{2m}-1}^2$ coming from $[\Lambda^{2s-\frac{1}{m}}, \zeta']$. Now, remember that $[L, \bar{L}] = g_{1\bar{1}} \partial_t$ without error terms, that is, combinations of L and \bar{L} ; recall also that $g_{1\bar{1}} \geq 0$. We get

$$(4.2) \quad \begin{aligned} \|\zeta |[L, \bar{L}]\|_{\frac{1}{2}} u\|_{s-\frac{1}{2m}}^2 &\sim \|\zeta g_{1\bar{1}}^{\frac{1}{2}} \Lambda_{\partial_t}^{\frac{1}{2}} u\|_{s-\frac{1}{2m}}^2 \\ &\lesssim sc \|\zeta u\|_s^2 + lc \|\zeta g_{1\bar{1}}^{\frac{1}{2} + \frac{k}{2(m-1)}} \Lambda_{\partial_t}^{\frac{1}{2}} u\|_{s+l}^2 \\ &\lesssim \text{absorbable} + \|\zeta g_{1\bar{1}}^{\frac{1}{2}} z^k \Lambda_{\partial_t}^{\frac{1}{2}} u\|_{s+l}^2 \\ &= \text{absorbable} + \|\zeta |[L, \bar{L}]\|_{\frac{1}{2}} z^k u\|_{s+l}^2 \\ &\leq \text{absorbable} + \|\zeta L(z^k u)\|_{s+l}^2 + \|\zeta \bar{L}(z^k u)\|_{s+l}^2 + \|\zeta' z^k u\|_{s+l}^2, \end{aligned}$$

where the first “ \sim ” is a way of rewriting the commutator, the second “ \lesssim ” follows from Lemma 3.2 (under the choice $n_1 = m - 1$, $n_2 = k$, $r = m - 1$, $\rho = \frac{1}{2m}$ and $f = g_{1\bar{1}}^{\frac{1}{2(m-1)}}$), the third “ \lesssim ” follows from $|g_{1\bar{1}}| \lesssim |z|^{2(m-1)}$, the fourth “ $=$ ” is obvious and the last “ \lesssim ” follows from Lemma 3.1. We go now to estimate, in the last line of (4.2), the two terms $\|\zeta \bar{L}(z^k u)\|_{s+l}^2$ and $\|\zeta' z^k u\|_{s+l}^2$. We start from

$$(4.3) \quad \|\zeta L(z^k u)\|_{s+l}^2 \leq \|\zeta z^k Lu\|_{s+l}^2 + \|\zeta z^{k-1} u\|_{s+l}^2,$$

where the last term is produced by the commutator $[L, z^k]$. By writing, in the scalar product, once z^{k-1} and once $[L, z^k]$, we get

$$(4.4) \quad \begin{aligned} \|\zeta z^{k-1} u\|_{s+l}^2 &= (\zeta z^{k-1} u, \zeta [L, z^k] u)_{s+l} \\ &= (\zeta z^{k-1} u, \zeta z^k Lu)_{s+l} + (\zeta z^{k-1} u, \zeta L z^k u)_{s+l}. \end{aligned}$$

Now,

$$(4.5) \quad \left\{ \begin{array}{l} (\zeta z^{k-1}u, \zeta z^k Lu)_{s+l} \leq \underbrace{sc\|\zeta z^{k-1}u\|_{s+l}^2}_{\text{absorbable}} + \underbrace{\|\zeta z^k Lu\|_{s+l}^2}_{\text{good}} \\ (\zeta z^{k-1}u, \zeta Lz^k u)_{s+l} = (\zeta z^{k-1}\bar{L}u, \zeta z^k u)_{s+l} + (\zeta z^{k-1}, \zeta' z^k u)_{s+l} \\ \leq \underbrace{\|\zeta z^k \bar{L}u\|_{s+l}^2}_{\text{good}} + \underbrace{sc\|\zeta z^{k-1}u\|_{s+l}^2}_{\text{absorbable}} + \|\zeta' z^k u\|_{s+l}^2. \end{array} \right.$$

Thus $\|\zeta z^{k-1}u\|_{s+l}^2$ has been estimated by $\|\zeta' z^k u\|_{s+l}^2$. What we have obtained so far is

$$(4.6) \quad \|\zeta u\|_s^2 \leq \|\zeta \bar{L}u\|_s^2 + \|\zeta z^k \bar{L}u\|_{s+l}^2 + \|\zeta z^k Lu\|_{s+l}^2 + \|\zeta' z^k u\|_{s+l}^2 + \|\zeta' u\|_{s-\frac{1}{2m}}^2.$$

Note that in this estimate, the terms coming with L and \bar{L} carry the same cut-off ζ as the left side; it is in this form that Theorem 1.1 will be applied for the proof of Theorems 1.2 and 1.4. Instead, to conclude the proof of Theorem 1.1, we have to go further with the estimation of $\|\zeta' z^k u\|_{s+l}^2$ (which also provides the estimate of the last term in (4.2)). We have, by subelliptic estimates

$$(4.7) \quad \|\zeta' z^k u\|_{s+l}^2 \leq \|\zeta' Lz^k u\|_{s+l-\frac{1}{2m}}^2 + \|\zeta' \bar{L}z^k u\|_{s+l-\frac{1}{2m}}^2 + \|\zeta'' z^k u\|_{s+l-\frac{1}{2m}}^2.$$

To $\|\zeta' Lz^k u\|_{s+l-\frac{1}{2m}}^2$ we apply (4.3) with $s+l$ replaced by $s+l-\frac{1}{2m}$. In turn, $\|\zeta' z^{k-1}u\|_{s+l-\frac{1}{2m}}^2$ can be estimated, by (4.4), (4.5) and (4.7) with Sobolev indices all lowered from $s+l$ to $s+l-\frac{1}{2m}$, by means of “good” + “absorbable” + $\|\zeta'' z^k u\|_{s+l-\frac{1}{2m}}^2$. (In fact, “good” even comes with lower index.) The conclusion (1.3) follows from induction over j such that $\frac{j}{2m} \geq s+l$. This completes the proof of Theorem 1.1. \square

Proof of Theorem 1.4. We first prove Theorem 1.4 instead of Theorem 1.2 because it is by far easier. As it has already been remarked in Section 1, it suffices to prove the theorem for $u = u^+$. Also, in this case, the full norm can be replaced by the totally real norm. So we write u for u^+ and $\|\cdot\|_s$ for $\|\cdot\|_{\mathbb{R}, s}$; however, in some crucial passage where Lemma 3.3 is on use, it is necessary to point attention to the kind of the norm. We start from (4.6); note that, for this estimate to hold, only finite type is required. We begin by noticing that the last term of (4.6) is neglectable. We then rewrite the third term in the right of (4.6) as

$$(4.8) \quad (\zeta z^k Lu, \zeta z^k Lu)_{s+l} = (\zeta \bar{L}|z|^{2k} Lu, \zeta u)_{s+l} + (\zeta z^k Lu, \zeta' z^k u)_{s+l},$$

where we recall that we are using the notation $l = \frac{k-1}{2m}$. (Note that the commutator $[L, \zeta]$ is not just ζ' but comes with an additional factor g_1 , the coefficient of L ; but we disregard this contribution here though it will play a crucial role in the proof of Theorem 1.2.) We keep the first term in the right of (4.8) as it stands and put together with the similar term coming from the first term in the right of (4.6) to form $\tilde{\square}^\kappa$. We then

apply Cauchy-Schwartz inequality and estimate the first term by $\|\zeta\tilde{\square}^k u\|_{s+2l}^2 + sc\|\zeta u\|_s^2$. As for the second term in the right of (4.8), it can be estimated, via Cauchy-Schwartz, by $sc\|\zeta z^k Lu\|_{s+l}^2 + lc\|\zeta' z^k u\|_{s+l}$. To this latter, we apply subelliptic estimates

$$(4.9) \quad \|\zeta' z^k u\|_{s+l}^2 \lesssim \|\zeta' z^k \bar{L}u\|_{s+l-\frac{1}{m}}^2 + \|\zeta' z^k Lu\|_{s+l-\frac{1}{2m}}^2 + \|\zeta' z^{k-1} u\|_{s+l-\frac{1}{2m}}^2 + \|\zeta'' z^k u\|_{s+l-\frac{1}{2m}}^2.$$

For the third term in the right, recalling (4.4) and (4.5), we get

$$(4.10) \quad \|\zeta' z^{k-1} u\|_{s+l-\frac{1}{2m}}^2 \lesssim \text{neglectable} + \|\zeta'' z^k u\|_{s+l-\frac{1}{2m}}^2.$$

Thus $\|\zeta' z^k u\|_{s+l}^2$ is controlled by induction over j with $\frac{j}{2m} \geq s+l$. (Recall, once more, that “good” is stable under passing from ζ' to ζ'' .) We notice that combination of (4.9) and (4.10) shows that $\|\zeta' z^k u\|_{s+l}^2$ is neglectable. We pass to $\|\zeta' z^k \bar{L}u\|_{s+l}^2$, the second term in the right of (4.6) and observe that it can be treated exactly in the same way as the third (with L instead of \bar{L}). We end with the first which does not carry the loss l ; we have

$$(4.11) \quad \begin{aligned} \|\zeta \bar{L}u\|_s^2 &= (\zeta L \bar{L}u, \zeta u)_s + (\zeta \bar{L}u, \zeta' g_1 u)_s \\ &= (\Lambda^{2l} \Lambda^{-2l} L \bar{L}u, \zeta u)_s + (\zeta \bar{L}u, \zeta' g_1 u)_s. \end{aligned}$$

The first term in the right combines to form $\tilde{\square}^k$. As for the second, we notice that $|g_1| \lesssim |z|$ and therefore applying Lemma 3.2 for $n_1 = k-1$ and $n_2 = 1$

$$(\zeta \bar{L}u, \zeta' g_1 u)_s \leq sc\|\zeta \bar{L}u\|_s^2 + lc(\|\zeta' z^k u\|_{s+l}^2 + \|\zeta' u\|_{s-\frac{1}{2m}}^2).$$

The first term in the right is absorbable, the last neglectable, the middle has already been proved to be neglectable by subelliptic estimates (4.9). This completes the proof. \square

Proof of Theorem 1.2. As before, we prove the theorem for $u = u^+$ and write $\|\cdot\|_s$ for $\|\cdot\|_{\mathbb{R},s}$ though, in some crucial passage, it is necessary to point the attention to the kind of the norm. Raising Sobolev indices, we rewrite (4.6) in a more symmetric fashion as

$$(4.12) \quad \|\zeta u\|_s^2 \lesssim \|\zeta \bar{L}u\|_{s+l}^2 + \|\zeta z^k Lu\|_{s+l}^2 + \|\zeta' u\|_{s-\frac{1}{2m}}^2 + \|\zeta' z^k u\|_{s+l}^2.$$

We handle all terms in the right as in Theorem 1.2 except from the first which comes now with the loss $s+l$. We point out that to control these terms, only finite type has been used. Instead, to control the remaining term, we need the additional hypothesis (1.5). We have

$$(4.13) \quad \|\zeta \bar{L}u\|_{s+l}^2 = (\zeta L \bar{L}u, \zeta u)_{s+l} + (\zeta \bar{L}u, \zeta' g_1 u)_{s+l}.$$

The first term combines to form \square^k . As for the second, we recall the estimate $|g_1| \lesssim |z|g_{1\bar{1}}$ and apply Lemma 3.3 for $h = zg_{1\bar{1}}$, $h_1 = g_{1\bar{1}}^{\frac{1}{2}}$ and $h_2 = zg_{1\bar{1}}^{\frac{1}{2}}$ to get

$$(4.14) \quad |(\zeta \bar{L}u, \zeta' g_1 u)|_{s+l} \leq sc\|\zeta g_{1\bar{1}}^{\frac{1}{2}} \bar{L}u\|_{s+2l}^2 + lc\|\zeta' z g_{1\bar{1}}^{\frac{1}{2}} u\|_s^2.$$

In the estimate above, we point our attention to the fact that the norms that we are considering are totally real norms (though we do not keep track in our notation) and therefore Lemma 3.3 can be applied. We start by estimating the second term in the right. By Lemma 3.1 and next, Lemma 3.2 for $n_1 = 1$, $n_2 = k - 1$

$$(4.15) \quad \begin{aligned} \|\zeta' g_{1\bar{1}}^{\frac{1}{2}} z u\|_s^2 &\lesssim \|\zeta' z L u\|_{s-\frac{1}{2}}^2 + \|\zeta' z \bar{L} u\|_{s-\frac{1}{2}}^2 + \text{neglectable} \\ &\leq \|z^k \zeta' L u\|_{s-\frac{1}{2}+l}^2 + \|\zeta' L u\|_{s-\frac{1}{2}-\frac{1}{2m}}^2 + \|\zeta' z \bar{L} u\|_{s-\frac{1}{2}}^2 + \text{neglectable}, \end{aligned}$$

where neglectable comes from the commutators $[L, z]$ and $[L, \zeta']$. Also, the first term in the second line of (4.15) is neglectable. As for the second term, we have, by (3.1)

$$(4.16) \quad \|\zeta' L u\|_{s-\frac{1}{2}-\frac{1}{2m}}^2 \lesssim \|\zeta' g_{1\bar{1}}^{\frac{1}{2}} u\|_{s-\frac{1}{2m}}^2 + \|\zeta' \bar{L} u\|_{s-\frac{1}{2}-\frac{1}{2m}}^2 + \text{neglectable}.$$

Since both terms in the right of (4.16) are neglectable, we conclude that $\|\zeta' z g_{1\bar{1}}^{\frac{1}{2}} u\|_s^2$ itself is neglectable. From now on, we follow closely the track of [1]. We pass to consider the last and most difficult term to estimate, that is, the first in the right of (4.14). Along with this term, that we denote by (a), we introduce three additional terms; we set therefore

$$\begin{cases} (a) := \|\zeta g_{1\bar{1}} \bar{L} u\|_{s+2l}^2, & (b) := \|\zeta z^{2k-1} g_{1\bar{1}}^{\frac{1}{2}} u\|_{s+2l}, \\ (c) := \|\zeta z^{2k-1} L u\|_{s+2l-\frac{1}{2}}^2, & (d) := \|L \zeta \bar{L} u\|_{s+2l-\frac{1}{2}}^2. \end{cases}$$

Because of these additional terms, that we are able to estimate, “neglectable” and “absorbable” take an extended range. We first show that (b) is controlled by (c). This is apparently as in [1] first half of 5.3 but more complicated because our (b) and (c) are different from their $(LHS)_5$ and $(LHS)_6$ respectively. Now, by Lemma 3.1 we get

$$(b) \lesssim (c) + \|\zeta' z^{2k-1} g_{1\bar{1}} u\|_{s+2l-\frac{1}{2}}^2 + \|\zeta z^{2k-2} u\|_{s+2l-\frac{1}{2}}^2 + \text{neglectable},$$

where the central terms in the right come from $[L, \zeta]$ and $[L, z^{2k-1}]$ respectively, and where neglectable, with respect to (a), is the term which involves $\bar{L} u$ and which comes lowered by $-\frac{1}{2}$. The first of the central terms is neglectable with respect to (b). As for the second, we have, using the notation $\# = s + 2l - \frac{1}{2} - \frac{1}{2m}$

$$\begin{aligned} \underbrace{\|\zeta z^{2k-2} u\|_{s+2l-\frac{1}{2}}^2}_{(i)} &\lesssim \underbrace{\|\zeta z^{2k-2} L u\|_{\#}^2}_{(ii)} + \underbrace{\|\zeta z^{2k-2} \bar{L} u\|_{\#}^2}_{(iii)} \\ &\quad + \underbrace{\|\zeta' z^{2k-2} g_{1\bar{1}} u\|_{\#}^2}_{(iv)} + \underbrace{\|\zeta z^{2k-3} u\|_{\#}^2}_{(v)}, \end{aligned}$$

where the two terms of the second line come from $[L, \zeta]$ and $[L, z^{2k-2}]$ respectively. First, (iv) is neglectable with respect to (i). Next, using Lemma 3.2 for $n_1 = 2k - 2$ and $n_2 = 1$

$$(ii) \lesssim \underbrace{sc \|\zeta Lu\|_{\# - \frac{2k-2}{2m}}^2}_{(ii)_1} + \underbrace{lc \|\zeta z^{2k-1} Lu\|_{\# + \frac{1}{2m}}^2}_{(ii)_2}.$$

Note that $\# - \frac{2k-2}{2m} = s - \frac{1}{2} - \frac{1}{2m}$ and $\# + \frac{1}{2m} = s + 2l - \frac{1}{2}$; thus $(ii)_1$ is absorbed by (4.16) and $(ii)_2$ is estimated by (c). Next, by Lemma 3.2 for $n_1 = 2k - 3$ and $n_2 = 1$

$$(v) \lesssim \underbrace{lc \|\zeta u\|_{\# - \frac{2k-3}{2m}}^2}_{(v)_1} + \underbrace{sc \|\zeta z^{2k-2} u\|_{\# + \frac{1}{2m}}^2}_{(v)_2}.$$

We have $\# - \frac{2k-3}{2m} = s - \frac{1}{2}$ and, again, $\# + \frac{1}{2m} = s + 2l - \frac{1}{2}$; thus $(v)_1$ is neglectable with respect to $\|\zeta u\|_s^2$, the term in the left of the estimate, and $(v)_2$ is absorbed by (i). Finally, by (3.1)

$$(iii) \leq \underbrace{\|\zeta z^{2k-2} Lu\|_{\#}}_{(iii)_1} + \underbrace{\|\zeta z^{2k-2} g_{1\bar{1}}^{\frac{1}{2}} u\|_{\# + \frac{1}{2}}^2}_{(iii)_2}.$$

Now, applying Lemma 3.2 for $n_1 = k - 2$, $n_2 = 1$ in the first line below and $n_1 = 2k - 2$ and $n_2 = 1$ in the second respectively, we get

$$\left\{ \begin{array}{l} (iii)_1 \lesssim \underbrace{\|\zeta z^{2k-1} Lu\|_{s+2l-\frac{1}{2}}}_{(c)} + \underbrace{\|\zeta z^k Lu\|_{s+l-\frac{1}{2}}}_{\text{neglectable w.r.to } \|\zeta z^k Lu\|_{s+l}^2} \\ (iii)_2 \lesssim \underbrace{sc \|\zeta z^{2k-1} g_{1\bar{1}}^{\frac{1}{2}} u\|_{s+2l}^2}_{(b)} + lc \underbrace{\|\zeta g_{1\bar{1}}^{\frac{1}{2}} u\|_{s-\frac{1}{2m}}^2}_{\text{neglectable w.r.to } \|\zeta u\|_s} \end{array} \right. .$$

Summarizing up,

$$(b) \lesssim (c) + \text{neglectable}.$$

We have to show now that

$$\left\{ \begin{array}{l} (c) \lesssim \|\square^k u\|_{s+2l-\frac{1}{2}}^2 + \text{absorbable} + \text{neglectable}, \\ (a) + (d) \lesssim \|\square^k u\|_{s+2l-\frac{1}{2}}^2 + \text{absorbable} + \text{neglectable}. \end{array} \right.$$

The first inequality is proved in the same way as the second part of 5.3 of [1]. The second as in 5.4 of [1] with the relevant change that we do not have at our disposal their estimate $|\bar{L}, |z|^{2k} g_{1\bar{1}}| \lesssim |z|^{2k-1-2(m-1)}$. Instead, we have to use, as a consequence of our key assumption (1.3)

$$\begin{aligned} |\bar{L}, |z|^{2k} g_{1\bar{1}}| &\lesssim |z|^{2k-1} g_{1\bar{1}} + |z|^{2k} |g_{1\bar{1}}| \\ &\lesssim |z|^{2k-1} g_{1\bar{1}}. \end{aligned}$$

Thus, when we arrive at the two error terms in the second displayed formula of p. 692 (second terms in the third and fourth lines), we have the factor $z^{2k-1}g_{1\bar{1}}$. With the notations of our Lemma 3.3, we split this factor as $h = h_1h_2$ for $h_1 = z^{2k-1}g_{1\bar{1}}^{\frac{1}{2}}$ and $h_2 = g_{1\bar{1}}^{\frac{1}{2}}$ respectively and then control these error terms as sc (a) and lc (b). The proof is complete. \square

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