arXiv:1009.0494v1 [math.AP] 2 Sep 2010

Asymptotic linear stability of solitary water waves

Robert L. Pego[1] and Shu-Ming Sun[2]

September 3, 2010

Abstract

We prove an asymptotic stability result for the water wave equations linearized around small solitary waves. The equations we consider govern irrotational flow of a fluid with constant density bounded below by a rigid horizontal bottom and above by a free surface under the influence of gravity neglecting surface tension. For sufficiently small amplitude waves, with waveform well-approximated by the well-known sech-squared shape of the KdV soliton, solutions of the linearized equations decay at an exponential rate in an energy norm with exponential weight translated with the wave profile. This holds for all solutions with no component in (i.e., symplectically orthogonal to) the two-dimensional neutral-mode space arising from infinitesimal translational and wave-speed variation of solitary waves. We also obtain spectral stability in an unweighted energy norm.

[1]Department of Mathematical Sciences and Center for Nonlinear Analysis, Carnegie Mellon University, Pittsburgh, PA 15213. Email: rpego@cmu.edu

[2]Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061. Email: sun@math.vt.edu

1 Introduction

The discovery of solitary water waves by J. Scott Russell in 1834 was a seminal event in nonlinear science. Russell's observations gave him immediate confidence in the significance of these waves, and led him to carry out an extensive program of experiments investigating solitary waves and their interactions [40]. But mathematical understanding was slow to develop. The first significant steps forward were made by Boussinesq [7, 8, 9, 10] and Rayleigh [38] by carefully balancing long-wave and small-amplitude approximations. The simplest useful model (derived by Boussinesq already in 1872, see [10, p. 360] and [30]) is the famous Korteweg-de Vries equation [25]. Its sech^2 soliton solution approximates the shape of small-amplitude solitary water waves.

Given the status of the KdV equation as an approximate model, it is important to understand whether the soliton solutions of the KdV equation are approximations of some solutions of a more exact water wave model with similar properties. In this paper, we focus on questions of stability for exact solitary wave solutions of the Euler equations that govern incompressible and irrotational motions of an inviscid, constant-density fluid of finite depth. The fluid occupies a two-dimensional domain whose lower boundary is a flat rigid bottom and whose upper boundary is a free surface that forms an interface with air of negligible density and viscosity. Surface tension on the free surface is neglected.

For these water wave equations, the existence of solitary wave solutions with shape well-approximated by the KdV soliton was proved by Lavrent'ev [27], Friedrichs and Hyers [13] and Beale [1]. If the surface tension is positive and small, finite-energy, single-hump solitary waves are not known to exist, and indeed, exact traveling waves approximated by the KdV soliton may not exist without 'ripples at infinity' [2, 42]. For large surface tension, solitary water waves of depression exist [42], but the relevant physical regime corresponds to water depth less than 0.5 cm.

Explaining the stability of solitary water waves mathematically remains a very challenging problem, despite considerable physical and numerical evidence. Remarkably, a valuable step forward was made already by Boussinesq [9, 10], who argued for their stability based on a quantity he called the 'moment of instability,' which he showed was invariant in time based on the KdV approximation. Over a century later, Benjamin [3] made use of the same quantity as a Hamiltonian energy, constrained by a time-invariant momentum functional, to develop a rigorous variational method to prove orbital stability for the set of solitary-wave solutions of the KdV equation. Benjamin's arguments were improved and perfected by Bona [5].

Variational methods for orbital stability and instability in Hamiltonian wave equations, based on the use of energy-momentum functionals, were subsequently greatly advanced by many authors. Notably, the general theory of Grillakis et al. [20, 21] has been applied extensively to many physical systems. Using variational methods of this type for the case of solitary water waves of depression for the Euler equations with large surface tension, orbital stability conditional on global existence was obtained by Mielke [29] and Buffoni [11]. For small surface tension, such variational stability results have also been obtained recently by Groves and Wahlen [22] for oscillatory traveling wave packets of finite energy (also called solitary waves by several authors).

For solitary waves with zero surface tension, however, it appears hopeless to study stability using variational methods based on constrained minimization. As remarked by Bona and Sachs [6], the usual energy-momentum functional is highly indefinite in this case—The second variation lacks the finite-dimensional indefiniteness property key to the success of current variational methods. Regarding the stability of solitary waves with zero surface tension, the only existing rigorous work appears to be the recent paper of Lin [28], which addresses the linear instability of large waves close to the wave of maximum height.

The present study involves a direct analysis of the Euler equations linearized about a small-amplitude solitary wave solution. The linearized equations have a natural two-dimensional space of neutral modes arising from infinitesimal shifts and changes in wave speed of solitary waves. We deduce asymptotic stability for solutions in a space of perturbations naturally constrained to omit these neutral-mode components, being symplectically orthogonal to them. Asymptotic stability is obtained in a norm that is weighted spatially to decay exponentially behind the wave profile. The time decay of such a norm corresponds to unidirectional scattering behavior for wave perturbations. The weighted-norm linear stability analysis is also used to obtain a spectral stability result in an unweighted energy norm. Our main results are stated precisely in section 3.

The use of exponential weights to obtain nonlinear asymptotic orbital stability for solitary waves was developed for KdV solitons by Pego and Weinstein [36], for regularized long-wave equations by Miller and Weinstein [31] and for Fermi-Pasta-Ulam lattice equations by Friesecke and Pego [14, 15, 16, 17]. Finiteness of an exponentially weighted norm imposes a condition of rapid decay in front of the wave profile. But Mizumachi [33, 32] recently showed how to prove asymptotic orbital stability for FPU solitary waves perturbed in the energy space, by using exponential weights together with dispersive wave propagation estimates as developed by Martel and Merle.

Nonlinear stability for solitary water waves remains an open problem. This issue would likely involve a general global existence theory for small-amplitude 2D fluid motions, which is not yet available despite the substantial progress on well-posedness questions by Wu [43, 44].

There are a number of other works on (in)stability for 2D solitary water waves that concern the case of waves of depression with large surface tension. These include results on 2D spectral stability for finite-wavelength perturbations [23], spectral instability for transverse (3D) perturbations [35], and a full nonlinear instability result for 3D perturbations by Rousset and Tzvetkov [39].

A convenient tool for singular perturbation theory, used in [35] and in the present paper to study spectrum in the KdV scaling limit of long time and length scales, is an operator-theoretic generalization of Rouché's theorem due to Gohberg and Sigal [19]. This use of the KdV scaling contrasts with works by Craig [12] and Schneider and Wayne [41] that concern the validity of the KdV approximation for water waves over time scales of order $O(\epsilon^{-3})$ for waves of amplitude $O(\epsilon^2)$ that are long with length scales of order ϵ^{-1}. Our use of the KdV approximation occurs in the spectral domain, where it is used to obtain partial information regarding the behavior of solutions to the linearized equations in the limit $t \to \infty$. To establish stability for time and space scales unrelated to the regime of validity of the KdV approximation requires a different technique for dealing with the linearized Euler equations, which resemble a wave equation with variable coefficients. We develop a method that obtains resolvent bounds from symmetrized weighted-norm energy estimates that use Fourier filters to cut off low frequencies.

2 Equations of motion and eigenvalue problem

In this section, we derive the equations of motion linearized around a solution steady in a frame moving at a constant speed c to the right, and formulate the associated eigenvalue problem.

Basic equations. We deal with an inviscid, incompressible and irrotational fluid of constant density ρ that is bounded above by a free surface $y = \eta(x,t)$ and below by a horizontal rigid bottom $y = -h$. The velocity field (u, v) is related to the velocity potential ϕ and the stream function ψ by

$$(u, v) = (\phi_x, \phi_y) = (\psi_y, -\psi_x). \tag{2.1}$$

On the free surface $y = \eta(x,t)$, the kinematic and Bernoulli equations are:

$$\partial_t \eta + u\eta_x = v, \tag{2.2}$$

$$\partial_t \phi + \tfrac{1}{2}(\phi_x^2 + \phi_y^2) + g\eta = 0. \tag{2.3}$$

To make the problem non-dimensional, we let

$$(x, y, t) = (h\tilde{x}, h\tilde{y}, h\tilde{t}/c), \qquad (\eta, u, v, \phi, \psi) = (h\tilde{\eta}, c\tilde{u}, c\tilde{v}, ch\tilde{\phi}, ch\tilde{\psi}). \tag{2.4}$$

After dropping the tildes, the equations take again the same form in the non-dimensional variables, with g replaced by

$$\gamma = \frac{gh}{c^2} = \frac{1}{\mathrm{Fr}^2}, \tag{2.5}$$

where $\mathrm{Fr} = c/\sqrt{gh}$ is the Froude number.

In the fluid region, where now $-1 < y < \eta(x,t)$, $-\infty < x < \infty$, the velocity potential and stream function are harmonic and are taken to satisfy the no-penetration boundary conditions

$$\phi_y(x, -1) = 0, \quad \psi(x, -1) = 0 \quad (-\infty < x < \infty). \tag{2.6}$$

The dynamics is described in terms of the surface traces defined by

$$\Phi(x,t) = \phi(x, \eta(x,t), t), \quad \Psi(x,t) = \psi(x, \eta(x,t), t). \tag{2.7}$$

Then

$$\begin{pmatrix} U \\ V \end{pmatrix} := \begin{pmatrix} \Phi_x \\ \Psi_x \end{pmatrix} = \begin{pmatrix} \phi_x + \eta_x \phi_y \\ \psi_x + \eta_x \psi_y \end{pmatrix} = \begin{pmatrix} 1 & \eta_x \\ \eta_x & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{2.8}$$

and we will write

$$M(\eta_x) = \begin{pmatrix} 1 & \eta_x \\ \eta_x & -1 \end{pmatrix}, \qquad M(\eta_x)^{-1} = \frac{M(\eta_x)}{1 + \eta_x^2}. \tag{2.9}$$

The non-dimensional equations of motion now take the form

$$\partial_t \eta = v - \eta_x u = -V, \tag{2.10}$$

$$\begin{aligned} \partial_t \Phi &= \partial_t \phi + \phi_y \partial_t \eta = -\gamma\eta - \tfrac{1}{2}(u^2 + v^2) - v(v - \eta_x u) \\ &= -\gamma\eta - \tfrac{1}{2}(U, V) M(\eta_x)^{-1} (U, V)^T. \end{aligned} \tag{2.11}$$

After transforming to a moving frame with $\hat{x} = x - t$ $(= (x - ct)/h$ dimensionally) and dropping the hats, the time derivative ∂_t is replaced by $\partial_t - \partial_x$. A solitary wave is a steady solution of the resulting equations.

It is convenient to regard the wave motion as determined by the evolution of the pair (η, Φ), with Ψ and $V = \Psi_x$ determined from (η, Φ) by solving for the stream function using Laplace's equation and the relevant boundary conditions, namely (suppressing the t variable)

$$\psi_{xx} + \psi_{yy} = 0 \qquad (-\infty < x < \infty,\ -1 < y < \eta(x)), \tag{2.12}$$

$$\psi(x, -1) = 0, \quad \psi_y - \eta_x \psi_x = U(x) \qquad (-\infty < x < \infty,\ y = \eta(x)). \tag{2.13}$$

We write

$$\Psi = \mathcal{H}_\eta \Phi = \psi(x, \eta(x)), \qquad V = \Psi_x. \tag{2.14}$$

Up to a normalization, $\mathcal{H}_\eta$ is a Hilbert transform for the fluid domain. (Note $\phi + i\psi$ is an analytic function of $x + iy$.) This map will be studied in detail in a later section.

Linearization. We linearize the equations in the moving frame about a steady solution, denoting linearized variables with a dot. These linearized equations of motion take the form

$$0 = (\partial_t - \partial_x)\dot{\eta} + \dot{V}, \tag{2.15}$$

$$0 = (\partial_t - \partial_x)\dot{\Phi} + \gamma\dot{\eta} + u\dot{U} + v\dot{V} - uv\partial_x\dot{\eta}. \tag{2.16}$$

Of course $\dot{U} = \partial_x \dot{\Phi}$. To relate $\dot{V}$ to $(\dot{\eta}, \dot{\Phi})$, we linearize the boundary-value problem (2.12)-(2.13) by formally differentiating with respect to a variational parameter. The variation $\dot{\psi}$ is harmonic in the fluid domain, zero on the bottom, and on the free surface $y = \eta(x)$ satisfies

$$\dot{U} = \dot{\psi}_y - \eta_x \dot{\psi}_x - \dot{\eta}_x \psi_x + (\psi_{yy} - \eta_x \psi_{xy})\dot{\eta}.$$

Since $\psi_{yy} - \eta_x \psi_{xy} = -\partial_x(\psi_x(x, \eta(x)))$ and $-\psi_x = v$, this means

$$\dot{\psi}_y - \eta_x \dot{\psi}_x = \partial_x(\dot{\Phi}(x) - \dot{\eta}(x) v(x, \eta(x))). \tag{2.17}$$

and by (2.14) this means $\dot{\psi}(x, \eta(x)) = \mathcal{H}_\eta(\dot{\Phi} - v\dot{\eta})$. Hence $\dot{V} = \partial_x \dot{\Psi}$ where

$$\dot{\Psi} = \psi(x, \eta(x))^{\cdot} = \dot{\psi}(x, \eta(x)) + \psi_y(x, \eta(x))\dot{\eta} = \mathcal{H}_\eta(\dot{\Phi} - v\dot{\eta}) + u\dot{\eta}. \tag{2.18}$$

We have found it to be important (much more than merely convenient) to study the linearized equations of motion in terms of the combination of $\dot{\eta}$ and $\dot{\Phi}$ expressed as

$$\dot{\phi} = \dot{\Phi} - v\dot{\eta}. \tag{2.19}$$

This is the *surface trace of the variation of velocity potential*, rather than the variation of the surface trace. A similar observation was made by Lannes [26] in his treatment of well-posedness for 3D water waves locally in time. In terms of the pair $(\dot{\eta}, \dot{\phi})$, the linearized equations of motion take the form

$$(\partial_t - \mathcal{A}_\eta) \begin{pmatrix} \dot{\eta} \\ \dot{\phi} \end{pmatrix} = 0, \qquad \mathcal{A}_\eta = \begin{pmatrix} \partial_x(1-u) & -\partial_x \mathcal{H}_\eta \\ -\gamma + (1-u)v' & (1-u)\partial_x \end{pmatrix}, \tag{2.20}$$

where v' is the multiplier $v'(x) = \partial_x(v(x,\eta(x)))$. Our analysis will show that the initial-value problem for the linear system (2.20) is well-posed and (conditionally) asymptotically stable in a certain weighted function space. The components $\dot\eta$ and $\dot\phi$ will belong to spaces of different order, however, and this complicates the problem of studying stability questions directly using the variables $(\dot\eta, \dot\Phi)$.

Eigenvalue problem. Looking for solutions of (2.20) in the form $(\dot\eta,\dot\phi) = e^{\lambda t}(\eta_1(x),\phi_1(x))$ leads to the associated eigenvalue problem

$$\begin{pmatrix} \lambda - \partial_x(1-u) & \partial_x \mathcal{H}_\eta \\ \gamma - (1-u)v' & \lambda - (1-u)\partial_x \end{pmatrix} \begin{pmatrix} \eta_1 \\ \phi_1 \end{pmatrix} = 0. \tag{2.21}$$

The hardest part of our analysis of the linearized dynamics involves showing that, in an appropriate function space, this equation has no nontrivial solutions for all nonzero λ in a half plane $\operatorname{Re}\lambda \ge -\beta$ for some $\beta > 0$ depending on the wave amplitude.

3 Main results

Our main result is an asymptotic linear stability result for the classical family of small-amplitude solitary water waves that exist for Froude number slightly more than 1, meaning $\gamma < 1$. Asymptotic stability is conditional on the absence of neutral-mode components arising from translational shifts of the solitary wave, and wave-speed variation, as is standard. The precise results involve L^2 spaces with exponential weights e^{ax} that decay to the left (having $a > 0$). For $a \in \mathbb{R}$, we define L^2_a to be the Hilbert space

$$L^2_a = \{f \mid e^{ax} f \in L^2(\mathbb{R})\},$$

with inner product and norm

$$\langle f, g\rangle_a = \int_{-\infty}^{\infty} f(x)\overline{g(x)} e^{2ax}\,dx, \qquad \|f\|_a = \|e^{ax} f\|_{L^2}.$$

Also, $H^s_a = \{f \mid e^{ax} f \in H^s(\mathbb{R})\}$ will denote a weighted Sobolev space with norm that is expressed in terms of the Fourier transform $\mathcal{F}f(k) = \hat f(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx$ as

$$\|f\|_{H^s_a} = \|e^{ax} f\|_{H^s} = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} (1+k^2)^s |\hat f(k+ia)|^2\,dk\right)^{1/2}.$$

Group velocity and weighted norms. The use of these weighted norms is motivated as follows. For linearization at the trivial solution $\eta = \Phi = 0$, the Hilbert transform for the fluid domain is the Fourier multiplier $\mathcal{H}_0 = i\tanh D$ with L^2 symbol $i\tanh k$ (see section 4). Then the dispersion relation for solutions of (2.20) with space-time dependence $e^{ikx - i\varpi t}$ is

$$\varpi = -k \pm \sqrt{\gamma k \tanh k}. \tag{3.1}$$

The solitary waves that we study travel faster than the speed of long gravity waves, meaning $c > \sqrt{gh}$ and so $\gamma < 1$. Thus, in this regime the group velocity of linear waves (relative to the solitary wave) is always negative:

$$\frac{d\varpi}{dk} < 0. \tag{3.2}$$

Heuristically, linear waves scatter to the left. Our analysis makes essential use of this directionality by measuring perturbation size using weights e^{ax} with $a > 0$. As a simple example, the solution $\eta(x,t) = f(x+t)$ of the transport equation $\partial_t\eta = \partial_x\eta$ satisfies $\|\eta(\cdot,t)\|_a = e^{-t}\|f\|_a$.

In analytic terms, the isomorphism $f \mapsto e^{ax}f$ from L^2_a to L^2 maps a Fourier multiplier $\mathcal{A}(D)$ acting on L^2_a to the weight-transformed operator $e^{ax}\mathcal{A}(D)e^{-ax} = \mathcal{A}(D+ia)$ acting on L^2. The L^2-symbol $\mathcal{A}(k)$ of the former is shifted to the symbol $\mathcal{A}(k+ia)$ of the latter. The L^2_a-spectrum of $\mathcal{A}(D)$ is the closure of the image of the latter symbol. This is so since the resolvent $(\lambda - \mathcal{A}(D))^{-1}$ is bounded in L^2_a exactly when the map $f_a \mapsto (\lambda - \mathcal{A}(k+ia))^{-1}\hat{f}_a(k)$ is bounded in L^2, where $f_a = e^{ax}f$. For the Fourier multipliers

$$\mathcal{A}_\pm(D) = iD \pm \sqrt{-\gamma D \tanh D},$$

which correspond to the branches of the dispersion relation (3.1) for our water-wave problem, the L^2_a-spectrum shifts from the imaginary axis into the left half-plane for small $a > 0$ exactly because the relative group velocity is negative. The same idea underlies the use of weights to obtain nonlinear asymptotic stability for solitary waves of the KdV equation in [36] and of FPU lattice equations in [15, 16, 17].

Energy and weighted norms. Zakharov [45] showed that the water wave equations have a canonical Hamiltonian structure in terms of (η, Φ) with (nondimensional) Hamiltonian

$$\frac{1}{2}\int_{-\infty}^{\infty}\int_{-1}^{\eta(x)} |\nabla\phi|^2\,dy\,dx + \frac{1}{2}\int_{-\infty}^{\infty}\gamma\eta^2\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\left(\Phi(-\partial_x\mathcal{H}_\eta)\Phi + \gamma\eta^2\right)dx. \tag{3.3}$$

The space that we use to study asymptotic stability of the linearized system (2.20) is equivalent to a weighted linearization of this Hamiltonian about a flat surface. Namely, stability will be studied with $(\dot\eta, \dot\phi)$ in the space $Z_a = L^2_a \times H^{1/2}_a$ with norm equivalent to the norm of $(\dot\eta, \sqrt{D\tanh D}\dot\phi)$ in $L^2_a \times L^2_a$.

Scaling. We study waves in the regime where the parameter

$$\epsilon = \sqrt{1-\gamma} \tag{3.4}$$

is small and positive. For all ϵ in this well-studied regime, there is an even solitary-wave surface elevation η with η and surface velocity (u, v) approximately given by

$$\eta(x) \sim u(x) \sim \epsilon^2\Theta(\epsilon x), \qquad v(x) \sim -\epsilon^3\Theta'(\epsilon x)$$

where

$$\Theta(x) = \operatorname{sech}^2(\sqrt{3}x/2). \tag{3.5}$$

For precise statements with estimates we use, see Theorems 5.1 and A.1. The significance of these results is that we use stability information for the KdV soliton with the profile (3.5) to study the eigenvalue problem (2.21) for $|\lambda|$ small, using the KdV scaling $\hat{x} = \epsilon x$, $\lambda = \epsilon^3 \tilde{\lambda}$. Because of this scaling, we take the weighted-norm exponent to have the form $a = \epsilon\alpha$, where α is required to satisfy $0 < \alpha < \sqrt{3}$ to have $\Theta \in L^2_\alpha$. For convenience in analysis, our stability results are formulated with the tighter restriction $0 < \alpha \le \frac{1}{2}$. The parameter α is taken as any fixed number in this range.

Neutral modes. The solitary waves we study belong to a two-parameter family, smoothly parameterized by translation and Froude number (equivalently translation and wave speed c). By consequence, as usual the value $\lambda = 0$ is an eigenvalue of $\mathcal{A}_\eta$ with algebraic multiplicity two, with generalized eigenfunctions produced by differentiation with respect to x and c. Denoting these functions with the notation

$$z_x = \begin{pmatrix} \eta_x \\ \phi_x \end{pmatrix}, \qquad z_c = \begin{pmatrix} \eta_c \\ \phi_c^+ \end{pmatrix},$$

we have $\mathcal{A}_\eta z_x = 0$, $-\mathcal{A}_\eta z_c = z_x$. The details are developed in Appendix B. (The notation $\phi_c^\pm$ indicates that different choices of an integration constant are made to ensure $\Phi^\pm = \partial_x^{-1} U \in H^{1/2}_{\pm a}$.)

Solutions of (2.20) that lie in the neutral-mode space spanned by z_x and z_c do not decay in time, naturally. A necessary condition that a solutions of (2.20) decay in time is that it should have no component in this neutral-mode space. The precise spectral meaning of this (being annihilated by the spectral projection for the eigenvalue $\lambda = 0$) can be expressed in a simple form, due to the canonical Hamiltonian structure of the problem. Namely, it turns out to be necessary that the solution be *symplectically orthogonal* to the neutral mode space, meaning that

$$0 = \int_{-\infty}^{\infty} \dot{\eta}\phi_x - \dot{\phi}\eta_x \, dx, \qquad 0 = \int_{-\infty}^{\infty} \dot{\eta}\phi_c^- - \dot{\phi}\eta_c \, dx. \tag{3.6}$$

Results. Our main results concern asymptotic stability for the linearized equations in a weighted norm, and spectral stability in an unweighted norm.

Theorem 3.1 *(Asymptotic stability with weights) Fix $\alpha \in (0, \frac{1}{2}]$ and set $a = \alpha\epsilon$. If $\epsilon > 0$ is sufficiently small and η, u, v correspond to the solitary wave profile given by Theorem 5.1, then the following hold.*

(i) With domain $H^1_a \times H^{3/2}_a$, $\mathcal{A}_\eta$ is the generator of a C^0-semigroup in $Z_a = L^2_a \times H^{1/2}_a$.

(ii) Whenever $\operatorname{Re}\lambda \ge -\frac{1}{6}\alpha\epsilon^3$ and $\lambda \ne 0$, λ is in the resolvent set of $\mathcal{A}_\eta$.

(iii) The value $\lambda = 0$ is a discrete eigenvalue of $\mathcal{A}_\eta$ with algebraic multiplicity 2.

(iv) There exist constants $K > 0$ and $\beta > \frac{1}{6}\alpha\epsilon^3$ depending on ϵ and α, such that for all $t \ge 0$,

$$\|\exp(t\mathcal{A}_\eta)\dot{z}\|_{Z_a} \le K e^{-\beta t} \|\dot{z}\|_{Z_a},$$

for every initial state $\dot{z} = (\dot{\eta}, \dot{\phi})$ that satisfies the symplectic orthogonality conditions (3.6).

Theorem 3.2 *(Spectral stability without weights) For $\epsilon > 0$ sufficiently small, in the space of pairs (η_1, ϕ_1) such that*

$$\int_{-\infty}^{\infty} \phi_1 (D \tanh D)\phi_1 + \eta_1^2 \, dx < \infty \tag{3.7}$$

the spectrum of the operator $\mathcal{A}_\eta$ is precisely the imaginary axis.

The asymptotic stability statement in part (iv) of Theorem 3.1 will be proved as a consequence of the Gearhart-Prüss spectral mapping theorem [37] by establishing that the operator $\mathcal{A}_\eta$ has uniformly bounded resolvent $(\lambda - \mathcal{A}_\eta)^{-1}$ for $|\lambda|$ large with $\operatorname{Re}\lambda \geq -\frac{1}{6}\alpha\epsilon^3$, and using parts (ii) and (iii) to infer the resolvent restricted to the spectral complement of the generalized kernel is uniformly bounded for all λ with $\operatorname{Re}\lambda \geq -\frac{1}{6}\alpha\epsilon^3$. The spectral stability result in Theorem 3.2 is proved using Theorem 3.1 and symmetry of the problem without weights under space and time reversal.

4 Riemann mapping and Hilbert transform for the fluid domain

4.1 Riemann stretch and strain

We will make much use of a Riemann mapping from the fluid domain Ω_η to the flat strip Ω_0, with

$$\begin{aligned} \Omega_\eta &= \{(x,y) : -\infty < x < \infty,\ -1 < y < \eta(x)\}, \\ \Omega_0 &= \{(\underline{x},\underline{y}) : -\infty < \underline{x} < \infty,\ -1 < \underline{y} < 0\}. \end{aligned}$$

To denote the corresponding Riemann mapping and its inverse, we write

$$(\underline{x},\underline{y}) = (Z_1(x,y), Z_2(x,y)), \qquad (x,y) = (z_1(\underline{x},\underline{y}), z_2(\underline{x},\underline{y})). \tag{4.1}$$

A key quantity is the 'Riemann stretch' ζ at the surface, given (with its inverse $h = \zeta^{-1}$) by

$$\zeta(\underline{x}) := z_1(\underline{x}, 0), \qquad h(x) = Z_1(x, \eta(x)). \tag{4.2}$$

The function z_2 is harmonic in the strip Ω_0, with boundary conditions

$$z_2(\underline{x}, -1) = -1, \qquad z_2(\underline{x}, 0) = \underline{\eta}(\underline{x}) := \eta \circ \zeta(\underline{x}). \tag{4.3}$$

Taking the Fourier transform in $\underline{x}$ leads to the formula

$$z_2(\underline{x},\underline{y}) = \underline{y} + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik\underline{x}} \frac{\sinh k(\underline{y}+1)}{\sinh k} \int_{-\infty}^{\infty} e^{-iks}\underline{\eta}(s)\, ds\, dk. \tag{4.4}$$

Using the Cauchy-Riemann equation $\partial_{\underline{x}} z_1 = \partial_{\underline{y}} z_2$, we find that the 'Riemann strain' defined by $\omega = \zeta' - 1$ satisfies

$$\omega(\underline{x}) = \zeta'(\underline{x}) - 1 = D \coth D\, \underline{\eta}(\underline{x}). \tag{4.5}$$

Integrating in $\underline{x}$ with an arbitrary constant of integration, we find that we can write

$$\zeta(\underline{x}) = \underline{x} - i\coth D\,\underline{\eta}(\underline{x}) + c_0 = \underline{x} - \int_{\underline{x}}^{\infty} \underline{\eta}(s)\,ds + Q_1(D)\underline{\eta}(\underline{x}) + c_0, \tag{4.6}$$

where $Q_1(D)$ is the Fourier multiplier with symbol bounded on $\mathbb{R}$ given by

$$Q_1(k) = \frac{k\cosh k - \sinh k}{ik\sinh k} = i(k^{-1} - \coth k).$$

If η is given, then since $\underline{\eta} = \eta\circ\zeta$, Eq. (4.6) is a fixed-point equation that determines ζ and therefore $h = \zeta^{-1}$. It will turn out to be more convenient in our analysis, however, to directly study the Riemann strain ω, and recover other quantities such as ζ and η from this.

4.2 Hilbert transform

The operator $\mathcal{H}_\eta$ admits a convenient expression in terms of the Riemann stretch ζ. To see this, first we introduce pullback operators $\zeta_\#$ and ζ_* via

$$\zeta_\# U(\underline{x}) = U\circ\zeta(\underline{x}), \qquad \zeta_* U(\underline{x}) = \zeta'\zeta_\# U(\underline{x}) = (U\circ\zeta)(\underline{x})\,\zeta'(\underline{x}). \tag{4.7}$$

For later use, note that since $h = \zeta^{-1}$, the chain rule yields the simple relations

$$\partial\zeta_\# = \zeta_*\partial, \quad \zeta_* = \zeta'\zeta_\# = \zeta_\#(1/h'), \quad h_*\zeta_* = \zeta_* h_* = \mathrm{id}\,. \tag{4.8}$$

Write $\underline{\psi}(\underline{x},\underline{y}) = \psi(x,y)$. Then $\underline{\psi}$ is harmonic in Ω_0, and the boundary conditions (2.13) transform to

$$\underline{\psi}(\underline{x},-1) = 0, \qquad \partial_{\underline{y}}\underline{\psi}(\underline{x},0) = \zeta_* U(\underline{x}) = \partial_{\underline{x}}(\Phi\circ\zeta)(\underline{x}). \tag{4.9}$$

By Fourier transform we find

$$\underline{\psi}(\underline{x},\underline{y}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik\underline{x}}\frac{\sinh k(\underline{y}+1)}{k\cosh k}\int_{-\infty}^{\infty} e^{-iks}\partial(\Phi\circ\zeta)(s)\,ds\,dk,$$

so since $\psi(x,\eta(x)) = \underline{\psi}(h(x),0)$, after an integration by parts we find

$$\Psi(x) = \psi(x,\eta(x)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikh(x)}(i\tanh k)\int_{-\infty}^{\infty} e^{-iks}\Phi\circ\zeta(s)\,ds\,dk. \tag{4.10}$$

In other words, since $\Psi = \mathcal{H}_\eta\Phi$ we have (with $\mathcal{F}$ denoting Fourier transform)

$$\mathcal{H}_\eta = \zeta_\#^{-1}\mathcal{H}_0\zeta_\#, \qquad \mathcal{H}_0 = i\tanh D = \mathcal{F}^{-1}(i\tanh k)\mathcal{F}. \tag{4.11}$$

Here $\mathcal{H}_0$ is the Hilbert transform for the top boundary of the strip. Though we will make no use of the fact, it is explicitly given in terms of an integral kernel by

$$\mathcal{H}_0 U(x) = \int_{-\infty}^{\infty} k_0(x-s)U(s)\,ds, \qquad k_0(x) = \frac{-1}{2\sinh(\pi x/2)}. \tag{4.12}$$

4.3 Linearization

To justify the later calculation of generalized eigenmodes (in Appendix B), we explain here how the formal linearization formula in (2.18) follows from the representation formula in (4.11).

To proceed, start with a family of (smooth) Riemann strains ω small in $L^2 \cap L^2_a$ and depending smoothly on a variational parameter, and compute

$$\zeta(\underline{x}) = \underline{x} + \partial^{-1}\omega + c_0, \quad \underline{\eta} = \frac{\tanh D}{D}\omega, \quad \eta = \underline{\eta} \circ \zeta^{-1}.$$

Determine conjugate harmonic functions z_1, z_2 in the strip Ω_0 such that (4.3) holds and $z_1(\underline{x}, 0) = \zeta(\underline{x})$. Then the function $\mathcal{Z}(\underline{x} + i\underline{y}) = z_1(\underline{x}, \underline{y}) + iz_2(\underline{x}, \underline{y})$ yields the Riemann mapping of Ω_0 to Ω_η as described above.

Also take a family of (smooth) functions Φ (free surface velocity potential) and introduce $\underline{\phi}$ as the harmonic extension of $\Phi \circ \zeta$ into Ω_0 satisfying $\partial_{\underline{y}}\underline{\phi} = 0$ at $y = -1$, and $\underline{\psi}$ as the harmonic function conjugate to $\underline{\phi}$ and satisfying $\underline{\psi} = 0$ at $y = -1$. Then

$$\underline{\phi}(\underline{x}, \underline{y}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik\underline{x}} \frac{\cosh k(\underline{y}+1)}{\cosh k} \int_{-\infty}^{\infty} e^{-iks}\Phi \circ \zeta(s)\, ds\, dk.$$

Write

$$\underline{\Upsilon}(\underline{x} + i\underline{y}) = \underline{\phi}(\underline{x}, \underline{y}) + i\underline{\psi}(\underline{x}, \underline{y}), \qquad \Upsilon = \underline{\Upsilon} \circ \mathcal{Z}^{-1}.$$

Regarding Ω_0 as a subset of the complex plane, we have that $\underline{\Upsilon}$ and $\mathcal{Z}$ are analytic in Ω_0 and that Υ is analytic in $\Omega_\eta = \mathcal{Z}(\Omega_0)$. Define ϕ and ψ to satisfy

$$\Upsilon(x + iy) = \phi(x, y) + i\psi(x, y), \qquad (x, y) \in \Omega_\eta.$$

$\Psi(x) = \psi(x, \eta(x))$ is the trace on the fluid surface and satisfies

$$\Psi \circ \zeta = i \tanh D\,(\Phi \circ \zeta),$$

due to $\Psi \circ \zeta(\underline{x}) = \underline{\psi}(\underline{x}, 0)$ and $\Phi \circ \zeta(\underline{x}) = \underline{\phi}(\underline{x}, 0)$ and the boundary condition $\operatorname{Im}\underline{\Upsilon} = 0$ at $y = -1$.

Denoting the derivative with respect to the variational parameter by a dot, we have

$$\dot\Psi \circ \zeta + \dot\zeta\Psi_x \circ \zeta = i \tanh D(\dot\Phi \circ \zeta + \dot\zeta\Phi_x \circ \zeta) \tag{4.13}$$

Note that $\dot{\mathcal{Z}}\Upsilon' \circ \mathcal{Z}$ is analytic in Ω_0 and has zero imaginary part on the bottom $y = -1$. This means that the real and imaginary parts of the surface trace are related by the Hilbert transform for the strip. But $\dot{\mathcal{Z}} = \dot z_1 + i\dot z_2 = \dot\zeta + i\dot{\underline{\eta}}$ on $y = 0$, whence (abusing notation to write ψ_x for $\psi_x(x, \eta(x))$ with $x = \zeta(\underline{x})$, etc.)

$$\psi_x\dot\zeta + \psi_y\dot{\underline{\eta}} = (i \tanh D)(\phi_x\dot\zeta + \phi_y\dot{\underline{\eta}}). \tag{4.14}$$

Using the formulas

$$\dot{\underline{\eta}} = \dot\eta \circ \zeta + \dot\zeta\eta_x \circ \zeta, \qquad \Phi_x = \phi_x + \phi_y\eta_x, \qquad \Psi_x = \psi_x + \psi_y\eta_x,$$

together with (4.13) now yields

$$\dot\zeta\Psi_x = -\dot\eta\psi_y + \dot\zeta\psi_x + \dot{\underline{\eta}}\psi_y$$

and similarly for Φ. Combining this with (4.14) yields

$$\dot\Psi \circ \zeta - \dot\eta\psi_y = i\tanh D(\dot\Phi \circ \zeta - \dot\eta\phi_y), \tag{4.15}$$

and composing with $h = \zeta^{-1}$ yields the desired linearization formula (2.18):

$$\dot\Psi - u\dot\eta = \mathcal{H}_\eta(\dot\Phi - v\dot\eta) \tag{4.16}$$

(with $(u,v) = (\phi_x, \phi_y) = (\psi_y, -\psi_x)$ on $y = \eta(x)$).

5 Solitary wave profiles

In this section and Appendix A, we will give a simple self-contained account of the existence of small solitary waves by fundamentally the same approach as Friedrichs and Hyers [13], establishing the estimates that we need regarding convergence of the scaled wave profiles in the KdV limit.

First, note that from (2.10)-(2.11), the steady equations for a solitary wave are

$$\partial_x\eta = V = \partial_x\mathcal{H}_\eta\Phi, \qquad U - \gamma\eta = \frac{1}{2}(U,V)M(V)^{-1}(U,V)^T = \frac{U^2 - V^2 + 2UV^2}{2(1+V^2)}, \tag{5.1}$$

whence

$$\eta = \mathcal{H}_\eta\Phi, \qquad U - \gamma\eta = \frac{1}{2}(U^2 - V^2) + \gamma\eta V^2. \tag{5.2}$$

Using (4.11) and changing variables by applying $\zeta_\#$ we must have $\underline{\eta} = i\tanh D(\Phi\circ\zeta)$, hence by (4.5),

$$\omega = \zeta' - 1 = (D\coth D)\underline{\eta} = \partial(\Phi\circ\zeta) = \zeta' U\circ\zeta. \tag{5.3}$$

Then we find

$$U\circ\zeta = \frac{\omega}{1+\omega} = \omega - \frac{\omega^2}{1+\omega}, \qquad V\circ\zeta = \frac{\partial\underline{\eta}}{\zeta'} = \frac{i\tanh D\,\omega}{1+\omega}. \tag{5.4}$$

It is convenient to apply $\zeta_\#$ to (5.2b), and isolate ω on the left-hand side. This turns (5.2b) into a fixed-point equation for the Riemann strain ω, in the form

$$\omega = \left(1 - \gamma\frac{\tanh D}{D}\right)^{-1}\left(\frac{\frac{3}{2}\omega^2 + \omega^3 - \frac{1}{2}(i\tanh D\,\omega)^2(1-2\gamma\underline{\eta})}{(1+\omega)^2}\right), \qquad \underline{\eta} = \frac{\tanh D}{D}\omega. \tag{5.5}$$

The following result provides scaled bounds for the fixed point approximating the sech^2 KdV profile from (3.5). The proof is given in appendix A.

Theorem 5.1 *Let $\alpha \in (0, \sqrt{3})$, $m \geq 2$, $\nu \in (0,1)$, and $\Theta(x) = \operatorname{sech}^2(\sqrt{3}x/2)$. Then for $\epsilon > 0$ sufficiently small, equation (5.5) has a unique even solution in H^1_a of the form*

$$\omega(\underline{x}) = \epsilon^2 \theta(\epsilon \underline{x}) \tag{5.6}$$

with $\|\theta - \Theta\|_{H^m_\alpha} < \epsilon^\nu$. Moreover, the map $\epsilon \mapsto \omega$ is smooth.

The coefficients that appear in the linearized system (2.16) can now be expressed as follows. Using (2.8), (5.1) and (5.4), on the fluid surface we have the formulas

$$u = \frac{U + V^2}{1 + V^2}, \qquad v = \frac{-V(1-U)}{1+V^2}, \tag{5.7}$$

$$\zeta_\# u = \frac{\omega + \omega^2 + (i \tanh D\omega)^2}{(1+\omega)^2 + (i\tanh D\omega)^2}, \quad \zeta_\# v = \frac{-i\tanh D\omega}{(1+\omega)^2 + (i\tanh D\omega)^2}, \quad \zeta_\# v' = \frac{\partial \zeta_\# v}{1+\omega}. \tag{5.8}$$

Note that $\underline{\eta}$, $\zeta_\# u$ and $\zeta_\# v'$ are even functions (since functions have the same parity as their Fourier transform). With the choice $c_0 = \int_0^\infty \underline{\eta}(s)\, ds$, ζ is odd, and η and u are even, with $v = (u-1)\eta_x$ odd. Formally, we have the leading order approximations

$$\omega(\underline{x}) \sim \underline{\eta}(\underline{x}) \sim \zeta_\# u(\underline{x}) \sim \epsilon^2 \Theta(\epsilon \underline{x}), \qquad \zeta_\# v'(\underline{x}) \sim -\epsilon^4 \Theta''(\epsilon \underline{x}) \tag{5.9}$$

For making estimates involving the quantities in (5.8) it is useful to note that

$$\|i \tanh \epsilon D \theta\|_{H^1} = \left\| \epsilon \partial \frac{\tanh \epsilon D}{\epsilon D} \theta \right\|_{H^1} \leq \epsilon \|\theta\|_{H^2}. \tag{5.10}$$

6 Transforming the system

Flattening. Given the form of $\mathcal{H}_\eta$ in (4.11), it appears convenient to transform the eigenvalue problem in (2.21) to work in variables associated with the flattened domain. We make a similarity transform of (2.21) by applying the operator $\zeta_* = \zeta' \zeta_\#$ from (4.7) to the first equation and $\zeta_\#$ to the second, introducing the variables

$$\begin{pmatrix} \eta_2 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \zeta_* \eta_1 \\ \zeta_\# \phi_1 \end{pmatrix}. \tag{6.1}$$

Noting that $\zeta_* \partial h_* = \partial \zeta_\# h' h_\# = \partial (1/\zeta')$, (2.21) becomes

$$\begin{pmatrix} \lambda - \partial \left(\frac{1-u_1}{\zeta'} \right) & -D \tanh D \\ \gamma \left(\frac{1-v_1}{\zeta'} \right) & \lambda - \left(\frac{1-u_1}{\zeta'} \right) \partial \end{pmatrix} \begin{pmatrix} \eta_2 \\ \phi_2 \end{pmatrix} = 0, \tag{6.2}$$

where (with formal leading order behavior indicated)

$$u_1 = \zeta_\# u \sim \omega, \qquad v_1 = \zeta_\# \gamma^{-1} (1-u) v' = \frac{(1 - \zeta_\# u) \partial \zeta_\# v}{\gamma(1+\omega)} \sim \zeta_\# v'. \tag{6.3}$$

Approximate diagonalization. In order to reduce the eigenvalue problem to the 'right' scalar equation, it is helpful to balance off-diagonal terms (up to a commutator) and diagonalize the leading part of the operator. Let us define p, q, u_p, u_q, and for later reference also ρ and u_ρ, so that

$$\frac{1-u_1}{\zeta'} = p = 1+u_p, \qquad \sqrt{\frac{1-v_1}{\zeta'}} = q = 1+u_q, \qquad \sqrt{q} = \rho = 1+u_\rho. \tag{6.4}$$

Asymptotically we expect

$$u_p \sim -2\omega, \qquad u_q \sim -\tfrac{1}{2}\omega \qquad u_\rho \sim -\tfrac{1}{4}\omega. \tag{6.5}$$

To make precise estimates, we write

$$u_p(x) = \epsilon^2 \tilde{u}_p(\epsilon x), \qquad u_q(x) = \epsilon^2 \tilde{u}_q(\epsilon x), \qquad u_\rho(x) = \epsilon^2 \tilde{u}_\rho(\epsilon x),$$

and apply the scaled H^2 bounds from Theorem 5.1 and (5.10) to the expressions in (5.8), using standard calculus inequalities. Straightforward computations yield the following.

Lemma 6.1 *For $\epsilon > 0$ sufficiently small, the H^2 norms of $\tilde{u}_p$, $\tilde{u}_q$ and $\tilde{u}_\rho$ are bounded by a constant K independent of ϵ, and the functions u_p, u_q, u_ρ satisfy the pointwise bounds*

$$|u_p| + |u_q| + |u_\rho| \le K\epsilon^2, \qquad |u_p'| + |u_q'| + |u_\rho'| \le K\epsilon^3. \tag{6.6}$$

Furthermore, as $\epsilon \to 0$ we have

$$\|\tilde{u}_p + 2\Theta\|_{H^1} \to 0, \qquad \|\tilde{u}_q + \tfrac{1}{2}\Theta\|_{H^1} \to 0. \tag{6.7}$$

Introduce the operator (Fourier multiplier)

$$\mathcal{S} = \sqrt{-\gamma D \tanh D}. \tag{6.8}$$

In order to balance orders of differentiation in the system, we change variables via

$$\begin{pmatrix} \eta_3 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} \gamma q \eta_2 \\ \mathcal{S}\phi_2 \end{pmatrix}. \tag{6.9}$$

The system (6.2) then takes the (partially symmetrized) form

$$\begin{pmatrix} \lambda - \partial p + R_1 & q\mathcal{S} \\ \mathcal{S}q & \lambda - \partial p + R_2 \end{pmatrix} \begin{pmatrix} \eta_3 \\ \phi_3 \end{pmatrix} = 0, \tag{6.10}$$

where R_1 and R_2 (which will both turn out to be negligible) are given by

$$R_1 = \partial p - q\partial p q^{-1} = (\partial q - q\partial)pq^{-1} = q'pq^{-1}, \tag{6.11}$$

$$R_2 = \partial p - \mathcal{S}p\partial\mathcal{S}^{-1} = p' + [p,\mathcal{S}]\mathcal{S}^{-1}\partial. \tag{6.12}$$

Finally, we approximately diagonalize by changing variables via

$$\begin{pmatrix} \eta_4 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \eta_3 \\ \phi_3 \end{pmatrix}, \qquad \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \eta_4 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} \eta_3 \\ \phi_3 \end{pmatrix}. \tag{6.13}$$

Then the system (6.10) takes the form

$$\left(\begin{pmatrix} \lambda - \partial p & 0 \\ 0 & \lambda - \partial p \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -\mathcal{S}q - q\mathcal{S} + R_1 + R_2 & -\mathcal{S}q + q\mathcal{S} + R_1 - R_2 \\ \mathcal{S}q - q\mathcal{S} + R_1 - R_2 & \mathcal{S}q + q\mathcal{S} + R_1 + R_2 \end{pmatrix}\right)\begin{pmatrix} \eta_4 \\ \phi_4 \end{pmatrix} = 0. \tag{6.14}$$

We make a few observations regarding the form of this system: First, the operators R_2 and $[\mathcal{S}, q] = \mathcal{S}q - q\mathcal{S}$ involve commutators and will turn out to be bounded, while R_1 is just a multiplier. So the off-diagonal terms are bounded operators, involving no derivatives. Second, as is needed for energy estimates, we will invoke the symmetrization identity

$$\partial p = \sqrt{p}\,\partial\sqrt{p} + \tfrac{1}{2}p'. \tag{6.15}$$

The operator $\frac{1}{2}(\mathcal{S}q + q\mathcal{S})$ can be explicitly symmetrized (for energy estimates) up to a (double) commutator in terms of $\rho = \sqrt{q}$:

$$\tfrac{1}{2}(\mathcal{S}q + q\mathcal{S}) = \sqrt{q}\,\mathcal{S}\sqrt{q} + \tfrac{1}{2}[[\mathcal{S}, \sqrt{q}], \sqrt{q}]. \tag{6.16}$$

Finally, note that the weight-transformed operator $e^{ax}\mathcal{S}e^{-ax}$ is a Fourier multiplier with symbol

$$\mathcal{S}(k + ia) = \sqrt{-\gamma\xi\tanh\xi}, \qquad \xi = k + ia. \tag{6.17}$$

The principal square root is used here and the real part is nonnegative. We define

$$\mathcal{A}_+ = \partial + \mathcal{S}, \qquad \mathcal{A}_- = \partial - \mathcal{S}. \tag{6.18}$$

It is easy to see that these formulae define closed operators in L^2_a with domain H^1_a and with spectrum given by the range of the weight-transformed Fourier multipliers

$$k \mapsto \mathcal{A}_\pm(\xi) = i\xi \pm \sqrt{-\gamma\xi\tanh\xi}, \qquad \xi = k + ia, \quad k \in \mathbb{R}.$$

Final form as system. Based on these observations, it will be convenient to write the eigenvalue problem as follows. We use (6.4) to write the operator in the (1,1) and (2,2) slots of (6.14) as $\lambda - \mathcal{A}_{11}$ and $\lambda - \mathcal{A}_{22}$ respectively, with

$$\mathcal{A}_{11} = \mathcal{A}_+ + \mathcal{U} + J_{11}, \qquad \mathcal{U} := \partial u_p + \mathcal{S}u_q, \tag{6.19}$$

$$\mathcal{A}_{22} = \mathcal{B}_- + J_{22}, \qquad \mathcal{B}_\pm := \sqrt{p}\,\partial\sqrt{p} \pm \sqrt{q}\,\mathcal{S}\sqrt{q}. \tag{6.20}$$

The system (6.14) then takes the form

$$(\lambda - \mathcal{A})\begin{pmatrix} \eta_4 \\ \phi_4 \end{pmatrix} = 0, \qquad \mathcal{A} = \begin{pmatrix} \mathcal{A}_{11} & J_{12} \\ J_{21} & \mathcal{A}_{22} \end{pmatrix}, \tag{6.21}$$

with the 'junk terms' J_{ij} given in terms of $R_1 = q'pq^{-1}$ and $R_2 = p' + [p, \mathcal{S}]\mathcal{S}^{-1}\partial$ by

$$\begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} R_2 + R_1 + [\mathcal{S}, q] & R_1 - R_2 - [\mathcal{S}, q] \\ R_1 - R_2 + [\mathcal{S}, q] & R_1 + [p, \mathcal{S}]\mathcal{S}^{-1}\partial + [[\mathcal{S}, \rho], \rho] \end{pmatrix}. \tag{6.22}$$

Another way we will sometimes use to write the (1,1) component of $\mathcal{A}$ is

$$\mathcal{A}_{11} = \mathcal{B}_+ + \tilde{J}_{11}, \qquad \tilde{J}_{11} = -\tfrac{1}{2}(R_1 + [p, \mathcal{S}]\mathcal{S}^{-1}\partial - [[\mathcal{S}, \rho], \rho]). \tag{6.23}$$

Our main results to be proved in this paper now amount to the following.

Theorem 6.2 *(Asymptotic stability with weights) Fix $\alpha \in (0, \frac{1}{2}]$ and set $a = \alpha\epsilon$. For $\epsilon > 0$ sufficiently small the following hold:*

(i) With domain $(H^1_a)^2$, $\mathcal{A}$ is the generator of a C^0 semigroup on $(L^2_a)^2$.

(ii) Whenever $\operatorname{Re}\lambda \geq -\frac{1}{6}\alpha\epsilon^3$ with $\lambda \neq 0$, λ is in the resolvent set of $\mathcal{A}$.

(iii) The value $\lambda = 0$ is a discrete eigenvalue of $\mathcal{A}$ with algebraic multiplicity 2.

(iv) Restricted to the $\mathcal{A}$-invariant spectral complement $\bar{Y}_a$ of the generalized kernel of $\mathcal{A}$, the semigroup e^{At} is asymptotically stable, satisfying

$$\|e^{\mathcal{A}t}z\|_a \leq Ke^{-\beta t}\|z\|_a$$

for all $t \geq 0$ and $z \in \bar{Y}_a$, with some constants $K > 0$ and $\beta > \frac{1}{6}\alpha\epsilon^3$ depending on ϵ and α.

Theorem 6.3 *(Spectral stability without weights) For $\epsilon > 0$ sufficiently small, with domain $(H^1)^2$ in the space $(L^2)^2$, the spectrum of the operator $\mathcal{A}$ is precisely the imaginary axis.*

Equivalences. The statements in these correspond directly to those in Theorem 3.1 due to the following facts. First, the map $(\eta_2, \phi_2) \mapsto (\eta_4, \phi_4)$ is clearly an isomorphism from $Z_a = L^2_a \times H^{1/2}_a$ to $(L^2_a)^2$ when $a > 0$. (Note the symbol $\mathcal{S}(k + ia)$ does not vanish at $k = 0$ in this case). Second, the map $\eta_1 \mapsto \eta_2$ from (6.1) is clearly an isomorphism on L^2_a (and on H^1_a). For example,

$$\|\eta_1\|^2_a = \int_{-\infty}^{\infty} e^{2as}|\eta_1 \circ \zeta(s)\zeta'(s)|^2 e^{2a(\zeta(s)-s)} \frac{ds}{\zeta'(s)} = (1 + O(\epsilon^2))\|\eta_2\|^2_a,$$

since pointwise $|\zeta' - 1| + \alpha\epsilon|\zeta(s) - s| = O(\epsilon^2)$ uniformly, due to Theorem 5.1. Next, the composition map $\zeta_\#$ is an isomorphism on H^s for $s = 0$, 1 and 2, hence also for $s = \frac{1}{2}$ and $\frac{3}{2}$ by interpolation (see [4], particularly Theorems 3.1.2 and 6.4.4). Therefore the map

$$\phi_1 \mapsto \phi_2 = e^{-ax}e^{-a(\zeta(x)-x)}\zeta_\# e^{ax}\phi_1$$

is an isomorphism on H^s_a. (Note that the multipliers $e^{\pm a(\zeta(x)-x)} = I + O(\epsilon^2)$ on H^s for $s = 0$, 1 and 2, as is easy to check using Theorem 5.1.)

Finally, we claim that the transformation steps (6.1), (6.9) and (6.13) map the space of pairs (η_1, ϕ_1) satisfying condition (3.7) of Theorem 3.2 isomorphically to the space of pairs $(\eta_4, \phi_4) \in L^2 \times L^2$. The first step to show this is to see that $(\eta_4, \phi_4) \in L^2 \times L^2$ is equivalent to finiteness of the linearized energy:

$$\int_{-\infty}^{\infty} \phi_1(-\partial_x \mathcal{H}_\eta)\phi_1 + \gamma\eta_1^2 \, dx < \infty. \tag{6.24}$$

The key point here is that since $\phi_2 = \zeta_\#\phi_1$ and $\mathcal{H}_\eta = \zeta_\#^{-1}\mathcal{H}_0\zeta_\#$, due to (4.8) the change of variables $x = \zeta(\underline{x})$, $dx = \zeta'(\underline{x})d\underline{x}$ yields

$$\int_{-\infty}^{\infty} \phi_1(-\partial_x\mathcal{H}_\eta)\phi_1 \, dx = \int_{-\infty}^{\infty} (\zeta_\#\phi_1)(-\zeta_\#\partial_x\mathcal{H}_\eta\phi_1)\zeta' d\underline{x} = \int_{-\infty}^{\infty} \phi_2(D\tanh D)\phi_2 \, d\underline{x} = \|\phi_3\|^2_{L^2}.$$

The second step is to demonstrate an equivalence of norms. We claim

Lemma 6.4 *For some positive constants c_- and c_+ independent of ϕ_1,*

$$c_- \int_{-\infty}^{\infty} \phi_1 (D \tanh D) \phi_1 \, dx \le \int_{-\infty}^{\infty} \phi_1 (-\partial_x \mathcal{H}_\eta) \phi_1 \, dx \le c_+ \int_{-\infty}^{\infty} \phi_1 (D \tanh D) \phi_1 \, dx. \tag{6.25}$$

For the proof, it is enough to consider ϕ_1 smooth and rapidly decaying on $\mathbb{R}$. Let ϕ and $\underline{\phi}$ be functions that are harmonic in the fluid domain Ω_η and the flat strip Ω_0 respectively, and satisfy the boundary conditions

$$\phi_1(x) = \phi(x, \eta(x)) = \underline{\phi}(x, 0), \qquad 0 = \phi_y(x, -1) = \underline{\phi}_y(x, -1). \tag{6.26}$$

Then

$$\int_{-\infty}^{\infty} \phi_1 (-\partial_x \mathcal{H}_\eta) \phi_1 \, dx = \int_{-\infty}^{\infty} \int_{-1}^{\eta(x)} |\nabla \phi|^2 \, dy \, dx, \tag{6.27}$$

$$\int_{-\infty}^{\infty} \phi_1 (D \tanh D) \phi_1 \, dx = \int_{-\infty}^{\infty} \int_{-1}^{0} |\nabla \underline{\phi}|^2 \, dy \, dx. \tag{6.28}$$

Moreover, the function ϕ (resp. $\underline{\phi}$) *minimizes* the double integral in (6.27) (resp. (6.28)) among functions satisfying the same Dirichlet boundary conditions. Let $X : \Omega_0 \to \Omega_\eta$ be a smooth (but non-conformal) change of variables of the form $X(x, y) = (x, \tilde{y}(x, y))$, such that $X(x, 0) = (x, \eta(x))$. The function $\tilde{\phi} = \phi \circ X$ is smooth on Ω_0 and satisfies $\tilde{\phi}(x, 0) = \phi(x, \eta(x)) = \phi_1(x)$. Using the minimizing property of $\underline{\phi}$, then changing variables and using that the gradient and (inverse) Jacobian of X are uniformly bounded, we find

$$\int_{-\infty}^{\infty} \int_{-1}^{0} |\nabla \underline{\phi}|^2 \, dy \, dx \le \int_{-\infty}^{\infty} \int_{-1}^{0} |\nabla \tilde{\phi}|^2 \, dy \, dx \le \frac{1}{c_-} \int_{-\infty}^{\infty} \int_{-1}^{\eta(x)} |\nabla \phi|^2 \, dy \, dx.$$

This establishes the first inequality in the Lemma. The other one is similar.

7 Estimates on commutators and junk

In order to bound the junk terms, we need to bound commutators of $\mathcal{S}$ with the multipliers p, q and ρ, or equivalently with u_p, u_q, and u_ρ, since $[\mathcal{S}, 1] = 0$. The functions u_p, u_q, u_ρ all have the scaled form $\epsilon^2 G(\epsilon x)$, where G depends on ϵ but remains bounded in H^2. The following result provides a general estimate for the commutator of a Fourier multiplier and a multiplier with this scaled form. Write $\langle k \rangle^s = (1 + k^2)^{s/2}$, $\langle D \rangle^s = (1 + D^2)^{s/2}$.

Proposition 7.1 *Let $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ be Fourier multipliers with symbols P, Q and R respectively, and let $s \ge 0$. Let $g(x) = \epsilon^2 G(\epsilon x)$ where $G\colon \mathbb{R} \to \mathbb{R}$ is smooth and exponentially decaying, and let $f\colon \mathbb{R} \to \mathbb{R}$ be smooth with compact support. Then*

$$\|\mathcal{P}[\mathcal{Q}, g]\mathcal{R} f\|_{L^2} \le C_* C_G \|f\|_{L^2},$$

where

$$C_* = \sup_{k, \hat{k} \in \mathbb{R}} \epsilon^2 \frac{P(\epsilon k) |Q(\epsilon k) - Q(\epsilon \hat{k})| R(\epsilon \hat{k})}{\langle k - \hat{k} \rangle^s}, \qquad C_G = \int_{-\infty}^{\infty} \langle k \rangle^s |\hat{G}(k)| \, \frac{dk}{2\pi}.$$

Proof. Using the Fourier transform and Young's inequality, since $\hat{g}(k) = \epsilon\hat{G}(k/\epsilon)$, we have

$$\begin{aligned}\|\mathcal{P}[\mathcal{Q},g]\mathcal{R}f\|_{L^2}^2 &= \int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty} P(k)(Q(k)-Q(\hat{k}))\epsilon\hat{G}\left(\frac{k-\hat{k}}{\epsilon}\right)R(\hat{k})\hat{f}(\hat{k})\,\frac{d\hat{k}}{2\pi}\right|^2\frac{dk}{2\pi}\\ &\le C_*^2\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left\langle\frac{k-\hat{k}}{\epsilon}\right\rangle^s\left|\hat{G}\left(\frac{k-\hat{k}}{\epsilon}\right)\right||\hat{f}(\hat{k})|\,\frac{d\hat{k}}{2\pi\epsilon}\right)^2\frac{dk}{2\pi}\\ &\le C_*^2C_G^2\|f\|_{L^2}^2.\end{aligned}$$

Corollary 7.2 *Suppose* $0\le a<\pi/4$. *With* $g=u_p$, u_q *or* u_ρ, *there exists* $K>0$ *such that for all small enough* $\epsilon>0$, *we have the* L_a^2 *operator norm estimates*

$$\|[\mathcal{S},g]\mathcal{S}^{-1}\partial\|_a \le K\epsilon^3, \tag{7.1}$$

$$\|J_{11}\|_a+\|J_{12}\|_a+\|J_{21}\|_a+\|J_{22}\|_a \le K\epsilon^3. \tag{7.2}$$

Proof. Observe that for each of the indicated choices for g, we have that G is uniformly bounded in H^2 as a consequence of Lemma 6.1. So, using $s=2-\frac{2}{3}$ we have

$$C_G \le \left(\int_{-\infty}^{\infty}\langle k\rangle^{-4/3}dk\right)^{1/2}\left(\int_{-\infty}^{\infty}\langle k\rangle^4|\hat{G}(k)|^2dk\right)^{1/2}\le K \tag{7.3}$$

independent of ϵ. And, the operator $\langle D\rangle^{-1/2}\mathcal{S}^{-1}\partial$ is uniformly bounded on L_a^2 since its weight-transformed symbol is $\langle\xi\rangle^{-1/2}i\xi/\sqrt{-\gamma\xi\tanh\xi}$, which is uniformly bounded. Hence it suffices to show that with the choices $P(k)=1$, $Q(k)=\sqrt{-\xi\tanh\xi}$, $R(k)=\langle\xi\rangle^{1/2}$ ($\xi=k+ia$), we have

$$C_*\le K\epsilon^3. \tag{7.4}$$

To prove this estimate the idea is to show that with $\xi=k+ia$, $\hat{\xi}=\hat{k}+ia$,

$$|Q(k)-Q(\hat{k})| = \left|\frac{\xi\tanh\xi-\hat{\xi}\tanh\hat{\xi}}{Q(k)+Q(\hat{k})}\right| \le \frac{K|k-\hat{k}|}{\max(1,|\hat{\xi}|^{1/2})}\le\frac{K|k-\hat{k}|}{\langle\hat{\xi}\rangle^{1/2}}, \tag{7.5}$$

and conclude through scaling by ϵ.

To prove (7.5), first note that $|Q(k)-Q(\hat{k})|\le K|k-\hat{k}|$, since $Q'(k)$ is uniformly bounded, as is easy to show. Suppose now that $\hat{k}>1$, without loss. If $k<0$ then

$$|Q(k)-Q(\hat{k})|\le|Q(k)-Q(0)|+|Q(0)-Q(\hat{k})|\le K(|k|+|\hat{k}|)=K|k-\hat{k}|.$$

If $k>0$, then one computes explicitly that

$$-\xi\tanh\xi=-(k+ia)\frac{\sinh 2k+i\sin 2a}{\cosh 2k+\cos 2a},$$

and finds that $Q(k)$ lies in the fourth quadrant of the complex plane together with $Q(\hat{k})$, and so

$$|Q(k) + Q(\hat{k})| \geq |Q(\hat{k})| \geq \langle \hat{\xi} \rangle^{1/2}/K,$$

since $\tanh \hat{\xi}$ is bounded away from zero. As the map $k \mapsto -\xi \tanh \xi$ is uniformly Lipschitz, the estimate (7.5) follows. This proves the commutator estimate (7.1). Using this together with (6.6) and the fact that $\mathcal{S}\partial^{-1}$ is bounded, the remaining estimates in (7.2) follow directly.

8 Symbol expansions and estimates

Here we develop basic approximations and key estimates that concern the Fourier multiplier $\mathcal{A}_+$.

8.1 Low-frequency expansion and KdV scaling

The Taylor expansion of tanh at zero,

$$\tanh \xi = \xi - \tfrac{1}{3}\xi^3 + O(\xi^5),$$

and the fact that for ξ with positive imaginary part one has $\sqrt{-\xi^2} = -i\xi$, yields

$$\sqrt{-\xi \tanh \xi} = -i\xi + \tfrac{1}{6} i\xi^3 + O(\xi^5). \tag{8.1}$$

We note that if $k \in \mathbb{R}$ and $|k|$ is sufficiently small,

$$\sqrt{\frac{\tanh k}{k}} \leq 1 - \tfrac{1}{9}k^2. \tag{8.2}$$

In the KdV long-wave scaling, one replaces ξ in (8.1) by $\epsilon\xi$, and λ by $\epsilon^3\tilde{\lambda}$. We find (recall $\gamma = 1 - \epsilon^2$)

$$\mathcal{A}_+(\epsilon\xi) = i\epsilon\xi + \sqrt{-\gamma\epsilon\xi \tanh \epsilon\xi} = \epsilon^3(\tfrac{1}{2}i\xi\gamma_1 + \tfrac{1}{6}i\xi^3\gamma + \xi^3 O(\epsilon^2\xi^2)), \tag{8.3}$$

where

$$\gamma_1 = 2\epsilon^{-2}(1 - \sqrt{1-\epsilon^2}) = 1 + O(\epsilon^2).$$

The KdV-scaled weight-transformed symbol of $-\lambda + \mathcal{A}_+$ with $\xi = k + i\alpha$, $\lambda = \epsilon^3\tilde{\lambda} = \epsilon^3(\tilde{\lambda}_r + i\tilde{\lambda}_i)$ is

$$\begin{aligned} \epsilon^{-3}(-\epsilon^3\tilde{\lambda} + i\epsilon\xi + \sqrt{-\gamma\epsilon\xi \tanh \epsilon\xi}) &= -\tilde{\lambda} + \tfrac{1}{2}i\xi\gamma_1 + \tfrac{1}{6}i\xi^3\gamma + \xi^3 O(\epsilon^2\xi^2) \\ &= (-\tilde{\lambda}_r - \tfrac{1}{2}\alpha\gamma_1 + \tfrac{1}{6}(\alpha^3 - 3\alpha k^2)\gamma) \ + \ i(-\tilde{\lambda}_i + \tfrac{1}{2}k\gamma_1 + \tfrac{1}{6}(k^3 - 3k\alpha^2)\gamma) + \xi^3 O(\epsilon^2\xi^2). \end{aligned} \tag{8.4}$$

This corresponds to the purely formal KdV approximation (writing $\mathcal{A}_+ = \mathcal{A}_+(D)$)

$$-\lambda + \mathcal{A}_+(\epsilon D) \sim \epsilon^3(-\tilde{\lambda} + \tfrac{1}{2}\partial - \tfrac{1}{6}\partial^3).$$

8.2 High-frequency estimates

Lemma 8.1 *If $z \in \mathbb{C}$, $a > 0$, then $\operatorname{Re}\sqrt{z} \le a$ if and only if $\frac{1}{2}(|z| + \operatorname{Re} z) \le a^2$.*

Proof. Write $\sqrt{z} = u + iv$ where $u \ge 0$. The result follows from

$$|z| = |u+iv|^2 = u^2 + v^2, \quad \operatorname{Re} z = u^2 - v^2.$$

Lemma 8.2 *Suppose $\xi = k + ia$ with $k \in \mathbb{R}$ and $0 < a < \pi/4$. Then*

$$0 < \operatorname{Re}\sqrt{-\xi \tanh \xi} \le \frac{a}{\sqrt{\cos 2a}}\sqrt{\frac{\tanh k}{k}}. \tag{8.5}$$

Proof. By the previous lemma, if $\beta > 0$ and $w = \sqrt{-\xi \tanh \xi}$, then $\operatorname{Re} w \le \beta$ if and only if

$$|\xi||\tanh \xi| - \operatorname{Re}(\xi \tanh \xi) \le 2\beta^2. \tag{8.6}$$

We may write

$$\tanh \xi = \frac{e^{\xi} - e^{-\xi}}{e^{\xi} + e^{-\xi}} = \frac{\sinh 2k + i \sin 2a}{\cosh 2k + \cos 2a} = \frac{u + iv}{D_0}$$

with $u = \sinh 2k$, $v = \sin 2a$, $D_0 = \cosh 2k + \cos 2a$. Then (8.6) is equivalent to

$$(k^2 + a^2)^{1/2}(u^2 + v^2)^{1/2} - (ku - av) \le 2\beta^2 D_0,$$

or (taking $k > 0$ without loss)

$$\left(1 + \frac{a^2}{k^2}\right)^{1/2}\left(1 + \frac{v^2}{u^2}\right)^{1/2} - 1 + \frac{av}{ku} \le 2\beta^2 \frac{D_0}{ku}. \tag{8.7}$$

Note that

$$\frac{v}{u} = \frac{\sin 2a}{\sinh 2k} \le \frac{a}{k}.$$

Using this bound on the left-hand side of (8.7), we find that (8.7) is implied by the bound

$$\frac{a^2}{k^2} \le \beta^2 \frac{D_0}{ku}. \tag{8.8}$$

Since $D_0 \ge (\cosh 2k + 1)\cos 2a = 2\cosh^2 k \cos 2a$ and $u = 2\cosh k \sinh k$, (8.8) is implied by

$$\frac{a^2}{\cos 2a}\frac{\tanh k}{k} = \beta^2. \tag{8.9}$$

This yields (8.5) as claimed.

Corollary 8.3 *Take* $\gamma = 1-\epsilon^2$, $a = \epsilon\alpha$ *with* $0 < \alpha \le \frac{1}{2}$. *For* $\epsilon > 0$ *sufficiently small, we have that for all real* k, *with* $\xi = k + ia$,

$$\operatorname{Re}\sqrt{-\gamma\xi\tanh\xi} \le \epsilon\alpha(1-\tfrac{1}{4}\epsilon^2)\sqrt{\frac{\tanh k}{k}},$$

$$\operatorname{Re}\mathcal{A}_+(\xi) = \operatorname{Re}(i\xi + \sqrt{-\gamma\xi\tanh\xi}) \le \epsilon\alpha\left(-1 + (1-\tfrac{1}{4}\epsilon^2)\sqrt{\frac{\tanh k}{k}}\right) \le -\frac{1}{4}\epsilon^3\alpha.$$

Moreover, uniformly for $\operatorname{Re}\lambda \ge -\frac{1}{6}\epsilon^3\alpha$, *the* L^2_a *operator norm of the resolvent of* $\mathcal{A}_+$ *satisfies*

$$\|(\lambda - \mathcal{A}_+)^{-1}\|_a \le \frac{12}{\alpha\epsilon^3}. \tag{8.10}$$

The first inequality follows since

$$\frac{\gamma}{\cos 2a} \le \frac{1-\epsilon^2}{1-2\epsilon^2\alpha^2} \le 1 - \tfrac{1}{2}\epsilon^2 \le (1-\tfrac{1}{4}\epsilon^2)^2,$$

and the resolvent bound follows since $|\lambda - \mathcal{A}_+(\xi)| \ge \operatorname{Re}(\lambda - \mathcal{A}_+(\xi)) \ge \frac{1}{12}\epsilon^3\alpha$ for all k.

For later reference, we note that for $\xi = k + ia$ with $|a| < \pi/8$, $k \in \mathbb{R}$,

$$|\tanh\xi| \le \frac{|\sinh 2k| + |\sin 2a|}{\cosh 2k + \cos 2a} \le 1. \tag{8.11}$$

9 Semigroup generation and scalar reduction by elimination

To start our analysis of the linearized dynamics governed by $\mathcal{A}$, we use energy estimates to establish resolvent bounds for the symmetrized operators $\mathcal{B}_\pm$ that dominate the diagonal of $\mathcal{A}$. By consequence, we show in this section that $\mathcal{A}$ generates a C^0 semigroup in $(L^2_a)^2$, with $a = \epsilon\alpha$ for $\alpha \in [0, \frac{1}{2}]$. Also we will show that if $\alpha \in (0, \frac{1}{2}]$, $\lambda - \mathcal{A}_{22}$ in (6.21) is uniformly invertible on L^2_a for all λ satisfying $\operatorname{Re}\lambda \ge -\frac{1}{2}\epsilon\alpha$. This allows us to eliminate ϕ_4 in the eigenvalue problem (6.21) and reduce to a scalar, nonlinear eigenvalue equation for η_4 in the form

$$(\lambda - \mathcal{A}_{11} - J_{12}(\lambda - \mathcal{A}_{22})^{-1}J_{21})\eta_4 = 0. \tag{9.1}$$

Lemma 9.1 *For some constant* K *independent of* ϵ, α *and* λ, *if* $\alpha \in [0, \frac{1}{2}]$, $\epsilon > 0$ *is sufficiently small, and* $\operatorname{Re}\lambda > -\epsilon\alpha(1-K\epsilon^2)$, *then* λ *is in the resolvent set of* $\mathcal{B}_-$, *with*

$$\|(\lambda - \mathcal{B}_-)^{-1}\|_a \le \frac{1}{\operatorname{Re}\lambda + \alpha\epsilon(1-K\epsilon^2)}, \tag{9.2}$$

and if $\operatorname{Re}\lambda > K\alpha\epsilon^3$ *then* λ *is in the resolvent set of* $\mathcal{B}_+$, *with*

$$\|(\lambda - \mathcal{B}_+)^{-1}\|_a \le \frac{1}{\operatorname{Re}\lambda - K\alpha\epsilon^3}. \tag{9.3}$$

Proof. The main step in the proof is to perform energy estimates for each term. Let z be smooth with compact support, and let $z_a = e^{ax}z$. Write $\partial_a = e^{ax}\partial e^{-ax} = \partial - a$, and recall $a = \alpha\epsilon$. We compute (using the fact that $p \geq 1 - K\epsilon^2$ in the last step)

$$\begin{aligned} \operatorname{Re}\langle -\sqrt{p}\partial\sqrt{p}\, z, z\rangle_a &= -\operatorname{Re}\int_{-\infty}^{\infty} (\sqrt{p}\partial\sqrt{p}\, z)\bar{z}\, e^{2ax} dx = -\operatorname{Re}\int_{-\infty}^{\infty} (\partial_a\sqrt{p}\, z_a)\overline{\sqrt{p}z_a}\, dx \\ &= a\int_{-\infty}^{\infty} p|z_a|^2\, dx \geq \alpha\epsilon(1-K\epsilon^2)\|z\|_a^2. \end{aligned} \tag{9.4}$$

Due to (6.17) above, with $\xi = k + ia$,

$$\operatorname{Re}\langle\sqrt{q}\mathcal{S}\sqrt{q}\, z, z\rangle_a = \operatorname{Re}\int_{-\infty}^{\infty} (\sqrt{q}\mathcal{S}\sqrt{q}z_a)\bar{z}_a e^{2ax}\, dx = \operatorname{Re}\int_{-\infty}^{\infty} \sqrt{-\gamma\xi\tanh\xi}\,|\mathcal{F}\sqrt{q}\, z_a|^2\, \frac{dk}{2\pi}. \tag{9.5}$$

By Corollary 8.3 we find $0 \leq \operatorname{Re}\langle\sqrt{q}\mathcal{S}\sqrt{q}\, z, z\rangle_a \leq \epsilon\alpha\|qz\|_a^2 \leq \epsilon\alpha(1+K\epsilon^2)\|z\|_a^2$. Hence it follows that for all smooth z with compact support

$$\frac{\operatorname{Re}\langle(\lambda - \mathcal{B}_-)z, z\rangle_a}{\|z\|_a^2} \geq \operatorname{Re}\lambda + \alpha\epsilon(1-K\epsilon^2), \tag{9.6}$$

$$\frac{\operatorname{Re}\langle(\lambda - \mathcal{B}_+)z, z\rangle_a}{\|z\|_a^2} \geq \operatorname{Re}\lambda - K\alpha\epsilon^3. \tag{9.7}$$

When the right-hand side is positive, this proves $\lambda - \mathcal{A}_\pm$ is uniformly invertible *on its range*, satisfying the respective estimates in (9.2) and (9.3).

To prove that λ is in the resolvent set of $\mathcal{B}_-$, what remains to prove is that the range of $\lambda - \mathcal{B}_\pm$ is all of L_a^2. To accomplish this, we use a perturbation estimate to establish that a fixed value $\lambda = 1$ is in the resolvent set for small enough ϵ, then invoke an analytic continuation property of resolvents. For $\lambda = 1 > 0$ fixed, if ϵ is small then we will show $1 - \mathcal{B}_\pm$ is a small relative perturbation of the Fourier multiplier $1 - \mathcal{A}_\pm$ from (6.18), with

$$\|(1-\mathcal{A}_\pm)^{-1}(\mathcal{B}_\pm - \mathcal{A}_\pm)\|_a \leq K\epsilon^2 < 1. \tag{9.8}$$

By perturbation it follows 1 is in the resolvent set of operator $\mathcal{B}_\pm$ and the range of $\lambda - \mathcal{B}_\pm$ is all of L_a^2 for $\lambda = 1$. Using the Neumann series for the resolvent, we see that the resolvent of any closed operator can be analytically continued to any set where the resolvent has a uniform apriori bound (see Theorem III.6.7 of [24]). By consequence, the resolvent set of $\mathcal{B}_-$ (resp. $\mathcal{B}_+$) contains the entire right half-plane where the right-hand side of (9.6) (resp. (9.7)) is positive.

We proceed to prove (9.8). We compute that

$$\mathcal{B}_\pm - \mathcal{A}_\pm = \partial u_p - \tfrac{1}{2}p' \pm \mathcal{S}u_q \pm [\rho, \mathcal{S}]\rho.$$

Since $\mathcal{S}\partial^{-1}$ is bounded, Corollary 7.2 and (6.6) imply

$$|u_p| + |u_q| + |p'| + \|[\mathcal{S}, \rho]\rho\|_a \leq K\epsilon^2. \tag{9.9}$$

We claim that for some constant K independent of ϵ and α,

$$\|(1-\mathcal{A}_\pm)^{-1}\partial^j\|_a \le K \quad \text{for } j=0 \text{ and } 1. \tag{9.10}$$

The symbol of the weight-transformed operator $-e^{-ax}(1-\mathcal{A}_\pm)^{-1}\partial^j e^{ax}$ is

$$m_j(\xi) = \frac{(i\xi)^j}{1-i\xi \mp \sqrt{-\gamma\xi\tanh\xi}}, \qquad \xi = k+ia, \tag{9.11}$$

for $j=0$ or 1. Since $-i\xi = -ik+a$, the real part of the denominator is always greater than 1, by Corollary 8.3. Hence $|m_0(\xi)| \le 1$ for all $k\in\mathbb{R}$, so (9.10) holds for $j=0$. For $j=1$, we have that $|m_1(\xi)| \le 9$ for $|\xi|\le 9$, while for $|\xi|>9$ the denominator is bounded below by

$$|1-i\xi| - |\xi|^{1/2} \ge |\xi|(1-|\xi|^{-1/2}) \ge \tfrac{2}{3}|\xi|,$$

because $|\tanh\xi| \le 1$ by (8.11). Thus $|m_1(\xi)| \le \frac{3}{2}$ for $|\xi|\ge 9$. Hence (9.10) holds also for $j=1$. The bound in (9.8) follows by combining (9.10) with (9.9). This completes the proof of the Lemma.

Proposition 9.2 *For $\epsilon>0$ sufficiently small, $\mathcal{A}$ is the generator of a C^0 semigroup on $(L^2_a)^2$.*

Proof. By the Lemma just proved and the Hille-Yosida theorem, the operator

$$\mathcal{A}_* = \begin{pmatrix} \mathcal{B}_+ & 0 \\ 0 & \mathcal{B}_- \end{pmatrix}$$

is the generator of a C^0 semigroup on $(L^2_a)^2$. But $\mathcal{A}-\mathcal{A}_*$ is bounded, so the result follows from a standard perturbation theorem (see Theorem IX.2.1 of [24]).

Lemma 9.3 *For some constant K independent of ϵ, α and λ and for $\alpha\in(0,\frac{1}{2}]$, if $\epsilon>0$ is sufficiently small then λ is in the resolvent set of $\mathcal{A}_{22}$ whenever* $\operatorname{Re}\lambda \ge -\frac{1}{2}\epsilon\alpha$, *with*

$$\|(\lambda-\mathcal{A}_{22})^{-1}\|_a \le \frac{K}{\epsilon\alpha}. \tag{9.12}$$

Proof. Due to the bound on J_{22} from Corollary 7.2, this result follows directly from the results in Lemma 9.1 concerning the resolvent of $\mathcal{B}_-$.

10 Resolvent bounds for $|\lambda|$ not too small

For the remainder of this paper we fix α satisfying $0<\alpha\le\frac{1}{2}$ and write $a=\epsilon\alpha$. Here we demonstrate a bound on the resolvent of the operator $\mathcal{A}$ from (6.21) that is uniform in λ, for λ in the right half-plane with $|\lambda|$ not too small. This bound, in combination with the Gearhart-Prüss spectral mapping theorem and our proof that the only eigenvalue of $\mathcal{A}$ in the right half-plane is $\lambda=0$ with algebraic multiplicity 2, will allow us to obtain linear asymptotic stability in L^2_a for the semigroup $e^{\mathcal{A}t}$, conditional for perturbations containing no neutral-mode components.

Proposition 10.1 *Let $0 < \nu < \hat{\nu} < 1$. For $\epsilon > 0$ sufficiently small, all λ satisfying*

$$|\lambda| \geq \epsilon^{\nu} \quad \textit{and} \quad \operatorname{Re}\lambda \geq -\tfrac{1}{12}\alpha\epsilon^{1+2\hat{\nu}} \tag{10.1}$$

belong to the resolvent set of $\mathcal{A}$, with $\|(\lambda - \mathcal{A})^{-1}\|_a \leq K/\epsilon^{1+2\hat{\nu}}$, for some constant K independent of ϵ and λ.

In the analysis we will make use of sharp Fourier cutoffs (Fourier filters) defined as follows. We specify a wavenumber threshold chosen to be $\hat{\kappa} = \epsilon^{\hat{\nu}}$, where we require $\nu < \hat{\nu} < 1$. Define projection operators π_o (low-pass), π_i (high-pass) on L^2_a as follows. First, on L^2, define low and high-pass filters by

$$\pi_{0o} = \mathcal{F}^{-1}\mathbb{1}_{[-\hat{\kappa},\hat{\kappa}]}\mathcal{F}, \qquad \pi_{0i} = I - \pi_{0o}. \tag{10.2}$$

These operators are orthogonal projections on L^2. Now in L^2_a, define orthogonal projections by

$$\pi_o = e^{-ax}\pi_{0o}e^{ax}, \qquad \pi_i = I - \pi_o. \tag{10.3}$$

10.1 Resolvent bounds for $\mathcal{A}_{11}$ for λ not small

Crucial for our estimate of $(\lambda - \mathcal{A})^{-1}$ is to demonstrate uniform invertibility of the operator

$$\lambda - \partial p - \mathcal{S}q = \lambda - \mathcal{A}_{11} - J_{11}$$

which appears as the dominant part of the operator in the $(1,1)$ slot of (6.21). The aim here is to establish uniform invertibility of the operator above with respect to the weighted norm with weight $a = \alpha\epsilon$. The estimate on the inverse will have the form

$$\|(\lambda - \partial p - \mathcal{S}q)^{-1}\|_a \leq \frac{K}{\epsilon^{1+2\hat{\nu}}} \tag{10.4}$$

and be valid for λ satisfying (10.1), provided $\epsilon > 0$ is smaller than some fixed positive constant. (Here and below, K is a generic constant independent of ϵ whose value may change from case to case.) Since we know $\|J_{11}\|_a = O(\epsilon^3)$ by Corollary 7.2, we infer that under conditions of the same form on λ and ϵ, $\lambda - \mathcal{A}_{11}$ is invertible with

$$\|(\lambda - \mathcal{A}_{11})^{-1}\|_a \leq \frac{K}{\epsilon^{1+2\hat{\nu}}}. \tag{10.5}$$

To prove the bound (10.4) we study the equation

$$(\lambda - \partial p - \mathcal{S}q)z = g$$

decomposing this equation in terms of $z_o = \pi_o z$, $z_i = \pi_i z$, $g_o = \pi_o g$, and $g_i = \pi_i g$. Apply π_o and note $\pi_o z_i = 0$, and the low-pass filter (nontrivially) commutes with derivatives and Fourier multipliers. We get

$$\begin{pmatrix} \mathcal{A}_{oo} & \mathcal{A}_{oi} \\ \mathcal{A}_{io} & \mathcal{A}_{ii} \end{pmatrix} \begin{pmatrix} z_o \\ z_i \end{pmatrix} = \begin{pmatrix} g_o \\ g_i \end{pmatrix}, \tag{10.6}$$

$$\mathcal{A}_{oo} = \lambda - \pi_o(\partial p + \mathcal{S}q)\pi_o, \qquad \mathcal{A}_{oi} = -\pi_o(\partial u_p + \mathcal{S}u_q)\pi_i,$$

$$\mathcal{A}_{io} = -\pi_i(\partial u_p + \mathcal{S}u_q)\pi_o, \qquad \mathcal{A}_{ii} = \lambda - \pi_i(\partial p + \mathcal{S}q)\pi_i.$$

Here recall $u_p = p-1$, $u_q = q-1$ satisfy the pointwise bounds in (6.6). The low-pass Fourier filter satisfies $\|\pi_o\partial\|_a \le |\hat\kappa + ia| \le 2\epsilon^{\hat\nu}$ and $\|\pi_o\mathcal{S}\|_a \le K_0\epsilon^{\hat\nu}$ since $|\xi\tanh\xi|^{1/2} = |\xi||\tanh\xi/\xi|^{1/2} \le K_0\epsilon^{\hat\nu}$ for $\xi = k+ia$ with $|k|\le\hat\kappa$, with some constant K_0 independent of ϵ. Also $\|p\pi_o\|_a + \|q\pi_o\|_a \le K_0$. Now clearly, if $|\lambda|\ge\epsilon^\nu$ and ϵ is small enough so $\epsilon^\nu > 2K_1\epsilon^{\hat\nu}$ with $K_1 = 2K_0 + K_0^2$ (we use $\nu<\hat\nu$ here), then $\mathcal{A}_{oo}$ is invertible and

$$\|\mathcal{A}_{oo}^{-1}\|_a \le \frac{1}{|\lambda| - K_1\epsilon^{\hat\nu}} \le \frac{2}{\epsilon^\nu}. \tag{10.7}$$

Since $\partial\tilde u\pi_o = (\tilde u' + \tilde u\partial)\pi_o$ for $\tilde u = u_p$ and u_q, we also find (since $\hat\nu\le 1$)

$$\|\mathcal{A}_{oi}\|_a \le K\epsilon^{2+\hat\nu}, \qquad \|\mathcal{A}_{io}\|_a \le K\epsilon^{2+\hat\nu}. \tag{10.8}$$

In order to establish the estimate (10.5), it suffices to show that whenever λ satisfies (10.1), $\mathcal{A}_{ii}$ is invertible on L^2_a with

$$\|\mathcal{A}_{ii}^{-1}\|_a \le \frac{K}{\epsilon^{1+2\hat\nu}} \tag{10.9}$$

This is because, after elimination of z_0, (10.6) reduces to

$$(I - \mathcal{A}_{ii}^{-1}\mathcal{A}_{io}\mathcal{A}_{oo}^{-1}\mathcal{A}_{oi})z_i = \mathcal{A}_{ii}^{-1}(g_i - \mathcal{A}_{io}\mathcal{A}_{oo}^{-1}g_o). \tag{10.10}$$

Since

$$\|\mathcal{A}_{ii}^{-1}\mathcal{A}_{io}\mathcal{A}_{oo}^{-1}\mathcal{A}_{oi}\|_a \le \frac{K}{\epsilon^{1+2\hat\nu}}(K\epsilon^{2+\hat\nu})^2\frac{K}{\epsilon^\nu} \le K\epsilon^{3-\nu},$$

the desired estimate (10.4) then follows for sufficiently small ϵ and large $|\lambda|$ from

$$\|z_o + z_i\|_a \le \|z_o\|_a + \|z_i\|_a \le \frac{K}{\epsilon^{1+2\hat\nu}}(\|g_i\|_a + \|g_o\|_a) \le \frac{2K}{\epsilon^{1+2\hat\nu}}\|g\|_a.$$

10.2 Uniform invertibility of $\mathcal{A}_{ii}$ by energy estimates

To prove the invertibility of $\mathcal{A}_{ii}$ with the estimate (10.9), since $z = z_o + z_i$, the main step is to prove the energy estimate

$$-\operatorname{Re}\langle\pi_i(\partial p + \mathcal{S}q)\pi_i z, z\rangle_a \ge \tfrac{1}{10}\epsilon^{1+2\hat\nu}\alpha\|z_i\|_a^2 \tag{10.11}$$

for all smooth z with compact support. Given such z, let $z_a = e^{ax}z$. Recall $\partial_a = e^{ax}\partial e^{-ax} = \partial - a$. Then $\pi_{0i}z_a$ is in H^m for all m since

$$\int_{|k|>\hat\kappa}(1+|k|^2)^m|\hat z_a|^2\,dk < \infty.$$

1. Recall $\partial p = \sqrt{p}\partial\sqrt{p} + \frac{1}{2}p'$ and $|p'| \le K\epsilon^3$. Similarly as for $\mathcal{B}_\pm$ which we treated before,

$$\begin{aligned} -\operatorname{Re}\langle \pi_i \sqrt{p}\partial\sqrt{p}\pi_i z, z\rangle_a &= -\operatorname{Re}\int_{-\infty}^{\infty} (\pi_i\sqrt{p}\partial\sqrt{p}\pi_i z)\bar{z}\, e^{2ax} dx \\ &= -\operatorname{Re}\int_{-\infty}^{\infty} (\partial_a \sqrt{p}\pi_{0i} z_a)\overline{\sqrt{p}\pi_{0i} z_a}\, dx \\ &= a\int_{-\infty}^{\infty} p|\pi_{0i} z_a|^2\, dx \ge \epsilon\alpha(1-K\epsilon^2)\|\pi_i z\|_a^2. \end{aligned} \tag{10.12}$$

The last inequality holds since $a = \epsilon\alpha$ and $p \ge 1 - K\epsilon^2$.

2. Now write $\rho = \sqrt{q}$ (as before) and compute

$$\pi_i \mathcal{S} q \pi_i z = \rho\pi_i\mathcal{S}\rho z_i + \mathcal{C}_{\rho i}\rho z_i, \tag{10.13}$$

where, with $u_\rho = \rho - 1$ ($= O(\epsilon^2)$), we can write $\mathcal{C}_{\rho i} = \mathcal{C}_\rho - \mathcal{C}_{\rho o}$ with

$$\mathcal{C}_\rho = [\mathcal{S},\rho] = \mathcal{S}u_\rho - u_\rho\mathcal{S}, \qquad \mathcal{C}_{\rho o} = \pi_o\mathcal{S}u_\rho - u_\rho\pi_o\mathcal{S}, \tag{10.14}$$

Now, since $\|\pi_o\mathcal{S}\|_a \le K\epsilon^{\hat\nu}$, evidently $\|\mathcal{C}_{\rho o}\|_a \le K\epsilon^{2+\hat\nu}$, and Corollary 7.2 implies $\|\mathcal{C}_\rho\|_a \le K\epsilon^3$. Recall that the weight-transformed operator $e^{ax}\mathcal{S}e^{-ax}$ is a Fourier multiplier with symbol given by (6.17). For this we will use the high-frequency dispersion estimate in Corollary 8.3. Note that for $|k| \ge \hat\kappa = \epsilon^{\hat\nu}$, if ϵ is small enough then by (8.2),

$$1 - \tfrac{1}{9}\epsilon^{2\hat\nu} > \sqrt{\frac{\tanh\hat\kappa}{\hat\kappa}} \ge \sqrt{\frac{\tanh k}{k}}. \tag{10.15}$$

Using this with Corollary 8.3, we find

$$\begin{aligned} -\operatorname{Re}\langle \rho\pi_i\mathcal{S}\rho z_i, z_i\rangle_a &= -\operatorname{Re}\int_{-\infty}^{\infty} (\rho\pi_i\mathcal{S}\rho z_i)\bar{z}_i\, e^{2ax} dx \\ &= -\operatorname{Re}\int_{|k|>\hat\kappa} \sqrt{-\gamma\xi\tanh\xi}\,|\mathcal{F}\rho z_{ia}|^2\,\frac{dk}{2\pi} \\ &\ge -\epsilon\hat\alpha(1-\tfrac{1}{9}\epsilon^{2\hat\nu})(1+K\epsilon^2)\|z_i\|_a^2. \end{aligned} \tag{10.16}$$

3. Combining (10.12) with (10.16) and $|p'| + \|\mathcal{C}_{\rho i}\rho\|_a \le K\epsilon^3$ yields (10.11), since for small ϵ,

$$\frac{-\operatorname{Re}\langle \pi_i(\partial p + \mathcal{S}q)\pi_i z, z\rangle_a}{\|z_i\|_a^2} \ge \epsilon\alpha\left(1 - K\epsilon^2 - (1-\tfrac{1}{9}\epsilon^{2\hat\nu})(1+K\epsilon^2)\right) - K\epsilon^3 \ge \tfrac{1}{10}\epsilon^{1+2\hat\nu}\alpha.$$

Since $\mathcal{A}_{ii} = \lambda\pi_o + \mathcal{A}_{ii}\pi_i$, it follows that if λ satisfies (10.1), then $\mathcal{A}_{ii}$ has bounded inverse on its range with bound given by (10.9)

4. To prove that the range of $\mathcal{A}_{ii}$ is all of L^2_a, we use the same continuation approach as previously given for $\mathcal{B}_\pm$. We can write

$$\mathcal{A}_{ii} = \lambda - (\partial + \mathcal{S})\pi_i - \pi_i(\partial u_p + \mathcal{S}u_q)\pi_i.$$

For $\lambda = 1$ fixed, the ϵ-independent bound

$$\|(1 - (\partial + \mathcal{S})\pi_i)^{-1}\partial\pi_i\|_a \leq K \tag{10.17}$$

follows by restricting the proof of (9.10) to frequencies $|k| \geq \hat{\kappa}$. Then we obtain

$$\|(1 - (\partial + \mathcal{S})\pi_i)^{-1}\pi_i(\partial u_p + \mathcal{S}u_q)\|_a \leq K\epsilon^2 < 1$$

for small enough ϵ, and the invertibility of $\mathcal{A}_{ii}$ whenever $\operatorname{Re}\lambda \geq 0$ and $|\lambda| \geq \epsilon/K$ now follows as before for $\mathcal{B}_\pm$, by continuation based on the energy estimate in step 3.

10.3 Bound on the resolvent of $\mathcal{A}$

Now we complete the proof of Proposition 10.1. We solve the resolvent equation

$$(\lambda - \mathcal{A})\begin{pmatrix}\eta_4\\ \phi_4\end{pmatrix} = \begin{pmatrix}g_1\\ g_2\end{pmatrix}$$

by simple elimination, writing

$$\begin{aligned}
\phi_4 &= (\lambda - \mathcal{A}_{22})^{-1}(g_2 + J_{21}\eta_4),\\
\eta_4 &= W_*(\lambda)^{-1}(\lambda - \mathcal{A}_{11})^{-1}(g_1 + J_{12}(\lambda - \mathcal{A}_{22})^{-1}g_2),\\
W_*(\lambda) &= I - (\lambda - \mathcal{A}_{11})^{-1}J_{12}(\lambda - \mathcal{A}_{22})^{-1}J_{21},
\end{aligned}$$

This is justified based on the estimates (9.12), (10.5), and the estimate

$$\|J_{12}(\lambda - \mathcal{A}_{22})^{-1}J_{21}\|_a \leq K\epsilon^5 \tag{10.18}$$

that follows from Corollary 7.2 together with (9.12) for $\operatorname{Re}\lambda \geq -\frac{1}{2}\epsilon\alpha$. For the solution of the system, one obtains the estimates

$$\|\eta_4\|_a \leq \frac{K}{\epsilon^{1+2\hat{\nu}}}\Big(\|g_1\|_a + K\epsilon^2\|g_2\|_a\Big), \qquad \|\phi_4\|_a \leq \frac{K}{\epsilon}\Big(\|g_2\|_a + K\epsilon^{2-2\hat{\nu}}\|g_1\|_a\Big),$$

whence the estimate $\|(\lambda - \mathcal{A})^{-1}\|_a \leq K/\epsilon^{1+2\hat{\nu}}$ follows.

11 KdV scaling and bundle limit

It remains to study the eigenvalue problem when $|\lambda|$ is small. satisfying $|\lambda| \leq \epsilon^\nu$. At this point we have shown that the eigenvalue system (6.21) can be reduced to the nonlinear eigenvalue equation (9.1) whenever $\operatorname{Re}\lambda \geq -\frac{1}{2}\epsilon\alpha$. For $\operatorname{Re}\lambda \geq -\frac{1}{6}\epsilon^3\alpha$, we may further apply the Fourier multiplier $(\lambda - \mathcal{A}_+)^{-1}$ to (9.1), by Corollary 8.3. This reduces the eigenvalue problem to the nonlinear eigenvalue equation

$$W(\lambda)\eta_4 := (I - (\lambda - \mathcal{A}_+)^{-1}\mathcal{U} - J_*)\eta_4 = 0, \tag{11.1}$$

where

$$J_* = (\lambda - \mathcal{A}_+)^{-1}(J_{11} + J_{12}(\lambda - \mathcal{A}_{22})^{-1}J_{21}).$$

The operator J_* will be shown to be negligible.

As we shall see, the bundle $W(\lambda)$ becomes singular at $\lambda = 0$ due to the fact that zero is an eigenvalue of the operator $\mathcal{A}$. To determine the multiplicity of this eigenvalue and establish the invertibility of $W(\lambda)$ for nonzero λ, we make use of the KdV long-wave scaling. We introduce scaled variables with tildes via

$$\tilde{x} = \epsilon x, \quad \lambda = \epsilon^3 \tilde{\lambda}. \tag{11.2}$$

Then $\partial = \epsilon\tilde{\partial}$, and in purely formal terms we have the following leading behavior (see (8.3) and note $\sqrt{-D\tanh D} \sim -\partial$):

$$\lambda - \mathcal{A}_+ \sim \epsilon^3(\tilde{\lambda} - \tfrac{1}{2}\tilde{\partial} + \tfrac{1}{6}\tilde{\partial}^3), \qquad \mathcal{U} = \partial u_p + \mathcal{S}u_q \sim \epsilon^3\tilde{\partial}(-2\Theta) - \epsilon^3\tilde{\partial}(-\tfrac{1}{2}\Theta).$$

Thus we expect $W(\lambda) \sim W_0(\tilde{\lambda})$ where (with tildes omitted on derivatives)

$$W_0(\tilde{\lambda}) = I + (\tilde{\lambda} - \tfrac{1}{2}\partial + \tfrac{1}{6}\partial^3)^{-1}\partial\left(\tfrac{3}{2}\Theta\right). \tag{11.3}$$

This bundle $W_0(\tilde{\lambda})$ is associated with the eigenvalue problem for the KdV equation scaled as

$$\partial_t f - \tfrac{1}{2}\partial_x f + \tfrac{3}{2} f\partial_x f + \tfrac{1}{6}\partial_x^3 f = 0,$$

linearized about the soliton profile $f = \Theta = \operatorname{sech}^2(\sqrt{3}x/2)$.

To be clear, what we are really doing when changing variables is using a similarity transform in terms of the dilation operator τ_ϵ defined by

$$(\tau_\epsilon f)(x) = f(x/\epsilon)/\sqrt{\epsilon} \tag{11.4}$$

which maps L^2_a isometrically onto L^2_α since $a = \alpha\epsilon$:

$$\int_{-\infty}^{\infty} |f(x)|^2 e^{2ax}\,dx = \int_{-\infty}^{\infty} |\tau_\epsilon f(y)|^2 e^{2\alpha y}\,dy, \qquad y = \epsilon x.$$

(Note that similarity transform does not change operator norms, but $\tau_\epsilon\partial\tau_\epsilon^{-1} = \epsilon\partial$.)

The formal discussion above involves uncontrolled approximations in terms of derivatives. But this motivates the following rigorous statement in terms of convergence of bundles. Based on this result, the scaled operator bundle will be studied using the Gohberg-Sigal-Rouché perturbation theorem [19].

Proposition 11.1 *Define the scaled bundle* $\widetilde{W}(\tilde{\lambda}) = \tau_\epsilon W(\epsilon^3\tilde{\lambda})\tau_\epsilon^{-1}$, *and let*

$$\tilde{\Omega}_\epsilon := \{\tilde{\lambda} \in \mathbb{C} : |\epsilon^3\tilde{\lambda}| \le 1,\ \operatorname{Re}\tilde{\lambda} \ge -\tfrac{1}{6}\alpha\}. \tag{11.5}$$

Then in operator norm on L^2_α, *we have*

$$\sup_{\tilde{\lambda}\in\tilde{\Omega}_\epsilon} \|\widetilde{W}(\tilde{\lambda}) - W_0(\tilde{\lambda})\|_\alpha \to 0 \quad \text{as } \epsilon \to 0. \tag{11.6}$$

We prove this proposition by studying pieces of the scaled bundle $\widetilde{W}(\tilde\lambda)$. Writing $\tau_\epsilon \mathcal{S}\tau_\epsilon^{-1} = \sqrt{-\gamma\epsilon D\tanh\epsilon D} = \epsilon\tilde{\mathcal{S}}$, $\tau_\epsilon\mathcal{A}_+\tau_\epsilon^{-1} = \epsilon^3\tilde{\mathcal{A}}_+$, and

$$u_p(x) = \epsilon^2\tilde u_p(\epsilon x), \qquad u_q(x) = \epsilon^2\tilde u_q(\epsilon x), \qquad \tau_\epsilon J_*\tau_\epsilon^{-1} = \tilde J_*,$$

the scaled bundle is written in the form

$$\widetilde{W}(\tilde\lambda) = I - (\tilde\lambda - \tilde{\mathcal{A}}_+)^{-1}(\partial\tilde u_p + \tilde{\mathcal{S}}\tilde u_q) - \tilde J_*. \tag{11.7}$$

The proposition is implied by following convergence results in operator norm on L^2_α, to hold as $\epsilon \to 0$, uniformly for $\tilde\lambda \in \tilde\Omega_\epsilon$:

$$\|(\tilde\lambda - \tilde{\mathcal{A}}_+)^{-1}\mathcal{B} - (\tilde\lambda - \tfrac12\partial + \tfrac16\partial^3)^{-1}\partial\|_\alpha \to 0 \quad \text{for both } \mathcal{B} = \partial \text{ and } \tilde{\mathcal{S}}, \tag{11.8}$$

$$\|\tilde u_p + 2\Theta\|_\alpha \to 0, \qquad \|\tilde u_q + \tfrac12\Theta\|_\alpha \to 0, \tag{11.9}$$

$$\|J_*\|_a \to 0. \tag{11.10}$$

In subsection 11.1, we will prove the first limit (11.8) by studying the corresponding weight-transformed symbols. The third limit (11.10) is treated in subsection 11.2. The limits in (11.9) are a simple consequence of the fact that $\tilde u_p + 2\Theta$ and $\tilde u_q + \frac12\Theta$ are pointwise multipliers, so the L^2_α operator norm is equal to the L^∞ norm as a function, and this is bounded by the H^1 norm, which tends to zero by Lemma 6.1.

11.1 KdV limit for symbols

Here we establish the main limit (11.8) needed to prove the operator limit in Theorem 11.1. Namely, we prove appropriate limits for the scaled, weight-transformed symbol of the operator $\mathcal{M}_\epsilon(\tilde\lambda, D) = (\tilde\lambda - \tilde{\mathcal{A}}_+)^{-1}\partial$. This symbol takes the form

$$\mathcal{M}_\epsilon(\tilde\lambda,\xi) = \frac{\epsilon^3 i\xi}{-\epsilon^3\tilde\lambda + i\epsilon\xi + \sqrt{-\gamma\epsilon\xi\tanh\epsilon\xi}}, \qquad \xi = k + i\alpha. \tag{11.11}$$

The corresponding symbol for the limiting operator $\mathcal{M}_0(\tilde\lambda, D) = (\tilde\lambda - \frac12\partial + \frac16\partial^3)^{-1}\partial$ is written

$$\mathcal{M}_0(\tilde\lambda,\xi) = \frac{i\xi}{-\tilde\lambda + \frac12 i\xi + \frac16 i\xi^3}. \tag{11.12}$$

The symbol limits that we need to prove both limits in (11.8) are:

$$|\mathcal{M}_\epsilon(\tilde\lambda,\xi) - \mathcal{M}_0(\tilde\lambda,\xi)| \to 0 \qquad \text{as } \epsilon \to 0, \tag{11.13}$$

$$\left|\mathcal{M}_\epsilon(\tilde\lambda,\xi)\sqrt{\frac{\gamma\tanh\epsilon\xi}{\epsilon\xi}} - \mathcal{M}_0(\tilde\lambda,\xi)\right| \to 0 \qquad \text{as } \epsilon \to 0. \tag{11.14}$$

These limits need to be established *uniformly for* $\xi \in \mathbb{R} + i\alpha$ *and* $\tilde\lambda \in \tilde\Omega_\epsilon$. (The second limit will follow easily once we establish the first.)

1. First we provide simple, preliminary bounds on the limiting symbol $\mathcal{M}_0$. As one sees from (8.4), for $\tilde\lambda = \tilde\lambda_r + i\tilde\lambda_i$ and $\xi = k + i\alpha$, the real part of the denominator of $\mathcal{M}_0$ is negative, with magnitude bounded below by

$$\tilde\lambda_r + \tfrac{1}{2}\alpha(1-\tfrac{1}{3}\alpha^2) + \tfrac{1}{2}\alpha k^2 \ge \tilde\lambda_r + \tfrac{1}{3}\alpha + \tfrac{1}{2}\alpha k^2 \ge \tfrac{1}{6}\alpha(1+k^2) \ge \tfrac{1}{6}\alpha|\xi|^2, \tag{11.15}$$

provided $\alpha \in (0,1)$ and $\tilde\lambda_r \ge -\frac{1}{6}\alpha$. Consequently we find

$$|\mathcal{M}_0(\tilde\lambda,\xi)| \le \frac{6}{\alpha|\xi|}, \qquad \xi \in \mathbb{R}+i\alpha,\ \operatorname{Re}\tilde\lambda \ge -\tfrac{1}{6}\alpha. \tag{11.16}$$

In particular, since $|\xi| \ge \alpha$, the left-hand side of (11.15) is uniformly bounded away from zero, and $\mathcal{M}_0$ is uniformly bounded.

2. (Low frequencies) Now we carefully identify a long-wave regime where the result of Taylor expansion in (8.4) yields the limits (11.13) and (11.14). Fix $\nu_0 \in (\frac{1}{3},\frac{1}{2})$ and let

$$I_0 = \{\xi \in \mathbb{R}+i\alpha : |\epsilon\xi| \le \epsilon^{\nu_0}\}. \tag{11.17}$$

Put $D_0 = \mathcal{M}_0(\tilde\lambda,\xi)^{-1}$, $E = \mathcal{M}_\epsilon(\tilde\lambda,\xi)^{-1} - D_0$. Then by (8.4), $E = \xi^2 O(\epsilon^2\xi^2)$ and since $|D_0| \ge \frac{1}{6}\alpha|\xi|$ we find that uniformly for $\xi \in I_0$, $\tilde\lambda \in \tilde\Omega_\epsilon$ we have

$$|\mathcal{M}_0(\tilde\lambda,\xi) - \mathcal{M}_\epsilon(\tilde\lambda,\xi)| = \left|\frac{E}{D_0(D_0+E)}\right| \le \left(\frac{6}{\alpha}\right)^2 \frac{K|\epsilon\xi|^2}{1-|\xi|K|\epsilon\xi|^2} \le \hat K\epsilon^{2\nu_0} \tag{11.18}$$

for small enough ϵ, since $\epsilon^2|\xi|^3 \le \epsilon^{3\nu_0-1} = o(1)$. Moreover, for $\xi \in I_0$ one has

$$\left|\sqrt{\frac{\gamma\tanh\epsilon\xi}{\epsilon\xi}} - 1\right| \le K\epsilon^{2\nu_0}.$$

It follows that (11.14) holds uniformly in this regime, as well.

3. For high frequencies the KdV limit is not relevant. In this regime, the symbols $\mathcal{M}_\epsilon$ and $\mathcal{M}_0$ must be shown separately to be small. Let us consider $\mathcal{M}_0$ first. When $\xi \in I_0^c := \mathbb{R}+i\alpha \setminus I_0$ we have $|\xi| \ge \epsilon^{\nu_0-1}$, and from the estimate (11.16) it is clear that

$$\sup_{\xi\in I_0^c,\,\tilde\lambda\in\tilde\Omega_\epsilon} |\mathcal{M}_0(\tilde\lambda,\xi)| \le \frac{6\epsilon^{1-\nu_0}}{\alpha}, \tag{11.19}$$

and this tends to zero as $\epsilon \to 0$.

What remains to show is that

$$\sup_{\xi\in I_0^c,\,\tilde\lambda\in\tilde\Omega_\epsilon} |\mathcal{M}_\epsilon(\tilde\lambda,\xi)| \to 0 \qquad \text{as } \epsilon \to 0. \tag{11.20}$$

Since the square-root factor in (11.14) is bounded, the proof of both (11.13) and (11.14) will be complete once (11.20) is established. This estimate is the most subtle of the symbol estimates. It has nothing to do with the KdV limit, but rather expresses a uniform stability property that holds at high frequency over moderately long time scales. Its proof breaks into two further regimes for $|\xi|$ and involves using Corollary 8.3 to interpolate between low frequencies and high.

4. (High frequencies) Here we consider the set

$$I_\infty = \{\xi \in \mathbb{R} + i\alpha : |\epsilon\xi| \ge K_2\}. \tag{11.21}$$

where K_2 is large. In this regime, the denominator of $\mathcal{M}_\epsilon$ is estimated from below by

$$|\epsilon\xi| - |\gamma\epsilon\xi \tanh \epsilon\xi|^{1/2} - |\epsilon^3\tilde{\lambda}| \ge |\epsilon\xi| - |\epsilon\xi|^{1/2} - 1 \ge \tfrac{1}{2}|\epsilon\xi|,$$

and consequently

$$\sup_{\xi\in I_\infty,\, \tilde{\lambda}\in\tilde{\Omega}_\epsilon} |\mathcal{M}_\epsilon(\tilde{\lambda}, \xi)| \le 2\epsilon^2. \tag{11.22}$$

5. (Transition frequencies) Fix ν_1 with $\nu_0 < \nu_1 < \frac{1}{2}$ and let

$$I_1 = \{\xi = k + i\alpha : 3\epsilon^{\nu_1} \le |\epsilon k| \le K_2\}. \tag{11.23}$$

Recalling $a = \epsilon\alpha$ and $\gamma < 1 - a^2$, we apply Corollary 8.3 together with the bound (8.2) valid for $|\kappa|$ small. Then we find that ϵ small enough, with $\xi = k + i\alpha \in I_1$ and $-\tilde{\lambda}_r < 1$, the real part of the denominator of $\mathcal{M}_\epsilon(\tilde{\lambda}, \xi)$ is negative and bounded (away from zero) by

$$\mathrm{Re}(-\epsilon^3\tilde{\lambda} + i\epsilon\xi + \sqrt{-\gamma\epsilon\xi \tanh \epsilon\xi}) \le -\epsilon^3 + \epsilon\alpha(-1 + 1 - \epsilon^{2\nu_1}) \le -\tfrac{1}{2}\epsilon^{1+2\nu_1}.$$

By consequence we find that

$$\sup_{\xi\in I_1,\, \tilde{\lambda}\in\tilde{\Omega}_\epsilon} |\mathcal{M}_\epsilon(\tilde{\lambda}, \xi)| \le 2K_2\epsilon^{2-1-2\nu_1}, \tag{11.24}$$

and this tends to zero since $\nu_1 < \frac{1}{2}$.

Now $\mathbb{R} + i\alpha = I_0 \cup I_1 \cup I_\infty$, and the outstanding estimate (11.20) is established. This finishes the proof of the limits (11.13)-(11.14).

Remark. The analysis of symbol limits in this section is simpler than the one carried out for lattice solitary waves in [17]. Partly this is due to the simple way that $\tilde{\lambda}$ appears in the denominator of $\mathcal{M}_\epsilon$ here. But partly it is due to the fact that here we must study the regime $|\epsilon^3\tilde{\lambda}| \ge \hat{K}$ by other means, since the symbol $\mathcal{M}_\epsilon$ in (11.11) is *not* bounded on the whole set where $\mathrm{Re}\,\tilde{\lambda} \ge 0$ and $\xi \in \mathbb{R} + i\alpha$.

11.2 Limit of junk terms

To complete the proof of Theorem 11.1 it suffices to prove (11.10), i.e., show that $\|J_*\|_a = o(1)$ in operator norm on L^2_a, uniformly for $\lambda \in \Omega_\epsilon$ where

$$\Omega_\epsilon = \epsilon^3\tilde{\Omega}_\epsilon = \{\lambda \in \mathbb{C} : |\lambda| \le 1,\ \mathrm{Re}\,\lambda \ge -\tfrac{1}{6}\alpha\epsilon^3\}. \tag{11.25}$$

By the bound (10.18) and the resolvent estimate for $\mathcal{A}_+$ in Corollary 8.3, we have

$$\|(\lambda-\mathcal{A}_+)^{-1}J_{12}(\lambda-\mathcal{A}_{22})^{-1}J_{21}\|_a \le K\epsilon^2. \tag{11.26}$$

Further, Corollaries 8.3 and 7.2 imply $\|(\lambda-\mathcal{A}_+)^{-1}J_{11}\|_a \le K$, hence

$$\|J_*\|_a \le K \tag{11.27}$$

uniformly for $\operatorname{Re}\lambda \ge -\frac{1}{6}\alpha\epsilon^3$. It remains to prove that uniformly for $\lambda\in\Omega_\epsilon$,

$$\|(\lambda-\mathcal{A}_+)^{-1}J_{11}\|_a \to 0 \quad \text{as } \epsilon\to 0. \tag{11.28}$$

Then $\|J_*\|_a = o(1)$ will follow. (We remark that it appears this may not be true uniformly for all λ satisfying $\operatorname{Re}\lambda\ge 0$, however.)

Because of the estimates (11.26) and (8.10) and the expression for J_{11} in (6.22), it suffices to prove that the operators

$$\begin{aligned} J_0 &= (\lambda-\mathcal{A}_+)^{-1}(u_q'\sqrt{\gamma}+[\mathcal{S},u_q]), && (11.29)\\ J_1 &= (\lambda-\mathcal{A}_+)^{-1}(u_p'-[\mathcal{S},u_p]\mathcal{S}^{-1}\partial), && (11.30) \end{aligned}$$

are $o(1)$ in L^2_a operator norm uniformly for $\lambda\in\Omega_\epsilon$ — note that $|q'pq^{-1}-\sqrt{\gamma}u_q'| = O(\epsilon^5)$.

In order to bound J_0 it would suffice to note $u_q' = [\partial,u_q]$ and apply Proposition 7.1 with $g=u_q$ and with appropriate symbols. However, the form of J_1 is slightly different, so what we do instead is observe that the weight-transformed operators $\tilde J_j = e^{-ax}J_je^{ax}$ $(j=0,1)$ act on a given smooth f with compact support via

$$(\mathcal{F}\tilde J_j f)(k) = \sqrt{\gamma^{1-j}}\int_{-\infty}^{\infty} P(k)\Big(i(k-\hat k)+(Q(k)-Q(\hat k))R(\hat k)^j\Big)\hat g(k-\hat k)\hat f(\hat k)\,\frac{d\hat k}{2\pi} \tag{11.31}$$

for $j=0$ and 1, with

$$P(k)=\frac{1}{-\lambda+i\xi+\sqrt{-\gamma\xi\tanh\xi}},\qquad Q(k)=\sqrt{-\xi\tanh\xi},\qquad R(k)=\sqrt{\frac{\xi}{\tanh\xi}}.$$

Here $\xi=k+ia$, and $g=u_p$ or u_q has the form $g(x)=\epsilon^2G(\epsilon x)$ with G bounded in H^2 as in section 7. By almost the same short proof as that of Proposition 7.1, we find that

$$\|J_jf\|_a \le C_*C_G\|f\|_a \tag{11.32}$$

where C_G is as in Proposition 7.1 (and is uniformly bounded), and

$$\begin{aligned} C_* &= \sup_{k,\hat k\in\mathbb{R}} \epsilon^2\frac{|P(\epsilon k)||i\epsilon(k-\hat k)+(Q(\epsilon k)-Q(\epsilon\hat k))R(\epsilon\hat k)^j|}{\langle k-\hat k\rangle^s}\\ &= \sup_{k,\hat k\in\mathbb{R}} \epsilon^3|P(\epsilon k)|\left|1+\frac{Q(\epsilon k)-Q(\epsilon\hat k)}{i(\epsilon k-\epsilon\hat k)}R(\epsilon\hat k)^j\right|\frac{|k-\hat k|}{\langle k-\hat k\rangle^s}. \qquad (11.33) \end{aligned}$$

Here we take $s = \frac{4}{3}$ as previously. This implies that the last factor in (11.33) is bounded. To bound the other factors we consider the case $|\epsilon\xi| \le \epsilon^{\hat{\nu}}$ and its opposite, for any $\hat{\nu} \in (0,1)$ fixed.

In the first case, $|\epsilon\xi| \le \epsilon^{\hat{\nu}}$, the factor $|P(\epsilon k)|\epsilon^3 \le K$ uniformly since by Corollary 8.3, the real part of the denominator of $P(\epsilon k)$ is bounded away from zero, satisfying $\operatorname{Re} P(\epsilon k)^{-1} \le -\frac{1}{4}\epsilon^3\alpha$. The middle factor is $O(\epsilon^{\hat{\nu}})$ and tends to zero uniformly, since the symbols Q and R are analytic near $\xi = 0$ and $Q'(k) \to -i$ and $R(k) \to 1$ as $\xi = k + ia \to 0$.

In the other case, $|\epsilon\xi| \ge \epsilon^{\hat{\nu}}$, we note that the middle factor is uniformly bounded due to the estimate (7.5). Due to Corollary 8.3,

$$\operatorname{Re} P(\epsilon k)^{-1} \le -\operatorname{Re}\lambda + \epsilon\alpha\left(-1 + \sqrt{\frac{\tanh \epsilon^{\hat{\nu}}}{\epsilon^{\hat{\nu}}}}\right) \le \alpha\epsilon^3 - \tfrac{1}{9}\alpha\epsilon^{1+2\hat{\nu}}.$$

Hence $|P(\epsilon k)|\epsilon^3 \le K\epsilon^{2-2\hat{\nu}}$ for small ϵ, and we conclude that

$$C_* \le K(\epsilon^{1+2\hat{\nu}} + \epsilon^{\hat{\nu}}) \tag{11.34}$$

which tends to zero uniformly for $\lambda \in \Omega_\epsilon$.

12 Analysis of the bundle limit

For the remainder of the proof of our asymptotic linear stability theorem, there are two approaches possible. One is to proceed in a fashion similar to the treatment of FPU lattice waves in the KdV limit in [17]. In that approach, one notes that any eigenfunction of $\mathcal{A}$ corresponding to a nonzero eigenvalue is orthogonal to two particular elements of the generalized kernel of the adjoint $\mathcal{A}^*$. (In [17] this was expressed in terms of symplectic orthogonality, using Hamiltonian structure.) This yields reduced orthogonality conditions that are necessary for elements of the kernel of the scalar bundle $W(\lambda)$. After an appropriate scaling, one proves convergence of these conditions to corresponding ones for the KdV bundle $W_0(\tilde{\lambda})$, in a dual space. Then uniform invertibility of $W(\lambda)$ on the codimension-2 subspace satisfying the orthogonality conditions follows by a straightforward perturbation argument.

We prefer to emphasize, however, that the required spectral properties follow from the bundle convergence theorem 11.1 by 'soft' arguments based on the GSR perturbation theorem, and does not require further convergence analysis of adjoint zero modes. There are essentially only two 'hard' points left. Namely, we need to show that (i) the bundle $W(\lambda)$ is Fredholm of index zero for relevant values of λ, and (ii) the solitary-wave degrees of freedom (translational shift and wave speed) naturally provide two independent elements in the generalized kernel of $\mathcal{A}$. In comparing the need for point (i) with the alternative approach, we observe that if one knows $\lambda - \mathcal{A}$ has empty kernel, one would likely need to prove a Fredholm property anyway to conclude that $\lambda - \mathcal{A}$ is surjective and λ is in the resolvent set of $\mathcal{A}$.

In this section we will establish point (i), and invoke Gohberg-Sigal-Rouché perturbation theory to characterize the null multiplicity of characteristic values of the bundle $W(\lambda)$. This is related to the algebraic multiplicity of eigenvalues of $\mathcal{A}$ in the following section. Point (ii) is dealt with in Appendix B.

12.1 Fredholm property of the bundle

Lemma 12.1 *For $\epsilon > 0$ sufficiently small, $W(\lambda)$ is Fredholm with index 0 for all $\lambda \in \Omega_\epsilon$.*

Proof. It suffices to show that we may write

$$W(\lambda) = W_i + W_c \tag{12.1}$$

where W_i is invertible and W_c is compact. To demonstrate this, we use Fourier filters π_o and π_i as defined in (10.2)-(10.3). It is convenient however to use a soft wavenumber cutoff in the range $[\hat\kappa, 2\hat\kappa]$ with $\hat\kappa = \epsilon^\nu$ where $0 < \nu < \frac{3}{5}$ (see (11.17)). More precisely, fix $\phi(k) = 1$ ($|k| \le 1$), $2 - |k|$ ($1 \le |k| \le 2$), 0 ($|k| \ge 2$) and set $\phi_\epsilon(k) = \phi(k/\epsilon^\nu)$ and in place of (10.2)-(10.3) define

$$\pi_{0o} = \mathcal{F}^{-1}\phi_\epsilon\mathcal{F} \qquad \pi_o = e^{-ax}\pi_{0o}e^{ax}, \qquad \pi_i = I - \pi_o. \tag{12.2}$$

We then define

$$W_i = I - \pi_i(\lambda - \mathcal{A}_+)^{-1}\mathcal{U} - J_*, \qquad W_c = -\pi_o(\lambda - \mathcal{A}_+)^{-1}\mathcal{U}. \tag{12.3}$$

The operator W_i is uniformly invertible for $\lambda \in \Omega_\epsilon$. This is so since $\|J_*\|_a$ is uniformly small and so is the middle term, for we have $\|\partial^{-1}\mathcal{U}\|_a \le K\epsilon^2$, while $e^{-ax}\pi_i(\lambda - \mathcal{A}_+)^{-1}\partial e^{ax}$ is a Fourier multiplier with symbol dominated by $\mathcal{M}_\epsilon$, with

$$\|\epsilon^2\pi_i(\lambda - \mathcal{A}_+)^{-1}\partial\|_a \le \sup_{\xi\in I_0^c,\,\tilde\lambda\in\Omega_\epsilon} |\mathcal{M}_\epsilon(\tilde\lambda, \xi)| \to 0 \tag{12.4}$$

as $\epsilon \to 0$ due to (11.20). Then, if ϵ is small enough, W_i is invertible for all $\lambda \in \Omega_\epsilon$.

On the other hand, the operator W_c on L^2_a is equivalent to the weight-transformed $e^{-ax}W_c e^{ax}$ on L^2. The latter operator is compact by the convenient compactness criterion of [34]—It is the sum of two terms of the form $\mathcal{F}^{-1}\phi_1\mathcal{F}\phi_2$, where ϕ_1 and ϕ_2 are multipliers by bounded continuous functions on $\mathbb{R}$ that approach zero at infinity. This finishes the proof of the Lemma.

Remark. We note that in the decomposition (12.1), both terms W_c and W_i are analytic functions of λ for $\lambda \in \Omega_\epsilon$. (This fact will be used in studying the full resolvent of $\mathcal{A}$.)

12.2 Characteristic values and the Gohberg-Sigal-Rouché theorem

We first recall some relevant basic information from [19]. (We change some terminology slightly for clarity. An alternative source is [18].) Let X be a Hilbert space, and suppose a function $\lambda \mapsto \mathcal{W}(\lambda)$ is analytic on a complex domain $\Omega_0 \subset \mathbb{C}$, taking values in the space of bounded linear operators on X, and all its values are Fredholm of index zero. A point λ_0 is a *characteristic value* of $\mathcal{W}$ if $\mathcal{W}(\lambda_0)$ has a nontrivial kernel. A *root vector* is an analytic function $z(\lambda)$ with values in X satisfying $\mathcal{W}(\lambda_0)z(\lambda_0) = 0$ with $z(\lambda_0) \ne 0$. The *order* of a root vector at λ_0 is the order of λ_0 as a zero of $\mathcal{W}(\lambda)z(\lambda)$. The *null multiplicity* of a characteristic value is a positive integer whose precise definition in general need not concern us here. The null multiplicity of λ_0 is always at least as large as the maximum order of any root vector. Furthermore, the null multiplicity equals this maximum order if and only if the kernel of $\mathcal{W}(\lambda_0)$ is one-dimensional.

Suppose Ω is a subdomain of Ω_0, with boundary Γ that is a simple closed rectifiable contour in Ω_0, and suppose $\mathcal{W}(\lambda)$ is invertible for all $\lambda \in \Gamma$. The sum of all null multiplicities for all characteristic values in Ω is denoted $n(\mathcal{W}, \Omega)$ and is called the *total multiplicity of* $\mathcal{W}$ *in* Ω. A simple corollary of a far-reaching generalization of Rouché's theorem proved by Gohberg and Sigal [19, 18] is the following.

Theorem 12.2 *Assume that for $j = 1$ and 2, $\mathcal{W}_j(\lambda)$ is analytic and Fredholm of index zero in $\Omega \cup \Gamma$. Assume that for all $\lambda \in \Gamma$, $\mathcal{W}_1(\lambda)$ is invertible and the operator norm*

$$\|\mathcal{W}_1(\lambda)^{-1}(\mathcal{W}_1(\lambda) - \mathcal{W}_2(\lambda)\|_X < 1.$$

Then $\mathcal{W}_2(\lambda)$ is invertible on Γ, and the total multiplicity $n(\mathcal{W}_2, \Omega) = n(\mathcal{W}_1, \Omega)$.

We apply this abstract result with $\mathcal{W}_1 = W_0$ as defined in (11.3), and $\mathcal{W}_2 = \widetilde{W}$ as defined in Proposition 11.1. We take $X = (L^2_a)^2$, and the contour Γ as the boundary of the set $\Omega = \tilde{\Omega}_\epsilon$. As a consequence of Proposition 11.1, for $\epsilon > 0$ sufficiently small, the null multiplicity $n(\widetilde{W}, \tilde{\Omega}_\epsilon) = n(W_0, \tilde{\Omega}_\epsilon)$. But the latter number is 2, as a consequence of the following result.

Proposition 12.3 *Suppose $0 < \alpha < \sqrt{3}$ and $\beta = \frac{1}{2}\alpha(1 - \frac{1}{3}\alpha^2)$. In L^2_α, the only characteristic value of $W_0(\tilde{\lambda})$ with $\operatorname{Re}\tilde{\lambda} > -\beta$ is $\tilde{\lambda} = 0$, and this value has null multiplicity 2.*

This Proposition mainly follows from known facts concerning the eigenvalue problem for the KdV soliton. We provide a self-contained proof in Appendix C for the reader's convenience.

Corollary 12.4 *For $\epsilon > 0$ sufficiently small, $W(\lambda)$ is invertible for all λ on the boundary of $\Omega_\epsilon = \epsilon^3\tilde{\Omega}_\epsilon$, and the total multiplicity of W in Ω_ϵ is 2.*

13 Analysis of resolvent and eigenvalues

It remains to complete the proof that for $\epsilon > 0$ sufficiently small, the operator $\mathcal{A}$ has no eigenvalue with $\operatorname{Re}\lambda \geq -\frac{1}{6}\alpha\epsilon^3$ other than $\lambda = 0$, which is a discrete eigenvalue with algebraic multiplicity 2. Conditional asymptotic stability will then follow directly from the Gearhart-Prüss theorem.

13.1 Resolvent and spectral projection

To begin, we note that by simple elimination, whenever $\lambda \in \Omega_\epsilon$, the resolvent equation

$$(\lambda - \mathcal{A})\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \tag{13.1}$$

is equivalent to

$$W(\lambda)f_1 = (\lambda - \mathcal{A}_+)^{-1}(g_1 + J_{12}(\lambda - \mathcal{A}_{22})^{-1}g_2), \tag{13.2}$$

$$f_2 = (\lambda - \mathcal{A}_{22})^{-1}(J_{21}f_1 + g_2). \tag{13.3}$$

Then clearly, λ is in the resolvent set of $\mathcal{A}$ if $W(\lambda)$ is invertible, so any point of the spectrum of $\mathcal{A}$ inside Ω_ϵ must be a characteristic value of $W(\lambda)$. (The converse is not so easy to argue, since a root vector in L^2_a need not be in H^1_a. We finesse this point in the following argument.)

For $\epsilon > 0$ small, as a consequence of Corollary 12.4, there are at most 2 points of Ω_ϵ in the spectrum of $\mathcal{A}$. We claim that each such spectral point λ_0 is a *discrete eigenvalue* of $\mathcal{A}$, which means that the associated spectral projection,

$$P_0 = \frac{1}{2\pi i}\int_{\Gamma_0} (\lambda - \mathcal{A})^{-1}\, d\lambda, \tag{13.4}$$

has finite rank. Here Γ_0 is a small enough circle about λ_0 enclosing no other point of the spectrum. To prove this, note that from the decomposition formula (12.1), we can write

$$W(\lambda)^{-1} = W_i^{-1} - W(\lambda)^{-1} W_c W_i^{-1}, \tag{13.5}$$

from which we easily deduce from (13.2)-(13.3) that we can write $(\lambda - \mathcal{A})^{-1} = \mathcal{R}_i + \mathcal{R}_c$ where $\mathcal{R}_i$ is analytic in Ω_ϵ and $\mathcal{R}_c$ is compact. Then the integral $\int_{\Gamma_0} \mathcal{R}_i\, d\lambda = 0$ and it follows that P_0 is compact. Since P_0 is a projection, it has finite rank, and it follows that its range consists entirely of generalized eigenvectors of $\mathcal{A}$.

13.2 Algebraic multiplicity of eigenvalues

It remains to relate the algebraic multiplicity of an eigenvalue λ_0 of $\mathcal{A}$ to the null multiplicity of λ_0 as a characteristic value of $W(\lambda)$. These quantities are in fact equal, but for present purposes it suffices to be brief and prove a simpler, weaker result.

Proposition 13.1 *For $\epsilon > 0$ sufficiently small, if $\lambda_0 \in \Omega_\epsilon$ is an eigenvalue of $\mathcal{A}$, then λ_0 is a characteristic value of W. Furthermore, if a Jordan chain $z_1, \ldots, z_k$ is a Jordan chain of elements in $(H^1_a)^2$ satisfying*

$$(\mathcal{A} - \lambda_0) z_j = z_{j-1} \quad \text{for } j = 1, \ldots, k, \text{ with } z_0 = 0,$$

then a root vector $\eta(\lambda)$ of order at least k exists for W at λ_0.

Proof. Supposing $z_1, \ldots, z_k$ is a Jordan chain for $\mathcal{A}$ of length k, let $f(\lambda) = \sum_{j=1}^k (\lambda - \lambda_0)^{j-1} z_j$. Then $f(\lambda)$ is analytic with values in $(H^1_a)^2$ (the domain of $\mathcal{A}$) and $(\lambda - \mathcal{A}) f(\lambda) = (\lambda - \lambda_0)^k z_k =: g(\lambda)$. By elimination, (13.2)-(13.3) hold, and consequently $W(\lambda) f_1(\lambda) = O(|\lambda - \lambda_0|^k)$. Thus there is a root vector $\eta(\lambda) = f_1(\lambda)$ of order at least k, and λ_0 is a characteristic value of W.

13.3 Proof of asymptotic stability

Recall $\alpha \in (0, \frac{1}{2}]$ is fixed, and take $\epsilon > 0$ sufficiently small. As a consequence of the last Proposition and the fact from Appendix B that $\lambda = 0$ has algebraic multiplicity *at least* 2 for $\mathcal{A}$, we conclude that the null multiplicity of the characteristic value $\lambda = 0$ for the bundle W is at least 2. Since the total multiplicity of the bundle W in Ω_ϵ is 2, we deduce that (i) there are no nonzero points of the spectrum of $\mathcal{A}$ in Ω_ϵ, and (ii) the algebraic multiplicity of $\lambda = 0$ is exactly 2. In particular, the kernel of $\mathcal{A}$ is simple (and the same is true for $W(0)$).

Consequently, the spectral projection P_0 for $\lambda = 0$ has rank 2, and restricted to the complementary invariant subspace $\bar{Y}_a = (I - P_0)(L^2_a)^2 = \ker P_0$, the resolvent $(\lambda - \mathcal{A})^{-1}$ is bounded uniformly for $\lambda \in \Omega_\epsilon$. By consequence of Proposition 10.1, this restricted resolvent is bounded uniformly for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq -\frac{1}{6}\alpha\epsilon^3$. It follows automatically that the restricted resolvent is bounded uniformly in a slightly larger half-plane $\operatorname{Re}\lambda \geq -\beta$ for some $\beta > \frac{1}{6}\alpha\epsilon^3$. Using the Gearhart-Prüss asymptotic stability criterion (see Corollary 4 in [37]) gives us the conditional linear asymptotic stability result claimed in Theorem 6.2.

14 Spectral stability without weight

In this section we prove Theorem 6.3, showing that in the unweighted space $(L^2)^2$, the spectrum of the operator $\mathcal{A}$ is the imaginary axis, if $\epsilon > 0$ is sufficiently small. The proof breaks into four steps. For $\operatorname{Re}\lambda > 0$, we show that (i) either λ is in the resolvent set or λ is an eigenvalue, and (ii) if λ is an eigenvalue in $(L^2)^2$, then it is an eigenvalue in $(L^2_a)^2$. Since by Theorem 3.1 there are no such eigenvalues, this proves that $\mathcal{A}$ has no spectrum in the right half-plane. Next we show that (iii) $\mathcal{A}$ has no spectrum in the left half-plane due to a symmetry under space and time reversal. Finally, we show (iv) each point of the imaginary axis does belong to the spectrum of $\mathcal{A}$, by a fairly standard construction of a sequence of approximate eigenfunctions.

1. Suppose $\operatorname{Re}\lambda > 0$. To accomplish the first step, as in section 9 we write

$$\mathcal{A} = \mathcal{A}_* + \mathcal{R}_*, \qquad \mathcal{A}_* = \begin{pmatrix} \mathcal{B}_+ & 0 \\ 0 & \mathcal{B}_- \end{pmatrix}, \qquad \mathcal{R}_* = \begin{pmatrix} \tilde{J}_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}.$$

By applying Lemma 9.1 with $\alpha = 0$, we infer that λ belongs to the resolvent set of $\mathcal{A}_*$ and

$$\lambda - \mathcal{A} = (I - \mathcal{R}_*(\lambda - \mathcal{A}_*)^{-1})(\lambda - \mathcal{A}_*)$$

We claim that $\mathcal{R}_*(\lambda - \mathcal{A}_*)^{-1}$ is compact, whence it follows that either λ is in the resolvent set of $\mathcal{A}$ or it is an eigenvalue. To prove the claim, it suffices to show that each entry is a sum of terms each of which is a product of bounded operators, at least one of which is compact. Let $L = I + \partial$ and note that since the domain of $\mathcal{B}_\pm$ is H^1, the operators $L(\lambda - \mathcal{B}_\pm)^{-1}$ are bounded on L^2. Thus it suffices to show that $\mathcal{R}_* L^{-1}$ is compact. By the criterion of [34], an operator of the form $g\mathcal{Q}$ or $\mathcal{Q}g$ is compact on L^2 provided that g is a pointwise multiplier by a continuous function satisfying $g(x) \to 0$ as $|x| \to \infty$, and $\mathcal{Q}$ is a Fourier multiplier with continuous symbol satisfying $\hat{\mathcal{Q}}(k) \to 0$ as $|k| \to \infty$.

We now deal with the various terms in $\mathcal{R}_* L^{-1}$ from (6.22)-(6.23). With the notation of section 6, note $q' = u_q'$ decays as $|x| \to \infty$, and L^{-1} has symbol $(1 + ik)^{-1}$ tending to 0 as $|k| \to \infty$. Hence the operator $R_1 L^{-1} = pq^{-1}q'L^{-1}$ is compact. Similarly $p'L^{-1}$ is compact.

To treat terms involving commutators, we consider first the worst term, $[p, \mathcal{S}]\mathcal{S}^{-1}\partial L^{-1}$. Note that the operator $\mathcal{S}^{-1}\partial L^{-1}$ is a Fourier multiplier with bounded continuous symbol $\sqrt{k/\tanh k}/(1+$

ik), hence is bounded. (The symbol decays, but we do not use this fact.) We now claim that

$$[p, \mathcal{S}] \text{ is compact.} \tag{14.1}$$

We will show, in fact, that $[p, \mathcal{S}]$ is the uniform-norm limit of a sequence $[p, \mathcal{S}_n]$, where $\mathcal{S}_n$ is a Fourier multiplier with continuous symbol of compact support. Let $\phi : \mathbb{R} \to [0, 1]$ be a smooth cutoff function, taking the value 1 on $[-1, 1]$ and 0 on $\mathbb{R} \setminus [-2, 2]$ and let $\psi = 1 - \phi$. Let $\mathcal{S}_n$ be the Fourier multiplier with symbol $\phi(k/n)\sqrt{-\gamma k \tanh k}$. Then $u_p \mathcal{S}_n$ and $\mathcal{S}_n u_p$ are each compact. As in the proof of Corollary 7.2, using Proposition 7.1, the L^2 operator norm of $[p, \mathcal{S}]-[p, \mathcal{S}_n] = [u_p, \mathcal{S}-\mathcal{S}_n]$ is bounded by $\epsilon^3 C_n C_G$, where C_G is bounded and

$$C_n = \sup_{k,\hat{k}\in\mathbb{R}} \frac{|Q(k)\psi(k/n) - Q(\hat{k})\psi(\hat{k}/n)|}{\langle k - \hat{k}\rangle^{4/3}}, \qquad Q(k) = \sqrt{k \tanh k}$$

Since Q is increasing and Q' decreasing for large k, we have the uniform derivative estimate

$$|(Q(k)\psi(k/n))'| = |Q'(k)\psi(k/n) + Q(k)\psi'(k/n)/n| \le |Q'(n)| + K|Q(2n)/n| \le K/\sqrt{n}.$$

Then it easily follows $C_n \le K/\sqrt{n} \to 0$ as $n \to \infty$. Hence $[p, \mathcal{S}]$ is compact on L^2.

Similarly the commutators $[\mathcal{S}, q]$ and $[\mathcal{S}, \rho]$ are compact, and it follows directly that $\mathcal{R}_* L^{-1}$ is compact. This finishes the first step.

2. For the second step, suppose λ is an eigenvalue with $\operatorname{Re}\lambda > 0$ and with eigenfunction $(\eta_4, \phi_4) \in (H^1)^2$, the domain of $\mathcal{A}$. We then can write (from (6.14))

$$\begin{pmatrix} \lambda - \mathcal{A}_+ & 0 \\ 0 & \lambda - \mathcal{A}_- \end{pmatrix} \begin{pmatrix} \eta_4 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix},$$

with

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} -\partial u_p - \mathcal{S}u_q - u_q\mathcal{S} + R_1 + R_2 & -\mathcal{S}u_q + u_q\mathcal{S} + R_1 - R_2 \\ \mathcal{S}u_q - u_q\mathcal{S} + R_1 - R_2 & -\partial u_p + \mathcal{S}u_q + u_q\mathcal{S} + R_1 + R_2 \end{pmatrix} \begin{pmatrix} \eta_4 \\ \phi_4 \end{pmatrix}.$$

We claim that g_1 and g_2 lie in L^2_a as well as L^2. This is not difficult to check, since $e^{-ax}u_p$ and $e^{-ax}u_q$ are in H^2. Since $\operatorname{Re}\lambda > 0$, the Fourier multipliers $(\lambda - \mathcal{A}_\pm)^{-1}$ are bounded on L^2 and on L^2_a. Indeed, they map the subspace $L^2 \cap L^2_a$ of L^2 into $H^1 \cap H^1_a$ (as one can check by approximation using smooth test functions and analyticity of the Fourier transform for $0 < \operatorname{Im}\xi < a$). It follows that $(\eta_4, \phi_4) \in (H^1_a)^2$, and that λ is an eigenvalue of $\mathcal{A}$ in the space $(L^2_a)^2$. But there is no such eigenvalue for $\epsilon > 0$ sufficiently small, by Theorem 3.1. This concludes the proof of spectral stability for $\mathcal{A}$ in $(L^2)^2$.

3. The resolvent equation for $\mathcal{A}$ has a symmetry under space and time reversal inherited from the original water wave equations. For present purposes, this is most easily studied in terms of the variables used in (6.10), for which the resolvent equation may be written (in $L^2 \times L^2$)

$$\begin{pmatrix} \lambda - q\partial p q^{-1} & q\mathcal{S} \\ \mathcal{S}q & \lambda - \mathcal{S}p\partial\mathcal{S}^{-1} \end{pmatrix} \begin{pmatrix} \eta_3 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \tag{14.2}$$

Recall that p and q are even functions. Let $\mathcal{C}$ be the space reversal operator, $\mathcal{C}f(x) = f(-x)$. Now, $\mathcal{S}$ preserves parity (since its symbol is even) while ∂ reverses it. Applying space reversal to (14.2) the problem is seen to be of the same type after the replacements

$$\lambda \mapsto -\lambda, \qquad \begin{pmatrix} \eta_3 \\ \phi_3 \end{pmatrix} \mapsto \begin{pmatrix} -\mathcal{C}\eta_3 \\ \mathcal{C}\phi_3 \end{pmatrix}.$$

Thus λ is in the resolvent set if and only if $-\lambda$ is. It follows at this point that the L^2 spectrum of $\mathcal{A}$ is contained in the imaginary axis.

4. Suppose $\operatorname{Re}\lambda = 0$. Then there exists $\hat{k} \in \mathbb{R}$ such that $\lambda = \mathcal{A}_+(\hat{k}) = i\hat{k} + i\sqrt{\gamma\hat{k}\tanh\hat{k}}$. Formally, $(\lambda - \mathcal{A}_+)e^{i\hat{k}x} = 0$. We construct a sequence of approximate eigenfunctions for $\mathcal{A}$ in $(L^2)^2$ by a cutoff and translation argument. Fix a smooth function ψ with compact support, and consider pairs (η_4, ϕ_4) of the form

$$\eta_4 = e^{i\hat{k}(x+\tau)}\psi(\nu(x+\tau))\sqrt{\nu}, \qquad \phi_4 = 0.$$

The L^2 norm of η_4 is independent of ν and τ. We claim that taking $\nu = 1/n$, we can choose τ depending on n such that $\|(\lambda - \mathcal{A})(\eta_4, 0)\|_{L^2} \to 0$ as $n \to \infty$. Due to the structure of $\mathcal{A}$ in (6.21) it suffices to show that as $n \to \infty$. in L^2 we have (a) $(\lambda - \mathcal{A}_+)\eta_4 \to 0$, and (b) $(\partial u_p + \mathcal{S}u_q + J_{11})\eta_4$ and $J_{21}\eta_4 \to 0$.

To prove (a), we simply note that the Fourier transform

$$\mathcal{F}((\lambda - \mathcal{A}_+)\eta_4)(k) = e^{ik\tau}(\mathcal{A}_+(\hat{k}) - \mathcal{A}_+(k))\hat{\psi}\left(\frac{k-\hat{k}}{\nu}\right)\frac{1}{\sqrt{\nu}},$$

and this tends to 0 in L^2 as $\nu \to 0$ uniformly in τ.

To prove (b), it is convenient to note that for any fixed $\nu > 0$, $e^{ax}\eta_4 \to 0$ in H^2 as $\tau \to \infty$. Moreover, $u_p e^{-ax}$ and $u_q e^{-ax}$ are bounded in H^1. Then it follows, for example, that $u_p\eta_4$ and $u_q\eta_4 \to 0$ in H^1 as $\tau \to \infty$, and

$$R_2\eta_4 = \Big(\partial(u_p e^{-ax}) - \mathcal{S}(u_p e^{-ax})(e^{ax}\partial\mathcal{S}L^{-1}e^{-ax})\Big)(e^{ax}\eta_4) \to 0$$

in L^2 as $\tau \to \infty$, since the weight-transformed operator $e^{ax}\partial\mathcal{S}L^{-1}e^{-ax}$ has bounded symbol and is bounded on H^1. ($L = 1 + \partial$ as above.) Similarly it follows $[S, u_q]\eta_4$ and $R_1\eta_4 \to 0$ in L^2 as $\tau \to \infty$. Choosing τ appropriately depending on ν, this finishes the proof of (b). Thus each point of the imaginary axis belongs to the L^2 spectrum of $\mathcal{A}$.

A Rigorous asymptotics for solitary wave profiles

Here we provide simple proofs of the estimates on the solitary wave profile needed for our analysis of the eigenvalue problem and resolvent. For a sharper treatment of solitary water waves in the limit $\epsilon \to 0$ see Beale's work [1].

We work with the scaled form of (5.5), written in terms of θ defined through

$$\omega(\underline{x}) = \epsilon^2\theta(\epsilon\underline{x}). \tag{A.1}$$

In terms of θ, we can write the fixed point equation (5.5) in the form

$$\theta = F(\theta) = QN(\theta), \tag{A.2}$$

with Fourier multiplier Q and nonlinearity N defined by

$$Q = \epsilon^2\left(1-\gamma\frac{\tanh\epsilon D}{\epsilon D}\right)^{-1}, \quad N(\theta) = \frac{\frac{3}{2}\theta^2 + \epsilon\theta^3 + \frac{1}{2}(\tanh\epsilon D\,\theta)^2\left(1-2\gamma\epsilon\frac{\tanh\epsilon D}{\epsilon D}\theta\right)}{(1+\epsilon\theta)^2}. \tag{A.3}$$

We study this equation in a weighted Sobolev space of even functions. For $\alpha > 0$ fixed, let

$$X_m = \{f\colon \mathbb{R}\to\mathbb{R} : e^{\alpha x}f \in H^m,\ f \text{ even}\}, \qquad Y_m = \{f\colon \mathbb{R}\to\mathbb{R} : e^{\alpha x}f \in H^m,\ f \text{ odd}\}, \tag{A.4}$$

with the same norm (recall $\langle k\rangle = (1+k^2)^{1/2}$)

$$\|f\|_{X_m} = \|f\|_{Y_m} = \|e^{\alpha x}f\|_{H^m} = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |\langle k\rangle^m \hat{f}(k+i\alpha)|^2\,dk\right)^{1/2}.$$

One has $f \in X_m$ (resp. Y_m) if and only if f is even (resp. odd) and $\cosh\alpha x\,\partial^j f \in L^2(\mathbb{R})$ for $j = 0,...,m$. For $m \geq 1$, the space X_m is a Banach algebra, while the bilinear product map $(f,g)\mapsto fg$ is continuous from $Y_m \times Y_m$ to X_m. The intersection of exponentially weighted H^m spaces is the direct sum of X_m and Y_m:

$$H^m_\alpha \cap H^m_{-\alpha} = X_m \oplus Y_m.$$

Due to the Taylor expansion of tanh, the symbol of Q has the expansion

$$\hat{Q}(\xi) = \frac{1}{1+\frac{1}{3}\gamma\xi^2 + \xi^2 O(\epsilon^2\xi^2)}, \qquad \xi = k + i\alpha. \tag{A.5}$$

Formally, the limit of the fixed point equation (A.2) is

$$\theta = Q_0N_0(\theta), \qquad Q_0 = (1-\tfrac{1}{3}\partial^2)^{-1}, \qquad N_0(\theta) = \tfrac{3}{2}\theta^2. \tag{A.6}$$

Provided $0 < \alpha < \sqrt{3}$, this fixed-point equation is satisfied by the KdV traveling-wave profile

$$\Theta(x) = \operatorname{sech}^2(\sqrt{3}x/2). \tag{A.7}$$

This fixed point is nondegenerate in the space X_m. Indeed, the linearized map $\theta \mapsto \theta - Q_0(3\Theta\theta)$ has bounded inverse on X_m, for the following reason. It is straightforward to show that the map $Q_0\Theta$ is compact on $H^m_\alpha \cap H^m_{-\alpha}$ (using [34]). So if $\theta \mapsto \theta - Q_0(3\Theta\theta)$ is not an isomorphism on X_m, then it vanishes for some nontrivial θ. By a simple bootstrapping argument, this θ must be a smooth function satisfying $(I - \frac{1}{3}\partial^2 + 3\Theta)\theta = 0$ with $e^{\alpha x}\theta \in L^2$. But only constant multiples of the odd function $\theta = \Theta'$ have this property; from standard results for asymptotic behavior in ordinary differential equations, any independent solution θ grows like $e^{\sqrt{3}|x|}$ as $x \to \pm\infty$.

Theorem A.1 *Fix $m \geq 2$, $\alpha \in (0, \sqrt{3})$, and $\nu \in (0,1)$. Then for $\epsilon > 0$ sufficiently small, equation (A.2) has a unique fixed point that belongs to X_m and satisfies $\|\theta - \Theta\|_{X_m} < \epsilon^\nu$. This fixed point θ depends smoothly on ϵ.*

To prove this result, we will invoke a standard fixed-point lemma in the simple quantitative form from the appendix of [14]. To make the estimates needed, we single out one difficult nonlinear term and let

$$N_1(\theta) = (\tanh \epsilon D\, \theta)^2. \tag{A.8}$$

Note $\tanh \epsilon D\, \theta$ is odd if θ is even, and $\tanh \epsilon D$ maps X_m to Y_m continuously. Then we write

$$Q = Q_0 + Q_1, \qquad N = N_0 + N_1 + N_2, \qquad F = F_0 + F_1 + F_2 + F_3, \tag{A.9}$$

with

$$F_0 = Q_0 N_0, \qquad F_1 = Q_0 N_1, \qquad F_2 = Q_0 N_2, \qquad F_3 = Q_1 N.$$

We will prove that for each $j = 0,1,2,3$, F_j is a smooth map on X_m, and will prove that for $\delta = \epsilon^\nu > 0$ small ($\nu \in (0,1)$ fixed) and B_δ a δ-ball about Θ in X_m,

$$\|F_j(\Theta)\|_{X_m} \leq \delta, \qquad \sup_{\theta \in B_\delta} \|F_j'(\theta)\|_{\mathcal{L}(X_m)} \leq \delta, \qquad j = 1,2,3. \tag{A.10}$$

(Here $\|\cdot\|_{\mathcal{L}(X_m)}$ denotes the operator norm on X_m.) The estimates in Theorem A.1 follow directly from Lemma A.1 of [14] by these estimates and the fact that F_0 is smooth and $I - F_0'(\Theta)$ has bounded inverse.

We proceed to prove the estimates in (A.10). It is clear that each N_j ($j = 0,1,2,3$) is smooth in B_δ. Also it is not hard to see that for some constant independent of ϵ, the remainder term in the nonlinearity satisfies

$$\|N_2(\Theta)\|_{X_m} + \sup_{\theta \in B_\delta} \|N_2'(\theta)\|_{\mathcal{L}(X_m)} \leq K\epsilon \tag{A.11}$$

Since Q_0 is bounded on X_m, the estimates (A.10) hold for F_2 with $\delta = K\epsilon$. We now need one more symbol estimate.

Lemma A.2 *Let $\nu \in (0, \frac{1}{2})$. Then there exists K such that for ϵ sufficiently small,*

$$\|Q_1\|_{\mathcal{L}(X_m)} = \sup_{k \in \mathbb{R}} |\hat{Q}_1(k + i\alpha)| \leq K\epsilon^{2\nu}.$$

Proof. We fix $\nu \in (0, \frac{1}{2})$ and consider separately the low-frequency case $|\epsilon\xi| < 3\epsilon^\nu$ and its high-frequency complement, with $\xi = k + i\alpha$.

1. Consider first the regime $|\epsilon\xi| < 3\epsilon^\nu$. Let

$$D_0 = \hat{Q}_0(\xi)^{-1} = 1 + \tfrac{1}{3}\xi^2, \qquad E(\xi) = \hat{Q}_0(\xi)^{-1} - \hat{Q}(\xi)^{-1} = \xi^2 O(\epsilon^2\xi^2),$$

so that

$$\hat{Q}_1(\xi) = \hat{Q}(\xi) - \hat{Q}_0(\xi) = \frac{1}{D_0}\frac{E/D_0}{1 - E/D_0}.$$

From the Taylor expansion used in (A.5) we infer that

$$|\hat{Q}_1(\xi)| \le K\left|\frac{E}{D_0}\right| \le \frac{K\epsilon^{2\nu}|\xi|^2}{|D_0|} \le K\epsilon^{2\nu}.$$

2. In the regime $|\epsilon\xi| > 3\epsilon^\nu$, we estimate $\hat{Q}_0$ and $\hat{Q}$ separately. First, for $\epsilon < 1$ we have $|\xi|^2 > 9$ and consequently $|\hat{Q}_0(\xi)|$ is handled by the estimate

$$|\hat{Q}_0(\xi)| \le \frac{6}{|\xi|^2} \le 2\epsilon^{2-2\nu}. \tag{A.12}$$

It remains to bound $|\hat{Q}(\xi)|$. We calculate that for $\xi = k + i\alpha$ with ϵ small,

$$\begin{aligned} \operatorname{Re}\frac{\tanh\epsilon\xi}{\epsilon\xi} &= \frac{k\sinh 2\epsilon k + \alpha\sin 2\epsilon\alpha}{\cosh 2\epsilon k + \cos 2\epsilon\alpha}\frac{\epsilon^{-1}}{k^2+\alpha^2} \\ &\le \frac{\sec\epsilon\alpha}{\epsilon k}\frac{\sinh 2\epsilon k}{\cosh 2\epsilon k + 1} + \frac{2\alpha^2}{|\xi|^2} \\ &= \sec\epsilon\alpha\frac{\tanh\epsilon k}{\epsilon k} + \frac{2\alpha^2}{|\xi|^2} \le (1+\epsilon^2\alpha^2)(1-\epsilon^{2\nu}) + \epsilon^{2-2\nu}. \end{aligned} \tag{A.13}$$

This implies that

$$\operatorname{Re}\hat{Q}(\xi)^{-1} \ge \epsilon^{-2}\gamma(1-(1+\epsilon^2\alpha^2)(1-\epsilon^{2\nu}) - \epsilon^{2-2\nu}) \ge \tfrac{1}{2}\epsilon^{2\nu-2}$$

for small enough ϵ (since $2\nu < 2 - 2\nu$), and hence

$$|\hat{Q}(\xi)| \le 2\epsilon^{2-2\nu}. \tag{A.14}$$

This finishes the proof of the Lemma.

From this Lemma, the estimates (A.10) for F_3 clearly follow with $\delta = K\epsilon^{2\nu}$.

It remains to prove (A.10) for F_1, with $\delta = K\epsilon^2$. To do this it is convenient to note that the operator Q_0 gains regularity—it is a bounded map from X_{m-1} to X_m. Since N_1 is quadratic, it then suffices to prove that for some constant K independent of ϵ, we have

$$\|(\tanh\epsilon D\,\theta_1)(\tanh\epsilon D\,\theta_2)\|_{X_{m-1}} \le K\epsilon^2\|\theta_1\|_{X_m}\|\theta_2\|_{X_m} \tag{A.15}$$

for all $\theta_1,\ \theta_2 \in X_m$. But this follows easily since the bilinear product map is continuous from $Y_{m-1}\times Y_{m-1}$ to X_{m-1}, and

$$\sup_{k\in\mathbb{R}}\frac{|\tanh\epsilon(k+i\alpha)|}{\langle k\rangle} \le \epsilon\sup_{k\in\mathbb{R}}\frac{|\tanh\epsilon(k+i\alpha)|}{\langle\epsilon k\rangle} \le K\epsilon \tag{A.16}$$

which implies that for all $\theta \in X_m$,

$$\|(\tanh\epsilon D\,\theta)\|_{Y_{m-1}} \le K\epsilon\|\theta\|_{X_m}.$$

This finishes the proof of the estimates in (A.10).

That the fixed point is a smooth function of ϵ is a standard consequence of the easily verified fact that the map $(\epsilon,\theta)\mapsto F(\theta)$ is smooth.

B Neutral modes and adjoints

Here we verify that the translational and wave-speed solitary-wave degrees of freedom naturally yield two independent elements of the generalized kernel of $\mathcal{A}$ in L^2_a, and we demonstrate that the symplectic orthogonality conditions (3.6) transform precisely to the condition that the initial data for the linearized equations lie in the spectral complement to this generalized kernel.

1. Recall that by Theorem 5.1, we have a smooth family of solitary waves (η, U) that are solutions of the equations (5.1). We exploit invariance with respect to translation by differentiating in x to obtain an eigenfunction of (2.21) corresponding to $\lambda = 0$. This is slightly tricky due to the meaning of the variable ϕ_1 in (2.21). We claim that the eigenfunction has the form

$$z_1 = \begin{pmatrix} \eta_1 \\ \phi_1 \end{pmatrix} = \begin{pmatrix} \eta_x \\ \phi_x \end{pmatrix}, \tag{B.1}$$

where $\phi_x = \Phi_x - v\eta_x = u$ is evaluated on the surface $(x, \eta(x))$. To justify this statement, we note that

$$(1-u)\eta_x = \mathcal{H}_\eta u = -v. \tag{B.2}$$

To see this, recall from (5.2) that $\eta = \Psi = \mathcal{H}_\eta \Phi$, and this equation continues to hold for translated wave profiles. Differentiating with respect to the translation parameter we have $\dot\eta = \eta_x = \dot\Psi$ and $\dot\Phi = \Phi_x = \phi_x + v\eta_x$. Then (B.2) follows from the linearization formula (2.18).

Using (B.2) together with direct differentiation of (5.1) (as in (2.20), noting $V = \eta_x$) yields

$$\begin{pmatrix} -\partial_x(1-u) & \partial_x \mathcal{H}_\eta \\ \gamma - (1-u)v' & -(1-u)\partial_x \end{pmatrix} \begin{pmatrix} \eta_x \\ \phi_x \end{pmatrix} = 0. \tag{B.3}$$

Thus $\mathcal{A}_\eta z_1 = 0$. Carrying out the transformations (6.1), (6.9), (6.13) that lead to (6.21), we let

$$z_4 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \gamma q \zeta_* & 0 \\ 0 & \mathcal{S}\zeta_\# \end{pmatrix} \begin{pmatrix} \eta_x \\ \phi_x \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \gamma q (i \tanh D)\omega \\ \mathcal{S}\zeta_\# u \end{pmatrix}. \tag{B.4}$$

Due to the regularity from Theorem A.1 and the formulae (5.8) and (6.4), $z_4 \in (H^1_a \cap H^1_{-a})^2$ with

$$\mathcal{A} z_4 = 0.$$

2. Next we exploit wave-speed variation to find a generalized eigenfunction in L^2_a for $\lambda = 0$. To calculate this, it is convenient to unscale the wave speed and keep γ at a fixed value $\hat\gamma$ when computing variations. For some fixed $\hat c > \sqrt{gh}$ set

$$\hat\gamma = \frac{gh}{\hat c^2}, \quad \tilde c = \frac{c}{\hat c} = \sqrt{\frac{\hat\gamma}{\gamma}}, \quad \eta_*(x;\tilde c) = \eta(x;\gamma), \quad U_*(x;\tilde c) = \tilde c U(x;\gamma), \tag{B.5}$$

$$\Phi^+_*(x;\tilde c) = \tilde c \Phi^+(x;\gamma) = \partial_x^{-1} U_* = \tilde c \int_{+\infty}^{\zeta^{-1}(x)} \omega(s;\gamma)\, ds. \tag{B.6}$$

Then the unscaled solitary-wave profile (η_*, Φ_*^+) is a smooth function of $\tilde{c}$ that takes values in $H_a^1 \times H_a^{3/2}$ and satisfies

$$-\tilde{c}\eta_* + \mathcal{H}_{\eta_*}\Phi_*^+ = 0, \quad -\tilde{c}U_* + \hat{\gamma}\eta_* + \frac{1}{2}(U_*, V_*)M(\eta_*)^{-1}(U_*, V_*)^T = 0, \tag{B.7}$$

where $V_* = \partial_x \mathcal{H}_{\eta_*}\Phi_*^+ = \tilde{c}\partial_x\eta_*$. For $\tilde{c} = 1$ this simply corresponds to (5.1)-(5.2) with $\gamma = \hat{\gamma}$.

We differentiate with respect to $\tilde{c}$, then set $\tilde{c} = 1$ and drop the star subscripts. Denoting the $\tilde{c}$-derivative by the subscript c, using the linearization formula (2.18) we find $V_c = \partial_x\eta + \partial_x\eta_c$ and

$$-\eta - \eta_c + \mathcal{H}_\eta(\Phi_c^+ - v\eta_c) + u\eta_c = 0,$$

$$-U - U_c + \gamma\eta_c + uU_c + v(\partial_x\eta + \partial_x\eta_c) - uv\partial_x\eta_c = 0.$$

Since $U - \eta_x v = u = \phi_x$, this yields

$$\begin{pmatrix} -\partial_x(1-u) & \partial_x\mathcal{H}_\eta \\ \gamma - (1-u)v' & -(1-u)\partial_x \end{pmatrix} \begin{pmatrix} \eta_c \\ \phi_c^+ \end{pmatrix} = \begin{pmatrix} \eta_x \\ \phi_x \end{pmatrix}, \qquad \phi_c^+ = \Phi_c^+ - v\eta_c. \tag{B.8}$$

With

$$y_1 = \begin{pmatrix} \eta_c \\ \phi_c^+ \end{pmatrix}, \qquad y_4 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \gamma q\zeta_* & 0 \\ 0 & \mathcal{S}\zeta_\# \end{pmatrix} y_1, \tag{B.9}$$

we have $-\mathcal{A}_\eta y_1 = z_1$, and find that $y_4 \in (H_a^1)^2$ with

$$-\mathcal{A}y_4 = z_4. \tag{B.10}$$

Adjoint modes. It is a standard fact of operator theory that the space $\bar{Y}_a$, the kernel of the spectral projection P_0 in (13.4), is the subspace annihilated by the generalized kernel of the adjoint $\mathcal{A}^*$. This generalized kernel is two-dimensional (since the generalized kernel of $\mathcal{A}$ is), and we aim to show that the annihilation conditions correspond to the symplectic orthogonality conditions (3.6).

We will work with the Banach space dual L_{-a}^2 of L_a^2, and note that for the Fourier multiplier $\mathcal{S} = \sqrt{-\gamma D \tanh D}$, the adjoint is given formally by $S^* = S$ acting in L_{-a}^2. To see this, take smooth test functions f and g and write $f_a = e^{ax}f$, $g_{-a} = e^{-ax}g$, and $\mathcal{S}_a = e^{ax}\mathcal{S}e^{-ax}$. Then since the symbol of $\mathcal{S}$ satisfies $\overline{\mathcal{S}(k+ia)} = \mathcal{S}(k-ia)$ we have

$$\begin{aligned} \int_{-\infty}^{\infty} (\mathcal{S}f)\overline{g}\,dx &= \int_{-\infty}^{\infty} \mathcal{S}_a f_a \overline{g_{-a}}\,dx = \int_{-\infty}^{\infty} \mathcal{S}(k+ia)\hat{f}_a(k)\overline{\hat{g}_{-a}(k)}\,\frac{dk}{2\pi} \\ &= \int_{-\infty}^{\infty} f_a\overline{\mathcal{S}_{-a}g_{-a}}\,dx = \int_{-\infty}^{\infty} f\overline{\mathcal{S}g}\,dx. \end{aligned}$$

To describe the generalized kernel of the adjoint $\mathcal{A}^*$, first note that with the definition

$$\Phi_*^-(x;\tilde{c}) = \tilde{c}\Phi^-(x;\gamma) = \tilde{c}\int_{-\infty}^{\zeta^{-1}(x)} \omega(s;\gamma)\,ds, \tag{B.11}$$

we can repeat the arguments leading up to (B.8) with Φ^- and H_{-a}^s replacing Φ^+ and H_a^s. Then (B.8) holds with $\phi_c^- = \Phi_c^- - v\eta_c$ replacing ϕ_c^+.

Next, it is convenient to work with the variables used in (6.9)-(6.10) and note that

$$\mathcal{A}_* = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \mathcal{A}_3 \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \frac{1}{2}, \qquad \mathcal{A}_3 = \begin{pmatrix} q\partial pq^{-1} & -q\mathcal{S} \\ -\mathcal{S}q & \mathcal{S}p\partial\mathcal{S}^{-1} \end{pmatrix}.$$

The operator $\mathcal{A}_3$ acts in $(L^2_a)^2$. As can be found by transformation from the original canonical Hamiltonian structure, this operator admits the factorization

$$\mathcal{A}_3 = \mathcal{J}\mathcal{L}, \qquad \mathcal{J} = \begin{pmatrix} 0 & q\mathcal{S} \\ -\mathcal{S}q & 0 \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} 1 & -q^{-1}p\partial\mathcal{S}^{-1} \\ \mathcal{S}^{-1}\partial pq^{-1} & -1 \end{pmatrix}. \tag{B.12}$$

The adjoints are given by $\mathcal{J}^* = -\mathcal{J}$, $\mathcal{L}^* = \mathcal{L}$, and $\mathcal{A}_3^* = -\mathcal{L}\mathcal{J}$, acting in $(L^2_{-a})^2$. Then with

$$z_3 = \begin{pmatrix} \gamma q\zeta_* & 0 \\ 0 & \mathcal{S}\zeta_\# \end{pmatrix} \begin{pmatrix} \eta_x \\ \phi_x \end{pmatrix}, \qquad y_3 = \begin{pmatrix} \gamma q\zeta_* & 0 \\ 0 & \mathcal{S}\zeta_\# \end{pmatrix} \begin{pmatrix} \eta_c \\ \phi_c^- \end{pmatrix}, \tag{B.13}$$

$$z_3^* = \gamma^{-1}\mathcal{J}^{-1}z_3 = \begin{pmatrix} -(\gamma q)^{-1}\zeta_\#\phi_x \\ \mathcal{S}^{-1}\zeta_*\eta_x \end{pmatrix}, \qquad y_3^* = \gamma^{-1}\mathcal{J}^{-1}y_3 = \begin{pmatrix} -(\gamma q)^{-1}\zeta_\#\phi_c^- \\ \mathcal{S}^{-1}\zeta_*\eta_c \end{pmatrix}, \tag{B.14}$$

one can check directly that z_3^*, $y_3^* \in (H^1_{-a})^2$ and

$$\begin{aligned} \mathcal{A}_3^* z_3^* &= \begin{pmatrix} -q^{-1}p\partial q & -q\mathcal{S} \\ -\mathcal{S}q & -\mathcal{S}^{-1}\partial p\mathcal{S} \end{pmatrix} \begin{pmatrix} -(\gamma q)^{-1}\zeta_\#\phi_x \\ \mathcal{S}^{-1}\zeta_*\eta_x \end{pmatrix} \\ &= \begin{pmatrix} (\gamma q)^{-1} & 0 \\ 0 & \mathcal{S}^{-1} \end{pmatrix} \begin{pmatrix} -p\partial & \gamma q^2 \\ D\tanh D & \partial p \end{pmatrix} \begin{pmatrix} \zeta_\#\phi_x \\ \zeta_*\eta_x \end{pmatrix} = 0, \end{aligned}$$

and similarly $-\mathcal{A}_3^* y_3^* = z_3^*$. Thus the generalized kernel of $\mathcal{A}_3^*$ is the span of z_3^* and y_3^*.

Now, corresponding to an arbitrary element $\dot z_1 = (\dot\eta, \dot\phi) \in Z_a = L^2_a \times H^{1/2}_a$ is $\dot z_3 = (\gamma q\zeta_*\dot\eta, \mathcal{S}\zeta_\#\dot\phi) \in (L^2_a)^2$. Then the conditions that z_3^* and y_3^* annihilate $\dot z_3$ transform as follows:

$$0 = -\langle \dot z_3, z_3^* \rangle = \int_{-\infty}^{\infty} (\gamma q\zeta_*\dot\eta)\overline{(\gamma q)^{-1}\zeta_\#\phi_x} - (S\zeta_\#\dot\phi)\overline{\mathcal{S}^{-1}\zeta_*\eta_x}\, d\underline{x} = \int_{-\infty}^{\infty} \dot\eta\phi_x - \dot\phi\eta_x\, dx, \tag{B.15}$$

$$0 = -\langle \dot z_3, y_3^* \rangle = \int_{-\infty}^{\infty} (\gamma q\zeta_*\dot\eta)\overline{(\gamma q)^{-1}\zeta_\#\phi_c^-} - (S\zeta_\#\dot\phi)\overline{\mathcal{S}^{-1}\zeta_*\eta_c}\, d\underline{x} = \int_{-\infty}^{\infty} \dot\eta\phi_c^- - \dot\phi\eta_c\, dx. \tag{B.16}$$

This shows that the symplectic orthogonality conditions (3.6) transform to the precise condition that the initial data for the semigroup $e^{\mathcal{A}t}$ lie in the space $\bar Y_a = \ker P_0$ that is the spectral complement of the generalized kernel of $\mathcal{A}$.

C Characteristic values for the KdV bundle

Here we provide the proof of Proposition 12.3 concerning the characteristic values of the bundle

$$W_0(\lambda) = I + (\lambda - \tfrac{1}{2}\partial + \tfrac{1}{6}\partial^3)^{-1}\partial(\tfrac{3}{2}\Theta), \qquad \Theta = \operatorname{sech}^2(\sqrt{3}x/2).$$

(In this section we will drop the tilde on λ for convenience.) For $\operatorname{Re}\lambda > \beta = \frac{1}{2}\alpha(1-\frac{1}{3}\alpha^2)$, the weight-transformed operator

$$e^{\alpha x}(I - W_0(\lambda))e^{-\alpha x} = (-\lambda + \tfrac{1}{2}(\partial-\alpha) + \tfrac{1}{6}(\partial-\alpha)^3)^{-1}(\partial-\alpha)(\tfrac{3}{2}\Theta)$$

is compact on L^2 and has range in H^1, due to estimates similar to (11.15)-(11.16). On L^2_α, therefore, $W_0(\lambda)$ is Fredholm of index zero. If $W_0(\lambda)f = 0$ for some nonzero $f \in L^2_\alpha$, then $e^{\alpha x}f \in H^m$ for all m by an easy bootstrapping argument, and f satisfies the ordinary differential equation

$$(\lambda - \tfrac{1}{2}\partial + \tfrac{1}{6}\partial^3)f + \partial\left(\tfrac{3}{2}\Theta f\right) = 0. \tag{C.1}$$

By standard results, such an equation has a solution $f \sim e^{\mu x}$ as $x \to \infty$ for each μ that satisfies

$$\lambda - \tfrac{1}{2}\mu + \tfrac{1}{6}\mu^3 = 0. \tag{C.2}$$

For $\lambda > 0$ large, this equation has one root with $\operatorname{Re}\mu < -\alpha$ and two with μ with $\operatorname{Re}\mu > -\alpha$. With $\mu = -\alpha + it$ ($t \in \mathbb{R}$), the curve

$$t \mapsto \mu - \tfrac{1}{3}\mu^3 = -\alpha + \tfrac{1}{3}\alpha^3 - \alpha t^2 + i(t + \tfrac{1}{3}t^3 - \alpha^2 t)$$

has increasing imaginary part and real part less than $-\alpha + \frac{1}{3}\alpha^3 = -2\beta$. For $\operatorname{Re}\lambda > -\beta$, then, (C.2) has a unique and simple root satisfying $\operatorname{Re}\mu < -\alpha$, hence (C.1) has a unique solution (up to a constant factor) satisfying $e^{\alpha x}f \to 0$ as $x \to \infty$. In particular, one may check explicitly (and easily by computer) that

$$f = \partial_x\left(e^{\mu x}((\sqrt{3}+\mu)^2 - (\sqrt{3}+\mu+\mu e^{\sqrt{3}x})\sqrt{3}\operatorname{sech}^2(\sqrt{3}x/2))\right). \tag{C.3}$$

Since $\operatorname{Re}\mu < -\alpha$, clearly $e^{\alpha x}f \in L^2$ is impossible unless $\sqrt{3}+\mu = 0$, meaning $\lambda = 0$ and $f = \partial_x\Theta$.

From the analysis so far, we see that the kernel of $W_0(0)$ in L^2_α is one-dimensional. To finish the proof, we need to show that there is a root vector at 0 with order 2, and no root vector of order 3. Any root vector $f(\lambda)$ at $\lambda = 0$ may be taken in the form $f(\lambda) = f_0 + \lambda f_1 + \lambda^2 f_2 + O(\lambda^3)$ with $f_0 = \partial_x\Theta$. And $W_0(\lambda) = W_0 + W_0'\lambda + \frac{1}{2}W_0''\lambda^2 + O(\lambda^3)$ where $W_0 = W_0(0)$ and

$$W_0' = -(-\tfrac{1}{2}\partial + \tfrac{1}{6}\partial^3)^{-2}\partial(\tfrac{3}{2}\Theta), \qquad \tfrac{1}{2}W_0'' = (-\tfrac{1}{2}\partial + \tfrac{1}{6}\partial^3)^{-3}\partial(\tfrac{3}{2}\Theta).$$

To find a root vector of order 2, it suffices to find $f_1 \in L^2_\alpha$ such that $W_0 f_1 + W_0' f_0 = 0$. Since $W_0 f_0 = 0$ we have

$$W_0' f_0 = (-\tfrac{1}{2}\partial + \tfrac{1}{6}\partial^3)^{-1} f_0,$$

so it suffices to find f_1 such that

$$(-\tfrac{1}{2}\partial + \tfrac{1}{6}\partial^3)f_1 + \partial(\tfrac{3}{2}\Theta f_1) + f_0 = 0.$$

Such a function can be found by differentiating the equation satisfied by the KdV wave profile with respect to wave speed. The function $\varphi_b(x) = b\operatorname{sech}^2\sqrt{3b}x/2$ satisfies $\varphi_1 = \Theta$ and

$$(-\tfrac{1}{2}b\partial_x + \tfrac{1}{6}\partial_x^3)\varphi_b + \partial_x(\tfrac{3}{4}\varphi_b^2) = 0.$$

Differentiating with respect to b and setting $b = 1$, we find that

$$(-\tfrac{1}{2}\partial_x + \tfrac{1}{6}\partial_x^3 + \tfrac{3}{2}\partial_x\Theta)\partial_b\varphi_1 - \tfrac{1}{2}\partial_x\varphi_1 = 0.$$

From this we see that the choice $f_1 = -\frac{1}{2}\partial_b\varphi_1$ works and yields a root vector of order 2. This choice is unique up to adding a scalar multiple of f_0.

To show that there is no root vector of order greater than 2, it suffices to show that with f_1 as above, there is no $f_2 \in L^2_\alpha$ such that

$$W_0 f_2 + W_0' f_1 + \tfrac{1}{2} W_0'' f_0 = 0. \tag{C.4}$$

If such an f_2 exists, then a bootstrapping argument involving the decay estimate (11.16) shows that $e^{\alpha x} f_2 \in H^m$ for all m. Because of the equations satisfied by f_1 and f_0, we find that

$$W_0' f_1 + \tfrac{1}{2} W_0'' f_0 = (-\tfrac{1}{2}\partial_x + \tfrac{1}{6}\partial_x^3)^{-1} f_1.$$

Therefore f_2 must be a smooth solution of

$$(-\tfrac{1}{2}\partial_x + \tfrac{1}{6}\partial_x^3 + \tfrac{3}{2}\partial_x\Theta) f_2 + f_1 = 0. \tag{C.5}$$

Now, the function $\varphi_1 = \Theta$ has $e^{-\alpha x}\varphi_1$ in H^m for all m and satisfies

$$(-\tfrac{1}{2}\partial_x + \tfrac{1}{6}\partial_x^3 + \tfrac{3}{2}\Theta\partial_x)\varphi_1 = 0.$$

Multiplying (C.5) by φ_1 and integrating by parts, we find that the terms involving f_2 vanish. Thus, for f_2 to exist, it is necessary that $\int_{-\infty}^{\infty} \varphi_1 \partial_b \varphi_1 = 0$. But

$$\int_{-\infty}^{\infty} \varphi_1 \partial_b \varphi_1 \, dx = \frac{d}{db} \int_{-\infty}^{\infty} \frac{1}{2} \varphi_b^2 \, dx = \frac{d}{db} b^{3/2} \int_{-\infty}^{\infty} \varphi_1^2 \, dx > 0.$$

Hence, f_2 cannot exist as required, and this proves that the characteristic value $\lambda = 0$ has null multiplicity 2.

Acknowledgments

RLP is grateful for discussions regarding this work with Michael Weinstein, and for partial support during the completion of this work by the Oxford Centre for Nonlinear PDE. This material is based upon work supported by the National Science Foundation under grant nos. DMS 06-04420, 09-05723, and 08-07597, and by the Center for Nonlinear Analysis under NSF grant DMS 06-35983.

References

[1] J. T. Beale, *The existence of solitary water waves*, Comm. Pure Appl. Math., 30 (1977), pp. 373–389.

[2] ——, *Exact solitary water waves with capillary ripples at infinity*, Comm. Pure Appl. Math., 44 (1991), pp. 211–257.

[3] T. B. Benjamin, *The stability of solitary waves*, Proc. Roy. Soc. (London) Ser. A, 328 (1972), pp. 153–183.

[4] J. Bergh and J. Löfström, *Interpolation spaces. An introduction*, Springer-Verlag, Berlin, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[5] J. Bona, *On the stability theory of solitary waves*, Proc. Roy. Soc. London Ser. A, 344 (1975), pp. 363–374.

[6] J. L. Bona and R. L. Sachs, *The existence of internal solitary waves in a two-fluid system near the KdV limit*, Geophys. Astrophys. Fluid Dynam., 48 (1989), pp. 25–51.

[7] J. Boussinesq, *Théorie de l'intumescence liquide appelée onde solitaire ou de translation se propageant dans un canal rectangulair*, C. R. Acad. Sci. Paris, 72 (1871), pp. 755–759.

[8] ——, *Théorie générale des mouvements qui sont propagés dans un canal rectangulaire horizontal*, C. R. Acad. Sci. Paris, 73 (1871), pp. 256–260.

[9] ——, *Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond*, J. Math. Pures et Appliq., 17 (1872), pp. 55–108.

[10] ——, *Essai sur la théorie des eaux courantes*, Mémoires présentés par divers savants á l'Académie des Sciences Inst. France (séries 2), 23 (1877), pp. 1–680.

[11] B. Buffoni, *Existence and conditional energetic stability of capillary-gravity solitary water waves by minimisation*, Arch. Ration. Mech. Anal., 173 (2004), pp. 25–68.

[12] W. Craig, *An existence theory for water waves and the Boussinesq and Korteweg-de Vries scaling limits*, Comm. Partial Differential Equations, 10 (1985), pp. 787–1003.

[13] K. O. Friedrichs and D. H. Hyers, *The existence of solitary waves*, Comm. Pure Appl. Math., 7 (1954), pp. 517–550.

[14] G. Friesecke and R. L. Pego, *Solitary waves on FPU lattices. I. Qualitative properties, renormalization and continuum limit*, Nonlinearity, 12 (1999), pp. 1601–1627.

[15] ——, *Solitary waves on FPU lattices. II. Linear implies nonlinear stability*, Nonlinearity, 15 (2002), pp. 1343–1359.

[16] ——, *Solitary waves on Fermi-Pasta-Ulam lattices. III. Howland-type Floquet theory*, Nonlinearity, 17 (2004), pp. 207–227.

[17] ——, *Solitary waves on Fermi-Pasta-Ulam lattices. IV. Proof of stability at low energy*, Nonlinearity, 17 (2004), pp. 229–251.

[18] I. Gohberg, S. Goldberg, and M. A. Kaashoek, *Classes of linear operators. Vol. I*, vol. 49 of Operator Theory: Advances and Applications, Birkhäuser Verlag, Basel, 1990.

[19] I. C. Gohberg and E. I. Sigal, *An operator generalization of the logarithmic residue theorem and Rouché's theorem*, Mat. Sb. (N.S.), 84(126) (1971), pp. 607–629.

[20] M. Grillakis, J. Shatah, and W. Strauss, *Stability theory of solitary waves in the presence of symmetry. I*, J. Funct. Anal., 74 (1987), pp. 160–197.

[21] ——, *Stability theory of solitary waves in the presence of symmetry. II*, J. Funct. Anal., 94 (1990), pp. 308–348.

[22] M. D. Groves and E. Wahlén, *On the existence and conditional energetic stability of solitary water waves with weak surface tension*, C. R. Math. Acad. Sci. Paris, 348 (2010), pp. 397–402.

[23] M. Haragus and A. Scheel, *Finite-wavelength stability of capillary-gravity solitary waves*, Comm. Math. Phys., 225 (2002), pp. 487–521.

[24] T. Kato, *Perturbation theory for linear operators*, Die Grundlehren der mathematischen Wissenschaften, Band 132, Springer-Verlag New York, Inc., New York, 1966.

[25] D. J. Korteweg and G. de Vries, *On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves*, Phil. Mag. (Ser. 5), 39 (1895), pp. 422–443.

[26] D. Lannes, *Well-posedness of the water-waves equations*, J. Amer. Math. Soc., 18 (2005), pp. 605–654 (electronic).

[27] M. A. Lavrent'ev, *On the theory of long waves; A contribution to the theory of long waves*, Amer. Math. Soc. Transl., 102 (1954), pp. 3–50.

[28] Z. Lin, *On linear instability of 2D solitary water waves*, Int. Math. Res. Not. IMRN, (2009), pp. 1247–1303.

[29] A. Mielke, *On the energetic stability of solitary water waves*, R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci., 360 (2002), pp. 2337–2358. Recent developments in the mathematical theory of water waves (Oberwolfach, 2001).

[30] J. W. Miles, *The Korteweg-de Vries equation: A historical essay*, J. Fluid Mech., 106 (1981), pp. 131–147.

[31] J. R. Miller and M. I. Weinstein, *Asymptotic stability of solitary waves for the regularized long-wave equation*, Comm. Pure Appl. Math., 49 (1996), pp. 399–441.

[32] T. Mizumachi, *Asymptotic stability of lattice solitons in the energy space*, Comm. Math. Phys., 288 (2009), pp. 125–144.

[33] T. Mizumachi and R. L. Pego, *Asymptotic stability of Toda lattice solitons*, Nonlinearity, 21 (2008), pp. 2099–2111.

[34] R. L. Pego, *Compactness in L^2 and the Fourier transform*, Proc. Amer. Math. Soc., 95 (1985), pp. 252–254.

[35] R. L. Pego and S. M. Sun, *On the transverse linear instability of solitary water waves with large surface tension*, Proc. Roy. Soc. Edinburgh Sect. A, 134 (2004), pp. 733–752.

[36] R. L. Pego and M. I. Weinstein, *Asymptotic stability of solitary waves*, Comm. Math. Phys., 164 (1994), pp. 305–349.

[37] J. Prüss, *On the spectrum of C_0-semigroups*, Trans. Amer. Math. Soc., 284 (1984), pp. 847–857.

[38] L. Rayleigh, *On waves*, Phil. Mag. (Ser. 5), 1 (1876), pp. 257–279.

[39] F. Rousset and N. Tzvetkov, *Transverse instability of the line solitary water waves.* Preprint at arXiv:0906.2487v2.

[40] J. S. Russell, *Report on waves*, in Report of the 14th Meeting of the British Association for the Advancement of Science, London, U.K., 1844, John Murray, pp. 311 – 390.

[41] G. Schneider and C. E. Wayne, *The long-wave limit for the water wave problem. I. The case of zero surface tension*, Comm. Pure Appl. Math., 53 (2000), pp. 1475–1535.

[42] S. M. Sun, *Non-existence of truly solitary waves in water with small surface tension*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci., 455 (1999), pp. 2191–2228.

[43] S. Wu, *Well-posedness in Sobolev spaces of the full water wave problem in 2-D*, Invent. Math., 130 (1997), pp. 39–72.

[44] ——, *Almost global wellposedness of the 2-D full water wave problem*, Invent. Math., 177 (2009), pp. 45–135.

[45] V. E. Zakharov, *Stability of periodic waves of finite amplitude on the surface of a deep fluid*, Zh. Prikl. Mekh. Fiz., 9 (1968), pp. 86–94.