EXACTNESS OF THE FOCK SPACE REPRESENTATION OF THE q-COMMUTATION RELATIONS

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Abstract. We show that for all q in the interval $(-1,1)$, the Fock representation of the q-commutation relations can be unitarily embedded into the Fock representation of the extended Cuntz algebra. In particular, this implies that the C*-algebra generated by the Fock representation of the q-commutation relations is exact. An immediate consequence is that the q-Gaussian von Neumann algebra is weakly exact for all q in the interval $(-1,1)$.

1. Introduction

The q-commutation relations provide a q-analogue of the bosonic ($q=1$) and the fermionic ($q=-1$) commutation relations from quantum mechanics. These relations have a natural representation on a deformed Fock space which was introduced by Bozejko and Speicher in [1], and was subsequently studied by a number of authors (see e.g. [2], [5], [6], [7], [9], [10]).

For the entirety of this paper, we fix an integer $d \geq 2$. Consider the usual full Fock space $\mathcal{F}$ over $\mathbb{C}^d$,

$$\mathcal{F} = \oplus_{n=0}^{\infty} \mathcal{F}_n \quad \text{(orthogonal direct sum)}, \tag{1.1}$$

where $\mathcal{F}_0 = \mathbb{C}\Omega$ and $\mathcal{F}_n = (\mathbb{C}^d)^{\otimes n}$ for $n \geq 1$.

Corresponding to the vectors in the standard orthonormal basis of $\mathbb{C}^d$, one has left creation operators $L_1, ..., L_d \in B(\mathcal{F})$. Define the C*-algebra $\mathcal{C}$ by

$$\mathcal{C} := C^*(L_1, \ldots, L_d) \subseteq B(\mathcal{F}). \tag{1.2}$$

It is well known that $\mathcal{C}$ is isomorphic to the extended Cuntz algebra. (Although it is customary to denote the extended Cuntz algebra by $\mathcal{E}$, we use $\mathcal{C}$ here to emphasize that we are working with a concrete C^*-algebra of operators.)

Now let $q \in (-1,1)$ be a deformation parameter. We consider the q-deformation $\mathcal{F}^{(q)}$ of $\mathcal{F}$ as defined in [1]. Thus

$$\mathcal{F}^{(q)} = \oplus_{n=0}^{\infty} \mathcal{F}_n^{(q)} \quad \text{(orthogonal direct sum)}, \tag{1.3}$$

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where every $\mathcal{F}_n^{(q)}$ is obtained by placing a certain deformed inner product on $(\mathbb{C}^d)^{\otimes n}$. (The precise definition will be reviewed in Subsection 2.1 below.) For $q = 0$, one obtains the usual non-deformed Fock space $\mathcal{F}$ from above.

In this deformed setting, one also has natural left creation operators $L_1^{(q)}, ..., L_d^{(q)} \in B(\mathcal{F}^{(q)})$, which satisfy the q-commutation relations

$$L_i^{(q)}(L_j^{(q)})^* = \delta_{ij}I + q(L_j^{(q)})^* L_i^{(q)}, \quad 1 \leq i, j \leq d.$$

Define the C*-algebra $\mathcal{C}^{(q)}$ by

$$\mathcal{C}^{(q)} := C^*(L_1^{(q)}, \ldots, L_d^{(q)}) \subseteq B(\mathcal{F}^{(q)}). \tag{1.4}$$

For $q = 0$, this construction yields the extended Cuntz algebra $\mathcal{C}$ from above.

It is widely believed that the algebra $\mathcal{C}$ and the deformed algebra $\mathcal{C}^{(q)}$ are actually unitarily equivalent. In fact, this is known for sufficiently small q. In [5], a unitary $U : \mathcal{F}^{(q)} \to \mathcal{F}$ was constructed which embeds $\mathcal{C}$ into $\mathcal{C}^{(q)}$ for all $q \in (-1, 1)$, i.e. $\mathcal{C} \subseteq U\mathcal{C}^{(q)}U^*$, and it was shown that for $|q| < 0.44$ this embedding is actually surjective, i.e. $\mathcal{C} = U\mathcal{C}^{(q)}U^*$.

The main purpose of the present paper is to show that it is possible to unitarily embed $\mathcal{C}^{(q)}$ into $\mathcal{C}$ for all $q \in (-1, 1)$. Specifically, we construct a unitary operator $U_{opp} : \mathcal{F}^{(q)} \to \mathcal{F}$ such that $U_{opp}\mathcal{C}^{(q)}U_{opp}^* \subseteq \mathcal{C}$. The unitary U_{opp} is closely related to the unitary U from [5], as we will now see.

Definition 1.1. Let $J : \mathcal{F} \to \mathcal{F}$ be the unitary conjugation operator which reverses the order of the components in a tensor in $(\mathbb{C}^d)^{\otimes n}$, i.e.

$$J(\eta_1 \otimes \cdots \otimes \eta_n) = \eta_n \otimes \cdots \otimes \eta_1, \quad \forall \eta_1, \ldots, \eta_n \in \mathbb{C}^d. \tag{1.5}$$

Note that for $n = 0$, Equation (1.5) says that $J(\Omega) = \Omega$.

Let $J^{(q)} : \mathcal{F}^{(q)} \to \mathcal{F}^{(q)}$ be the operator which acts as in Equation (1.5), where the tensor is now viewed as an element of the space $\mathcal{F}_n^{(q)}$. It is known that $J^{(q)}$ is also unitary operator (see the review in Subsection 2.1).

Definition 1.2. Let $q \in (-1, 1)$ be a deformation parameter and let $U : \mathcal{F}^{(q)} \to \mathcal{F}$ be the unitary defined in [5]. Define a new unitary $U_{opp} : \mathcal{F}^{(q)} \to \mathcal{F}$ by

$$U_{opp} = JUJ^{(q)}.$$

The following theorem is the main result of this paper.

Theorem 1.3. *For every $q \in (-1, 1)$ the unitary U_{opp} from Definition 1.2 satisfies*

$$U_{opp}\mathcal{C}^{(q)}U_{opp}^* \subseteq \mathcal{C}.$$

The following corollary follows immediately from Theorem 1.3.

Corollary 1.4. *For every $q \in (-1, 1)$ the C^*-algebra $\mathcal{C}^{(q)}$ is exact.*

To prove Theorem 1.3, we first consider the more general question of how to verify that an operator $T \in B(\mathcal{F})$ belongs to the algebra $\mathcal{C}$. It is well known that a necessary condition for T to be in $\mathcal{C}$ is that it commutes modulo the compact operators with the C*-algebra generated by right creation operators on $\mathcal{F}$. Unfortunately, this condition isn't sufficient (and wouldn't be sufficient even if we were to set d equal to 1, cf. [4]). Nonetheless, by restricting our attention to a $*$-subalgebra of "band-limited operators" on $\mathcal{F}$ and considering commutators modulo a suitable ideal of compact operators in this algebra, we do obtain a sufficient condition for T to belong to $\mathcal{C}$. This bicommutant-type result is strong enough to help in the proof of Theorem 1.3.

In addition to this introduction, the paper has four other sections. In Section 2, we provide a brief review of the requisite background material. In Section 3, we prove the above-mentioned bicommutant-type result, Theorem 3.8. In Section 4, we establish the main results, Theorem 1.3 and Corollary 1.4. In Section 5, we apply these results to the family of q-Gaussian von Neumann algebras, showing in Theorem 5.1 that these algebras are weakly exact for every $q \in (-1, 1)$.

2. Review of background

2.1. **Basic facts about the q-deformed Fock space.** As explained in the introduction, there is a fairly large body of research devoted to the q-deformed Fock framework and its generalizations. Here we provide only a brief review of the terminology and facts which will be needed in Section 4.

2.1.1. *The q-deformed inner product.* As mentioned above, the integer $d \geq 2$ will remain fixed throughout this paper. Also fixed throughout this paper will be an orthonormal basis $\xi_1, \ldots, \xi_d$ for $\mathbb{C}^d$. For every $n \geq 1$ this gives us a preferred basis for $(\mathbb{C}^d)^{\otimes n}$, namely

$$\{\xi_{i_1} \otimes \cdots \otimes \xi_{i_n} \mid 1 \leq i_1, \ldots, i_n \leq d\}. \tag{2.1}$$

This basis is orthonormal with respect to the usual inner product on $(\mathbb{C}^d)^{\otimes n}$ (obtained by tensoring n copies of the standard inner product on $\mathbb{C}^d$). As in the introduction, we will use $\mathcal{F}_n$ to denote the Hilbert space $(\mathbb{C}^d)^{\otimes n}$ endowed with this inner product. The full Fock space over $\mathbb{C}^d$ is then the Hilbert space $\mathcal{F}$ from Equation (1.1), with the convention that $\mathcal{F}_0 = \mathbb{C}\Omega$ for a distinguished unit vector Ω, referred to as the "vacuum vector".

Now let $q \in (-1, 1)$ be a deformation parameter. It was shown in [1] that there exists a positive definite inner product $\langle \cdot, \cdot \rangle_q$ on $(\mathbb{C}^d)^{\otimes n}$, uniquely determined by the requirement that for vectors in the natural basis (2.1), one has the formula

$$\langle \xi_{i_1} \otimes \cdots \otimes \xi_{i_n}, \xi_{j_1} \otimes \cdots \otimes \xi_{j_n} \rangle_q = \sum_{\sigma} q^{\mathrm{inv}(\sigma)} \delta_{i_1, \sigma(j_1)} \cdots \delta_{i_n, \sigma(j_n)}. \tag{2.2}$$

The sum on the right-hand side of Equation (2.2) is taken over all permutations σ of $\{1, \ldots, n\}$, and $\mathrm{inv}(\sigma)$ denotes the number of inversions of σ, i.e.

$$\mathrm{inv}(\sigma) := |\{(i,j) \mid 1 \leq i < j \leq n,\ \sigma(i) > \sigma(j)\}|.$$

Note that under this new inner product, the natural basis (2.1) will typically no longer be orthogonal.

We will use $\mathcal{F}_n^{(q)}$ to denote the Hilbert space $(\mathbb{C}^d)^{\otimes n}$ endowed with this deformed inner product. In addition, we will use the convention that $\mathcal{F}_0^{(q)}$ is the same as $\mathcal{F}_0$, i.e. it is spanned by the same vacuum vector Ω. The q-deformed Fock space over $\mathbb{C}^d$ is then the Hilbert space $\mathcal{F}^{(q)}$ from Equation (1.3). For $q = 0$, the construction of $\mathcal{F}^{(q)}$ yields the usual non-deformed Fock space $\mathcal{F}$ from Equation (1.1).

2.1.2. *The deformed creation and annihilation operators.* For every $1 \leq j \leq d$, one has deformed left creation operators $L_j^{(q)} \in \mathcal{B}(\mathcal{F}^{(q)})$ and deformed right creation operators $R_j^{(q)} \in \mathcal{B}(\mathcal{F}^{(q)})$, which act on the natural basis of $\mathcal{F}_n^{(q)}$ by $L_j^{(q)}(\Omega) = R_j^{(q)}(\Omega) = \xi_j$ and

$$\left\{ \begin{array}{l} L_j^{(q)}(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) = \xi_j \otimes \xi_{i_1} \otimes \cdots \otimes \xi_{i_n}, \\ R_j^{(q)}(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) = \xi_{i_1} \otimes \cdots \otimes \xi_{i_n} \otimes \xi_j. \end{array} \right. \tag{2.3}$$

Their adjoints are the deformed left annihilation operators $(L_j^{(q)})^*$ and the deformed right annihilation operators $(R_j^{(q)})^*$, which act on the natural basis of $\mathcal{F}_n^{(q)}$ by

$$\left\{ \begin{array}{l} (L_j^{(q)})^*(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) \\ \qquad = \displaystyle\sum_{m=1}^{n} q^{m-1} \delta_{j,i_m} \xi_{i_1} \otimes \cdots \otimes \widehat{\xi_{i_m}} \otimes \cdots \otimes \xi_{i_n}, \\ (R_j^{(q)})^*(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) \\ \qquad = \displaystyle\sum_{m=1}^{n} q^{n-m} \delta_{i_m,j} \xi_{i_1} \otimes \cdots \otimes \widehat{\xi_{i_m}} \otimes \cdots \otimes \xi_{i_n}, \end{array} \right. \tag{2.4}$$

where the "hat" symbol over the component ξ_{i_m} means that it is deleted from the tensor (e.g. $\xi_{i_1} \otimes \widehat{\xi_{i_2}} \otimes \xi_{i_3} = \xi_{i_1} \otimes \xi_{i_3}$).

It's clear from these formulas that the left creation (left annihilation) operators commute with the right creation (right annihilation) operators. For the commutator of a left annihilation operator and a right creation operator, a direct calculation (see also Lemma 3.1 from [10]) gives the formula

$$[(L_i^{(q)})^*, R_j^{(q)}] \mid_{\mathcal{F}_n^{(q)}} = \delta_{ij} q^n I_{\mathcal{F}_n^{(q)}}, \quad \forall n \geq 1. \tag{2.5}$$

Taking adjoints gives the formula for the commutator of a left creation operator and a right annihilation operator.

When we are working on the non-deformed Fock space $\mathcal{F}$ corresponding to the case when $q = 0$, it will be convenient to suppress the superscripts and write L_j and R_j for the left and right creation operators respectively.

Note that in this case, Equation (2.3) and Equation (2.4) imply that

$$\sum_{j=1}^{d} L_j L_j^* = \sum_{j=1}^{d} R_j R_j^* = 1 - P_0, \tag{2.6}$$

where P_0 is the orthogonal projection onto $\mathcal{F}_0$.

2.1.3. *The unitary conjugation operator.* For every $n \geq 1$, let $J_n^{(q)} : \mathcal{F}_n^{(q)} \to \mathcal{F}_n^{(q)}$ be the operator which reverses the order of the components in a tensor in $(\mathbb{C}^d)^{\otimes n}$, i.e, $J_n^{(q)}$ acts by the formula in Equation (1.5) of the Introduction. A consequence of Equation (2.2), which defines the inner product $\langle \cdot, \cdot \rangle_q$, is that $J_n^{(q)}$ is a unitary operator in $B(\mathcal{F}_n^{(q)})$. Indeed, this is easily seen to follow from Equation (2.2) and the following basic fact about inversions of permutations: if θ denotes the special permutation which reverses the order on $\{1, \ldots, n\}$, then one has $\mathrm{inv}(\theta\tau\theta) = \mathrm{inv}(\tau)$ for every permutation τ of $\{1, \ldots, n\}$.

Therefore, we can speak of the unitary operator $J^{(q)} \in B(\mathcal{F}^{(q)})$ from Definition 1.1, which is obtained as $J^{(q)} := \oplus_{n=0}^{\infty} J_n^{(q)}$. Note that $J^{(q)}$ is an involution, i.e. $(J^{(q)})^2 = I_{\mathcal{F}^{(q)}}$, and that it intertwines the left and right creation operators, i.e.

$$R_j^{(q)} = J^{(q)} L_j^{(q)} J^{(q)}, \quad 1 \leq j \leq d. \tag{2.7}$$

2.2. **The original unitary operator.** In this subsection, we review the construction of the unitary $U : \mathcal{F}^{(q)} \to \mathcal{F}$ from [5], which appears in Definition 1.2. An important role in the construction of this unitary is played by the positive operator

$$M^{(q)} := \sum_{j=1}^{d} L_j^{(q)} (L_j^{(q)})^* \in B(\mathcal{F}^{(q)}).$$

Clearly $M^{(q)}$ can be written as a direct sum $M^{(q)} = \oplus_{n=0}^{\infty} M_n^{(q)}$, where $M_n^{(q)}$ is a positive operator on $\mathcal{F}_n^{(q)}$, for every $n \geq 0$. Using Equation (2.3) and Equation (2.4), one can show that $M_n^{(q)}$ acts on the natural basis of $\mathcal{F}_n^{(q)}$ by

$$M_n^{(q)}(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) = \sum_{m=1}^{n} q^{m-1} \xi_{i_m} \otimes \xi_{i_1} \otimes \cdots \otimes \widehat{\xi_{i_m}} \otimes \cdots \otimes \xi_{i_n}. \tag{2.8}$$

(Recall that the "hat" symbol over the component ξ_{i_m} means that it is deleted from the tensor.)

With the exception of $M_0^{(q)}$ (which is zero), the operators $M_n^{(q)}$ are invertible. This is implied by Lemma 4.1 of [5], which also gives the estimate

$$\|(M_n^{(q)})^{-1}\| \leq (1 - |q|) \prod_{k=1}^{\infty} \frac{1 + |q|^k}{1 - |q|^k} < \infty, \quad \forall n \geq 1. \tag{2.9}$$

An important thing to note about Equation (2.9) is that the upper bound on the right-hand side is independent of n.

The unitary operator U is defined as a direct sum, $U := \oplus_{n=0}^{\infty} U_n$, where the unitaries $U_n : \mathcal{F}_n^{(q)} \to \mathcal{F}_n$ are defined recursively as follows: we first define U_0 by $U_0(\Omega) = \Omega$, and for every $n \geq 1$ we define U_n by

$$U_n := (I \otimes U_{n-1})(M_n^{(q)})^{1/2}. \tag{2.10}$$

In Proposition 3.2 of [5] it was shown that U_n as defined in Equation (2.10) is actually a unitary operator, and hence that U is a unitary operator. Moreover, in Section 4 of [5] it was shown that $\mathcal{C} \subseteq U\mathcal{C}^{(q)}U^*$ for every $q \in (-1, 1)$.

2.3. **Summable band-limited operators.** Throughout this section, we fix a Hilbert space $\mathcal{H}$, and in addition we fix an orthogonal direct sum decomposition of $\mathcal{H}$ as

$$\mathcal{H} = \oplus_{n=0}^{\infty} \mathcal{H}_n. \tag{2.11}$$

We will study certain properties an operator $T \in B(\mathcal{H})$ can have with respect to this decomposition of $\mathcal{H}$. We would like to emphasize that the concepts considered here depend not only on $\mathcal{H}$, but also on the orthogonal decomposition for $\mathcal{H}$ in Equation (2.11).

Definition 2.1. Let T be an operator in $B(\mathcal{H})$. If there exists a non-negative integer b such that

$$T(\mathcal{H}_n) \subseteq \bigoplus_{\substack{m \geq 0 \\ |m-n| \leq b}} \mathcal{H}_m, \quad \forall n \geq 0, \tag{2.12}$$

then we will say that T is *band-limited*. A number b as in Equation (2.12) will be called a *band limit* for T. The set of all band-limited operators in $B(\mathcal{H})$ will be denoted by $\mathcal{B}$.

Definition 2.2. Let T be an operator in $\mathcal{B}$. We will say that T is *summable* when it has the property that

$$\sum_{n=0}^{\infty} \|T \mid_{\mathcal{H}_n}\| < \infty,$$

where we have used $T \mid_{\mathcal{H}_n} \in B(\mathcal{H}_n, \mathcal{H})$ to denote the restriction of T to $\mathcal{H}_n$. The set of all summable band-limited operators in $B(\mathcal{H})$ will be denoted by $\mathcal{S}$.

Proposition 2.3. *With respect to the preceding definitions,*

1. *$\mathcal{B}$ is a unital $*$-subalgebra of $B(\mathcal{H})$ and*
2. *$\mathcal{S}$ is a two-sided ideal of $\mathcal{B}$ which is closed under taking adjoints.*

Proof. The proof of (1) is left as an easy exercise for the reader. To verify (2), we first show that $\mathcal{S}$ is closed under taking adjoints. Suppose $T \in \mathcal{S}$, and let b be a band limit for T. By examining the matrix representations of

T and of T^* with respect to the orthogonal decomposition (2.11), it is easily verified that

$$\|T^* \mid_{\mathcal{H}_n} \| \leq \sum_{\substack{m \geq 0 \\ |m-n| \leq b}} \|T \mid_{\mathcal{H}_m} \|, \quad \forall n \geq 0.$$

This implies that

$$\sum_{n=0}^{\infty} \|T^* \mid_{\mathcal{H}_n} \| \leq (2b+1) \sum_{m=0}^{\infty} \|T \mid_{\mathcal{H}_m} \| < \infty,$$

which gives $T^* \in \mathcal{S}$. Next, we show that $\mathcal{S}$ is a two-sided ideal of $\mathcal{B}$. Since $\mathcal{S}$ was proved to be self-adjoint, it will suffice to show that it is a left ideal. It is clear that $\mathcal{S}$ is closed under linear combinations. The fact that $\mathcal{S}$ is a left ideal now follows from the simple observation that for $T \in \mathcal{B}$ and $S \in \mathcal{S}$ we have

$$\sum_{n=0}^{\infty} \|TS \mid_{\mathcal{H}_n}\| \leq \|T\| \sum_{n=0}^{\infty} \|S \mid_{\mathcal{H}_n}\|, < \infty,$$

which implies $TS \in \mathcal{S}$. □

In the following definition, we identify some special types of band-limited operators.

Definition 2.4. Let T be an operator in $\mathcal{B}$.

(1) If T satisfies $T(\mathcal{H}_n) \subseteq \mathcal{H}_n$ for all $n \geq 0$, then we will say that T is *block-diagonal*.
(2) If there is $k \geq 0$ such that T satisfies $T(\mathcal{H}_n) \subseteq \mathcal{H}_{n+k}$ for $n \geq 0$, then we will say that T is *k-raising*.
(3) If there is $k \geq 0$ such that T satisfies $T(\mathcal{H}_n) \subseteq \mathcal{H}_{n-k}$ for $n \geq k$ and $T(\mathcal{H}_n) = \{0\}$ for $n < k$, then we will say that T *is k-lowering.*

Note that a block-diagonal operator is both 0-raising and 0-lowering.

The following proposition gives a Fourier-type decomposition for band-limited operators.

Proposition 2.5. *Let T be an operator in $\mathcal{B}$ with a band-limit $b \geq 0$, as in Definition 2.1. Then we can decompose T as*

$$T = \sum_{k=0}^{b} X_k + \sum_{k=1}^{b} Y_k, \tag{2.13}$$

where each X_k is a k-raising operator for $0 \leq k \leq b$, and each Y_k is a k-lowering operator for $1 \leq k \leq b$. This decomposition is unique. Moreover, if T is summable in the sense of Definition 2.2, then each of the X_k and Y_k are summable.

Proof. First, fix an integer k satisfying $0 \leq k \leq b$. For each $n \geq 0$, consider the linear operator $P_{n+k}T \mid_{\mathcal{H}_n} \in B(\mathcal{H}_n, \mathcal{H}_{n+k})$ which results from composing the orthogonal projection P_{n+k} onto $\mathcal{H}_{n+k}$ with the restriction

$T\mid_{\mathcal{H}_n}$. Clearly $\|P_{n+k}T\mid_{\mathcal{H}_n}\| \leq \|T\|$. This allows us to define an operator $X_k \in B(\mathcal{H})$ which acts on $\mathcal{H}_n$ by

$$X_k\xi = P_{n+k}T\xi, \quad \forall \xi \in \mathcal{H}_n. \tag{2.14}$$

It follows from this definition that X_k is a k-raising operator.

Similarly, for an integer k satisfying $1 \leq k \leq b$, we can define a k-lowering operator $Y_k \in B(\mathcal{H})$ which acts on $\xi \in \mathcal{H}_n$ by

$$Y_k\xi = \begin{cases} P_{n-k}T\xi & \text{if } k \leq n, \\ 0 & \text{if } k > n. \end{cases} \tag{2.15}$$

It's clear that Equation (2.13) holds with each X_k and Y_k defined as above. Conversely, if Equation (2.13) holds, then it's clear that each X_k and Y_k is completely determined as in Equation (2.14) and Equation (2.15) respectively. This implies the uniqueness of this decomposition.

Finally, suppose T is summable. The fact that each X_k and Y_k is summable then follows from the observation that Equation (2.14) and Equation (2.15) imply $\|X_k\mid_{\mathcal{H}_n}\| \leq \|T\mid_{\mathcal{H}_n}\|$ and $\|Y_k\mid_{\mathcal{H}_n}\| \leq \|T\mid_{\mathcal{H}_n}\|$ for every $n \geq 0$. □

The following result about commutators will be needed in Section 4.

Proposition 2.6. *Let $T \in \mathcal{B}$ be a positive block-diagonal operator, and let $V \in \mathcal{B}$ be a 1-raising operator. Suppose that the commutator $[T, V]$ satisfies*

$$\sum_{n=0}^{\infty} \|[T, V]\mid_{\mathcal{H}_n}\|^{1/2} < \infty. \tag{2.16}$$

Then the commutator $[T^{1/2}, V]$ is a summable 1-raising operator.

Proof. For every $n \geq 0$, let $T_n = T\mid_{\mathcal{H}_n} \in B(\mathcal{H}_n)$ and let $V_n = V\mid_{\mathcal{H}_n} \in B(\mathcal{H}_n, \mathcal{H}_{n+1})$. Since T is block-diagonal and V is 1-raising, it's clear that $[T, V]$ and $[T^{1/2}, V]$ are 1-raising operators which satisfy

$$[T, V]\mid_{\mathcal{H}_n} = T_{n+1}V_n - V_nT_n, \quad \forall n \geq 0,$$

and

$$[T^{1/2}, V]\mid_{\mathcal{H}_n} = T_{n+1}^{1/2}V_n - V_nT_n^{1/2}, \quad \forall n \geq 0.$$

It follows that the hypothesis (2.16) can be rewritten as

$$\sum_{n=0}^{\infty} \|T_{n+1}V_n - V_nT_n\|^{1/2} < \infty,$$

while the required conclusion that $[T^{1/2}, V] \in \mathcal{S}$ is equivalent to

$$\sum_{n=0}^{\infty} \|T_{n+1}^{1/2}V_n - V_nT_n^{1/2}\| < \infty.$$

We will prove that this holds by showing that for every $n \geq 0$,

$$\|T_{n+1}^{1/2}V_n - V_nT_n^{1/2}\| \leq \frac{5}{4}\|V\|^{1/2}\|T_{n+1}V_n - V_nT_n\|^{1/2}. \tag{2.17}$$

For the rest of the proof, fix $n \geq 0$. Consider the operators $A, B \in B(\mathcal{H}_n \oplus \mathcal{H}_{n+1})$ which, written as 2×2 matrices, are given by

$$A := \begin{bmatrix} T_n & 0 \\ 0 & T_{n+1} \end{bmatrix}, \qquad B := \begin{bmatrix} 0 & V_n^* \\ V_n & 0 \end{bmatrix}.$$

Since T is positive, it follows that A is positive, with

$$A^{1/2} = \begin{bmatrix} T_n^{1/2} & 0 \\ 0 & T_{n+1}^{1/2} \end{bmatrix}.$$

A well-known commutator inequality (see e.g. [8]) gives

$$\|[A^{1/2}, B]\| \leq \frac{5}{4}\|B\|^{1/2}\|[A, B]\|^{1/2}. \tag{2.18}$$

From the definitions of A and B, we compute

$$[A, B] = \begin{bmatrix} 0 & (T_{n+1}V_n - V_nT_n)^* \\ T_{n+1}V_n - V_nT_n & 0 \end{bmatrix},$$

and this implies $\|[A, B]\| = \|T_{n+1}V_n - V_nT_n\|$. Similarly, $\|[A^{1/2}, B]\| = \|T_{n+1}^{1/2}V_n - V_nT_n^{1/2}\|$, and it's clear that $\|B\| = \|V_n\|$. By substituting these equalities into (2.18) we obtain

$$\|T_{n+1}^{1/2}V_n - V_nT_n^{1/2}\| \leq \frac{5}{4}\|V_n\|^{1/2}\|T_{n+1}V_n - V_nT_n\|^{1/2}.$$

Since $\|V_n\| \leq \|V\|$, this clearly implies that (2.17) holds. □

3. An inclusion criterion

In this section, we work exclusively in the framework of the (non-deformed) extended Cuntz algebra $\mathcal{C}$. We will use the terminology of Subsection 2.3 with respect to the natural decomposition $\mathcal{F} = \oplus_{n=0}^{\infty}\mathcal{F}_n$. In particular, we will refer to the unital $*$-subalgebra $\mathcal{B} \subseteq B(\mathcal{F})$ which consists of band-limited operators as in Definition 2.1, and to the ideal $\mathcal{S}$ of $\mathcal{B}$ which consists of summable band-limited operators as in Definition 2.2.

The main result of this section is Theorem 3.8. This is an analogue in the C*-framework of the bicommutant theorem from von Neumann algebra theory, where we restrict our attention to the $*$-algebra $\mathcal{B}$ and consider commutators modulo the ideal $\mathcal{S}$. In this framework, the role of "commutant" is played by the C*-algebra generated by right creation operators on $\mathcal{F}$.

For clarity, we will first consider the special case of a block-diagonal operator.

Definition 3.1. Let $T \in \mathcal{B}$ be a block-diagonal operator. The sequence of $\mathcal{C}$-*approximants* for T is the sequence $(A_n)_{n=0}^{\infty}$ of block-diagonal elements of $\mathcal{C}$ defined recursively as follows: we first define A_0 by $A_0 = \langle T(\Omega), \Omega\rangle I_{\mathcal{F}}$,

and for every $n \geq 0$ we define A_{n+1} by
(3.1)
$$A_{n+1} := A_n + \sum_{\substack{1 \leq i_1, \ldots, i_{n+1} \leq d \\ 1 \leq j_1, \ldots, j_{n+1} \leq d}} c_{i_1,\ldots,i_{n+1};j_1,\ldots,j_{n+1}} \big(L_{i_1} \cdots L_{i_{n+1}}\big)\big(L_{j_1} \cdots L_{j_{n+1}}\big)^*,$$

where the coefficients $c_{i_1,\ldots,i_{n+1};j_1,\ldots,j_{n+1}}$ are defined by
(3.2)
$$\begin{aligned} c_{i_1,\ldots,i_{n+1};j_1,\ldots,j_{n+1}} := \; & \langle T(\xi_{j_1} \otimes \cdots \otimes \xi_{j_{n+1}}), \xi_{i_1} \otimes \cdots \otimes \xi_{i_{1+1n}} \rangle \\ & -\delta_{i_{n+1},j_{n+1}} \cdot \langle T(\xi_{j_1} \otimes \cdots \otimes \xi_{j_n}), \xi_{i_1} \otimes \cdots \otimes \xi_{i_n} \rangle. \end{aligned}$$

The main property of the approximant A_n is that it agrees with the operator T on each subspace $\mathcal{F}_m$ for $m \leq n$. More precisely, we have the following lemma.

Lemma 3.2. *Let $T \in \mathcal{B}$ be a block-diagonal operator, and let $(A_n)_{n=0}^{\infty}$ be the sequence of $\mathcal{C}$-approximants for T, as in Definition 3.1. Then for every $m \geq 0$,*

$$A_n \mid_{\mathcal{F}_m} = \begin{cases} T \mid_{\mathcal{F}_m} & \text{if } m \leq n, \\ (T \mid_{\mathcal{F}_n}) \otimes I_{m-n} & \text{if } m > n. \end{cases} \tag{3.3}$$

Proof. We will show that for every fixed $n \geq 0$, Equation (3.3) holds for all $m \geq 0$. The proof of this statment will proceed by induction on n. The base case $n = 0$ is left as an easy exercise for the reader. The remainder of the proof is devoted to the induction step. Fix $n \geq 0$ and assume that Equation (3.3) holds for this n and for all $m \geq 0$. We will prove the analogous statement for $n + 1$.

From Equation (3.1), it is immediate that

$$A_{n+1} \mid_{\mathcal{F}_m} = A_n \mid_{\mathcal{F}_m} = T \mid_{\mathcal{F}_m}, \quad \forall m \leq n.$$

Thus it remains to fix $m \geq n + 1$ and verify that

$$A_{n+1} \mid_{\mathcal{F}_m} = (T \mid_{\mathcal{F}_{n+1}}) \otimes I_{m-n-1} \in B(\mathcal{F}_m).$$

In light of how $(T \mid_{\mathcal{F}_{n+1}}) \otimes I_{m-n-1}$ acts on the canonical basis of $\mathcal{F}_m$, this amounts to showing that for every $1 \leq k_1, \ldots, k_m, \ell_1, \ldots, \ell_m \leq d$, one has

$$\begin{aligned} & \langle A_{n+1}(\xi_{\ell_1} \otimes \cdots \otimes \xi_{\ell_m}), \xi_{k_1} \otimes \cdots \otimes \xi_{k_m} \rangle \\ & \quad = \delta_{k_{n+2},\ell_{n+2}} \cdots \delta_{k_m,\ell_m} \langle T(\xi_{\ell_1} \otimes \cdots \otimes \xi_{\ell_{n+1}}), \xi_{k_1} \otimes \cdots \otimes \xi_{k_{n+1}} \rangle. \end{aligned} \tag{3.4}$$

On the left-hand side of Equation (3.4) we substitute for A_{n+1} using the recursive definition given by Equation (3.1). This gives

$$\begin{aligned} & \langle A_{n+1}(\xi_{\ell_1} \otimes \cdots \otimes \xi_{\ell_m}), \xi_{k_1} \otimes \cdots \otimes \xi_{k_m} \rangle \\ & \quad = \langle A_n \xi_{l_1} \otimes \cdots \otimes \xi_{l_m}), \xi_{k_1} \otimes \cdots \xi_{k_m} \rangle \\ & \quad + \sum_{\substack{i_1,\ldots,i_{n+1} \\ j_1,\ldots,j_{n+1}}} c_{i_1,\ldots,i_{n+1};j_1,\ldots,j_{n+1}} \alpha(i_1, \ldots, i_{n+1}; j_1, \ldots, j_{n+1}), \end{aligned} \tag{3.5}$$

where for every $1 \leq i_1, \dots i_{n+1}, j_1, \dots, j_{n+1} \leq d$, we have written

$$\begin{aligned}&\alpha(i_1, \dots, i_{n+1}; j_1, \dots, j_{n+1})\\ &\quad = \langle \big(L_{i_1} \cdots L_{i_{n+1}}\big)\big(L_{j_1} \cdots L_{j_{n+1}}\big)^*(\xi_{\ell_1} \otimes \cdots \otimes \xi_{\ell_m}), (\xi_{k_1} \otimes \cdots \otimes \xi_{k_m})\rangle.\end{aligned}$$

It is clear that an inner product like the one just written simplifies as follows:

$$\begin{aligned}&\langle \big(L_{i_1} \cdots L_{i_{n+1}}\big)\big(L_{j_1} \cdots L_{j_{n+1}}\big)^*(\xi_{\ell_1} \otimes \cdots \otimes \xi_{\ell_m}), (\xi_{k_1} \otimes \cdots \otimes \xi_{k_m})\rangle\\ &\ = \langle \big(L_{j_1} \cdots L_{j_{n+1}}\big)^*(\xi_{\ell_1} \otimes \cdots \otimes \xi_{\ell_m}), (L_{i_1} \cdots L_{i_{n+1}})^*(\xi_{k_1} \otimes \cdots \otimes \xi_{k_m})\rangle\\ &\ = \delta_{i_1,k_1} \cdots \delta_{i_{n+1},k_{n+1}} \delta_{j_1,\ell_1} \cdots \delta_{j_{n+1},\ell_{n+1}} \langle \xi_{\ell_{n+2}} \otimes \cdots \otimes \xi_{\ell_m}, \xi_{k_{n+2}} \otimes \cdots \otimes \xi_{k_m}\rangle\\ &\ = \delta_{i_1,k_1} \cdots \delta_{i_{n+1},k_{n+1}} \delta_{j_1,\ell_1} \cdots \delta_{j_{n+1},\ell_{n+1}} \delta_{\ell_{n+2},k_{n+2}} \cdots \delta_{\ell_m,k_m}.\end{aligned}$$

Thus in the sum on the right-hand side of Equation (3.5), the only term that survives is the one corresponding to $i_1 = k_1, \dots, i_{n+1} = k_{n+1}$ and $j_1 = \ell_1, \dots, j_{n+1} = \ell_{n+1}$, and we obtain that

$$\begin{aligned}\langle A_{n+1}(\xi_{\ell_1} \otimes \cdots \otimes \xi_{\ell_m}), \xi_{k_1} \otimes \cdots \otimes \xi_{k_m}\rangle &= \langle A_n(\xi_{\ell_1} \otimes \cdots \otimes \xi_{\ell_m}), \xi_{k_1} \otimes \cdots \otimes \xi_{k_m}\rangle\\ &\quad + \delta_{\ell_{n+2},k_{n+2}} \cdots \delta_{\ell_m,k_m} c_{k_1,\dots,k_{n+1};\ell_1,\dots,\ell_{n+1}}.\end{aligned} \tag{3.6}$$

Finally, we remember our induction hypothesis, which gives

$$\begin{aligned}&\langle A_n(\xi_{\ell_1} \otimes \cdots \otimes \xi_{\ell_m}), \xi_{k_1} \otimes \cdots \otimes \xi_{k_m}\rangle\\ &\quad = \delta_{k_{n+1},\ell_{n+1}} \cdots \delta_{k_m,\ell_m} \langle T(\xi_{\ell_1} \otimes \cdots \otimes \xi_{\ell_n}), \xi_{k_1} \otimes \cdots \otimes \xi_{k_n}\rangle.\end{aligned} \tag{3.7}$$

A straightforward calculation shows that if we substitute Equation (3.7) into Equation (3.6) and use Formula (3.2) which defines the coefficient $c_{k_1,\dots,k_{n+1};\ell_1,\dots,\ell_{n+1}}$, then we arrive at the right-hand side of Equation (3.4). This completes the induction argument. □

Lemma 3.3. *Let $T \in \mathcal{B}$ be a block-diagonal operator, and let $(A_n)_{n=1}^{\infty}$ be the sequence of $\mathcal{C}$-approximants for T, as in Definition 3.1. Then for every $n \geq 1$,*

$$\|A_{n+1} - A_n\| = \|T \mid_{\mathcal{F}_{n+1}} - (T \mid_{\mathcal{F}_n}) \otimes I\|. \tag{3.8}$$

Proof. Note that since $A_{n+1} - A_n$ is block-diagonal,

$$\|A_{n+1} - A_n\| = \sup_{m \geq 0} \|A_{n+1} |_{\mathcal{F}_m} - A_n |_{\mathcal{F}_m}\|.$$

To compute this supremum, there are three cases to consider. In each case we apply Lemma 3.2. First, for $m \leq n$,

$$\|A_{n+1} |_{\mathcal{F}_m} - A_n |_{\mathcal{F}_m}\| = 0.$$

Next, for $m = n + 1$,

$$\|A_{n+1} |_{\mathcal{F}_{n+1}} - A_n |_{\mathcal{F}_{n+1}}\| = \|T |_{\mathcal{F}_{n+1}} - (T |_{\mathcal{F}_n}) \otimes I\|.$$

Finally, for $m > n+1$,

$$\begin{aligned}\|A_{n+1}|_{\mathcal{F}_m} - A_n|_{\mathcal{F}_m}\| &= \|(T|_{\mathcal{F}_{n+1}}) \otimes I_{m-n-1} - (T|_{\mathcal{F}_n}) \otimes I_{m-n}\| \\ &= \|(T|_{\mathcal{F}_{n+1}} - (T|_{\mathcal{F}_n}) \otimes I) \otimes I_{m-n-1}\| \\ &= \|T|_{\mathcal{F}_{n+1}} - (T|_{\mathcal{F}_n}) \otimes I\|.\end{aligned}$$

This makes it clear that the supremum over all $m \geq 0$ is equal to the right hand side of Equation (3.8), as required. □

Lemma 3.4. *Let T be a block-diagonal operator. If T satisfies*

$$\sum_{n=1}^{\infty} \|(T|_{\mathcal{F}_{n+1}}) - (T|_{\mathcal{F}_n}) \otimes I\| < \infty,$$

then $T \in \mathcal{C}$.

Proof. Let $(A_n)_{n=1}^{\infty}$ be the sequence of $\mathcal{C}$-approximants for T, as in Definition 3.1. In view of Lemma 3.3, the hypothesis of the present lemma implies that the sum $\sum_{n=1}^{\infty} \|A_{n+1} - A_n\|$ is finite. This in turn implies that the sequence $(A_n)_{n=1}^{\infty}$ converges in norm to an operator A. Since each A_n belongs to $\mathcal{C}$, it follows that A belongs to $\mathcal{C}$. But we must have $A = T$, as Lemma 3.2 implies that

$$A|_{\mathcal{F}_m} = \lim_{n\to\infty} A_n|_{\mathcal{F}_m} = T|_{\mathcal{F}_m}, \quad \forall\, m \geq 0.$$

Hence $T \in \mathcal{C}$, as required. □

Proposition 3.5. *Let T be a block-diagonal operator. If the block-diagonal operator $T - \sum_{i=1}^{d} R_i T R_i^*$ belongs to the ideal $\mathcal{S}$, then $T \in \mathcal{C}$.*

Proof. The hypothesis is equivalent to

$$\sum_{n=1}^{\infty} \|(T - \textstyle\sum_{i=1}^{d} R_i T R_i^*)|_{\mathcal{F}_n}\| < \infty. \tag{3.9}$$

It's easy to verify that for $n \geq 1$,

$$(\textstyle\sum_{i=1}^{d} R_i T R_i^*)|_{\mathcal{F}_n} = (T|_{\mathcal{F}_{n-1}}) \otimes I,$$

which gives

$$\|(T - \textstyle\sum_{i=1}^{d} R_i T R_i^*)|_{\mathcal{F}_n}\| = \|T|_{\mathcal{F}_n} - (T|_{\mathcal{F}_{n-1}}) \otimes I\|.$$

Therefore, (3.9) implies that the hypothesis of Lemma 3.4 holds, and the result follows by applying the said lemma. □

Corollary 3.6. *Let $T \in \mathcal{B}$ be a block-diagonal operator such that $[T, R_i^*] \in \mathcal{S}$ for $1 \leq i \leq d$. Then $T \in \mathcal{C}$.*

Proof. By Proposition 3.5, it suffices to show that $T - \sum_{i=1}^{d} R_i T R_i^* \in \mathcal{S}$. We can write

$$\begin{aligned}T - \textstyle\sum_{i=1}^{d} R_i T R_i^* &= (P_0 + \textstyle\sum_{i=1}^{d} R_i R_i^*)T - \textstyle\sum_{i=1}^{d} R_i T R_i^* \\ &= P_0 T - \textstyle\sum_{i=1}^{d} R_i [T, R_i^*],\end{aligned}$$

where P_0 is the orthogonal projection onto $\mathcal{F}_0$, and where we have used Equation (2.6). Since P_0 and $[T, R_i^*]$ belong to $\mathcal{S}$, and since T and R_i belong to $\mathcal{B}$, the result follows from the fact that $\mathcal{S}$ is a two-sided ideal of $\mathcal{B}$. □

We now apply the above results on block-diagonal operators in order to bootstrap the case of general band-limited operators. It is convenient to first consider the case of k-raising/lowering operators, which were introduced in Definition 2.4.

Proposition 3.7. *Let $T \in \mathcal{B}$ be a k-raising or k-lowering operator for some $k \geq 0$. If T satisfies* $[T, R_j^*] \in \mathcal{S}$ for $1 \leq j \leq d$, then $T \in \mathcal{S}$.

Proof. First, suppose that T is k-raising. For every $1 \leq i_1, \ldots, i_k \leq d$, the fact that the left and right annihilation operators commute implies that

$$[(L_{i_1} \ldots L_{i_k})^* T, R_j^*] = (L_{i_1} \ldots L_{i_k})^* [T, R_j^*], \quad \forall 1 \leq j \leq d.$$

Since $[T, R_j^*] \in \mathcal{S}$ by hypothesis, and since $\mathcal{S}$ is a two-sided ideal of $\mathcal{B}$, it follows that $[(L_{i_1} \ldots L_{i_k})^* T, R_j^*] \in \mathcal{S}$. The operator $(L_{i_1} \ldots L_{i_k})^* T$ is block-diagonal, hence Corollary 3.6 gives $(L_{i_1} \ldots L_{i_k})^* T \in \mathcal{C}$.

Since T is k-raising, the range of T is orthogonal to the subspace $\mathcal{F}_\ell$ whenever $\ell < k$. This implies that

$$\Big(I - \sum_{1 \leq i_1, \ldots, i_k \leq d} L_{i_1} \cdots L_{i_k} (L_{i_1} \cdots L_{i_k})^*\Big) T = 0.$$

Hence

$$T = \sum_{1 \leq i_1, \ldots, i_k \leq d} L_{i_1} \cdots L_{i_k} \big((L_{i_1} \cdots L_{i_k})^* T \big),$$

and it follows that $T \in \mathcal{C}$.

The case when T is k-lowering is handled in a similar way by considering the operators $T L_{i_1} \ldots L_{i_k}$ for every $1 \leq i_1, \ldots, i_k \leq d$. □

Theorem 3.8. *Let $T \in \mathcal{B}$ be an operator such that either $[T, R_j^*] \in \mathcal{S}$ for all $1 \leq j \leq d$, or $[T, R_j] \in \mathcal{S}$ for all $1 \leq j \leq d$. Then $T \in \mathcal{C}$.*

Proof. First, suppose that T satisfies $[T, R_j^*] \in \mathcal{S}$ for every $1 \leq j \leq d$. Let $b \geq 0$ be a band-limit for T. By Proposition 2.5, we can decompose T as

$$T = \sum_{k=0}^{b} X_k + \sum_{k=1}^{b} Y_k,$$

where each X_k is a k-raising operator, and each Y_k is a k-lowering operator. We will prove that each $X_k \in \mathcal{C}$ and each $Y_k \in \mathcal{C}$.

Fix for the moment $1 \leq j \leq d$. We have

$$
\begin{aligned}
[T, R_j^*] &= \sum_{k=0}^{b} [X_k, R_j^*] + \sum_{k=1}^{b} [Y_k, R_j^*] \\
&= \sum_{k=0}^{b+1} X_k' + \sum_{k=0}^{b+1} Y_k', \qquad (3.10)
\end{aligned}
$$

where

$$
X_k' = \begin{cases} [X_{k+1}, R_j^*] & \text{if } 0 \leq k \leq b-1, \\ 0 & \text{if } k = b \text{ or } k = b+1, \end{cases}
$$

and

$$
Y_k' = \begin{cases} [X_0, R_j^*] & \text{if } k = 1, \\ [Y_{k-1}, R_j^*] & \text{if } 2 \leq k \leq b+1. \end{cases}
$$

It is clear that each X_k' is a k-raising operator, and that each Y_k' is a k-lowering operator. Hence Equation (3.10) provides the (unique) Fourier-type decomposition for $[T, R_j^*]$, as in Proposition 2.5. Since it is given that $[T, R_j^*] \in \mathcal{S}$, Proposition 2.5 implies that each $X_k' \in \mathcal{S}$ and each $Y_k' \in \mathcal{S}$. This in turn implies that $[X_k, R_j^*] \in \mathcal{S}$ for every $0 \leq k \leq b$, and that $[Y_k, R_j^*] \in \mathcal{S}$ for every $1 \leq k \leq b$.

Now let us unfix the index j from the preceding paragraph. For every $0 \leq k \leq b$, we have proved that $[X_k, R_j^*] \in \mathcal{S}$ for all $1 \leq j \leq d$, hence Proposition 3.7 implies that $X_k \in \mathcal{C}$. The fact that $Y_k \in \mathcal{C}$ for every $1 \leq k \leq b$ is obtained in the same way. This concludes the proof in the case when the hypothesis on T is that $[T, R_j^*] \in \mathcal{S}$ for all $1 \leq j \leq d$.

If T satisfies $[T, R_j] \in \mathcal{S}$ for all $1 \leq j \leq d$, then since the ideal $\mathcal{S}$ is closed under taking adjoints, it follows that $[T^*, R_j^*] \in \mathcal{S}$ for all $1 \leq j \leq d$. The above arguments therefore apply to T^*, and lead to the conclusion that $T^* \in \mathcal{C}$, which gives $T \in \mathcal{C}$. □

4. Construction of the embedding

In this section we fix a deformation parameter $q \in (-1, 1)$ and consider the C^*-algebra $\mathcal{C}^{(q)} = C^*(L_1^{(q)}, \ldots, L_d^{(q)}) \subseteq B(\mathcal{F}^{(q)})$ from Equation (1.4). The main result of this section (and also this paper), Theorem 1.3, shows that it is possible to unitarily embed $\mathcal{C}^{(q)}$ into the C*-algebra $\mathcal{C} = \mathrm{C}^*(L_1, \ldots, L_d) \subseteq B(\mathcal{F})$ from Equation 1.2.

We will once again utilize the terminology of Subsection 2.3 with respect to the natural decomposition $\mathcal{F} = \oplus_{n=0}^{\infty} \mathcal{F}_n$. In particular, we will refer to the unital $*$-algebra $\mathcal{B} \subseteq B(\mathcal{F})$ consisting of band-limited operators, and to the ideal $\mathcal{S}$ of $\mathcal{B}$ consisting of summable band-limited operators.

The deformed Fock space $\mathcal{F}^{(q)}$ also has a natural decomposition $\mathcal{F}^{(q)} = \oplus_{n=0}^{\infty} \mathcal{F}_n^{(q)}$, and we will also need to utilize the terminology of Subsection 2.3 with respect to this decomposition. We will let $\mathcal{B}^{(q)} \subseteq B(\mathcal{F}^{(q)})$ denote the unital $*$-algebra consisting of band-limited operators, and we will let $\mathcal{S}^{(q)}$ denote the ideal of $\mathcal{B}^{(q)}$ which consists of summable band-limited operators.

Remark 4.1. Recall the positive block-diagonal operator $M^{(q)} = \oplus_{n=0}^{\infty} M_n^{(q)} \in \mathcal{B}^{(q)}$, which was reviewed in Subsection 2.2. It was recorded there that for $n \geq 1$, $M_n^{(q)}$ is an invertible operator on $\mathcal{F}_n^{(q)}$. Moreover, for every $n \geq 1$, one has the upper bound (2.9) for the norm $\|(M_n^{(q)})^{-1}\|$, and this upper bound is independent of n.

Therefore, the only obstruction to the operator $M^{(q)}$ being invertible on $\mathcal{F}^{(q)}$ is the fact that $M_0^{(q)} = 0$. We can overcome this obstruction by working instead with the operator $\widehat{M}^{(q)}$ defined by

$$\widehat{M}^{(q)} := P_0^{(q)} + M^{(q)}, \tag{4.1}$$

where $P_0^{(q)} \in B(\mathcal{F}^{(q)})$ is the orthogonal projection onto the subspace $\mathcal{F}_0^{(q)}$. It's clear that $\widehat{M}^{(q)}$ is invertible, and that the bound from (2.9) applies to $\|(\widehat{M}^{(q)})^{-1}\|$.

Lemma 4.2. *The operator* $\widehat{M}^{(q)}$ *satisfies* $[(\widehat{M}^{(q)})^{-1/2}, R_j^{(q)}] \in \mathcal{S}^{(q)}$ *for all* $1 \leq j \leq d$.

Proof. First, we will show that $\widehat{M}^{(q)}$ and $R^{(q)}$ satisfy the hypotheses of Proposition 2.6. It's clear that $\widehat{M}^{(q)}$ is block-diagonal and that $R^{(q)}$ is 1-raising, but it will require a bit of work to check that

$$\sum_{n=0}^{\infty} \|[\widehat{M}^{(q)}, R_j^{(q)}] \mid_{\mathcal{F}_n^{(q)}} \|^{1/2} < \infty, \quad \forall 1 \leq j \leq d. \tag{4.2}$$

In order to show that (4.2) holds, fix $1 \leq j \leq d$. Using Equation (4.1), which defines $\widehat{M}^{(q)}$, we can write

$$\begin{aligned} [\widehat{M}^{(q)}, R_j^{(q)}] &= [P_0^{(q)}, R^{(q)}] + \sum_{i=1}^{d} [L_i^{(q)} (L_i^{(q)})^*, R_j^{(q)}] \\ &= [P_0^{(q)}, R^{(q)}] + \sum_{i=1}^{d} L_i^{(q)} [(L_i^{(q)})^*, R_j^{(q)}], \end{aligned}$$

where the last equality follows from the fact that $L_i^{(q)}$ and $R_j^{(q)}$ commute. The sum in this equation has only a single non-zero term. Indeed, as a consequence of Equation (2.5), we have $[(L_i^{(q)})^*, R_j^{(q)}] = 0$ whenever $i \neq j$. Thus we arrive at the following formula:

$$[\widehat{M}^{(q)}, R_j^{(q)}] = [P_0^{(q)}, R^{(q)}] + L_j^{(q)} [(L_j^{(q)})^*, R_j^{(q)}]. \tag{4.3}$$

We next restrict the operators on both sides of (4.3) to a subspace $\mathcal{F}_n^{(q)}$, for $n \geq 1$. Noting that $[P_0^{(q)}, R_j^{(q)}] = -R_j^{(q)} P_0^{(q)}$ vanishes on $\mathcal{F}_n^{(q)}$, we obtain that

$$[\widehat{M}^{(q)}, R_j^{(q)}] \mid_{\mathcal{F}_n^{(q)}} = L_j^{(q)}[(L_j^{(q)})^*, R_j^{(q)}] \mid_{\mathcal{F}_n^{(q)}}, \quad \forall n \geq 1. \tag{4.4}$$

Finally, we take norms in Equation (4.4) and invoke Equation (2.5) once more to obtain that

$$\|[\widehat{M}^{(q)}, R_j^{(q)}] \mid_{\mathcal{F}_n^{(q)}} \| \leq |q|^n \, \|L_j^{(q)}\|, \quad \forall n \geq 1.$$

The conclusion that (4.2) holds follows from here, since $\sum_{n=1}^{\infty} |q|^{n/2} < \infty$.

Therefore, we can apply Proposition 2.6 to $\widehat{M}^{(q)}$ and $R_j^{(q)}$, and conclude that $[(\widehat{M}^{(q)})^{1/2}, R_j^{(q)}] \in \mathcal{S}^{(q)}$. Note that the operator $(\widehat{M}^{(q)})^{-1/2}$ is bounded and block-diagonal, meaning in particular that it belongs to the $*$-algebra $\mathcal{B}^{(q)}$. The desired result now follows from the obvious identity

$$[(\widehat{M}^{(q)})^{-1/2}, R_j^{(q)}] = -(\widehat{M}^{(q)})^{-1/2}[(\widehat{M}^{(q)})^{1/2}, R_j^{(q)}](\widehat{M}^{(q)})^{-1/2},$$

and the fact that $\mathcal{S}^{(q)}$ is a two-sided ideal of $\mathcal{B}^{(q)}$. □

Lemma 4.3. *For $1 \leq j \leq d$, the unitary $U = \oplus_{n=0}^{\infty} U_n$ from Subsection 2.2 satisfies*

$$U_{n-1}^* L_j^* U_n = (L_j^{(q)})^* (M_n^{(q)})^{-1/2}, \quad \forall n \geq 1. \tag{4.5}$$

(Note that on the left-hand side of Equation (4.5), we view L_j^* as an operator in $B(\mathcal{F}_n, \mathcal{F}_{n-1})$. On the right-hand side of Equation (4.5), we view $(L_j^{(q)})^*$ as an operator in $B(\mathcal{F}_n^{(q)}, \mathcal{F}_{n-1}^{(q)})$.)

Proof. Consider the operator $A_j^{(q)} : \mathcal{F}_n^{(q)} \to \mathcal{F}_{n-1}^{(q)}$ which acts on the natural basis of $\mathcal{F}_n^{(q)}$ by

$$A_j^{(q)}(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) = \delta_{j,i_1} \xi_{i_2} \otimes \cdots \otimes \xi_{i_n}, \quad \forall 1 \leq i_1, \ldots, i_n \leq d.$$

We claim that $A_j^{(q)}$ satisfies

$$A_j^{(q)} = (L_j^{(q)})^* \, (M_n^{(q)})^{-1} \tag{4.6}$$

To see this, note that for $1 \leq i_1, \ldots, i_n \leq d$,

$$\begin{aligned} A_j^{(q)} M_n^{(q)}(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) &= A_j^{(q)} \sum_{m=1}^{n} q^{m-1} \xi_{i_m} \otimes \xi_{i_1} \otimes \cdots \otimes \widehat{\xi_{i_m}} \otimes \cdots \otimes \xi_{i_n} \\ &= \sum_{m=1}^{n-1} q^{m-1} \delta_{j,i_m} \xi_{i_1} \otimes \cdots \otimes \widehat{\xi_{i_m}} \otimes \cdots \otimes \xi_{i_n} \\ &= (L_j^{(q)})^*(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}), \end{aligned}$$

where the first and last equalities follow from Equation (2.8) and Equation (2.4) respectively. Hence $A_j^{(q)} M_n^{(q)} = (L_j^{(q)})^* \mid_{\mathcal{F}_n^{(q)}}$, so multiplying on the right by $(M_n^{(q)})^{-1}$ establishes the claim.

Now, from Equation (2.10), which defines U_n, we see that

$$U_{n-1}^* L_j^* U_n = U_{n-1}^* L_j^* (I \otimes U_{n-1})(M_n^{(q)})^{1/2},$$

and from the definition of $A_j^{(q)}$ it's immediate that

$$L_j^*(I \otimes U_{n-1}) = U_{n-1} A_j^{(q)}.$$

Together, this allows us to write

$$\begin{aligned} U_{n-1}^* L_j^* U_n &= U_{n-1}^* U_{n-1} A_j^{(q)} (M_n^{(q)})^{1/2} \\ &= A_j^{(q)} (M_n^{(q)})^{1/2}. \end{aligned}$$

Applying Equation (4.6) now gives Equation (4.5), as required. □

Proposition 4.4. *For $1 \leq i, j \leq d$, the unitary U from Subsection 2.2 satisfies $[U^* L_j^* U, R_i^{(q)}] \in \mathcal{S}^{(q)}$.*

Proof. Fix i and j and let C denote the commutator $C = [U^* L_j^* U, R_i^{(q)}]$. It's clear that C is a block-diagonal operator on $\mathcal{F}^{(q)}$. In order to show that $C \in \mathcal{S}^{(q)}$, we will need to estimate the norm of its diagonal blocks.

For $n \geq 1$, Lemma 4.3 gives

$$\begin{aligned} C\mid_{\mathcal{F}_n^{(q)}} &= U_n^* L_j^* U_{n+1} R_i^{(q)} - R_i^{(q)} U_{n-1}^* L_j^* U_n \\ &= (L_j^{(q)})^* (M_{n+1}^{(q)})^{-1/2} R_i^{(q)} - R_i^{(q)} (L_j^{(q)})^* (M_n^{(q)})^{-1/2} \\ &= (L_j^{(q)})^* (((M_{n+1}^{(q)})^{-1/2} R_i^{(q)} - R_i^{(q)} (M_n^{(q)})^{-1/2}) \\ &\quad + ((L_j^{(q)})^* R_i^{(q)} - R_i^{(q)} (L_j^{(q)})^*)(M_n^{(q)})^{-1/2}. \end{aligned}$$

Since C is block-diagonal, this gives

$$C = (L_j^{(q)})^* [(\widehat{M}^{(q)})^{-1/2}, R_i^{(q)}] + [(L_j^{(q)})^*, R_i^{(q)}](\widehat{M}^{(q)})^{-1/2}.$$

Now, $[(\widehat{M}^{(q)})^{-1/2}, R_i^{(q)}] \in \mathcal{S}^{(q)}$ by Lemma 4.2. By Equation (2.5),

$$[(L_j^{(q)})^*, R_i^{(q)}]\mid_{\mathcal{F}_n^{(q)}} = \delta_{ij} q^n I_{\mathcal{F}_n^{(q)}},$$

and since the operator $[(L_j^{(q)})^*, R_i^{(q)}]$ is block-diagonal, this implies that it also belongs to $\mathcal{S}^{(q)}$. Since $(L_j^{(q)})^*$ and $(\widehat{M}^{(q)})^{-1/2}$ both belong to $\mathcal{B}^{(q)}$, and since $\mathcal{S}^{(q)}$ is a two-sided ideal of $\mathcal{B}^{(q)}$, it follows that $C \in \mathcal{S}^{(q)}$. □

We are now able to complete the proof of the embedding theorem.

Proof of Theorem 1.3. It suffices to show that $U_{opp} L_i^{(q)} U_{opp}^* \in \mathcal{C}$, for $1 \leq i \leq d$. Since $U_{opp} L_i^{(q)} U_{opp}^*$ belongs to the algebra $\mathcal{B}$ of all band-limited operators, by Theorem 3.8 it will actually be sufficient to verify that

$$[U_{opp} L_i^{(q)} U_{opp}^*, R_j^*] \in \mathcal{S}, \quad \forall 1 \leq i, j \leq d.$$

By Definition 1.1, we can write

$$\begin{aligned} U_{opp}L_i^{(q)}U_{opp}^* &= JUJ^{(q)}L_i^{(q)}J^{(q)}U^*J \\ &= JUR_i^{(q)}U^*J, \end{aligned}$$

where the last equality follows from Equation (2.7). This gives

$$\begin{aligned} [U_{opp}L_i^{(q)}U_{opp}^*, R_j^*] &= [JUR_i^{(q)}U^*J, R_j^*] \\ &= JU[R_i^{(q)}, U^*JR_j^*JU]U^*J \\ &= JU[R_i^{(q)}, U^*L_j^*U](JU)^*, \end{aligned}$$

and we know from Proposition 4.4 that $[R_i^{(q)}, U^*L_j^*U] \in \mathcal{S}^{(q)}$. It is clear that conjugation by the unitary JU takes $\mathcal{S}^{(q)}$ onto $\mathcal{S}$, so this gives the desired result. □

The proof that $\mathcal{C}^{(q)}$ is exact now follows from some simple observations about nuclear and exact C*-algebras (see e.g. [3]).

Proof of Corollary 1.4. The extended Cuntz algebra $\mathcal{C}$ is (isomorphic to) an extension of the Cuntz algebra. Since the Cuntz algebra is nuclear, this implies that $\mathcal{C}$ is nuclear, and in particular that $\mathcal{C}$ is exact. Since exactness is inherited by subalgebras (see e.g. Chapter 2 of [3]), it follows from Theorem 1.3 that $U_{opp}\mathcal{C}^{(q)}U_{opp}^*$ is exact, and hence that $\mathcal{C}^{(q)}$ is exact. □

Remark 4.5. Since Theorem 1.3 holds for all $q \in (-1, 1)$, a natural thought is that the methods used above could also be applied to establish the inclusion $U\mathcal{C}^{(q)}U^* \subseteq \mathcal{C}$ for all $q \in (-1, 1)$, and hence (since the opposite inclusion was shown in [5]) that $U\mathcal{C}^{(q)}U^* = \mathcal{C}$. To do this, it would be necessary to establish that

$$[UL^{(q)}U^*, R_j^*] \in \mathcal{S}, \quad \forall 1 \leq i, j \leq d. \tag{4.7}$$

This condition looks superficially similar to the condition from Proposition 4.4, but this is deceptive. We believe that establishing (4.7) will require a deeper understanding of the combinatorics which underlie the q-commutation relations.

The algebra $\mathcal{C}^{(q)}$ arises as a representation of the the univeral algebra $\mathcal{E}^{(q)}$ corresponding to the q-commutation relations. It was shown in [6] that for $|q| < \sqrt{2} - 1$, $\mathcal{C}^{(q)}$ and $\mathcal{E}^{(q)}$ are isomorphic (and in particular that they are both isomorphic to the extended Cuntz algebra). It is believed that this is the case for all $q \in (-1, 1)$.

5. An application to the q-Gaussian von Neumann algebras

The q-Gaussian von Neumann algebra $\mathcal{M}^{(q)}$ is the von Neumann algebra generated by $\{L_i^{(q)} + (L_i^{(q)})^* \mid 1 \leq i \leq d\}$. This algebra can be considered as a type of deformation of $L(\mathbb{F}_d)$, the von Neumann algebra of the free group on d generators. Indeed, for $q = 0$, a basic result in free probability states that $\mathcal{M}^{(q)}$ is precisely the realization of $L(\mathbb{F}_d)$ as the von Neumann algebra generated by a free semicircular family (see e.g. Section 2.6 of [11] for the details).

For general $q \in (-1, 1)$ it is known that $\mathcal{M}^{(q)}$ is a von Neumann algebra in standard form, with Ω being a cyclic and separating trace-vector. The commutant of $\mathcal{M}^{(q)}$ is the von Neumann algebra generated by $\{R_i^{(q)} + (R_i^{(q)})^* \mid 1 \leq i \leq d\}$ (see Section 2 of [2]).

Not much is known about the isomorphism class of the algebras $\mathcal{M}^{(q)}$ for $q \neq 0$. The major open problem is to determine the extent to which they behave like $L(\mathbb{F}_d)$. The best results to date show that $\mathcal{M}^{(q)}$ does share certain properties with $L(\mathbb{F}_d)$. Nou showed in [7] that $\mathcal{M}^{(q)}$ is non-injective, and Ricard showed in [9] that it is a II_1 factor. Shlyakhtenko showed in [10] that if we assume $|q| < 0.44$, then the results in [6] and [5] can be used to obtain that $\mathcal{M}^{(q)}$ is solid in the sense of Ozawa.

Based on the results in Section 4, we show here that $\mathcal{M}^{(q)}$ is weakly exact. For more details on weak exactness, we refer the reader to Chapter 14 of [3].

Theorem 5.1. *For every q in the interval $(-1, 1)$, the q-Gaussian von Neumann algebra $\mathcal{M}^{(q)}$ is weakly exact.*

Proof. It is known that a von Neumann algebra is weakly exact if it contains a weakly dense C*-algebra which is exact (see e.g. Theorem 14.1.2 of [3]). Consider the C*-algebra $\mathcal{A}^{(q)}$ generated by $\{L_i^{(q)} + (L_i^{(q)})^* \mid 1 \leq i \leq d\}$. It is clear that $\mathcal{A}^{(q)}$ is weakly dense in $\mathcal{M}^{(q)}$, while on the other hand, we have $\mathcal{A}^{(q)} \subseteq \mathcal{C}^{(q)}$. Therefore, the exactness of $\mathcal{A}^{(q)}$ follows from Corollary 1.4, combined with the fact that exactness is inherited by subalgebras. □

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