On the Semisimplicity of the Action of the Frobenius on Etale Cohomology

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Abstract

I give a proof of the semisimplicity of the action of the geometric frobenius on etale cohomology. The proof is based on [MGM10] and on the Weil Conjectures, ie on the Riemann Hypothesis for non singular projective varieties over finite fields.

Keywords Local Spectra. Algebraic and Topological K Theory l-adic Completion of Spectra.

Introduction:

I am going to work with the abelian category $\mathcal{B}(l)$ See [B83]. In [B83] Bousfield defined an universal functor $\mathcal{U}$: $Z_{(l)} - modules \mapsto \mathcal{B}(l)$ which will be crucial here. I start with the isomorphism ([T89])

(1)
$$\pi_0([L_{E(1)}K(X_\infty)]^l) \otimes Q \simeq \oplus_{i=0}^{d} H_{et}^{2i}(X_\infty, Q_{(l)}(i))$$

Here $L_{(E(1)}$ is is the Bousfield localization of $E(1)$ where $E(1)$ is such that $\mathcal{K}_{(l)} = \vee_{i=1}^{l-2}\Sigma^{2i}E(1)$, where $\mathcal{K}_{(l)}$ is the l localized topological K spectrum $\mathcal{K}$. In (1), $K(X_\infty)$ is the Quillen's algebraic K theory spectrum on $X_\infty = X \otimes F$, where F is the algebraic closure of the finite field F_q with $q = p^s$ and $p \neq l$. X is a nonsingular projective variety over F_q with finite dimension and d is the dimension of X_∞ . $L_{E(1)}$ denotes $E(1)$ localization, and $[L_{E(1)}K(X_\infty)]^l$ denotes the l-adic completion of the spectra $L_{E(1)}K(X_\infty)$.Here $H_{et}^{2i}(X_\infty, Q_l(i))$ are the etale cohomology groups with coefficients in Q_l,which are Q_l vector spaces.

I show that $\mathcal{U}(\pi_0([L_{E(1)}K(X_\infty)]^l) \otimes Q) \simeq [E_0(1)(K(X_\infty))]^l \otimes Q$ (2)

where on the right hand side of the equality, $[E_0(1)(K(X_\infty))]^l$ is the l-adic completion of the $Z_{(l)}$-module $E_0(1)(K(X_\infty))$ which is an object in $\mathcal{B}(l)$. This isomorphism seems to be dependent on highly non trivial facts, such as the fact that $\pi_0 L_{E(1)}K(X_\infty)$ is an l-reduced group, which follows from [MGM10]. $\pi_0 L_{E(1)}K(X_\infty)$ being l-reduced group means:

$$Hom(Q/Z_{(l)}, \pi_0(L_{E(1)}K(X_\infty))) = 0$$

We then obtain the isomorphism: $[(E_0(1)K(X_\infty))]^l \otimes Q \simeq \mathcal{U}(\oplus_{i=0}^{2d} H_{et}(X_\infty, Q_{(l)}(i)))$. Therefore the functor $\mathcal{U}$ allows to study the action of the geometric frobenius $\Phi_X : X \mapsto X$ on $\oplus_{i=0}^{2d} H_{et}(X_\infty, Q_{(l)}(i))$ through the action of $\mathcal{U}(\Phi_X)$ on $[E_0(1)(K(X_\infty))]^l \otimes Q$. I prove that $(E_0(1)(K(X_\infty))) \otimes Q$ is dense in $[E_0(1)(K(X_\infty))]^l \otimes Q$ with the l-adic topology and show that $\mathcal{U}(\Phi_X \circ {\Phi_X}^\dagger)$ where ${\Phi_X}^\dagger$ is the complex conjugate to Φ_X is an Adams operation on $[E_0(1)(K(X_\infty))]^l \otimes Q$ because of the Weil Conjectures. Finally I conclude that the action of Φ_X on the etale cohomological spaces $H^{2i}_{et}(X_\infty, Q_{(l)}(i))$ is semisimple.

Remarks 1.

a) We know that $\pi_*(E(1)) \simeq Z_{(l)}(\nu, \nu^{-1})$ and therefore $\pi_0(E(1)) \simeq Z_{(l)}$ Then the functor $\mathcal{U}$ verifies $\mathcal{U}(Z_{(l)}) \simeq \mathcal{U}(\pi_0(E(1))) \simeq E_0E(1) \simeq Z_{(l)}[t]$ where $Z_{(l)}[t]$ are the polinomials in one variable and coefficients in $Z_{(l)}$

b) Also, $\mathcal{U}(Z_{(l)} \otimes Q) = \mathcal{U}(Z_{(l)}) \otimes Q \simeq Q_{(l)}[t]$, $\mathcal{U}(Z_l) \simeq [E_0E(1)]^l \simeq Z_l[[t]]$ and $\mathcal{U}(Q_l) \simeq Q_l[[t]]$ where $Z_l[[t]]$ and $Q_l[[t]]$ are the l-adic completions of $Z_{(l)}[t]$ and $Z_{(l)}[t] \otimes Q$ respectively.

Proposition 1:

$$\pi_0([L_{E(1)}K(X_\infty)]^l) \otimes Q \simeq [\pi_0(L_{E(1)}K(X_\infty))]^l \otimes Q$$

Proof: Let $G_\nu = \pi_0(Y \wedge M(Z/l^\nu)) = \pi_0(T_\nu) = \pi_0(Y/l^\nu)$, where $Y = L_{E(1)}K(X_\infty)$.There is an exact sequence ([BK72] Chap 9)

(3)

$$0 \mapsto lim^1(G_\nu) \mapsto \pi_0(homlimT_\nu) \mapsto lim(G_\nu) \mapsto 0$$

and (4)

$$G_\nu = \pi_0(Y \wedge M(Z/l^\nu)) \simeq (\pi_0(Y) \otimes Z/l^\nu) \oplus Tor^1(Z/l^\nu, Y)$$

Now $limTor^1(Z/l^\nu, Y) = 0$ since the limit is equal to

$$\Pi_{i=1}^{\infty}\{g_i = l^i - torsion - element \in \pi_0(L_{E(1)}K(X_\infty))/lg_{i+1} = g_i\}$$

and each coordinate in this limit is 0,for it belongs to the intersection of all $l^l(\pi_0 L_{E(1)}K(X_\infty))$ which is 0 because $\pi_0 L_{E(1)}K(X_\infty)$ is reduced.Then by (4), I get:

(5)
$$lim(G_\nu) = lim(\pi_0(Y) \otimes Z/l^\nu) = [\pi_0(Y)]^l$$

.

Obviously (6): $lim^1(\pi_0(Y) \otimes Z/l^\nu) = 0$. On the other hand, $lim^1 Tor^1(Z/l^\nu, Y)$ has bounded l-torsion. Let me show why this is so:

Let $M_\nu = Tor^1(Z/l^\nu, Y)$.The map $M_{\nu+1} \mapsto M_\nu$ is the map which goes from the $l^{\nu+1}$-torsion elements of Y to the l^ν-torsion elements of Y given by $x \mapsto lx$.It is in general not surjective,so that it is difficult to prove that $lim^1 M_\nu = 0$. Anyway,(3) has simplified because of (5) and (6) to

(7)
$$0 \mapsto lim^1(Tor^1(Z/l^\nu, \pi_0(Y)) \mapsto \pi_0(homlim Y \wedge M(Z/l^\nu)) = \pi_0(Y^l) \mapsto lim(G_\nu) = [\pi_0(Y)]^l \mapsto 0$$

where $Y = L_{E(1)}K(X_\infty)$ and $lim(G_\nu) = [\pi_0(Y)]^l$ is reduced since it is the projective limit of the reduced groups G_ν See [MGM10]. $[\pi_0(Y)]^l$ is also a cotorsion group (see [F1]), since it is the epimorphic image of a cotorsion group in the exact sequence (7): $\pi_0(Y^l)$ is equal to the cotorsion reduced group $Ext^1(Q/Z_{(l)}, Y)$ since(see [B79])

$$Ext^1(Q/Z_{(l)}, Y) \mapsto \pi_0(Y^l) \mapsto Hom(Q/Z_{(l)}, \pi_{-1}(Y))$$

and $Hom(Q/Z_{(l)}, \pi_{-1}(Y)) = 0$ since $\pi_{-1}(Y)$is l-reduced, see ([MGM10]) and therefore, $Ext^1(Q/Z_{(l)}, Y) \simeq \pi_0(Y^l)$ and $Ext^1(Q/Z_{(l)}, Y)$ is a cotorsion group.The torsion group of the cotorsion reduced group $[\pi_0(Y)]^l$ noted $T([\pi_0(Y)]^l)$ is in the terminology of [F2] an l-complete torsion group.Now $[\pi_0(Y)]^l$ is a reduced algebraically compact group since it is complete in the terminology of[R08] page 440.It is complete because it is the closure in the l-adic topology of the topological Hausdorff group $\pi_0(Y)]$ Being reduced and algebraically compact implies it is a direct summand of a direct product of cyclic l-groups by [F2] Corollary 38.2 page 161.Henceforth, $T([\pi_0(Y)]^l)$ is contained in a direct product of cyclic l-groups, and so the torsion part of $\pi_0(Y)$, noted $T(\pi_0(Y))$ is contained in a direct sum of cyclic l-groups. Since $T(\pi_0(Y))$ is reduced, then it has bounded l-torsion, ie there exists ν_0 such that $l^{\nu_0}T(\pi_0(Y) = 0$ Then by definition of lim^1, $lim^1(Tor^1(Z/l^\nu, \pi_0(Y))$ has bounded l-torsion, as wanted. Then,tensoring with Q in the exact sequence (7) becomes,

(8)
$$0 \mapsto \pi_0(Y^l) \otimes Q \mapsto [\pi_0(Y)]^l \otimes Q \mapsto 0$$

and therefore Proposition 1 has been proved.

Remark 2: I conjecture that $\pi_0(Y^l)$ and therefore also $[\pi_0(Y)]^l$ are without torsion, in which case by [R08] page 445 $[\pi_0(Y)]^l$, is a direct summand of copies of Z_l. Since tensored by Q, ie $[\pi_0(Y)]^l \otimes Q$, using (1) and the above Proposition 1, is a finite direct sum of copies of Q_l, $[\pi_0(Y)]^l$ has to be a finite direct sum of copies of Z_l.

Theorem 1: $\mathcal{U}([\pi_0(L_{E(1)}K(X_\infty))]^l \otimes Q) \simeq [E(1)_0(K(X_\infty)]^l \otimes Q$

Corollary 2: $\mathcal{U}(\oplus_{i=0}^d H_{et}^{2i}(X_\infty, Q_{(l)}(i))) \simeq [E(1)_0(K(X_\infty)]^l \otimes Q$.

Proof: $\mathcal{U}(\oplus_{i=0}^d H_{et}^{2i}(X_\infty, Q_{(l)}(i))) \simeq [\mathcal{U}(\pi_0(L_{E(1)}K(X_\infty))]^l \otimes Q$.This fact follows from Proposition 1 and from (1). Hence, corollary 2 follows immediately from theorem 1.

Proof of theorem 1:

Since $\pi_0(L_{E(1)}K(X_\infty))$ is l-reduced the kernel of the l-completion $\pi_0(L_{E(1)}K(X_\infty)) \mapsto [\pi_0(L_{E(1)}K(X_\infty))]^l$ is equal to 0 and the completion map is injective. Also the l-adic topology in $\pi_0(L_{E(1)}K(X_\infty))$ is Hausdorff and this space is dense in its l-completed space. The functor $\mathcal{U}$ is exact and $0 \mapsto \mathcal{U}(\pi_0(L_{E(1)}K(X_\infty))) = E(1)_0K(X_\infty) \mapsto \mathcal{U}([\pi_0(L_{E(1)}K(X_\infty))]^l)$ has dense image. On the other hand since $\pi_0(L_{E(1)}K(X_\infty))$ is l-reduced, $\mathcal{U}(\pi_0(L_{E(1)}K(X_\infty))) = E(1)_0K(X_\infty)$ is l-reduced See [B83]. Then,

$$0 \mapsto E(1)_0K(X_\infty) \mapsto [E(1)_0K(X_\infty)]^l$$

with dense image. By uniqueness of the l-completed space, (9):$\mathcal{U}([\pi_0(L_{E(1)}K(X_\infty))]^l) \simeq [E(1)_0(K(X_\infty)]^l$, and the theorem follows.

Remark 3: If the conjecture stated in remark 2 holds, then by (9), remark 2, and remark 1 b), $[E(1)_0(K(X_\infty)]^l$ is a finite direct sum of copies of $Z_l[[t]]$, a fact which was proved for nonsingular complete curves in [DM95].

Theorem 2: *The geometric frobenius Φ_X acts semisimply in $\oplus_{i=0}^d H_{et}^{2i}(X_\infty, Q_l(i))$.*

Proof: By the Weil Conjectures the eigenvalues of Φ_X acting on $H^i_{et}(X_\infty, Q_l)$ are algebraic numbers with absolute value $q^{i/2}$. Then $\mathcal{U}((\Phi_X) \circ (\Phi_X)^\dagger) = \mathcal{U}(\Phi_X) \circ \mathcal{U}(\Phi^\dagger_X)$ acting on $\mathcal{U}(\oplus^d_{i=0} H^{2i}_{et}(X_\infty, Q_l))$ has eigenvalues q^{2i},$i \in 1, 2, ...d$ though those eigenvalues do not represent the complete set of its eigenvalues. Hence this composite map can be identified with the Adams operation $\psi^{(q^2)}$ on $[E(1)_0 K(X_\infty)]^l$ which is an object in $\mathcal{B}(l)$. Therefore the Adams operation is diagonalizable in $\mathcal{U}(\oplus^d_{i=0} H^{2i}_{et}(X_\infty, Q_l))$. This fact implies that $\mathcal{U}(\Phi_X)$ is also diagonalizable in $\mathcal{U}(\oplus^d_{i=0} H^{2i}_{et}(X_\infty, Q_l))$, which in turn implies that Φ_X is diagonalizable in $\oplus^d_{i=0} H^{2i}_{et}(X_\infty, Q_l)$ as wanted.

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