

# HAMILTON TYPE ESTIMATES FOR HEAT EQUATIONS ON MANIFOLDS

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ABSTRACT. In this paper, we study the gradient estimates of Hamilton type for positive solutions to both drifting heat equation and the simple nonlinear heat equation problem

$$u_t - \Delta u = au \log u, \quad u > 0$$

on the compact Riemannian manifold  $(M, g)$  of dimension  $n$  and with non-negative (Bakry-Emery)-Ricci curvature. Here  $a \leq 0$  is a constant. The latter heat equation is a basic evolution equation which is the negative gradient heat flow to the functional of Log-Sobolev inequality on the Riemannian manifold. An open question concerning the Hamilton type gradient estimate is proposed.

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**Keywords:** drifting Laplacian, nonlinear heat equation, gradient estimate

## 1. INTRODUCTION

In this paper we give two gradient estimates of Hamilton type [7] for heat equations, which have been studied recently by many researchers (see [18], [6], [17],[2], [8], [12], [4],[3],[19],[5],etc). The equations under consideration have their deep background from the fundamental gap of the Schrodinger equation and the Ricci flow (see [9], [15], [21], etc). Interesting gradient estimates of Hamilton type for heat equation associated to Ricci flow have been obtained in [3], [2], and [22]. We derive the Hamilton type gradient estimate for the drifting heat equation and the simple nonlinear heat equation from the view-point of the Bernstein type estimates. This is a new observation. Our argument is shorter than previous ones.

We first derive the Hamilton type gradient estimate for the drifting heat equation

$$(1) \quad u_t - \Delta u = -\nabla\phi \cdot \nabla u, \quad u > 0$$

on the compact Riemannian manifold  $(M, g)$  of dimension  $n$ . Here  $\phi$  is a smooth function on  $M$  and  $\nabla\phi$  is the gradient of  $\phi$  in the metric  $g$ . We shall denote  $D^2\phi$  the hessian matrix of  $\phi$ .

We have the following gradient estimate.

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**Theorem 1.** *Assume that the compact Riemannian manifold  $(M, g)$  has the non-negative Bakry-Emery-Ricci curvature in the sense that*

$$Rc + D^2\phi \geq -K$$

*on  $M$  for some constant  $K \geq 0$ . Let  $u > 0$  be a positive smooth solution to (1). Assume that  $\sup_M u = 1$ . Let  $f = -\log u$ . Then we have, for all  $t > 0$ ,*

$$t|\nabla f|^2 \leq (2Kt + 1)f.$$

*The same estimate is true for (1) on complete Riemannian manifolds when the maximum principle can be applied.*

We also have the following result for (1) on the manifold with smooth boundary.

**Theorem 2.** *Assume that the compact Riemannian manifold  $(M, g)$  with convex boundary has the curvature condition about the Bakry-Emery-Ricci tensor that*

$$Rc + D^2\phi \geq -K$$

*on  $M$  for some constant  $K \geq 0$ . Let  $u > 0$  be a positive smooth solution to (1) with Neumann boundary condition  $u_\nu = 0$ , where  $\nu$  is the outward unit normal to the boundary. Assume that  $\sup_M u = 1$ . Let  $f = -\log u$ . Then we have, for all  $t > 0$ ,*

$$(2) \quad t|\nabla f|^2 \leq (2Kt + 1)f.$$

Assume that  $K = 0$  and  $u$  is any bounded smooth solution to (1). Assume that  $A = \sup_M u > 0$ . Let  $v = (A - u)/A$ . Then  $v$  is a positive solution to (1) and the above gradient estimate is

$$(3) \quad t|\nabla u|^2 \leq (A - u)^2 \log \frac{A}{A - u},$$

which is the usual form of Hamilton type gradient estimate.

The drifting heat equation is closely related to the fundamental gap of the Schrodinger operator on convex domains (so  $K = 0$ ). Namely, Let  $\lambda = \lambda_2 - \lambda_1$  be the fundamental gap of the Laplacian operator  $-\Delta$  and let  $f_j$  be the eigenfunctions corresponding to  $\lambda_j$ ,  $j = 1, 2$ . Let  $u := u(x) := f_2/f_1$ . Then we have ([21])

$$\Delta u = -\lambda u - 2(\nabla u \cdot \nabla \log f_1).$$

Set

$$\phi = -2 \log f_1,$$

which is convex by the well-known result of Brascamp-Lieb [1]. Let

$$v(x, t) = \exp(-\lambda t)u(x).$$

Then  $v$  satisfies (1) with the Neumann boundary condition.

The other interesting problem to us is to derive the Hamilton type gradient estimate for the following nonlinear heat equation problem

$$(4) \quad u_t - \Delta u = au \log u, \quad u > 0$$

on the compact Riemannian manifold  $(M, g)$  of dimension  $n$ . Here  $a \in \mathbf{R}$  is some constant. This heat equation can be considered as the negative gradient heat flow to  $W$ -functional [18], which is closely related to the Log-Sobolev inequalities on the Riemannian manifold. In [14], we propose the study of the local gradient estimates for solutions to (4) based on its relation with Ricci solitons. Soon after, Y. Yang gives a nice answer in [20] and his result is Li-Yau type [11].

We have the following result for (4).

**Theorem 3.** *Assume that the compact Riemannian manifold  $(M, g)$  has the non-negative Ricci curvature condition, i.e.,  $Rc \geq 0$ . Let  $u > 0$  be a positive smooth solution to (4). Assume that  $\sup_M u < 1$  at the initial time and  $a \leq 0$ . Let  $f = -\log u$ . Then we have, for all  $t > 0$ ,  $\sup_M u < 1$  and*

$$t|\nabla f|^2 \leq f.$$

*The same estimate is true for (1) on complete Riemannian manifolds when the maximum principle can be applied.*

Similar to Theorem 2, we also have the following result for (4) on the manifold with smooth boundary.

**Theorem 4.** *Assume that the compact Riemannian manifold  $(M, g)$  with convex boundary has the non-negative Ricci curvature condition. Let  $u > 0$  be a positive smooth solution to (1) with Neumann boundary condition  $u_\nu = 0$ , where  $\nu$  is the outward unit normal to the boundary. Assume that  $\sup_M u < 1$  at the initial time and  $a \leq 0$ . Let  $f = -\log u$ . Then we have, for all  $t > 0$ ,  $\sup_M u < 1$  and*

$$(5) \quad t|\nabla f|^2 \leq f.$$

It is quite clear that our results to (4) are not satisfied by us because of the assumption  $\sup_M u < 1$  at the initial time. So we leave it open to derive the Hamilton type gradient estimate for positive solutions to (4).

The plan of our paper is below. In section 2, we give the proofs of Theorems 1 and 2. In section 3 we study (4).

## 2. HAMILTON TYPE ESTIMATE FOR DRIFTING HEAT EQUATION

Assume that  $u > 0$  is a positive solution to (1). Let  $f = -\log u$ . Then

$$f_j = -u_j/u, \quad \Delta f = -\Delta u/u + |\nabla f|^2.$$

Then we have

$$(6) \quad (\partial_t - \Delta)f + \nabla\phi \cdot \nabla f = -|\nabla f|^2.$$

Let  $L = \partial_t - \Delta + \nabla\phi \cdot$ . We compute  $L|\nabla f|^2$ .

Note that

$$(|\nabla f|^2)_t = 2 \langle \nabla f, \nabla f_t \rangle.$$

Recall the Bochner formula that

$$\Delta|\nabla f|^2 = 2|D^2 f|^2 + \langle \nabla f, \nabla \Delta f \rangle + 2Rc(\nabla f, \nabla f).$$

Then we have

$$(7) \quad L|\nabla f|^2 = 2 \langle \nabla f, \nabla Lf \rangle - 2|D^2 f|^2 - 2(Rc + D^2 \phi)(\nabla f, \nabla f).$$

By the Ricci curvature bound assumption, we have

$$L|\nabla f|^2 \leq -2|D^2 f|^2 + 2K|\nabla f|^2.$$

Dropping the term  $-2|D^2 f|^2$  (comment: Hamilton type estimate is not as sharp as Li-Yau's estimate ) we have

$$L|\nabla f|^2 \leq 2K|\nabla f|^2.$$

Then we have

$$(8) \quad L(t|\nabla f|^2) \leq (1 + 2Kt)|\nabla f|^2.$$

Using (6), we get from (8) that

$$(9) \quad L(t|\nabla f|^2 - (2Kt + 1)f) \leq -2Kf.$$

We may re-write (9) as

$$L(t|\nabla f|^2 - (2Kt + 1)f) \leq \frac{2K}{2K + 1}(t|\nabla f|^2 - (2K + 1)f) - \frac{2K}{2K + 1}(t|\nabla f|^2).$$

Then we have

$$L(t|\nabla f|^2 - (2Kt + 1)f) \leq \frac{2K}{2K + 1}(t|\nabla f|^2 - (2K + 1)f).$$

Applying the Maximum principle we obtain that

$$t|\nabla f|^2 - (2Kt + 1)f \leq 0.$$

This completes the proof of Theorem 1.

We now prove Theorem 2. We need to treat the boundary term. Note that  $f_\nu = 0$  on the boundary. Then on the boundary,

$$[t|\nabla f|^2 - (2Kt + 1)f]_\nu = 2tf_j f_{j\nu} = -2II(\nabla f, \nabla f) \leq 0.$$

Hence by the strong maximum principle we know that the maximum point of  $t|\nabla f|^2 - (2Kt + 1)f$  can not occur at the boundary point and then we have

$$t|\nabla f|^2 - (2Kt + 1)f \leq 0.$$

This completes the proof of Theorem 2.

### 3. HAMILTON TYPE ESTIMATE FOR THE SIMPLE NONLINEAR HEAT EQUATION

As before, we let  $f = -\log u$ . Then we have

$$(10) \quad (\partial_t - \Delta)f = af - |\nabla f|^2.$$

Using  $a \leq 0$  and the maximum principle we know that if  $\inf_M f > 0$  at the initial time, then it is always positive for  $t > 0$ .

Let  $L := \partial_t - \Delta$  in this section. Compute,

$$(11) \quad L|\nabla f|^2 = 2 \langle \nabla f, \nabla Lf \rangle - 2|D^2 f|^2 - 2Rc(\nabla f, \nabla f).$$

Then we have

$$L|\nabla f|^2 = 2a|\nabla f|^2 - 2 \langle \nabla f, \nabla |\nabla f|^2 \rangle - 2|D^2 f|^2 - 2\text{Rc}(\nabla f, \nabla f).$$

Using the non-negative Ricci curvature assumption we have

$$L|\nabla f|^2 \leq 2a|\nabla f|^2 - 2 \langle \nabla f, \nabla |\nabla f|^2 \rangle.$$

Then

$$L(t|\nabla f|^2) \leq (2at + 1)|\nabla f|^2 - 2 \langle \nabla f, \nabla (t|\nabla f|^2) \rangle.$$

Using (10) we get that

$$L(t|\nabla f|^2 - f) \leq 2at|\nabla f|^2 - af - 2 \langle \nabla f, \nabla (t|\nabla f|^2 - f) \rangle.$$

Let  $H = t|\nabla f|^2 - f$ . Then  $f = t|\nabla f|^2 - H$ . Hence we have

$$LH \leq at|\nabla f|^2 + aH - 2 \langle \nabla f, \nabla H \rangle.$$

Using the assumption that  $a \leq 0$ , we obtain that

$$LH \leq aH - 2 \langle \nabla f, \nabla H \rangle.$$

Applying the maximum principle to  $H$  we know that  $H \leq 0$ . That is,

$$t|\nabla f|^2 - f \leq 0,$$

which is the desired gradient estimate of Hamilton type. Then we complete the proof of Theorem 3.

Using the same argument as in the proof of Theorem 2, we can prove Theorem 4.

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