GRADIENT ESTIMATES FOR A SIMPLE NONLINEAR HEAT EQUATION ON MANIFOLDS

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ABSTRACT. In this paper, we study the gradient estimate for positive solutions to the following nonlinear heat equation problem

$$u_t - \Delta u = au \log u + Vu, \quad u > 0$$

on the compact Riemannian manifold (M, g) of dimension n and with non-negative Ricci curvature. Here $a \leq 0$ is a constant, V is a smooth function on M with $-\Delta V \leq A$ for some positive constant A. This heat equation is a basic evolution equation and it can be considered as the negative gradient heat flow to W-functional (introduced by G.Perelman), which is the Log-Sobolev inequalities on the Riemannian manifold and V corresponds to the scalar curvature.

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1. Introduction

The study of the heat equation

$$u_t - \Delta u = 0, \quad u > 0 \tag{1}$$

on the compact Riemannian manifold (M, g) of dimension n plays an important role in the field of Partial differential equations, Geometric analysis, and Differential Geometry ([4]). Rather than mentioning a lot references, we prefer to give a comment about our appendix studying the quantity introduced by G.Perelman for the gradient estimate of the fundamental solution to the conjugate heat equation associated with the Ricci flow. From it, one can see that the choices of differential Harnack quantities should depend on the geometric background of eveolution equations. Because Perelman's gradient estimate for heat kernel plays an important role in Ricci flow, people are motivated to find extensions of his result to various problems. In [10] L. Ni made a clever observation that the same gradient estimate for the fundamental solution to the heat equation on a compact Riemannian manifold with fixed metric is also true provided the Ricci curvature is non-negative. Recently attention is focused to a nature question, which asks if such a gradient estimate is still true for all positive solutions to heat equation or conjugate heat equation associated with the Ricci flow. Cao-Hamilton[1]

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and Kuang-Zhang [6] then showed independently that the Perelman's gradient estimate is true for the conjugate heat equation associated with the Ricci flow. In our appendix, we notice that the Perelman type gradient estimate is an easy consequence of Li-Yau gradient estimate for any positive solution to the heat equation on a compact Riemannian manifold with non-negative Ricci curvature. Actually the above estimate can be obtained easily from Li-Yau's gradient estimate ([7]), namely,

$$2t\Delta f \leq n.$$

Just note that

$$tP \leq tP + t|\nabla f|^2 = 2t\Delta f \leq n,$$

where the constant n is the best. Other method may not give such a sharp constant. Following our works [8] and [9], we continue to study the gradient estimate for positive solutions to nonlinear evolution equations on manifolds.

The main topic of this note is to study the following nonlinear heat equation problem

$$u_t - \Delta u = au \log u + Vu, \quad u > 0 \tag{2}$$

on the compact Riemannian manifold (M, g) of dimension n. Here $a \leq 0$ is some constant and V is a given smooth function on M. This heat equation can be considered as the negative gradient heat flow to W-functional [11], which is the Log-Sobolev inequalities on the Riemannian manifold and V corresponds to the scalar curvature.

Our result is below.

Theorem 1. *Assume that the compact Riemannian manifold (M, g) has non-negative Ricci curvature. Assume $-\Delta V \leq A$ for some constant $A \geq 0$. Let $u > 0$ be a positive smooth solution to (2). Let $f = \log u$. Then we have, for all $t > 0$,*

$$\Delta f - At - \frac{n}{2t} \leq 0,$$

and in other words,

$$f_t - af + V + |\nabla f|^2 \leq At + \frac{n}{2t}.$$

The same result is also true for complete Riemannian manifold provided the maximum principle is applicable.

The important part in our result is that we only assume the lower bound of ΔV on M. However, our gradient estimate may not be the best one because of the term At. In our paper [8], we propose the study the nonlinear heat equation (2). Then Y.Yang [12] finds a nice gradient estimate for (2). G.Huang [5], Chen and Chen [3], and others also find more interesting results for (2). See also [9] for related rsults.

We can extend the result above to compact Riemannian manifold with smooth convex boundary.

Theorem 2. *Assume that the compact Riemannian manifold (M, g) with smooth convex boundary has non-negative Ricci curvature. Assume $-\Delta V \leq A$ for some constant $A \geq 0$. Let $u > 0$ be a positive smooth solution to (2) with Neumann boundary condition u_ν where ν is the outward unit normal to the boundary. Assume that $V_\nu \leq 0$ on the boundary ∂M. Let $f = \log u$. Then we have, for all $t > 0$,*

$$\Delta f - At - \frac{n}{2t} \leq 0.$$

From the geometric view-point, the boundary condition about V is not nature. We point out that it is interesting question to extend the results above to the equation (2) on general Riemannian manifolds.

2. Proofs of Theorems 1 and 2

We shall follow Li-Yau's method [7].

Consider the positive solution u to the equation (2). Let $f = -\log u$. Then we have

$$Lf := f_t - \Delta f = af - V - |\nabla f|^2.$$

Then we have

$$L\Delta f = a\Delta f - \Delta V - \Delta|\nabla f|^2.$$

By the Bochner formula and the non-negative Ricci curvature assumption we know that

$$\Delta|\nabla f|^2 \geq \frac{2}{n}|\Delta f|^2 + 2g(\nabla f, \nabla \Delta f).$$

Then we have

$$L\Delta f - a\Delta f \leq -\Delta V - \frac{2}{n}|\Delta f|^2 - 2g(\nabla f, \nabla \Delta f).$$

Recall that $-\Delta V \leq A$ for some constant $A \geq 0$. Hence we have

$$L\Delta f - a\Delta f \leq A - \frac{2}{n}|\Delta f|^2 - 2g(\nabla f, \nabla \Delta f).$$

Define

$$Q = \Delta f - At - \frac{n}{2t}.$$

We then have

$$LQ - aQ + 2g(\nabla f, \nabla Q) \leq a(At + \frac{n}{2t}) + \frac{2}{n}[(\frac{n}{2t})^2 - |\Delta f|^2].$$

Since $a \leq 0$, we have

$$LQ - aQ + 2g(\nabla f, \nabla Q) \leq \frac{2}{n}[(\frac{n}{2t})^2 - |Q + At + \frac{n}{2t}|^2].$$

Note that

$$[(\frac{n}{2t})^2 - |Q + At + \frac{n}{2t}|^2] = -(Q + At)(Q + At + \frac{n}{t}).$$

Case 1. At point (x, t) where

$$Q + At + \frac{n}{t} \leq 0,$$

we have $Q + At \leq 0$ too. Then we have

$$LQ - aQ + 2g(\nabla f, \nabla Q) \leq 0.$$

Case 2. At point (x, t) where

$$Q + At + \frac{n}{t} \geq 0,$$

we have

$$-(Q + At)(Q + At + \frac{n}{t}) \leq -Q(Q + At + \frac{n}{t})$$

and then

$$LQ - aQ + 2g(\nabla f, \nabla Q) \leq -Q(Q + At + \frac{n}{t}).$$

Define the function $B = 0$ at the point where $Q + At + \frac{n}{t} \leq 0$ and $B = -Q(Q + At + \frac{n}{t})$ at the point where $Q + At + \frac{n}{t} \geq 0$. Then using the maximum principle we obtain that

$$Q := \Delta f - At - \frac{n}{2t} \leq 0.$$

This completes the proof of Theorem 1.

The proof of Theorem 2 is similar. We need only to exclude the possibility of the maximum point of Q at boundary points. If the maximum occurs at the boundary point (x_0, t_0), then by the strong maximum principle we have

$$Q_\nu > 0$$

at this point. Note that

$$\Delta f = f_t - af + V + |\nabla f|^2.$$

So, at (x_0, t_0),

$$Q_\nu = (\Delta f)_\nu = V_\nu + (|\nabla f|^2)_\nu \leq -2II(\nabla f, \nabla f) \leq 0.$$

A contradiction. This proves Theorem 2.

3. Appendix

To compare the Perelman and Li-Yau gradient estimate for heat equation, we follow Perelman's approach to study the equation (1). Precisely, we prove the following.

Theorem 3. *Assume that the compact Riemannian manifold (M, g) has non-negative Ricci curvature. Let $u > 0$ be a positive smooth solution to (1). Let $f = \log u$. Then we have, for all $t > 0$,*

$$P := 2\Delta f - |\nabla f|^2 \leq 2n/t.$$

We now prove Theorem 3. Recall that $f = -\log u$. Then

$$f_j = -u_j/u, \quad \Delta f = -\Delta u/u + |\nabla f|^2.$$

Then we have

$$(\partial_t - \Delta)f = -|\nabla f|^2.$$

Let $L = \partial_t - \Delta$. Set $P = 2y - z$ with $y = \Delta f$ and $z = |\nabla f|^2$.

Note that

$$Lf = -z.$$

Then $Ly = -\Delta z$. Compute

$$\begin{aligned} Lz &= -\Delta z + 2g(\nabla f, \nabla f_t) \\ &= -\Delta z + 2g(\nabla f, \nabla (y - z)). \end{aligned}$$

Using the Bochner formula

$$\Delta z = 2|D^2 f|^2 + 2g(\nabla f, \nabla y) + 2Rc(\nabla f, \nabla f),$$

we get that

$$LP = -2g(\nabla f, \nabla P) - 2|D^2 f|^2 - 2Rc(\nabla f, \nabla f).$$

Using the Cauchy-Schwartz inequality, we obtain that

$$|D^2 f|^2 \geq \frac{1}{4n}(P + z)^2.$$

Hence using $Rc \geq 0$, we have

$$LP + 2g(\nabla f, \nabla P) \leq -\frac{1}{2n}(P + z)^2$$

and

$$L(P - 2nt^{-1}) + 2g(\nabla f, \nabla(P - 2nt^{-1})) \leq -\frac{1}{2n}[(P + z)^2 - (2n/t)^2].$$

Note that

$$I := [(P + z)^2 - (2n/t)^2] = [P + z - (2n/t)][P + z + (2n/t)].$$

Then we have

$$I = [P + z - (2n/t)]^2 + \frac{4n}{t}[P + z - (2n/t)] \geq \frac{4n}{t}[P + z - (2n/t)].$$

Note that $z = |\nabla f|^2 \geq 0$. We then have

$$I \geq \frac{2}{t}[P - (2n/t)].$$

Then we have

$$L(P - 2nt^{-1}) + 2g(\nabla f, \nabla(P - 2nt^{-1})) \leq -\frac{1}{2n}\frac{2}{t}[P - (2n/t)].$$

Using the maximum principle on $M \times [\delta, T)$ for $\delta > 0$ small, we conclude that

$$P - (2n/t) \leq 0.$$

This completes the proof of Theorem 3.

We remark that one may also use the argument in [6] in the following way.

Case 1. Assume that

$$P + z + (2n/t) \leq 0.$$

Then

$$P + z - (2n/t) \leq 0$$

too and $I \geq 0$.

Case 2.

Assume that

$$P + z + (2n/t) \geq 0.$$

Then

$$I = z[P + z + (2n/t)] + [P - (2n/t)][P + z + (2n/t)]$$
$$\geq [P - (2n/t)][P + z + (2n/t)].$$

Define a new function $B = 0$ on the set where $P + z + (2n/t) \leq 0$ and $B = P + z + (2n/t)$ on the set where $P + z + (2n/t) \geq 0$. Then we have

$$L(P - 2nt^{-1}) + 2g(\nabla f, \nabla(P - 2nt^{-1})) \leq -\frac{1}{2n}B[P - (2n/t)].$$

Using the maximum principle again to get the conclusion as before.

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