Solvable Models Of Infrared Gupta-Bleuler Quantum Electrodynamics

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Abstract

Solvable hamiltonian models are employed to investigate the extent and limitations of the procedures adopted in the perturbative treatment of the infrared divergences, occurring in the Feynman-Dyson expansion of Quantum Electrodynamics. Isometric Möller operators are obtained in the presence of an infrared regularization, after the removal of an adiabatic switching, with the aid of a suitable mass renormalization. We gain an hamiltonian control of the Yennie-Frautschi-Suura infrared factors and discuss the implications on the perturbative prescriptions for inclusive cross-sections.

Key words: Infrared Problem, Quantum Electrodynamics, Local Gauge Quantization, Solvable Models

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Introduction

In Quantum Electrodynamics (QED) , the description of states at asymptotic times and the derivation of the scattering matrix are still open issues.

At the perturbative level, transition amplitudes between states containing a finite number of photons are ill-defined, since radiative corrections due to soft photons typically exhibit logarithmic divergences ([7, 8]). As a consequence, in contrast with ordinary quantum field theories, Dyson's $S - matrix$ ([4]) is only defined in the presence of a low-energy cutoff and the problem of a proper identification of asymptotic states arises.

Already in 1937, Bloch and Nordsieck proved that the occurrence of infrared (IR) singularities in the perturbative expansion is due to basic physical facts. In their fundamental paper on the subject ([1]), they pointed out that on the basis of the correspondence principle one has to expect a vanishing probability for the emission of a finite number of photons in any collision process involving electrically charged particles. They showed that this prediction is indeed realized within a quantum-mechanical setting, at least under a suitable approximation, not requiring to treat the interaction of the electron with the low-energy modes of the electromagnetic field as a small perturbation, and allowing to get an explicit solution.

The occurrence of IR divergences can therefore be traced back to the assumption, at the basis of the expansion in powers of the electric charge, that the rate for the emission of radiation decreases with the number of photons.

Exponentiation of the low-energy photon contributions was first conjectured by Schwinger ([18]) and proved by Yennie, Frautschi and Suura (YFS) in [6] within the framework of the local and covariant Gupta-Bleuler formulation ([5]) of QED .

This led to a pragmatic approach to the problem; in the presence of an infrared cutoff, one sums the transition rates over all possible final photon-states with energy below the threshold of the experimental arrangement, and finally removes the regularization. The finiteness of the result is ensured by the well-known cancellation between the virtual infrared divergences and those due to soft-photon emission.

Such a procedure, nowadays at the basis of the perturbative-theoretic treatment of the infrared problem, merely gives inclusive cross-sections, depending on the energy resolution of the photon detectors; therefore the question of whether it is possible to obtain a full quantum mechanical scattering theory, yielding transition amplitudes between pure states, remains open. Furthermore, since the calculations have to be performed in an intermediate, infrared-regularized theory, their outcomes, although finite, do not seem to be founded on solid theoretical basis and in fact have been questioned in the literature ([9]). In particular, the time limit is taken *before* removing the regularization and in principle such an exchange of limits may lead to incorrect results.

It is also important to stress that the standard recipes to handle the low-energy photon singularities somehow avoid to directly take into account the non-perturbative properties involved in the characterization of physical charged states. In this respect, several model-independent investigations ([11, 12, 13, 14, 15, 16]) made it clear that besides the occurrence of representations of the asymptotic electromagnetic field "containing an infinite number of photons", the description of scattering states of abelian gauge theories requires to take into account further peculiarities.

In particular, it was proven that as a consequence of Gauss' law states carrying an electric charge cannot be obtained by applying local operators to the vacuum and that the state space admits uncountably many charged superselection sectors, non-invariant under Lorentz boosts and with single-particle subspaces not containing proper eigenstates of the mass operator.

Such results were obtained in general settings, independently of perturbation theory; quite generally, since its early days the Gupta-Bleuler formulation has been mostly applied to extract physical predictions while it has seldom been employed as a tool to gain insights into the structural properties of Quantum Electrodynamics. A careful analysis of the principles of local and covariant quantizations of (abelian) gauge theories has been given in [27],[24]. The problem of the identification of physical charged states in the Gupta-Bleuler formulation has been addressed in [25],[26]. An approach especially focused on the analicity properties of the scattering matrix has been pursued in ([19]).

More recently, Steinmann has developed a perturbative formulation in terms of Coulomb-type physical fields ([9]), using an adapted version of Wightman's formalism ([17]). Steinmann's strategy is noteworthy since it does not employ an intermediate infrared-regularized theory and does not rely on the introduction of asymptotic fields, which is problematic in QED; however, it also raises questions of principle and difficulties at the level of practical computations.

In fact, on the one hand the use of specific physical fields demands to employ a non-trivial generalization of Feynman's rules in the calculation of transition amplitudes, on the other it is limited with respect to the large arbitrariness involved in the description of charged states ([29]); moreover, it leads to cancellations ([20]) and in principle may also imply other physical effects which are still not understood.

The above discussion provides motivations to understand and possibly fill the gap that separates the standard methods from a collision theory in which the non-perturbative aspects of the infrared problem are taken into account.

As a starting point to address this issue, we analyze hamiltonian models, characterized by suitable infrared approximations, in order to seek for a formalization of the procedures involved in the local and covariant treatment of the IR divergences occurring in QED.

In particular, we shall compare the amplitudes obtained via the infrared diagrammatic in the Feynman-Gupta-Bleuler (FGB) gauge with the outcome of solvable hamiltonian models, based on two different approximations, which from a physical point of view seem to be equally suited for an analysis of the low-energy contributions; the electric dipole approximation and the expansion around a fixed (asymptotic) charged particle four-momentum. Perhaps surprisingly, such a comparison has never been carried out explicitly and IR models have been used to confirm the problems arising in the formulation of scattering, rather than for a systematic analysis of the procedures adopted in the standard approach.

Our main result is that it is possible to reproduce the infrared diagrammatic of QED within an hamiltonian framework, in terms of Möller operators, whose existence can be established with the aid of an adiabatic switching of the interaction and of a suitable mass counterterm, in the presence of a low-energy regularization.

The structure of the paper is as follows. Section 1 is devoted to the analysis of the Pauli-Fierz model ([2]), a non-relativistic model based on the dipole approximation, taking Blanchard's treatment ([3]) as a starting point. It is shown that by introducing an adiabatic switching and a mass-renormalization counterterm in the Pauli-Fierz hamiltonian, one obtains Möller operators as strong limits of the corresponding evolution operator (in the interaction representation), for each value of an infrared cutoff.

Since the model is formulated in the Coulomb-gauge, while the diagrammatic expansion is performed in local and covariant gauges, in order to set up a comparison with the perturbative expansion we introduce a four-vector model, retaining the approximations of the Pauli-Fierz hamiltonian.

We prove that Möller operators can be defined as asymptotic time-limits of the evolution operator in a suitable topology, and by means of them we obtain the exponentiation of the infrared contributions and the compatibility of the low-energy approximations with the renormalization procedure. However, it is also shown that the dipole approximation necessarily prevents the model from fully reproducing the IR behaviour of Feynman's amplitudes.

In order to cope with these difficulties, in section 2 we introduce hamiltonian models based on an expansion already implicit in [1] and hereafter referred to as Bloch-Nordsieck (BN) models. We will first consider a model formulated in the Coulomb-gauge and then seek for a suitable four-vector version of the latter.

In section 3, we prove that the infrared diagrammatic in the FGB gauge is reproduced, by means of the Möller operators of the four-vector BN model, for each value of the low-energy cutoff.

Moreover, within the same hamiltonian framework the expressions of the inclusive cross-sections, obtained by the standard perturbative recipes, are recovered and the extent and limitations of such a formulation are discussed. In particular, we prove that if the charged particle states are described by wave packets the removal of the infrared cutoff in the corresponding inclusive cross-section yields a vanishing result unless suitable coherent representations of the asymptotic photon field are employed, and finally outline the relationship of such a treatment with Chung's approach ([10]).

These results stem from an analysis of infrared models of QED also addressing the problems connected with the definition of a scattering matrix and the construction and characterization of physical charged states in the Gupta-Bleuler formulation ([28]).

Notations

The metric $g^{\mu\nu} = diag\,(1, -1, -1, -1)$ of Minkowski-space is adopted and natural units are used $(\hbar = c = 1)$.
A four-vector is indicated with v^μ or simply by v, while the symbol $\mathbf{v}$ denotes a three-dimensional vector. We use the symbol $c \cdot d$ for the indefinite inner product between four-vectors c and d.

The norm of a vector $\phi \in L^2$ is indicated by $\| \phi \|_2$, the commutator between the operators A and B by $[A , B]$, the trace of A by $Tr\, A$ and the projection on the vector ψ by P_ψ.
We denote by $\mathscr{F}$ the symmetric Fock space, by $\phi^{(n)} \equiv S_n \phi$ the projection of $\phi \in \mathscr{F}$ on the $n-$ particle space, with S_n the symmetric $n-$ dimensional projector, and by ω_F the Fock vacuum functional.
In the Coulomb-gauge formulation, $a_s\, (a_s^*)$ will stand for the photon annihilation (creation) operator-valued distribution, fulfilling the canonical commutation relations (CCR)

$$[\, a_s (\mathbf{k}) , a^*_{s'} (\mathbf{k}')] = \delta_{s s'} \, \delta (\mathbf{k} - \mathbf{k}') ,$$

with s and s' polarization indices.
In the same gauge, the hamiltonian of the free electromagnetic field will be denoted by $H_{0,tr}^{e.m.}$ and the vector potential at time $t = 0$ by

$$\mathbf{A}_{tr} (\hat{\mathbf{x}}) \equiv \sum_s \int \frac{d^3 k}{\sqrt{2\, \omega_{\mathbf{k}}}} \; \epsilon_s (\mathbf{k}) \, [\, a_s (\mathbf{k}) \; e^{i\, \mathbf{k} \cdot \hat{\mathbf{x}}} + h.c. \,] ,$$

with ϵ_s, $s = 1, 2$, orthonormal polarization vectors, fulfilling the transversality condition $\mathbf{k} \cdot \epsilon_s (\mathbf{k}) = 0$.
The annihilation and creation operator-valued distributions in the FGB gauge, denoted respectively by $a^\mu (\mathbf{k})$ and $a^{\mu *} (\mathbf{k})$, fulfill the CCR

$$[\, a^\mu (\mathbf{k}) , a^{\nu *} (\mathbf{k}')] = - g^{\mu\nu} \, \delta (\mathbf{k} - \mathbf{k}') .$$

In the same gauge, the hamiltonian of the free e.m. field shall be denoted by $H_0^{e.m.}$ and the vector potential at time $t = 0$ by

$$A^\mu (\hat{\mathbf{x}}) \equiv \int \frac{d^3 k}{\sqrt{2\, \omega_{\mathbf{k}}}} \; [\, a^\mu (\mathbf{k}) \; e^{i\, \mathbf{k} \cdot \hat{\mathbf{x}}} + h.c. \,] .$$

The convolution with a form factor ρ is indicated by

$$A^\mu (\rho , \hat{\mathbf{x}}) \equiv \int d^3 \xi \;\; \rho (\xi) \;\; A^\mu (\hat{\mathbf{x}} - \xi) ,$$

and similarly for $\mathbf{A}_{tr}$. For brevity we write

$$a\, (f\, (t)) \equiv \int d^3 k \;\; a_\mu (\mathbf{k}) \;\; f^\mu (\mathbf{k} , t)$$

and denote by $a_{tr}\, (f\, (t))$ the corresponding sum in the Coulomb-gauge.
We shall make use of the operator

$$W (f , g) \equiv \exp\, (- \frac{i}{2} \, \langle f , g \rangle) \;\; U (f) \;\; V (g) ,$$

where U and V are the Weyl exponentials

$$U (f) \equiv e^{-i / \sqrt{2}\; ((a + a^*) (f))} , \; V (g) \equiv e^{1 / \sqrt{2}\; ((a^* - a) (g))} .$$

1 Pauli-Fierz-Blanchard Models

This section is devoted to the introduction of the Pauli-Fierz-Blanchard (PFB) model. The model ([2]) describes the interaction of a spin-less Schrödinger particle with the quantum electromagnetic field under suitable infrared approximations.

It was reconsidered three decades later by Blanchard, who investigated the questions connected with a mathematical formulation of the fundamental fact that an infinite number of photons is emitted in any collision process involving electrically charged particles. In [3], he proved the existence of the dynamics and constructed the asymptotic states of the quantum system. In detail, he showed that an unitary operator can be obtained as the limit of the evolution operator in the sense of morphisms of a suitable $(C^{*} -)$ algebra, defined on the infinite tensor product starting from the algebra of linear continuous operators on a finite product of (Fock) spaces. Furthermore, he established the existence of Möller operators, interpolating between the Pauli-Fierz hamiltonian and its perturbation by a potential term, for a large class of potentials.

Our treatment will require some changes with respect with Blanchard's setting. In fact, since we wish to employ the model in order to investigate the methods at the basis of the local and covariant perturbative treatment of the infrared divergences, a question not addressed in Reference [3], we shall introduce an IR cutoff from the start.

In the presence of an infrared regularization, the limits considered by Blanchard exist with respect to the weak topology. However, as we shall see in the sequel, in order to recover the results of the diagrammatic analysis of the infrared contributions it is necessary first to get strong convergence for the asymptotic time-limits of the evolution operator; in a four-vector formulation of the model, Möller operators will then be shown to be definable in a suitable topology.

We consider the infrared-regularized PFB hamiltonian

$$H_{\lambda}^{(PFB)} = \frac{\hat{\mathbf{p}}^{2}}{2m} + H_{0,tr}^{e.m.} + H_{int} \equiv H_{0} + H_{int} \ , \tag{1}$$

$$H_{int} \equiv -\frac{e}{m} \ \hat{\mathbf{p}} \cdot \mathbf{A}_{tr}(\rho, \hat{\mathbf{x}} = 0). \tag{2}$$

The particle, of mass m, charge e and a spherically symmetric distribution of charge ρ, will also be called electron. The photon-mass method is employed as an IR regularization, by setting $\omega_{\mathbf{k}}^{2} \equiv \mathbf{k}^{2} + \lambda^{2}$, the $\lambda-$dependence in $H_{0,tr}^{e.m.}$ and in (2) being understood. The functional form of the interaction is dictated by the electric dipole approximation and implies that the electron momentum is conserved, while the total one is not.

The Hilbert space of states of the model is $\mathscr{H} = L^{2}(\mathbb{R}^{3}) \otimes \mathscr{F}$, with L^{2} the one-particle space and $\mathscr{F}$ the Fock space of photons of mass λ.
Since H_{0} acts as a multiplication it is essentially self-adjoint (e.s.a.) on the domain $D_{0} = \mathscr{S}(\mathbb{R}^{3}) \otimes D_{F_{0}}$, with $\mathscr{S}(\mathbb{R}^{3})$ the Schwartz space of C^{∞} functions of rapid decrease, F_{0} the dense set of finite-particle vectors belonging to $\mathscr{F}$ and $D_{F_{0}} \equiv (\psi \in F_{0} ; \psi^{(n)} \in S_{n} \bigotimes_{k=1}^{n} \mathscr{S}(\mathbb{R}^{3}), \forall n)$. As the operator (2) is small in the sense of Kato with respect to H_{0}, the hamiltonian

is e.s.a. on D_0 ([22]); by Stone's theorem ([21]) it then follows the existence of the one-parameter group $U(t) = \exp(-iHt)$.

The evolution operator in the interaction representation obeys

$$i\frac{dU_I(t)}{dt} \equiv H_I(t)\; U_I(t)\,. \tag{3}$$

Since the commutator of H_I evaluated at different times is a multiple of the identity operator in each definite-momentum restriction of the single-particle sector, in order to solve equation (3) one can apply the Baker-Hausdorff (BH) formula $e^{A+B} = e^A\; e^B\; e^{-\frac{1}{2}[A,B]}$; one obtains

$$U_I(t) = \exp(-i\int_0^t dt'\; H_I(t'))\; \exp(-\frac{1}{2}\int_0^t dt' \times \\ \times \int_0^{t'} dt''\; [H_I(t'), H_I(t'')])\,, \tag{4}$$

and

$$U(t) = \exp(-iH_0 t)\; U_I(t)\,. \tag{5}$$

An explicit calculation yields

$$U_I(t) = c(t)\; \exp(\,ie\,(a^*_{tr}(f_{\hat{\mathbf{p}}}(t)) + a_{tr}(\overline{f}_{\hat{\mathbf{p}}}(t))))\,, \tag{6}$$

$$c(t) = \exp(\frac{ie^2\,\hat{\mathbf{p}}^2}{3m^2}\int\frac{d^3k\;\tilde{\rho}^2(\mathbf{k})}{\omega^2_{\mathbf{k}}}(t - \frac{\sin\omega_{\mathbf{k}}t}{\omega_{\mathbf{k}}}))\,, \tag{7}$$

$$f_{\mathbf{p}s}(\mathbf{k},t) = \frac{\tilde{\rho}(\mathbf{k})}{\sqrt{2\,\omega_{\mathbf{k}}}}\;\frac{\mathbf{p}\cdot\epsilon_s(\mathbf{k})}{m}\;\frac{e^{i\omega_{\mathbf{k}}} - 1}{i\,\omega_{\mathbf{k}}}\,. \tag{8}$$

It is immediate to see that $U_I(t)$ does not converge for large times; in fact the operator (7), which is obtained by evaluating the commutator in (4), has divergent eigenvalues on plane-wave single-particle states in the limit $|t| \to \infty$. We shall see below that in order to cope with this problem it is necessary to introduce a suitable mass counterterm. Moreover, since the oscillating terms occurring in (7)-(8) only allow for the existence of asymptotic weak limits, they have to be regularized by a suitable time mean; this can be accomplished for instance by replacing the electric coupling by $e^{(ad)}(t) \equiv e\; e^{-\epsilon|t|}$.

An hamiltonian fulfilling the above requirements is

$$H^{(PFB)}_{\lambda,R} = \frac{\hat{\mathbf{p}}^2}{2m} + H^{e.m.}_{0,tr} + H^{(\epsilon)}_{int,R}\,, \tag{9}$$

$$H^{(\epsilon)}_{int,R} \equiv H_{int}\; e^{-\epsilon|t|} + z\,e^2\,\frac{\hat{\mathbf{p}}^2}{2m}\; e^{-2\epsilon|t|}\,, \tag{10}$$

$$z = \frac{2}{3m}\int\frac{d^3k}{\omega^2_{\mathbf{k}}}\;\tilde{\rho}^2(\mathbf{k})\,. \tag{11}$$

By substituting (9)-(11) in the r.h.s of equation (4) we obtain, for positive times,

$$U^{(\epsilon)}_{I,\lambda}(t) \equiv c^{(\epsilon)}_z(t)\; \exp(\,ie\,(a^*_{tr}(f^{(\epsilon)}_{\hat{\mathbf{p}}}(t)) + h.c.))\,, \tag{12}$$

with

$$c_z^{(\epsilon)}(t) \equiv \exp\left(\frac{i e^2 \hat{\mathbf{p}}^2}{3 m^2} \, d^{(\epsilon)}(t)\right) \exp\left(i e^2 z \, \frac{\hat{\mathbf{p}}^2}{2 m} \, \frac{e^{-2 \epsilon t} - 1}{2 \epsilon}\right), \tag{13}$$

$$d^{(\epsilon)}(t) = -\int \frac{d^3 k \;\, \tilde{\rho}^2(\mathbf{k})}{\omega_{\mathbf{k}} (\omega_{\mathbf{k}}^2 + \epsilon^2)} \left(e^{-\epsilon t} \sin \omega_{\mathbf{k}} t + \right. \\ \left. + \frac{\omega_{\mathbf{k}}}{2 \epsilon} (e^{-2 \epsilon t} - 1)\right), \tag{14}$$

$$f_{\mathbf{p} s}^{(\epsilon)}(\mathbf{k}, t) = \frac{\tilde{\rho}(\mathbf{k})}{\sqrt{2 \omega_{\mathbf{k}}}} \; \frac{\mathbf{p} \cdot \epsilon_s(\mathbf{k})}{m} \; \frac{e^{(i \omega_{\mathbf{k}} - \epsilon) t} - 1}{i \omega_{\mathbf{k}} - \epsilon} \,. \tag{15}$$

Equations (13), (14) provide a regularization of (7). In fact, the oscillating contribution from the first line on the right-hand side of (14) vanishes for $|t| \to \infty$ due to the presence of the adiabatic factor. Furthermore, the residual contribution of order $1/\epsilon$ arising from the second line of (14) is canceled in (13), for $\epsilon \to 0$, by the $z-$ dependent exponent; hence the existence of the large-time limits and of the adiabatic limit of $c_z^{(\epsilon)}(t)$ is proved.

The convergence of the evolution operator (12) is then established in the following

Lemma 1. *By choosing the coefficient of the mass counterterm as in (11), both the large-time limits and the adiabatic limit of the evolution operator (12), defining the Möller operators, exist in the strong topology:*

$$\Omega_\pm = s - \lim_{\epsilon \to 0} \lim_{t \to \mp \infty} U_{I,\lambda}^{(\epsilon)}(-t) = \\ = \exp\left(-i e \sum_s [a_s^*(f_{\hat{\mathbf{p}} s}) + a_s(\overline{f}_{\hat{\mathbf{p}} s})]\right), \tag{16}$$

$$f_{\mathbf{p} s}(\mathbf{k}) \equiv L^2 - \lim_{\epsilon \to 0} \lim_{t \to \mp \infty} f_{\mathbf{p} s}^{(\epsilon)}(\mathbf{k}, t) \,. \tag{17}$$

Proof. By the Riemann-Lebesgue lemma,

$$L^2 - \lim_{\epsilon \to 0} \lim_{t \to \mp \infty} f_{\mathbf{p} s}^{(\epsilon)}(\mathbf{k}, t) \equiv f_{\mathbf{p} s}(\mathbf{k}) \,. \tag{18}$$

With the aid of standard Fock-space estimates one can immediately prove that the Segal operator $\Phi(f_{\mathbf{p}}^{(\epsilon)}(\mathbf{k}, t)) \equiv a_{tr}(f_{\mathbf{p}}^{(\epsilon)}) + h.c.$ converges strongly to $\Phi(f_{\mathbf{p}}(\mathbf{k}))$, on its (common with the limiting operator) domain F_0 of essential self-adjointness, spanned by the vectors of $\mathscr{F}$ with a finite number of non-zero components; one has therefore also convergence in the strong generalized sense. The statement finally follows by the continuity and boundedness of $U_{I,\lambda}^{(\epsilon)}(t)$ as a function of $\Phi(f_{\mathbf{p}}^{(\epsilon)}(\mathbf{k}, t))$. □

We can now introduce, for a fixed value of λ, the $S-$ *matrix*

$$S_\lambda \equiv s - \lim_{\epsilon \to 0} \lim_{t, t' \to +\infty} U_{I,\lambda}^{(\epsilon)}(t) \, \mathscr{W} \, U_{I,\lambda}^{(\epsilon)}(t') \,. \tag{19}$$

In the above formula, $\mathscr{W}$ is a unitary operator on $\mathscr{H}$ with improper one-particle momentum eigenstates in its domain, able to connect particle states with different momenta and thus allowing for a non-trivial scattering.

In order to compare the expansion of the Möller operators with the infrared diagrammatic of QED, it is necessary to formulate a hamiltonian model in a gauge employing four independent photon degrees of freedom, such as Feynman's gauge. In the following, we introduce a four-vector model, which will be referred to as $PFBR$ model, demanding that it should retain the approximations of the Pauli-Fierz hamiltonian. An hamiltonian fulfilling the above requirements is

$$H_{\lambda}^{(PFBR)} = m + \frac{\hat{\mathbf{p}}^2}{2m} + H_0^{e.m.} + e\ \tilde{v} \cdot A(\rho, \hat{\mathbf{x}} = 0), \tag{20}$$

with $\tilde{v}^{\mu} = (1 + \frac{\mathbf{v}^2}{2}, \mathbf{v}), \mathbf{v} \equiv \hat{\mathbf{p}}/m$. Since, as for the PFB model, the commutator of H_I evaluated at different times is a multiple of the identity operator for a fixed four-velocity $\tilde{v}$ of the charge, we can still employ the BH formula in order to find a (formal) solution of equation (3) for the evolution operator in the interaction representation. For an assigned value of the four-velocity, we get

$$U_I(t) \sim \exp(-i e (a^*(f_{\tilde{v}}(t)) + a(\overline{f}_{\tilde{v}}(t)))), \tag{21}$$

$$f_{\tilde{v}}^{\mu}(\mathbf{k}, t) = \frac{\tilde{\rho}(\mathbf{k})\ \tilde{v}^{\mu}}{\sqrt{2\omega_{\mathbf{k}}}} \frac{e^{i\omega_{\mathbf{k}} t} - 1}{i\omega_{\mathbf{k}}}. \tag{22}$$

The space obtained by applying the canonical photon variables to the Fock vacuum is only equipped with an indefinite inner product; this raises the questions of how to prove the existence of the dynamics and whether it is possible to define the asymptotic limits of the evolution operator.

In the presence of a positive metric, one would perform the GNS construction ([23]) over the Fock functional, defined on the polynomial algebra of the (smeared) canonical variables of the system, and would then obtain a unique Hilbert space by taking completions and quotients. On the other hand, in the absence of positivity the GNS space is not closed, nor there exists a univocally determined closure.

However, since the model is solvable we do not need to address the problem in full generality. In particular, as (21) is an exponential of the canonical variables of the electromagnetic field, the indefinite-metric linear space obtained by means of the GNS construction over the polynomial algebra generated by a and a^* can be extended simply by applying the latter construction to a larger algebra, also containing the exponentials of the photon canonical variables.

Accordingly, we consider the algebra $\mathscr{A}_{ext}^{e.m.}$ generated by polynomials and exponentials of the photon annihilation and creation operator-valued distributions, smeared with test functions in $\mathscr{S}(\mathbb{R}^3)$, and denote by $\mathscr{D}_0$ the space, stable under the time evolution, obtained by applying the GNS theorem to $\omega_F(\mathscr{A}_{ext}^{e.m.})$.

The model is then defined on the linear space $L^2(\mathbb{R}^3) \otimes \mathscr{D}_0$. For fixed $\tilde{v}$, the dynamics in the interaction representation is determined by an isometric operator $U_I : \mathscr{D}_0 \to \mathscr{D}_0$, whose time-derivative on $\mathscr{D}_0$ fulfills equation (3) in the weak sense. The solution (21) clearly fulfills these requirements and it is also unique, since it leaves $\mathscr{D}_0$ stable.

Proceeding as for the PFB model, we introduce the adiabatic mean and the renormalization counterterm; let

$$H^{(\epsilon)}_{\lambda , R} = m + \frac{\hat{\mathbf{p}}^2}{2m} + H_0^{e.m.} + e \ \tilde{v} \cdot A(\rho , \hat{\mathbf{x}} = 0) \ e^{-\epsilon |t|} + \\ - \frac{4}{3} \ m \ z \ e^2 \ e^{-2 \epsilon |t|} . \qquad (23)$$

Denoting by $U^{(\epsilon)}_{I , \lambda}(t)$ the evolution operator in the interaction representation, corresponding to the hamiltonian (23), one can prove the following

Proposition 1. *The Möller operators of the* $PFBR$ *model are given by*

$$\Omega_{\pm , \lambda} = \tau - \lim_{\epsilon \to 0} \lim_{t \to \mp \infty} U^{(\epsilon)}_{I , \lambda}(-t) = \\ = h_{\mp , z} \ \exp\left(i e \left[a^*(f_{\tilde{v}}) + a(\overline{f}_{\tilde{v}}) \right] \right) , \qquad (24)$$

$$h_{\mp , z} = 1 + O(\mathbf{v}^2) , \ f^{\mu}_{\tilde{v}}(\mathbf{k}) = \frac{\tilde{\rho}(\mathbf{k}) \ \tilde{v}^{\mu}}{\sqrt{2 \ \omega_{\mathbf{k}}}} \ \frac{i}{\omega_{\mathbf{k}}} \cdot \qquad (25)$$

In order to control the above limits we have considered the larger algebra $\mathscr{A}^{e.m.}_{ext , L^2}$, obtained by allowing for test functions in L^2, and introduced a suitable topology τ on $\overline{\mathscr{D}}_0$, the completion of $\mathscr{D}_0$ given by GNS construction over $\omega_F(\mathscr{A}^{e.m.}_{ext , L^2})$. We can now define the scattering operator

$$S^{(PFBR)}_{\lambda} = \tau - \lim_{\epsilon \to 0} \lim_{t , t' \to + \infty} U^{(\epsilon)}_{I , \lambda}(t) \ \mathscr{W} \ U^{(\epsilon)}_{I , \lambda}(t') . \qquad (26)$$

In (26), $\mathscr{W}$ is an isometric operator on $L^2(\mathbb{R}^3) \otimes \overline{\mathscr{D}}_0$, playing the same role as in (19); in the comparison with the diagrammatic expansion, its matrix elements are interpreted as the non-infrared contributions to the corresponding scattering process.

The expansion in powers of the electric charge of suitable matrix elements of the Möller operators, obtained on the basis of the $PFBR$ hamiltonian, can be shown to reproduce qualitatively the infrared contributions of Dyson's power series, under the dipole approximation and in the non-relativistic limit. For instance, consider the transition amplitude for the scattering between two single particle states ψ_v and $\psi_{v'}$, without external (massive) photons,

$$(S^{(PFBR)}_{\lambda})_{v', v} \equiv \langle \psi_{v'} \otimes \Psi_F , \Omega^*_{- , \lambda} \ \mathscr{W} \ \Omega_{+ , \lambda}(\psi_v \otimes \Psi_F) \rangle = \\ = \omega_F\left(\Omega^{*(\lambda)}_{- , \tilde{v}'} \ \Omega^{(\lambda)}_{+ , \tilde{v}} \right) \ \mathscr{W}_{v', v} , \qquad (27)$$

where we have employed the relation $\Omega^{(\lambda)}_{\pm}(\psi_v \otimes \Psi_F) = \psi_v \otimes \Omega^{(\lambda)}_{\pm , v} \Psi_F$, which holds true since (24) act as multiplication operators on plane-wave states. The radiative soft-photon corrections to the basic process are reproduced by the Fock expectation in the second line of equation (27). In particular, the exponentiation of the low-energy radiative corrections turns out to be a direct consequence of the BH formula and the compatibility of the dipole approximation with the renormalization procedure is displayed non-perturbatively.

However, by a direct calculation one can check that contributions from unphysical polarizations occur in the (infrared-regularized) transition amplitudes. In order to clarify this issue, we point out that the dipole approximation requires the transversality of the electron current and therefore prevents the local conservation of the electric charge in a gauge involving four independent photon degrees of freedom. The standard argument leading to the cancellation of the photon contributions to Feynman's amplitudes arising from the unphysical polarizations is therefore no longer valid, since it is based on the free-field character of $\partial \cdot A$, which in turn derives from the continuity equation.

Furthermore, one can verify that the lack of local charge-conservation of the $PFBR$ model leads indeed to infrared effects. As a matter of fact, the power series expansion of Dyson's $S - matrix$ elements for, say, the scattering of an electron by a potential, contains powers of the scalar photon contribution $(\frac{p'^0}{p' \cdot k} - \frac{p^0}{p \cdot k})^2 \simeq \omega_{\mathbf{k}}^{-2} [(\mathbf{v}' - \mathbf{v}) \cdot \hat{\mathbf{k}}]^2$, which do not appear in the presence of the electric dipole approximation, thus leading to the occurrence of residual contributions from longitudinal photons in the transition amplitudes.

Therefore, the dipole approximation is not suitable to fully reproduce the results obtained in the perturbative-theoretic treatment of the infrared divergences. The analysis carried out so far will nonetheless prove useful; in fact, the methods and the procedures upon which it is based will be also used in the investigation of the models that will be introduced in the next section.

2 Bloch-Nordsieck Models

In the present section we introduce hamiltonian models based on an approximation first devised by Bloch and Nordsieck. Such an approximation amounts to a first-order expansion around a fixed four-momentum of each charge, with respect to the energy-momentum transfer.
Consider the one-particle Dirac hamiltonian with minimal coupling,

$$H = \alpha \cdot (\hat{\mathbf{p}} - e \mathbf{A}) + \beta m + e A^0 \equiv H_D - e \alpha \cdot \mathbf{A} + e A^0 , \quad (28)$$

and an eigenstate of H_D with momentum $\mathbf{p}$ and positive energy $E_{\mathbf{p}}$, denoted by $\psi_{+, p} (x) = e^{-i p \cdot x} u_r (\mathbf{p})$, $u_r (\mathbf{p})$ being a spinor with elicity r.
Let $u_r (\mathbf{p}) = u_r (\mathbf{p}_0) + O (\mathbf{p} - \mathbf{p}_0)$; by the algebraic relations of Dirac's matrices one finds

$$\begin{aligned} H_D \psi_{0, p} (x) = [\mathbf{v} \cdot \mathbf{p} + (1 - \mathbf{v}^2)^{1/2} m] \psi_{0, p} (x) + \\ + O (\mathbf{p} - \mathbf{p}_0) , \psi_{0, p} (x) \equiv e^{-i p \cdot x} u_r (\mathbf{p}_0) , \mathbf{v} \equiv E_{\mathbf{p}_0} / \mathbf{p}_0 . \quad (29) \end{aligned}$$

The $\mathbf{v} -$ dependent terms on the first line of (29) could also be obtained by formally replacing the matrices α and β in H_D respectively by the (diagonal in the spinor indices) matrices $\mathbf{v}$ and $(1 - \mathbf{v}^2)^{1/2}$. Although it may seem to rely on the linearity of H_D with respect to the α matrices, this result is indeed more general; for instance, it would also be obtained by performing the same expansion on the relativistic energy of a free particle.

According to the above discussion, we introduce the models defined by the hamiltonians, respectively in the Coulomb-gauge and in the FGB gauge,

$$H_\lambda^{(\mathbf{v})} = \hat{\mathbf{p}} \cdot \mathbf{v} + H_{0,\,tr}^{e.m.} - e\ \mathbf{v} \cdot \mathbf{A}_{tr}(\rho, \hat{\mathbf{x}}) \equiv H_0^{(\mathbf{v})} + H_{int}^{(\mathbf{v})}, \qquad (30)$$

$$H_\lambda^{(v)} = \hat{\mathbf{p}} \cdot \mathbf{v} + H_0^{e.m.} + e\ v \cdot A(\rho, \hat{\mathbf{x}}) \equiv H_0^{(v)} + H_{int}^{(v)}. \qquad (31)$$

with $v \equiv (1, \mathbf{v})$. The e.m. potentials occurring in (30), (31) will be interpreted as describing soft-photon degrees of freedom and accordingly the observable algebra generated by each of them will be denoted by $\mathscr{A}_{e.m.}^{soft}$. Moreover, $\mathbf{v}$ is a self-adjoint operator, commuting with the algebra $\mathscr{A}_{ch}$ generated by the canonical variables of the electron and with $\mathscr{A}_{e.m.}^{soft}$, to be interpreted as the observable corresponding to the asymptotic velocity of the particle. For brevity we shall write $\mathscr{A}^{soft} \equiv \mathscr{A}_{ch} \cup \mathscr{A}_{e.m.}^{soft}$.

First we consider the model formulated in the Coulomb-gauge. The Hilbert space of states corresponding to an irreducible representation of the algebra $\mathscr{A}^{soft}$ at positive times is $\mathscr{H}_+ \equiv L^2(\mathbb{R}^3; \mathbf{v}_+) \otimes \mathscr{F}$, with $\mathbf{v}_+$ the constant value taken by $\mathbf{v}$, and with obvious notations $\mathscr{H}_-$ is the space of an irreducible representation of $\mathscr{A}^{soft}$ at negative times. Since for $\mathbf{v}$ a multiple of the identity the hamiltonian (30) is e.s.a. on $D = \sum_{n<+\infty} \phi_{\mathbf{v}} \otimes \psi^{(n)}$, with $\phi_{\mathbf{v}} \in \mathscr{S}(\mathbb{R}^3)$, $\psi^{(n)} \in S_n \bigotimes_{k=1}^n \mathscr{S}^{(k)}(\mathbb{R}^3)$, the existence of the dynamics follows by Stone's theorem.

The equations of the motion in the interaction representation can still be solved with the aid of the BH formula. Moreover, by the same treatment as in the first section, we introduce the adiabatic renormalized hamiltonian

$$\begin{aligned} H_{\lambda,R}^{(\mathbf{v}),(\epsilon)} = {} & H_0^{(\mathbf{v})} + H_{int}^{(\mathbf{v})}\ e^{-\epsilon|t|} + \\ & + e^2\ z_1(v)\ \mathbf{v}^2\ e^{-2\epsilon|t|} \equiv H_0^{(\mathbf{v})} + H_{int,R}^{(\mathbf{v}),(\epsilon)}, \end{aligned} \qquad (32)$$

$$z_1(v) = \frac{1}{3} \int \frac{d^3k}{\omega_{\mathbf{k}}} \frac{\tilde{\rho}^2(\mathbf{k})}{v \cdot k}. \qquad (33)$$

One can then prove the following

Lemma 2. *The evolution operator in the interaction representation, corresponding to the hamiltonian (32), (33), admits well-defined asymptotic limits, yielding the Möller operators in the Coulomb-gauge*

$$\begin{aligned} \Omega_{\pm,\lambda} = {} & s - \lim_{\epsilon\to 0} \lim_{t\to\mp\infty} U_{I,\lambda}^{(\epsilon)}(-t) = \\ & = \exp\left(-ie \sum_s [\, a_s^*(f_{\mathbf{v}_\mp s \hat{\mathbf{x}}}) + a_s(\overline{f}_{\mathbf{v}_\mp s \hat{\mathbf{x}}})\,]\right), \end{aligned} \qquad (34)$$

$$f_{\mathbf{v} s \hat{\mathbf{x}}}(\mathbf{k}) = e^{-i\,\mathbf{k}\cdot\hat{\mathbf{x}}} \frac{\tilde{\rho}(\mathbf{k})}{\sqrt{2\,\omega_{\mathbf{k}}}} \frac{i\ \mathbf{v} \cdot \epsilon_s(\mathbf{k})}{v \cdot k}. \qquad (35)$$

The $t \to -\infty$ limit in (34) exists in the strong topology of $\mathscr{H}_-$, and likewise for $t \to +\infty$ one has strong convergence on $\mathscr{H}_+$.

Proof. As for lemma 1. □

In the sequel we shall consider representations of the algebras $\mathscr{A}_{out}$ ($\mathscr{A}_{in}$) generated by the variables $\hat{\mathbf{x}}_{out}$ ($\hat{\mathbf{x}}_{in}$) $\equiv \hat{\mathbf{x}}$ and $\mathbf{v}$, describing the particle for positive (negative) asymptotic times in the interaction picture, and by the corresponding asymptotic soft-photon variables.
We define the scattering operator of the model as

$$S_\lambda = s - \lim_{\epsilon \to 0} \lim_{t, t' \to +\infty} U^{(\epsilon)}_{I,\lambda}(t)\ \mathscr{W}\ U^{(\epsilon)}_{I,\lambda}(t'), \tag{36}$$

with $\mathscr{W}$ a unitary operator with one-particle states of definite $\mathbf{v}$ in its domain, able to intertwine between (irreducible) representations of $\mathscr{A}_{in}$ and $\mathscr{A}_{out}$ with different values of $\mathbf{v}$.

We shall now turn to the analysis of the model defined by the hamiltonian (31), which will be also referred to as four-vector BN model.

Concerning the problems posed by the absence of a positive scalar product, we refer to the discussion made in the first section. With the aid of the BH formula, one can determine a formal solution for the evolution operator in the interaction representation. In order to prove its existence and uniqueness, we need to construct a suitable space. Let $a(f_{v_\mp \hat{\mathbf{x}}}) \equiv \int d^3k\ a^\mu(\mathbf{k})\ f_{v_\mp \hat{\mathbf{x}}}(\mathbf{k})$, with

$$f^{\mu}_{v\,\hat{\mathbf{x}}}(\mathbf{k}) = e^{-i\,\mathbf{k}\cdot\hat{\mathbf{x}}}\ \frac{\tilde{\rho}(\mathbf{k})}{\sqrt{2\,\omega_{\mathbf{k}}}}\ \frac{v^\mu}{-i\ v\cdot k}\ , \tag{37}$$

and let $\mathscr{A}^{\pm}_{ext}$ be the algebras generated by polynomials of the canonical variables of the system, by exponentials of the position and momentum of the charge and by the Weyl operators $U(f_{v_\pm \hat{\mathbf{x}}})$ and $V(f_{v_\pm \hat{\mathbf{x}}})$. By definition, $\mathscr{A}^{\pm}_{ext}$ contain the algebra $\mathscr{A}^{soft}$.

For positive times, the model is defined on the linear space $\mathscr{D}_+$ obtained via the GNS construction over $\omega_F(\mathscr{A}^{+}_{ext})$; for $t < 0$, the corresponding space is given by an analogous procedure in terms of the algebra $\mathscr{A}^{-}_{ext}$ and is denoted by $\mathscr{D}_-$. The dynamics in the interaction representation is determined for $t > 0$ by an operator $U_I : \mathscr{D}_+ \to \mathscr{D}_+$, unitary with respect to the indefinite inner product of $\mathscr{D}_+$ and with time-derivative satisfying (3) in the weak sense. The existence and uniqueness of the time evolution can be proved as for the $PFBR$ model; a solution is determined by the BH formula and its uniqueness follows since it leaves stable $\mathscr{D}_+$. For negative times, one has the same results for $U_I : \mathscr{D}_- \to \mathscr{D}_-$.

In order to construct Möller operators, we introduce the renormalized hamiltonian, in the presence of the adiabatic approximation,

$$\begin{aligned} H^{(v),(\epsilon)}_{\lambda,R} &= \hat{\mathbf{p}}\cdot\mathbf{v} + H^{e.m.}_{0,\lambda} + e\ v\cdot A(\rho,\hat{\mathbf{x}})\ e^{-\epsilon|t|} + \\ &\quad - e^2\ z_2(v)\ v^2\ e^{-2\epsilon|t|} = H^{(v)}_{0,\lambda} + H^{(v),(\epsilon)}_{int,R}\ , \end{aligned} \tag{38}$$

$$z_2(v) = \frac{1}{2}\int \frac{d^3k}{\omega_{\mathbf{k}}}\ \frac{\tilde{\rho}^2(\mathbf{k})}{v\cdot k} = \frac{3}{2}\ z_1(v)\,, \tag{39}$$

and determine the evolution operator in the interaction representation:

$$U^{(\epsilon)}_{I,R}(t) = h^{(\epsilon)}_{z_2}(t) \; \exp\left(i e \left(a^*(f^{(\epsilon)}_{v \hat{\mathbf{x}}}(t)) + a(\overline{f}^{(\epsilon)}_{v \hat{\mathbf{x}}}(t))\right)\right) , \quad (40)$$

$$h^{(\epsilon)}_{z_2}(t) \equiv c^{(\epsilon)}_{2}(t) \; \exp\left(i e^2 \; z_2(v) \; v^2 \int_0^t d t' \, e^{-2 \epsilon t'} \right) . \quad (41)$$

It is then possible to prove the following

Proposition 2. *Both the large-time limits and the adiabatic limit of the evolution operator (40), defining the Möller operators of the four-vector BN model, exist and are given by*

$$\Omega_{\pm,\lambda} \equiv \; \tilde{\tau} - \lim_{\epsilon \to 0} \; \lim_{t \to \mp\infty} \; U^{(\epsilon)}_{I,\lambda}(-t) =$$
$$= \exp\left(i e \left[a^*(f_{v_{\mp} \hat{\mathbf{x}}}) + a(\overline{f}_{v_{\mp} \hat{\mathbf{x}}})\right]\right) . \quad (42)$$

As a starting point to control the above limits, we have considered the algebras $\mathscr{A}^{\pm}_{ext, L^2}$, containing $\mathscr{A}^{\pm}_{ext}$, obtained by smearing the photon variables with functions in L^2; we have then chosen as a completion of $\mathscr{D}_+$ the space $\overline{\mathscr{D}}_+$ obtained through the GNS construction over $\omega_F(\mathscr{A}^{+}_{ext, L^2})$, and likewise for $\mathscr{D}_-$; finally we have introduced a suitable topology $\tilde{\tau}$ on $\overline{\mathscr{D}}_{\pm}$. The details regarding the definition of the $\tilde{\tau}-$ topology are given in the appendix.
We can now introduce the scattering operator of the four-vector BN model,

$$S^{(FGB)}_{\lambda} = \; \tilde{\tau} - \lim_{\epsilon \to 0} \; \lim_{t, t' \to +\infty} \; U^{(\epsilon)}_{I,\lambda}(t) \; \mathscr{W} \; U^{(\epsilon)}_{I,\lambda}(t') \equiv$$
$$\equiv \; \tilde{\tau} - \lim_{\epsilon \to 0} \; S^{(FGB)}_{\lambda,(\epsilon)} = \Omega^*_{-,\lambda} \; \mathscr{W} \; \Omega_{+,\lambda} , \quad (43)$$

with $\mathscr{W}$ an isometric operator with one-particle states of definite v in its domain, able to intertwine between (irreducible) representations of $\mathscr{A}_{in}$ and $\mathscr{A}_{out}$ labeled by different values of v, and thus allowing to get non-vanishing transition matrix elements. In the comparison of (43) with the Feynman-Dyson expansion, to be carried out in the next section, $\mathscr{W}$ will be interpreted as yielding the infrared-finite contributions, obtained by factoring out the YFS exponentials from the scattering operator.

3 Hamiltonian Control Of The Infrared Contributions

This section is devoted to the proof that the Möller operators of the four-vector BN model reproduce the YFS infrared factors and to the hamiltonian formulation of the recipes leading to infrared-finite inclusive cross-sections. It is important to recall that the perturbative calculations are performed in the interaction representation, in the presence of an infrared cutoff, and that the Fock property of the state of the incoming photon field is required to hold also after the removal of the regularization.

For definiteness, we consider as a basic process $\alpha \to \beta$ the scattering of an electron by a potential and suppose that the incoming (outgoing) particle state is described by a plane wave, with four-velocity v (v'). In the sequel, the symbol Λ will stand for an energy scale conventionally dividing the soft photons from the high-energy photons; accordingly, the subspaces of $\mathscr{F}$ spanned by the photon states of energy below and above Λ will be denoted respectively by $\mathscr{F}_{soft}$ and $\mathscr{F}_{hard}$.

The process $\alpha \to \beta$ is supposed not to involve low-energy photons, while it may possibly involve hard photons. We denote by $\eta^{in}_{e.m.}$ $(\eta^{out}_{e.m.})$ the state of the incoming (outgoing) e.m. field and by $\xi^{in}_{e.m.}$ $(\xi^{out}_{e.m.})$ its restriction to $\mathscr{F}_{hard}$; under the above assumptions, $\eta^{in\,(out)}_{e.m.} = \Psi_F \otimes \xi^{in\,(out)}_{e.m.}$, with Ψ_F the vacuum vector of $\mathscr{F}_{soft}$.
The isometric operator $\mathscr{W}$ in (43) will be interpreted as the hard-photon part of the scattering operator, hence it is supposed to act as the identity operator on $\mathscr{F}_{soft}$, and the same assumption is made for the restrictions of $\Omega^{(\lambda)}_{\pm}$ to $\mathscr{F}_{hard}$. The transition amplitude for the reaction $\alpha \to \beta$ is therefore given by

$$\mathscr{W}_{\beta\alpha} \equiv \langle \psi_{v'} \otimes \xi^{out}_{e.m.}\,,\, \mathscr{W}\,(\psi_v \otimes \xi^{in}_{e.m.}) \rangle\,. \tag{44}$$

First of all we want to show that the IR radiative corrections to the basic process are reproduced by the contributions of the Möller operators to the matrix element

$$(S^{(FGB)}_{\lambda})_{\beta\alpha} = \langle \psi_{v'} \otimes \eta^{out}_{e.m.}\,,\, \Omega^{*(\lambda)}_{-}\ \mathscr{W}\ \Omega^{(\lambda)}_{+}\,(\psi_v \otimes \eta^{in}_{e.m.}) \rangle\,, \tag{45}$$

that is, by the Fock expectation

$$\omega_F\,(\Omega^{*(\lambda)}_{-,\,v'}\ \Omega^{(\lambda)}_{+,\,v}) = \langle \Psi_F\,,\, \Omega^{*(\lambda)}_{-,\,v'}\ \Psi_F \rangle\, \langle \Psi_F\,,\, \Omega^{(\lambda)}_{+,\,v}\ \Psi_F \rangle \times \\ \times \exp\,(e^2\,[\,a\,(\overline{f}_{v'\hat{\mathbf{x}}})\,,\, a^*\,(f_{v\,\hat{\mathbf{x}}})\,])\,. \tag{46}$$

In fact, the last term on the r.h.s. of equation (46) provides the corrections due to soft photons emitted and absorbed from different charged lines; by explicit computation, it equals the perturbative result, summed to all orders:

$$\exp\,(e^2\ v \cdot v' \int \frac{d^3k}{2\,\omega_{\mathbf{k}}}\ \tilde{\rho}^{\,2}(\mathbf{k})\ \frac{i}{-\,v \cdot k}\ \frac{i}{-\,v' \cdot k}\,)\,. \tag{47}$$

Moreover, the expressions in brackets in (46) yield residual infrared contributions, due to renormalization. In particular, the first (second) expression is related to the infrared-singular part $Z_{2,IR}$ of the wave-function renormalization constant relative to the outgoing (incoming) external fermion leg; for instance, for the final external line one has

$$Z^{1/2}_{2,IR}(\lambda) = \exp\,(-\,\frac{e^2}{2}\ \frac{\partial\,\Sigma^{(\epsilon)}_{IR,\,\lambda}(v')}{\partial\,(i\,\epsilon)}\ |_{\,\epsilon=0}) = \lim_{\epsilon \to 0}\ \omega_F\,(\Omega^{(\epsilon)\,*}_{-,\,v',\,\lambda})\,. \tag{48}$$

Equation (48) yields the non-perturbative relation between $Z_{2,IR}$ and the IR contribution $\Sigma_{IR,\,\lambda}$ from the radiative correction due to the second-order electron self-energy part. In this respect, we recall that an infrared-singular term as (48) arises from renormalization in the FGB gauge, for each external fermion line, whenever on-shell renormalization conditions are imposed. As it can be inferred from the analyses carried out in References [7, 8], it is non-trivial to obtain such a result in an order-by-order diagrammatic treatment.

We can now turn to the examination of the effects due to the emission of low-energy radiation. The corrections to the basic process due to the emission of n soft photons with polarization indices $\mu_1 \ldots \mu_n$ and four-momenta $k_1 \ldots k_n$ from, say, the outgoing fermion line are given, for a fixed ϵ, by the matrix element

$$\langle \Psi^{\mu_1 \ldots \mu_n}_{k_1 \ldots k_n} , \Omega^{(\epsilon)\,*}_{-,\, v',\, \lambda} \Psi_F \rangle = \Pi^n_{j=1} \frac{e \, \tilde{\rho}(\mathbf{k}_j)}{\sqrt{2\,\omega_{\mathbf{k}_j}}} \frac{v'^{\,\mu_j} \, e^{-i\,\mathbf{k}\cdot\hat{\mathbf{x}}_j}}{-v'\cdot k_j - i\,\epsilon} \cdot \quad (49)$$

The transition amplitude involving corrections from both legs can be written as

$$\mathscr{W}^{\mu_1 \ldots \mu_n}_{\beta\,\alpha} \equiv \langle \psi_{v'} \otimes \Psi^{\mu_1 \ldots \mu_n}_{k_1 \ldots k_n} \otimes \xi^{out}_{e.m.} , S^{(FGB)}_{\lambda,(\epsilon)} (\psi_v \otimes \Psi_F \otimes \xi^{in}_{e.m.}) \rangle =$$
$$= \mathscr{W}_{\beta\,\alpha} \, \Pi^n_{j=1} \frac{\tilde{\rho}(\mathbf{k}_j)}{\sqrt{2\,\omega_{\mathbf{k}_j}}} \sum_{l=1}^{2} \frac{\eta_l \, e \, v^{\mu_j}_l}{-v_l \cdot k_j - i\,\eta_l\,\epsilon} , \quad (50)$$

with $v_1 \equiv v$, $v_2 \equiv v'$, $\eta_2 = 1 = -\eta_1$.
The last important property emerging from the infrared diagrammatic that we have to reproduce is the occurrence of infrared-divergent phases in the transition amplitude for a process described by the sum of one-particle hamiltonians, with at least two charged particles in either the initial or the final state and with asymptotic dynamics supposed to be governed by the sum of free hamiltonians.

In order to obtain the phases for a given process arising from, say, n electrons in the final state, we first evaluate $\omega_F(\Omega^{(\epsilon)\,*}_{-,(\hat{\mathbf{x}})})$, with $\Omega^{(\epsilon)}_{\pm,(\hat{\mathbf{x}})}$ the corresponding Möller operators, for a fixed ϵ. In particular, one obtains the term

$$\exp\,(-\frac{e^2}{2} \sum_{j<l=2}^{n} v_j \cdot v_l \int \frac{d^3k}{\omega^2_{\mathbf{k}} - (\mathbf{v}_j \cdot \mathbf{k})^2} \times$$
$$\times \frac{e^{-i\,\mathbf{k}\cdot(\hat{\mathbf{x}}_j - \hat{\mathbf{x}}_l)} \, \tilde{\rho}^2(\mathbf{k})}{(\mathbf{v}_j - \mathbf{v}_l)\cdot\mathbf{k} - 2\,i\,\epsilon}) + O(\epsilon) , \quad (51)$$

and, by integrating over the directions of the photon momentum $\mathbf{k}$,

$$\exp\,(-\frac{i\,e^2}{2} \sum_{j<l=2}^{n} \frac{1}{4\,\pi\,\beta_{j\,l}} \int_0^{k_{max}} d\,|\mathbf{k}| \, \frac{|\mathbf{k}|}{|\mathbf{k}|^2 + \lambda^2}) , \quad (52)$$

with $\beta_{j\,l}$ the relative four-velocity of the charges j and l in the rest frame of either. In order to employ covariance in the calculation, the form factor $\tilde{\rho}(\mathbf{k})$ occurring in (51) has been replaced by a covariant ultraviolet cutoff, indicated by k_{max}. The exponent in (52) yields the sum of infrared-divergent phases, one for each couple of outgoing electrons, that is obtained within the perturbative framework from the evaluation of Feynman's amplitudes in the multiparticle case.

We can also express in operatorial terms the prescriptions yielding infrared-finite inclusive cross-sections. Let E be the threshold energy of the photon detectors and assume that the experimental arrangement is such that not more than a total energy E_T goes into unobserved photons. The corresponding inclusive cross-section is given by (equation (13.3.11) in [7])

$$\sigma(\alpha \to \beta ; E, E_T) = b(E/E_T ; A_{v'v})\ \Gamma_{\beta\alpha}(E) = \\ = b(E/E_T ; A_{v'v})\ (\frac{E}{\Lambda})^{A_{v'v}}\ \Gamma_{\beta\alpha}(\Lambda), \quad (53)$$

where $\Gamma_{\beta\alpha}(E)$ is the transition rate for the process $\alpha \to \beta$ involving the corrections due to virtual soft-photons with energy above E, $A_{v'v}$ is a positive function of the four-velocities of the charges and b is a kinematical contribution due to the constraint on E_T, such that $b(0;0) = 1 = b_{max}$.

In order to recover equation (53) within the hamiltonian framework, first we introduce further notations. We indicate respectively by $\mathscr{F}_<$ and $\mathscr{F}_>$ the subspaces of $\mathscr{F}_{soft}$ spanned by the real photons of mass λ, with energy below and above E, omitting the $\lambda -$ dependence. The restriction of $\Omega_\pm$ to $\mathscr{F}_<$ ($\mathscr{F}_>$) is denoted by $\Omega_\pm^{(<)}$ ($\Omega_\pm^{(>)}$) and the symbol $\chi_{\pm,v}^{(<)}(\lambda)$ ($\overline{\chi}_{\pm,v}^{(<)}(\lambda)$) stands for the coherent state $\Omega_{\pm,v,\lambda}^{(<)}\Psi_0$ ($\Omega_{\pm,v,\lambda}^{*(<)}\Psi_0$), with Ψ_0 the vacuum state of $\mathscr{F}_<$. The restriction of $\eta_{e.m.}^{out\,(in)}$ to the subspace $\mathscr{F}_> \otimes \mathscr{F}_{hard}$ spanned by the photons with energy above E is denoted by $\eta_{e.m.}^{out\,(in)\,(>)}$.

The $S-matrix$ operator including the corrections due to the hard photons and to the soft photons with energy above E is $\overline{\mathscr{W}}(E) \equiv \Omega_-^{*(>)}\ \mathscr{W}\ \Omega_+^{(>)}$. For a fixed value of λ one evaluates integrals "over the phase space of the unobserved photons", which will be henceforth written as sums over the states e_k of a basis of $\mathscr{F}_<$, with the primed summation $\sum_k'$ denoting that the constraint on E_T is taken into account.

The infrared-regularized transition rate involving the soft-photon radiative corrections to the basic process is

$$\Gamma_{\beta\alpha}(\lambda) \equiv |\langle \psi_{v'} \otimes \eta_{e.m.}^{out}, S_\lambda^{(FGB)}(\psi_v \otimes \eta_{e.m.}^{in})\rangle|^2 = \\ = (\frac{\lambda}{\Lambda})^{A_{v'v}}\ \Gamma_{\beta\alpha}(\Lambda), \quad (54)$$

with $\Gamma_{\beta\alpha}(\Lambda) = |\mathscr{W}_{\beta\alpha}|^2$. The hamiltonian expression of the bulk of the inclusive cross-section (53) is

$$\Gamma_{\beta\alpha}(E) \equiv |\langle \psi_{v'} \otimes \eta_{e.m.}^{out\,(>)}, \overline{\mathscr{W}}(E)(\psi_v \otimes \eta_{e.m.}^{in\,(>)})\rangle|^2 = |\overline{\mathscr{W}}_{\beta\alpha}(E)|^2, \quad (55)$$

where the dependence of the matrix element of $\overline{\mathscr{W}}$ on the effective infrared cutoff E has been indicated explicitly.

This result implies that *the finiteness of the inclusive cross-sections is a consequence of the exponentiation of the infrared contributions.*

Moreover,

$$b\,(E/E_T\,;\,A_{\beta\alpha}) \equiv \lim_{\lambda\to 0} \sum_k{}' \,|\langle e_k\,,\,\Omega^{*\,(<)}_{-,\,v',\,\lambda}\ \Omega^{(<)}_{+,\,v,\,\lambda}\ \Psi_0\rangle|^2 =$$
$$= \lim_{\lambda\to 0}\ \mathscr{N}_{E,E_T}\ Tr\,(\rho_{E,E_T}\ P_{\Xi^{(<)}_{v'v}(\lambda)}). \tag{56}$$

In the second line of the above equation, which yields the operatorial expression of the contribution due to the constraint on E_T, there appear the density matrix $\rho_{E,E_T} \equiv \mathscr{N}^{-1}_{E,E_T} \sum'_k P_{e_k}$, with $\mathscr{N}_{E,E_T}$ a normalization factor, and the projection operator on the vector $\Xi^{(<)}_{v'v}(\lambda) \equiv \Omega^{*\,(<)}_{-,\,v',\,\lambda}\ \chi^{(<)}_{+,\,v}(\lambda)$.
It may be useful to point out that although such a term has to be taken into account in order to reproduce quantitatively the experimental data, it does not play a relevant role from a theoretical point of view.

The limitations of the perturbative recipes lay in the fact that if wave packets are employed in the description of the charged particle states, *even* the inclusive cross-sections vanish in the infrared limit, under the same hypothesis for the representation of the e.m. field. We prove this statement for a particular process; in order to establish the same result for any given process a simple generalization of the proof given in the following is required.

We consider an incoming wave-packet $\phi = \int d^3v\ \phi(\mathbf{v})\ \psi_v$ and allow for a final state described in terms of a density matrix on $L^2(\mathbb{R}^3) \otimes \mathscr{F}_<$. The representation of the *in* photon field is assumed to be Fock, also in the limit $\lambda \to 0$; in particular, we choose the same state as in (54). Moreover, without loss of generality we also suppose that the observed outgoing photons are still described by the state $\eta^{out\,(>)}_{e.m.}$.

The corresponding inclusive cross-section is given for fixed λ by

$$\sum_k{}' \int d^3v'\ |\zeta(\mathbf{v}')|^2\ |\langle \psi_{v'} \otimes e_k \otimes \eta^{out\,(>)}_{e.m.}\,,\ S^{(FGB)}_\lambda\,(\phi \otimes \eta^{in}_{e.m.})\rangle|^2 \tag{57}$$

and its behaviour for small λ is governed by the term

$$Tr\,(\rho_{E,E_T}\ P_{\Phi^{(<)}_{v'\phi}(\lambda)})\,, \tag{58}$$

with $\Phi^{(<)}_{v'\phi}(\lambda) \equiv \Omega^{*\,(<)}_{-,\,v',\,\lambda} \int d^3v\ \ \phi(\mathbf{v})\ \chi^{(<)}_{+,\,v}(\lambda)$.

We can easily show that (58) approaches zero for $\lambda \to 0$ and therefore that the expression (57) vanishes in the same limit. It is enough to prove the statement for the upper bound of (58) given by $\lim_{E_T\to\infty} Tr\,(\rho_{E,E_T}\ P_{\Phi^{(<)}_{v'\phi}(\lambda)})$. Taking into account that for $E_T \to \infty$ the density matrix ρ_{E,E_T} approaches the identity operator on $\mathscr{F}_<$ and that for the inner product between coherent states one has

$$\lim_{\lambda\to 0}\ \langle \chi^{(<)}_{+,\,w}(\lambda)\,,\,\chi^{(<)}_{+,\,v}(\lambda)\rangle = 0\,,\ v \neq w\,, \tag{59}$$

it follows

$$\lim_{\lambda\to 0}\lim_{E_T\to\infty} Tr\,(\rho_{E,E_T}\,P_{\Phi^{(<)}_{v'\phi}(\lambda)}) = \lim_{\lambda\to 0}\langle\,\Phi^{(<)}_{v'\phi}(\lambda)\,,\,\Phi^{(<)}_{v'\phi}(\lambda)\rangle =$$
$$= \lim_{\lambda\to 0}\int d^3v\;d^3w\;\phi(\mathbf{v})\;\overline{\phi}(\mathbf{w})\,\langle\,\chi^{(<)}_{+,w}(\lambda)\,,\,\chi^{(<)}_{+,v}(\lambda)\rangle = 0\,. \tag{60}$$

The vanishing of the inclusive cross-section can be understood in terms of the superselection of the asymptotic particle momenta. In fact, by solving the Heisenberg equations of the model for fixed λ, one can check that the standard assumption that the Fock property of the incoming photon field also holds for $\lambda\to 0$ leads, in the same limit, to a non-Fock representation of the e.m. field at finite times, indexed by the four-momentum p_{in} of the charge. The superselection of p_{in} then follows by an argument as that devised and exploited in [11] and [16]; therefore, interference terms between different momentum components of a wave packet cannot contribute to the $\lambda\to 0$ limit of (57).

Accordingly, we prove below that in order to obtain an infrared-finite inclusive cross-section it is necessary to choose suitable non-Fock (coherent) representations of the low-energy modes of the incoming e.m. field.
With the same assumptions for the final state, we perform the calculations for an in state whose restriction to $L^2\otimes\mathscr{F}_<$ is given by

$$\Phi^{(<)}(\lambda)\equiv\int d^3v\;\phi(\mathbf{v})\;\psi_v\otimes\overline{\chi}^{(<)}_{+,v}(\lambda)\,. \tag{61}$$

We can suppose, without losing generality, that the incoming photons with energy above E are described by the state $\eta^{in\,(>)}_{e.m.}$. Under these hypotheses, the inclusive cross-section has the same behaviour, for small λ, as the expression obtained by replacing $\Phi^{(<)}_{v'\phi}(\lambda)$ with $\Psi^{(<)}_{v'\phi}(\lambda)\equiv\int d^3v\;\phi(\mathbf{v})\;\overline{\chi}^{(<)}_{-,v'}(\lambda)$ in (58). Note that the previously considered upper bound has now the non-vanishing limit

$$\lim_{\lambda\to 0}\lim_{E_T\to\infty} Tr\,(\rho_{E,E_T}\,P_{\Psi^{(<)}_{v'\phi}(\lambda)}) = \int d^3v\;d^3w\;\phi(\mathbf{v})\;\overline{\phi}(\mathbf{w})\,. \tag{62}$$

For the cross-section one obtains indeed the infrared-finite result

$$\int d^3v'\;|\,\zeta(\mathbf{v}')\,|^2\;|\langle\,\psi_{v'}\otimes\eta^{out\,(>)}_{e.m.}\,,\,\overline{\mathscr{W}}(E)\,(\phi\otimes\eta^{in\,(>)}_{e.m.})\rangle|^2\,\times$$
$$\times\,\mathscr{N}_{E,E_T}\;Tr\,(\rho_{E,E_T}\,P_{\overline{\chi}^{(<)}_{-,v'}})\,, \tag{63}$$

with $\overline{\chi}^{(<)}_{-,v'}\equiv\lim_{\lambda\to 0}\overline{\chi}^{(<)}_{-,v'}(\lambda)$.

We stress that while the term (58) depends both on the incoming wave packet and on the velocity v' of the outgoing particle, the second line of (63) only depends on v'. As a result, the overall expression turns out to be an integral over the final states of the squared modulus of an infrared-finite amplitude; hence, as we will check below, one can obtain an amplitude free of

IR divergences for the transition between the same initial state and a suitably chosen final (coherent) state.

Since hereafter we do not consider inclusive cross-sections, the symbol $<$ will no longer denote quantities related to the undetected photons but will refer to the overall soft-photon contributions. By the separation of the infrared factors we can then write $\overline{\chi}_{\pm, v}(\lambda) \equiv \overline{\chi}^{(<)}_{\pm, v}(\lambda) = \Omega^{*(<)}_{\pm, v, \lambda} \Psi_F$.

Let ϕ and ξ be the initial and final particle wave-packet states; it is straightforward to check that transition amplitudes of the form

$$\langle \Xi^{(<)}(\lambda) \otimes \beta_>, S^{(FGB)}_\lambda (\Phi^{(<)}(\lambda) \otimes \alpha_>) \rangle, \tag{64}$$

with $\Phi^{(<)}(\lambda)$ given by (61), $\Xi^{(<)}(\lambda) \equiv \int d^3v\ \xi(\mathbf{v})\ \psi_v \otimes \overline{\chi}_{-, v}(\lambda)$ and $\alpha_>, \beta_>$ hard-photon states, admit a finite and non-vanishing infrared limit. As matter of fact, it follows from (43) that

$$\lim_{\lambda \to 0} \langle \Xi^{(<)}(\lambda) \otimes \beta_>, S^{(FGB)}_\lambda (\Phi^{(<)}(\lambda) \otimes \alpha_>) \rangle = \\ = \langle \xi \otimes \beta_>, \mathscr{W}(\phi \otimes \alpha_>) \rangle. \tag{65}$$

Therefore, the representations of the photon field required to obtain finite inclusive cross-sections in the presence of wave-packets allow indeed for finite transition amplitudes, which are expressed as matrix elements of the hard-photon scattering operator $\mathscr{W}$. The latter can then be identified with Chung's scattering matrix; as a matter of fact, the main result of the approach pursued in [10] is that it is possible to obtain infrared-finite transition amplitudes between suitable coherent scattering states.

We wish to remark that it is possible to carry out an analysis of the main properties of the four-vector BN model, independently of a comparison with the standard approach, by dropping the adiabatic approximation and the infrared cutoff and by employing the Heisenberg representation ([28]).

Within this framework, one can consider the algebra of observables, obeying the relativistically covariant Heisenberg equations of the model, and construct its Haag-Ruelle ([30]) asymptotic limits. In particular, the automorphism between the *in* and *out* algebras, defining the scattering matrix for $\lambda = 0$, turns out to be the limit of a one-parameter group of automorphisms constructed in terms of the family of infrared-regularized scattering matrices (43). Moreover, it is implementable, uniquely up to an element of the commutant of the algebra generated by the asymptotic vector potential, and an implementer can be shown to be the operator $\mathscr{W}$. Finally, we point out that the identification of $\mathscr{W}$ as Chung's $S - matrix$ agrees with the aforementioned results; in fact, Chung's approach is based on the decomposition of asymptotic states as coherent states indexed by the asymptotic momenta in the Gupta-Bleuler formulation.

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Appendix

The construction of the Möller operators in the four-vector BN model, as well as in the $PFBR$ model, requires to specify a suitable notion of convergence on an indefinite-metric linear space.

Concerning the four-vector BN model, let Ψ_0 be the GNS vector obtained by applying the GNS theorem to $\omega_F(\mathscr{A}_{ext,\,L^2})$.
Denoting by $\mathscr{P}_{M,N}(x, y)$ a polynomial of degree M in the first variable and of degree N in the second one, let

$$\mathscr{V}_{M,N} \equiv (\phi \in \overline{\mathscr{D}}\,,\ \phi = \mathscr{P}_{M,N}(W(f, g), b(h))\,\Psi_0)\,, \tag{66}$$

with f, g, h collectively standing for the corresponding sets of test functions. The expression

$$p_{M,N}(\phi) \equiv \sup_{i,\,j,\,k} (\| f_i \|_2\,, \| g_j \|_2\,, \| h_k \|_2) \tag{67}$$

defines a seminorm on $\mathscr{V}_{M,N}$.

We have denoted by $\tilde{\tau}$ the topology induced on $\overline{\mathscr{D}}$ by the family of seminorms $p_{M,N}$. Such a topology ensures that the limits

$$\lim_{m,\,n,\,k} \langle \Phi\,, \mathscr{P}_{M,N}(W(f_m, g_n), b(h_l))\,\Psi \rangle\,, \tag{68}$$

$\Phi, \Psi \in \overline{\mathscr{D}}$, exist for fixed M, N independently of the order, whenever the sequences of test functions $f_m^{(i)}$, $g_n^{(j)}$, $h_l^{(k)}$ converge in the L^2 norm, and also that they are matrix elements of operators on $\overline{\mathscr{D}}$.

The $\tau -$ topology employed in the $PFBR$ model is induced by seminorms $p_{N,M}$, defined as above, on the space $\overline{\mathscr{D}}_0$ obtained via the GNS construction over $\omega_F(\mathscr{A}^{e.m.}_{ext,\,L^2})$.

References

[1] F. Bloch and A. Nordsieck, Phys. Rev. **52**, 54 (1937).

[2] M. Fierz and W. Pauli, Nuovo Cimento **15**, 167 (1938).

[3] P. Blanchard, Discussion mathématique du modèle de Pauli et Fierz relatif à la catastrofe infrarouge, Comm. Math. Phys. **19**, 301 (1970).

[4] F.J. Dyson, Phys. Rev. **75**, 1736 (1949); Phys. Rev. **82**, 428 (1951).

[5] S. Gupta, Proc. Phys. Soc. London A**63**, 681 (1950); K. Bleuler, Helv. Phys. Acta **23**, 567 (1950).

[6] D.R. Yennie, S.C. Frautschi and H. Suura, Ann. Phys. **13**, 379 (1961).

[7] S. Weinberg, *The Quantum Theory of Fields*, Vol. I, Chapt. 13 (Cambridge University Press, Cambridge, 1995).

[8] J.M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons*, Second Expanded Edition, Chapt. 16 (Springer, New York, 1976).

[9] O. Steinmann, *Perturbative Quantum Electrodynamics and Axiomatic Field Theory* (Springer Verlag, New York, 2000).

[10] V. Chung, Phys. Rev. **140**B, 1110 (1965).

[11] F. Strocchi and A.S. Wightman, Jour. Math. Phys. **15**, 2198 (1974).

[12] R. Ferrari, L.E. Picasso and F. Strocchi, Comm. Math. Phys. **35**, 25 (1974); Nuovo Cimento Soc. Ital. Fis., A**39**, 1 (1977).

[13] J. Fröhlich, G. Morchio and F. Strocchi, Charged sectors and scattering states in quantum electrodynamics, Ann. Phys. **119**, 241 (1979).

[14] J. Fröhlich, G. Morchio and F. Strocchi, Spontaneous breaking of the Lorentz group in QED, Phys. Lett. **89B**, 61 (1979).

[15] D. Buchholz, The Physical State Space of Quantum Electrodynamics, Comm. Math. Phys. **85**, 49 (1982).

[16] D. Buchholz, Gauss' law and the infraparticle problem, Phys. Lett. **B 174**, 331 (1986).

[17] R.F. Streater and A.S. Wightman, *PCT, spin and statistics and all that* (Benjamin, New York, 1968).

[18] J. Schwinger, Phys. Rev. **76**, 790 (1949).

[19] H.P. Stapp, Phys. Rev. Lett. **50**, 467 (1983), Phys. Rev. **D 28**, 1386 (1983); T. Kawai and H.P. Stapp, Phys. Rev. **D 52**, 2484 (1995).

[20] O. Steinmann, Comm. Math. Phys. **237**, 181 (2003).

[21] M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Vol. I. Functional Analysis, Academic Press 1972.

[22] T. Kato, *Perturbation Theory for Linear Operators*, Springer 1966.

[23] S. Sakai, C^*- algebras and W^*- algebras, Berlin, Heidelberg, New York: Springer 1971.

[24] K. Symanzik, *Lectures on Lagrangian Field Theory*, DESY report T-71/1.

[25] G. Morchio and F. Strocchi, Nucl. Phys. B**211**, 471 (1983); **232**, 547 (1984).

[26] G. Morchio and F. Strocchi, Ann. Phys. **168**, 27 (1986).

[27] F. Strocchi, Gauge Problem in Quantum Field Theory, Phys. Rev. **162**, 1429 (1967).

[28] S. Zerella, Scattering Theories In Models Of Quantum Electrodynamics, PhD Thesis, Università degli Studi di Pisa, unpublished (2009).

[29] D. Buchholz, S. Doplicher, G. Morchio, J.E. Roberts, F. Strocchi, Asymptotic abelianess and braided tensor C* categories in *Rigorous Quantum Field Theory*, pages 49-64, A. Boutet de Monvel et al. Eds., Birkhauser 2006.

[30] R. Haag, Phys. Rev. **112**, 669 (1958), D. Ruelle, Helv. Phys. Acta **35**, 147 (1962).