A Bijection between Atomic Partitions and Unsplitable Partitions

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Abstract

In the study of the algebra NCSym of symmetric functions in noncommutative variables, Bergeron and Zabrocki found a free generating set consisting of power sum symmetric functions indexed by atomic partitions. On the other hand, Bergeron, Reutenauer, Rosas, and Zabrocki studied another free generating set of NCSym consisting of monomial symmetric functions indexed by unsplitable partitions. Can and Sagan raised the question of finding a bijection between atomic partitions and unsplitable partitions. In this paper, we provide such a bijection.

1 Introduction

In their study of the algebra NCSym of symmetric functions in noncommutative variables, Rosas and Sagan [5] introduced a vector space with a basis

$$\{p_\pi \,|\, \pi \text{ is a set partition}\},$$

where p_π is the power sum symmetric function in noncommutative variables. Bergeron, Hohlweg, Rosas, and Zabrocki [1] obtained the following formula

$$p_{\pi|\sigma} = p_\pi\, p_\sigma,$$

where $\pi|\sigma$ denotes the slash product of π and σ. It follows that, as an algebra, NCSym is freely generated by p_π with π atomic, see Bergeron and Zabrocki [3]. It should be noted that Wolf [6] showed that NCSym is freely generated by another basis. A combinatorial characterization of the generating set of Wolf has been found by Bergeron, Reutenauer, Rosas, and Zabrocki [2]. More precisely, they introduced the notion of unsplitable partitions and proved that the generating set of Wolf can be described as the set of monomial symmetric functions in noncommutative variables indexed by unsplitable partitions.

Let $[n]$ denote the set $\{1, 2, \ldots, n\}$. Taking the degree into account, one sees that the number of atomic partitions of $[n]$ equals the number of unsplitable partitions of $[n]$. Recently, Can and Sagan [4] raised the question of finding a combinatorial proof of this fact. The objective of this paper is to present such a proof.

2 The bijection

In this section we construct a bijection between the set of atomic partitions of $[n]$ and the set of unsplitable partitions of $[n]$.

Let us begin with an overview of terminology. Let X be an totally ordered set. A partition π of X is a family $\{B_1, B_2, \ldots, B_k\}$ of disjoint nonempty subsets of X whose union is X. The subsets B_i are called blocks of π. Without loss of generality, we may assume that the blocks of a partition are arranged in the increasing order of their minimal elements, and that the elements in each block are written in increasing order.

Let π be a partition of X and $S \subseteq X$. We say that σ is the restriction of π on S, denoted by $\sigma = \pi_S$, if σ is a partition of S such that any two elements lie in the same block of σ if and only if they are in the same block of π. In other words, π_S is obtained from π by removing all elements that do not belong to S. For two positive integers i and j with $i < j$, we use $[i, j]$ to denote the set $\{i, i+1, \ldots, j\}$. For example, if

$$\pi = \big\{\{1, 3, 5, 6\}, \{2, 7, 9\}, \{4, 8, 10\}\big\}, \tag{2.1}$$

then

$$\pi_{[5,10]} = \big\{\{5, 6\}, \{7, 9\}, \{8, 10\}\big\}. \tag{2.2}$$

Let Π_n be the set of partitions of $[n]$. Assume that

$$\pi = \{B_1, B_2, \ldots, B_k\} \in \Pi_m, \qquad \sigma = \{C_1, C_2, \ldots, C_l\} \in \Pi_n.$$

The slash product of π and σ, denoted by $\pi|\sigma$, is defined to be the partition obtained by joining the blocks of π and the blocks of the partition

$$\sigma + m = \{C_1 + m,\, C_2 + m,\, \ldots,\, C_l + m\},$$

that is,

$$\pi|\sigma = \{B_1,\, B_2,\, \ldots,\, B_k,\, C_1 + m,\, C_2 + m,\, \ldots,\, C_l + m\},$$

where $C_i + m$ denotes the block obtained by adding m to each element in C_i. It can be seen that $\pi|\sigma \in \Pi_{m+n}$. A partition π is said to be atomic if there are no nonempty partitions σ and τ such that $\pi = \sigma|\tau$. Let $\mathcal{A}_n$ be the set of atomic partitions of $[n]$. For example, for $n = 3$ there are two atomic partitions $\big\{\{1, 3\}, \{2\}\big\}$ and $\big\{\{1, 2, 3\}\big\}$.

The split product of π and σ, denoted by $\pi \circ \sigma$, is given by

$$\pi \circ \sigma = \begin{cases} \{B_1 \cup (C_1 + m),\, \ldots,\, B_k \cup (C_k + m),\, C_{k+1} + m,\, \ldots,\, C_l + m\}, & \text{if } k \leq l; \\ \{B_1 \cup (C_1 + m),\, \ldots,\, B_l \cup (C_l + m),\, B_{l+1},\, \ldots,\, B_k\}, & \text{if } k > l. \end{cases}$$

Clearly, $\pi \circ \sigma \in \Pi_{m+n}$. A partition is said to be splitable if it is the split product of two nonempty partitions. Otherwise, it is said to be unsplitable. Denote by $\mathcal{US}_n$ the set of unsplitable partitions of $[n]$. For example, for $n = 3$ there are two unsplitable partitions $\big\{\{1\}, \{2,3\}\big\}$ and $\big\{\{1\}, \{2\}, \{3\}\big\}$.

To describe our bijection, we first notice that it is possible for a partition to be atomic and unsplitable at the same time. For example, the partition

$$\big\{\{1,3,7\}, \{2,6\}, \{4,5,8\}\big\}$$

is both atomic and unsplitable. Our bijection will be concerned with atomic partitions that are splitable and unsplitable partitions that are not atomic. In other words, we shall establish a bijection

$$\varphi\colon \mathcal{A}_n \backslash \mathcal{US}_n \longrightarrow \mathcal{US}_n \backslash \mathcal{A}_n.$$

For the sake of presentation, let us introduce a notation. Let $X = \{x_1, x_2, \ldots, x_n\}$ under the assumption that $x_1 < x_2 < \cdots < x_n$. Assume that $\pi = \{B_1, B_2, \ldots, B_k\}$ is a partition of X. Let r be the largest integer j such that

$$B_j \cup B_{j+1} \cup \cdots \cup B_k = \{x_t,\, x_{t+1},\, \ldots,\, x_n\}$$

for some t. The existence of such an integer r is evident. We define

$$R(\pi) = \{B_r,\, B_{r+1},\, \ldots,\, B_k\}.$$

Given the partition $\pi = \big\{\{1,3,5,6\}, \{2,7,9\}, \{4,8,10\}\big\}$ as in (2.1), we have

$$R(\pi_{[5,10]}) = \big\{\{7,9\}, \{8,10\}\big\}. \tag{2.3}$$

In the above notation, we see that π is atomic if and only if $\pi = R(\pi)$.

We are now ready to present the map φ. Suppose that $\pi = \{B_1, B_2, \ldots, B_k\} \in \mathcal{A}_n \backslash \mathcal{US}_n$. It consists of three steps.

Step 1. Let i be the smallest element in B_1 such that $\pi = \pi_{[i-1]} \circ (\pi_{[i,\,n]} - i + 1)$. The existence of the element i is guaranteed by the condition that π is splitable.

Step 2. Let j be the smallest element in the underlying set of the partition $R(\pi_{[i,\,n]})$. We see that $2 \le i \le j \le n$ and $R(\pi_{[i,\,n]}) = \pi_{[j,\,n]}$.

Step 3. Set $\varphi(\pi)$ to be the partition $\pi_{[j-1]} \,\big|\, (\pi_{[j,\,n]} - j + 1)$.

Theorem 2.1 *The map φ is a bijection from $\mathcal{A}_n \backslash \mathcal{US}_n$ to $\mathcal{US}_n \backslash \mathcal{A}_n$.*

Proof. First, we claim that $\varphi(\pi) \in \mathcal{US}_n \backslash \mathcal{A}_n$. Since $2 \le j \le n$, both $\pi_{[j-1]}$ and $\pi_{[j,\,n]}$ are nonempty partitions. This implies that $\varphi(\pi) \notin \mathcal{A}_n$.

We next proceed to show that $\varphi(\pi)$ is unsplitable. To this end, let

$$\pi_{[j-1]} = \{C_1, C_2, \ldots, C_s\}, \qquad \pi_{[j,n]} = \{D_1, D_2, \ldots, D_t\}.$$

Then

$$\varphi(\pi) = \{C_1,\, C_2,\, \ldots,\, C_s,\, D_1,\, D_2,\, \ldots,\, D_t\}.$$

Suppose to the contrary that $\varphi(\pi)$ is splitable, namely, there exists an element $l \in C_1$ such that

$$\varphi(\pi) = \varphi(\pi)_{[l-1]} \circ (\varphi(\pi)_{[l,\, n]} - l + 1).$$

Since n belongs to some block D_h, by the definition of the split product, we deduce that

$$C_p \cap [l,\, n] \neq \emptyset, \quad \text{for each } 1 \leq p \leq s. \tag{2.4}$$

By the choice of i, we find that $l \geq i$. Recall that $\pi = \{B_1, B_2, \ldots, B_k\}$. By the definition of $\pi_{[j,n]}$, we may assume that the block D_1 of $\pi_{[j,n]}$ is contained in some block B_r of π. If $D_1 = B_r$, then the smallest element of B_r is j. Therefore all elements in $B_{r+1}, B_{r+2}, \ldots, B_k$ are larger than j. Now, by the choice of j, we deduce that

$$B_r \cup B_{r+1} \cup \cdots \cup B_k = [j, n].$$

Consequently,

$$\pi = \pi_{[j-1]} \,\big|\, (\pi_{[j,\, n]} - j + 1),$$

which contradicts the assumption that π is atomic. Hence we have $D_1 \neq B_r$, and so $C_r = B_r \backslash D_1 \neq \emptyset$. Since D_1 is a block of the partition $\pi_{[i,n]}$, it consists of all the elements in B_r that are larger than or equal to i. In other words, each element in C_r is less than i. This yields that $C_r \cap [l,\, n] = \emptyset$, a contradiction to (2.4). Thus we have proved the claim that $\varphi(\pi) \in \mathcal{US}_n \backslash \mathcal{A}_n$.

We now define a map

$$\psi\colon \mathcal{US}_n \backslash \mathcal{A}_n \longrightarrow \mathcal{A}_n \backslash \mathcal{US}_n,$$

and shall show that ψ is the inverse of φ. Let $\sigma = \{B_1, B_2, \ldots, B_k\} \in \mathcal{US}_n \backslash \mathcal{A}_n$.

Step 1. Let j be the smallest element in the underlying set of the partition $R(\sigma)$.

Step 2. Let B_r be the first block in the partition $R(\sigma)$. We consider two cases.

Case 1. If $\sigma_{[j-1]}$ is unsplitable, then set

$$\psi(\sigma) = \sigma_{[j-1]} \circ (\sigma_{[j,\, n]} - j + 1).$$

Case 2. If $\sigma_{[j-1]}$ is splitable, then choose i to be the smallest element in B_1 such that

$$\sigma_{[j-1]} = \sigma_{[i-1]} \circ (\sigma_{[i,\, j-1]} - i + 1). \tag{2.5}$$

Let $q = \min\{l \mid B_l \subseteq [i-1]\}$. If $2r - q - 1 \leq k$, then set

$$\psi(\sigma) = \{B_1,\, \ldots,\, B_{q-1},\, B_q \cup B_r,\, \ldots,\, B_{r-1} \cup B_{2r-q-1},\, B_{2r-q},\, \ldots,\, B_k\}.$$

If $2r - q - 1 > k$, then set

$$\psi(\sigma) = \{B_1,\, \ldots,\, B_{q-1},\, B_q \cup B_r,\, \ldots,\, B_{q+k-r} \cup B_k,\, B_{q+k-r+1},\, \ldots,\, B_{r-1}\}.$$

It remains to show that the map ψ is well-defined and it is indeed the inverse of the map φ.

For any $\sigma \in \mathcal{US}_n \backslash \mathcal{A}_n$, we notice that in Step 1 of the above construction of ψ, the element j always exists since σ is not atomic. Moreover, we observe that $j \geq 2$. In Step 2, by the choice of j, we have

$$\sigma_{[j-1]} = \{B_1, B_2, \ldots, B_{r-1}\},$$
$$\sigma_{[j,n]} = R(\sigma) = \{B_r, B_{r+1}, \ldots, B_k\}.$$

Since σ is unsplitable, we can always find the element q. Otherwise, if every block $B_1, B_2, \ldots, B_k$ contains an element in $[i, n]$, by the assumption (2.5), we have $B_p \cap [i, n] \neq \emptyset$ for any $1 \leq p \leq k$, and

$$\min(B_1 \cap [i, n]) < \min(B_2 \cap [i, n]) < \cdots < \min(B_k \cap [i, n]).$$

This implies that

$$\sigma = \sigma_{[i-1]} \circ (\sigma_{[i,n]} - i + 1),$$

a contradiction to the fact that σ is unsplitable. This confirms the existence of the element q. At this point, we still need to show that $\psi(\sigma) \in \mathcal{A}_n \backslash \mathcal{US}_n$. It is clear from the above construction that $\psi(\sigma)$ is splitable. For the case when $\sigma_{[j-1]}$ is unsplitable, it is easily seen that $\psi(\sigma)$ is atomic. When $\sigma_{[j-1]}$ is splitable, since $i \in B_1$ and $B_q \subseteq [i-1]$, we find that $\psi(\sigma)$ is atomic. Thus we have shown that $\psi(\sigma) \in \mathcal{A}_n \backslash \mathcal{US}_n$. Consequently, ψ is well-defined.

It is not difficult to verify that ψ is the inverse map of φ. The details are omitted. This completes the proof. ∎

The following example is an illustration of the maps φ and ψ. Let

$$\pi = \big\{\{1, 3, 5, 6\}, \{2, 7, 9\}, \{4, 8, 10\}\big\}$$

be the partition as given in (2.1). In Step 1 of the map φ, we have $i = 5$. By (2.3), we get

$$\varphi(\pi) = \{\{1, 3, 5, 6\}, \{2\}, \{4\}, \{7, 9\}, \{8, 10\}\}. \tag{2.6}$$

Conversely, assume that σ is the partition given in (2.6). Then $\psi(\sigma)$ is determined as follows. First, we have $R(\sigma) = \{\{7, 9\}, \{8, 10\}\}$ and $j = 7$. Then,

$$\sigma_{[j-1]} = \{\{1, 3, 5, 6\}, \{2\}, \{4\}\} = \{B_1, B_2, B_3\}$$

is splitable, and $i = 5$ is the smallest element in $\{1, 3, 5, 6\}$ such that

$$\sigma_{[j-1]} = \sigma_{[i-1]} \circ (\sigma_{[i,j-1]} - i + 1).$$

Since B_2 is the first block of $\sigma_{[j-1]}$ that is contained in $[i - 1]$, we get

$$\psi(\sigma) = \{\{1, 3, 5, 6\}, \{2, 7, 9\}, \{4, 8, 10\}\} = \pi.$$

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