ON A SPECTRAL ANALOGUE OF THE STRONG MULTIPLICITY ONE THEOREM

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ABSTRACT. We prove spectral analogues of the classical strong multiplicity one theorem for newforms. Let Γ_1 and Γ_2 be uniform lattices in a semisimple group G. Suppose all but finitely many irreducible unitary representations (resp. spherical) of G occur with equal multiplicities in $L^2(\Gamma_1\backslash G)$ and $L^2(\Gamma_2\backslash G)$. Then $L^2(\Gamma_1\backslash G) \cong L^2(\Gamma_2\backslash G)$ as G - modules (resp. the spherical spectra of $L^2(\Gamma_1\backslash G)$ and $L^2(\Gamma_2\backslash G)$ are equal).

1. Introduction

The beginnings of the analogy between the spectrum and arithmetic of Riemannian locally symmetric spaces can be attributed to Maass, who defined non-analytic modular forms as eigenfunctions of the Laplacian satisfying suitable modularity and growth conditions. From the viewpoint of Gelfand, the theory of Maass forms can be re-interpreted in terms of the representation theory of $PSL(2,\mathbb{R})$ on $L^2(\Gamma\backslash PSL(2,\mathbb{R}))$ for a lattice Γ in $PSL(2,\mathbb{R})$. Subsequently, the analogy between the spectrum and arithmetic has been extended by the work of A. Selberg, P. Sarnak, M. F. Vigneras and T. Sunada amongst others.

In this paper, our aim is to establish an analogue in the spectral context of the classical strong multiplicity one theorem for cusp forms. Suppose f and g are newforms for some Hecke congruence subgroup $\Gamma_0(N)$ such that the eigenvalues of the Hecke operator at a prime p are equal for all but finitely many primes p. Then the strong multiplicity one theorem of Atkin and Lehner states that f and g are equal (*cf.* [La, p.125]).

Now, let G be a semisimple Lie group and Γ be a uniform lattice (a discrete cocompact subgroup) in G. Let R_Γ be the right regular representation of G on $L^2(\Gamma\backslash G)$:

$$R_\Gamma(g)(\phi)(y) = \phi(yg) \quad \forall\, g,\, y \in G \text{ and } \phi \in\ L^2(\Gamma\backslash G)$$

This defines a unitary representation of G. It is known ([GGP, p.23]) that R_Γ decomposes discretely as a direct sum of irreducible unitary representations of G occurring with finite multiplicities. Let $\widehat{G}$ be the set of equivalence classes of irreducible unitary representations of G. For $\pi \in \widehat{G}$, let $m(\pi, \Gamma)$ be the multiplicity of π in R_Γ.

Definition 1.1. Let Γ_1 and Γ_2 be uniform lattices in G. The lattices Γ_1 and Γ_2 are said to be representation equivalent in G if

$$L^2(\Gamma_1\backslash G) \cong L^2(\Gamma_2\backslash G)$$

as representations of G, i.e. for every $\pi \in\ \widehat{G}$,

$$m(\pi, \Gamma_1) = m(\pi, \Gamma_2)$$

In this article we prove the following result :

Theorem 1.1. *Let Γ_1 and Γ_2 be uniform lattices in a semisimple Lie group G. Suppose all but finitely many irreducible unitary representations of G occur with equal multiplicities in $L^2(\Gamma_1\backslash G)$ and $L^2(\Gamma_2\backslash G)$. Then the lattices Γ_1 and Γ_2 are representation equivalent in G.*

The proof of Theorem 1.1 uses the Selberg trace formula and fundamental results of Harish Chandra on the character distributions of irreducible unitary representations of G; in particular, we make crucial use of a deep and difficult result of Harish Chandra that the character distribution of an irreducible unitary representation of G is given by a locally integrable function on G.

We now consider an analogue of Theorem 1.1 for the spherical spectrum of uniform lattices. Let K be a maximal compact subgroup of G. An irreducible unitary representation π of G is said to be spherical if there exists a non-zero vector $v \in \pi$ such that

$$\pi(k)v = v \quad \forall\, k \in K.$$

The spherical spectrum $\widehat{G}_s$ of G is the subset of $\widehat{G}$ consisting of equivalence classes of irreducible unitary spherical representations of G.

Theorem 1.2. *Let G be a connected, semisimple Lie group. Suppose Γ_1, Γ_2 are uniform torsion-free lattices in G such that*

$$m(\pi, \Gamma_1) = m(\pi, \Gamma_2)$$

for all but finitely many representations π in $\widehat{G}_s$. Then

$$m(\pi, \Gamma_1) = m(\pi, \Gamma_2)$$

for all representations π in $\widehat{G}_s$.

The proof of this theorem follows the broad outline of Theorem 1.1, but requires a more delicate control of $K \times K$-saturation of a conjugacy class of an element of Γ (see Proposition 4.2). This is achieved by looking at the behaviour of the conjugacy classes of elements of Γ in a neighbourhood of identity.

We now relate the spherical spectrum with the spectrum of G-invariant differential operators on the associated symmetric space $X = G/K$. For a torsion-free uniform lattice Γ in G, let $X_\Gamma = \Gamma \backslash G/K$ be the associated compact Riemannian locally symmetric space. The space of smooth functions on X_Γ can be considered as the space of smooth functions on X invariant under the action of Γ. Let $D(G/K)$ be the algebra of G-invariant differential operators on X. For a character λ of $D(G/K)$ (i.e. an algebra homomorphism of $D(G/K)$ into $\mathbb{C}$), consider the eigenspace of λ,

(1) $\quad V(\lambda, \Gamma) = \{f \in C^\infty(X_\Gamma) \; : \; D(f) = \lambda(D)(f) \quad \forall \, D \in D(G/K)\} \,.$

It is known that the space $V(\lambda, \Gamma)$ is of finite dimension (see Section 5).

Definition 1.2. Let Γ_1 and Γ_2 be torsion-free uniform lattices in G. The locally symmetric spaces $X_{\Gamma_1} = \Gamma_1 \backslash G/K$ and $X_{\Gamma_2} = \Gamma_2 \backslash G/K$ are said to be *compatibly isospectral* if

$$\dim(V(\lambda, \Gamma_1)) = \dim\,(V(\lambda, \Gamma_2))$$

for every character λ of $D(G/K)$.

Remark 1. From the generalized Sunada criterion proved by Berard [Be, p.566] and DeTurck - Gordon [DG], it can be seen that if two uniform lattices in G are representation equivalent, then the associated compact locally symmetric Riemannian spaces X_{Γ_1} and X_{Γ_2} are compatibly isospectral.

We prove the following result in Section 4 :

Theorem 1.3. *Let G be a connected, semisimple Lie group. Suppose Γ_1, Γ_2 are uniform torsion-free lattices in G. Suppose*

$$dim \ (V(\lambda, \Gamma_1)) = dim(V(\lambda, \Gamma_2))$$

for all but finitely many characters λ, then X_{Γ_1} and X_{Γ_2} are compatibly isospectral.

If X is of rank one, the algebra $D(G/K)$ is the polynomial algebra in the Laplace-Beltrami operator Δ on G/K (see [He, p.397]). Hence the eigenvalues of Δ determine the characters of $D(G/K)$. Consequently we get :

Corollary 1. *Let X_1 and X_2 be two locally symmetric Riemannian spaces of rank one and Δ_1, Δ_2 be the Laplace-Beltrami operators acting on the space of smooth functions on X_1 and X_2 respectively. If all but finitely many eigenvalues occur with equal multiplicities in the spectra of Δ_1 and Δ_2, then the spaces are isospectral with respect to the Laplace-Beltrami operators.*

Remark 2. Using an analytic version of the Selberg Trace formula, J. Elstrodt, F. Grunewald, and J. Mennicke (on a suggestion of M. F. Vigneras) proved Corollary 1 for $G = PSL(2, \mathbb{R})$ and $G = PSL(2, \mathbb{C})$ ([EGM, Theorem 3.3, p.203]).

Remark 3. When $G = PSL(2, \mathbb{R})$, it can be seen that the spherical spectrum of determines the full spectrum $L^2(\Gamma \backslash G)$ ([Pe]). One can raise the question whether such a result will be true in general. This fits in with the conjectures linking spectrum and arithmetic in the context of automorphic forms (see [Ra, Conjecture 3]).

2. Preliminaries

2.1. **Representations of semisimple groups.** We recall some facts about representations of semisimple groups. Let G be a semisimple Lie group with a Haar measure μ. Let π be a unitary representation of G on a Hilbert space V. For a compactly supported smooth function f on G, define the convolution operator $\pi(f)$ on V as follows :

$$\pi(f)(v) = \int_G f(g)\, \pi(g)v\; d\mu(g)$$

This defines a bounded linear operator on V. We recall the following result from [Kn, Theorem 10.2; p.334] :

Proposition 2.1. *Let G be a semisimple group and π be an irreducible unitary representation of G. Then the convolution operator $\pi(f)$ is of trace class for every compactly supported smooth function f on G.*

Let π be an irreducible unitary representation of G. Let $C_c^\infty(G)$ be the space of compactly supported smooth functions on G. Define the character distribution χ_π by,

$$\chi_\pi(f) = \text{trace } (\pi(f)) \quad \forall f \in C_c^\infty(G).$$

2.2. **Some results of Harish Chandra.** We recall some results of Harish Chandra on the characters of irreducible unitary representations of G.

Theorem 2.2. [Kn, Theorem 10.6; p.336] *Let $\{\pi_i\}$ be a finite collection of mutually inequivalent irreducible unitary representations of G. Then their characters $\{\chi_{\pi_i}\}$ are linearly independent distributions on $C_c^\infty(G)$.*

Let $L_{loc}^1(G)$ be the space of all complex valued measurable functions f on G such that

$$\int_C |f(g)|\, d\mu(g)\; < \infty \quad \text{for all compact subset } C \text{ of } G.$$

The following deep result of Harish Chandra will be crucially used in the proof of Theorem 1.1.

Theorem 2.3. [Kn, Theorem 10.25; p.356] *Let π be an irreducible unitary representation of G. The distribution character χ_π is given by a locally integrable function h on G. i.e. there exists $h \in L^1_{loc}(G)$ such that*

$$\chi_\pi(f) = \int_G f(g) \, h(g) \, d\mu(g) \quad \forall f \in C^\infty_c(G).$$

2.3. **Selberg trace formula for compact quotient.** We recall the Selberg trace formula for compact quotient. (For details, see Wallach [Wa, p.171-172]). Let μ' be the normalized G-invariant measure on the quotient space $\Gamma\backslash G$. For a compactly supported smooth function f on G, the convolution operator $R_\Gamma(f)$ on $L^2(\Gamma\backslash G)$ is given by :

$$R_\Gamma(f)(\phi)(y) \quad = \int_G f(x) \, \phi(yx) \, d\mu(x) \qquad \forall \, \phi \in L^2(\Gamma\backslash G) \text{ and } y \in G.$$

$$= \int_{\Gamma\backslash G} \left[\sum_{\gamma \, \in \, \Gamma} f(y^{-1}\gamma x) \right] \phi(x) \, d\mu'(x).$$

Since f is a smooth and compactly supported function on G and Γ is uniform lattice, the sum $K_f(y, x) = \sum_{\gamma \, \in \, \Gamma} f(y^{-1}\gamma x)$ is a finite sum, and hence it follows that the operator $R_\Gamma(f)$ is of Hilbert-Schmidt class. The trace of $R_\Gamma(f)$ is defined and it is given by integrating the kernel function $K_f(y, x)$,

$$\mathrm{tr}(R_\Gamma(f)) = \int_{\Gamma\backslash G} \left[\sum_{\gamma \, \in \, \Gamma} f(x^{-1}\gamma x) \right] d\mu'(x)$$

Let $[\gamma]_G$ (resp. $[\gamma]_\Gamma$) be the conjugacy class of γ in G (resp. in Γ). Let $[\Gamma]$ (resp. $[\Gamma]_G$) be the set of conjugacy classes in Γ (resp. the G-conjugacy classes of elements in Γ). For $\gamma \in \Gamma$, let G_γ be the centralizer of γ in G. Put $\Gamma_\gamma = \Gamma\cap G_\gamma$. It can be seen that Γ_γ is a lattice in G_γ and the quotient $\Gamma_\gamma\backslash G_\gamma$ is compact. Since G_γ is unimodular, there exists a G-invariant measure on $G_\gamma\backslash G$, denoted by $d_\gamma x$. After normalizing the measures on G_γ and $G_\gamma\backslash G$ appropriately and rearranging the terms on the right hand side of above equation, we get :

$$(2) \qquad \operatorname{tr}(R_\Gamma(f)) \quad = \sum_{[\gamma] \, \in \, [\Gamma]} \operatorname{vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} f(x^{-1}\gamma x) \, d_\gamma x$$

$$= \sum_{[\gamma] \, \in \, [\Gamma]_G} a(\gamma, \Gamma) \, O_\gamma(f)$$

where $O_\gamma(f)$ is the orbital integral of f at γ defined by,

$$O_\gamma(f) = \int_{G_\gamma \backslash G} f(x^{-1}\gamma x) \, d_\gamma x.$$

Here

$$a(\gamma, \Gamma) = \sum_{[\gamma']_\Gamma \, \subseteq \, [\gamma]_G} \operatorname{vol}\left(\Gamma_{\gamma'} \backslash G_{\gamma'}\right).$$

If γ is not conjugate to an element in Γ, we define $a(\gamma, \Gamma) = 0$. On the other hand, the trace of $R_\Gamma(f)$ on the spectral side can be written as an absolutely convergent series as,

$$(3) \qquad \operatorname{tr}(R_\Gamma(f)) = \sum_{\pi \, \in \, \widehat{G}} m(\pi, \Gamma)\chi_\pi(f)$$

Hence from (2) and (3), we obtain the Selberg trace formula:

$$(4) \qquad \sum_{\pi \, \in \, \widehat{G}} m(\pi, \Gamma)\chi_\pi(f) = \sum_{[\gamma] \, \in \, [\Gamma]_G} a(\gamma, \Gamma) \, O_\gamma(f).$$

3. Proof of Theorem 1.1

3.1. **Some preliminary lemmas.** We first recall some known results about the geometry of conjugacy classes in G.

Lemma 3.1. *Let Γ be a uniform lattice in G. Let $\gamma \in \Gamma$. Then the G-conjugacy class $[\gamma]_G$ is a closed subset of measure zero in G.*

Proof. Let $\{g_n^{-1}\gamma g_n\}_{n=1}^{\infty}$ be a sequence of points in $[\gamma]_G$ which converges to h in G. Since $\Gamma \backslash G$ is compact, there exists a relatively compact set D of G such that $G = \Gamma D$. Write $g_n = \gamma_n \, d_n$ where $\gamma_n \in \Gamma$ and $d_n \in D$. Hence,

$$g_n^{-1} \, \gamma \, g_n = d_n^{-1} \, \gamma_n^{-1} \, \gamma \, \gamma_n \, d_n.$$

Since D is relatively compact, there is a convergent subsequence of $\{d_n\}_{n=1}^{\infty}$, which converges to some element d of G. Hence we get,

$$\lim_{n\to\infty} \gamma_n^{-1}\, \gamma\, \gamma_n = d^{-1}hd.$$

Since Γ is discrete, for large n,

$$\gamma_n^{-1}\, \gamma\, \gamma_n = d^{-1}hd.$$

Hence $h \in [\gamma]_G$. Thus $[\gamma]_G$ is closed in G.

The conjugacy class $[\gamma]_G$ is homeomorphic to the homogeneous space $G_\gamma\backslash G$. Hence there exists a natural structure of a smooth manifold on it such that it is a submanifold of G. Since G_γ is non trivial (it contains a Cartan subgroup of G), it is of lower dimension than G and hence of measure zero with respect to the Haar measure μ on G.

$\square$

Lemma 3.2. *Let Ω be a relatively compact subset of G. Then the set*

$$A_\Omega = \{ [\gamma]_G \ : \ \gamma \in \Gamma \ and \ [\gamma]_G \cap \Omega \neq \emptyset \}$$

is finite.

Proof. Let $x \in G$ be such that $x^{-1}\gamma x \in \Omega$. As in Lemma 3.1, write $x = \gamma_1.\delta$ where $\gamma_1 \in \Gamma$ and $\delta \in D$. Hence $\gamma_1^{-1}\gamma\gamma_1 \in D\Omega D^{-1}$ which is relatively compact in G. Hence $\gamma_1^{-1}\gamma\gamma_1 \in D\Omega D^{-1} \cap \Gamma$ which is a finite set. $\square$

Corollary 2. *Let E be the union of the conjugacy classes $[\gamma]_G$ such that $\gamma \in \Gamma_1 \cup \Gamma_2$. Then E is a closed subset of measure zero in G.*

Proof. By using above two lemmas, it follows that $E \cap C$ is finite for every compact subset $C \subseteq G$. Hence E is closed in G. It is of measure zero since it is a countable union of sets of measure zero. $\square$

3.2. **Proof of Theorem 1.1.** For $\pi \in \widehat{G}$, let $t_\pi = m(\pi, \Gamma_1) - m(\pi, \Gamma_2)$. Let $f \in C_c^\infty(G)$. Since the series in equation (4) converges absolutely, by comparing equation (4) for Γ_1 and Γ_2, we obtain:

$$\sum_{\pi \, \in \, \widehat{G}} t_\pi\, \chi_\pi(f) = \sum_{[\gamma] \, \in \, [\Gamma_1]_G \, \cup \, [\Gamma_2]_G} (a(\gamma, \Gamma_1) - a(\gamma, \Gamma_2))\, O_\gamma(f).$$

By hypothesis, $t_\pi = 0$ for all but finitely many $\pi \in \hat{G}$. Hence there exists a finite subset S of $\hat{G}$ such that,

$$(5) \qquad \sum_{\pi \in S} t_\pi \chi_\pi(f) = \sum_{[\gamma] \in [\Gamma_1]_G \cup [\Gamma_2]_G} (a(\gamma, \Gamma_1) - a(\gamma, \Gamma_2))\, O_\gamma(f).$$

Since S is a finite set, by Harish Chandra's Theorem 2.3, there exists a function $\phi \in L^1_{loc}(G)$ such that

$$(6) \qquad \sum_{\pi \in S} t_\pi \cdot \chi_\pi(f) = \int_G f(g)\, \phi(g)\, d\mu(g) \quad \forall\, f \in C_c^\infty(G).$$

Let E be as in Corollary 2 above. Let $g \in G$ be any point outside E. Since E is closed in G, there exists a relatively compact neighborhood U of g such that $U \cap E = \emptyset$. Hence, if $f \in C_c^\infty(G)$ is supported on U, we have

$$O_\gamma(f) = 0 \qquad \forall\, \gamma \in \Gamma_1 \cup \Gamma_2.$$

Hence from equations (5) and (6) above, we get :

$$\int_G f(g)\, \phi(g)\, d\mu(g) = 0,$$

for all smooth compactly supported functions f supported in U.

But this means that $\phi(g)$ is essentially 0 on U. Since U was a neighborhood of an arbitrary point g outside E, and E is a closed subset of measure zero, we conclude that $\phi(g)$ is essentially 0 on G. By equation (6) above :

$$\sum_{\pi \in S} t_\pi \chi_\pi(f) = 0 \quad \forall\, f \in C_c^\infty(G).$$

From the linear independence of characters (Theorem 2.2), we get that $t_\pi = 0$ for any $\pi \in S$. Hence,

$$m(\pi, \Gamma_1) = m(\pi, \Gamma_2) \quad \forall\, \pi \in \hat{G}.$$

i.e., the lattices Γ_1 and Γ_2 are representation equivalent in G.

4. Proof of Theorem 1.2

In this section we give a proof of Theorem 1.2, following the broad outline of the proof of Theorem 1.1. Since the analogue of Corollary 2 does not seem available to us, we need to establish a more delicate proposition concerning the $K \times K$-saturation $KC_\gamma K$ of the conjugacy class of elements $\gamma \in \Gamma$. Corresponding to use of Harish Chandra's theorem on the local integrability of the character of an irreducible unitary representation of G, we instead use the analyticity of the spherical functions on G.

Definition 4.1. [GV, p.399] A complex valued function ϕ on G is called a spherical function if

(1) $\phi(e) = 1$.
(2) $\phi(k_1 x k_2) = \phi(x) \quad \forall \ k_1, \ k_2 \in K$ and $x \in G$.
(3) ϕ is a common eigenfunction for all D in the space $D(G/K)$ of G-invariant differential operators on G/K with eigenvalue $\lambda(D)$:

$$D\phi = \lambda(D)\phi \ \ \forall \ D \in D(G/K).$$

The map $D \to \lambda(D)$ defines a algebra homomorphism of $D(G/K)$ into $\mathbb{C}$. Denote by $C_c^\infty(G//K)$ the space of smooth and compactly supported bi-K-invariant functions on G.

Let π be a spherical unitary representation of G. The space π^K of K-fixed vectors is one dimensional (*cf.* Helgason [He, p.416]). Let ϕ_π be the associated elementary spherical function defined by

$$\phi_\pi(x) = \langle \ \pi(x) \ e_\pi, e_\pi \ \rangle,$$

where e_π is a K-fixed vector of the representation space of π such that $\|e_\pi\| = 1$.

We have the following proposition:

Proposition 4.1. *Let π be an irreducible unitary spherical representation of G. Then the following hold:*

(i) The associated elementary spherical functions ϕ_π are analytic on G.

(ii) *The relationship of the elementary spherical function ϕ_π to character χ_π is given by the following equation:*

$$\chi_\pi(f) = \int_G f(g) \; \phi_\pi(g) \; dg \quad f \in C_c^\infty(G//K).$$

(iii) *Let $\{\pi_j : 1 \leq j \leq k\}$ be a finite collection of mutually inequivalent irreducible spherical representations of G. The associated elementary spherical functions $\{\phi_{\pi_j} : 1 \leq j \leq k\}$ are linearly independent.*

Proof. (i) Since the algebra $D(G/K)$ contains the Laplace-Beltrami operator which is an elliptic, essentially self adjoint differential operator, it follows that the elementary spherical functions ϕ_π are analytic on G.

(ii) Let V be the space underlying the representation π. Given $f \in C_c^\infty(G//K)$, the image $\pi(f)(V)$ of the convolution operator $\pi(f)$ lands in the space V^K of K-invariants. Hence the trace is given by,

$$\chi_\pi(f) = \text{trace } (\pi(f)) = \langle \pi(f)e_\pi, e_\pi \rangle = \int_G f(g) \; \phi_\pi(g) \; dg.$$

(iii) The function ϕ_{π_j} is an eigenvector for the character λ_{π_j}. Since the representations π_j are mutually inequivalent, the homomorphisms λ_{π_j} are distinct and hence the corresponding eigenvectors are linearly independent. $\qquad \square$

Now we turn to the geometric aspects of the Selberg trace formula. Let $G//K$ denote the collection of orbits under the action of $K \times K$ acting on G by,

$$(7) \qquad\qquad (k,l)g = k^{-1}gl \qquad k,l \in K, \; g \in G.$$

Lemma 4.1. *The space $C_c^\infty(G//K)$ consisting of bi-K-invariant compactly supported smooth functions on G separate points on $G//K$.*

Proof. The orbits of $K \times K$ being compact are closed subsets of G. Given two orbits KxK, KyK choose a compactly supported, smooth function which is positive on KxK and vanishes on KyK. Then,

$$F(g) = \int_{K \times K} f(kgl)dkdl,$$

is a bi-K-invariant compactly supported smooth function on G which separates the two orbits.   □

Lemma 4.2. *Let Γ be a torsion-free uniform lattice in G. For a non-trivial element $\gamma \in \Gamma$, the conjugacy class C_γ is disjoint from K.*

Proof. The group $x^{-1}\gamma x \cap K$ is discrete and contained in the compact group K, hence finite. Consequently, $x^{-1}\gamma x$ is of finite order in G and hence γ is of finite order. Since Γ is torsion-free, γ is the identity element of G.   □

Lemma 4.3. *If $\gamma \neq e$, then $e \notin KC_\gamma K$.*

Proof. Let $x \in G$ and $k, l \in K$ be such that $kx^{-1}\gamma xl = e$. Then $x^{-1}\gamma x \in K$, which is not possible by Lemma 4.2.   □

Proposition 4.2. *There exists an open set B in G such that $C_\gamma \cap B$ is empty for all $\gamma \in \Gamma_1 \cup \Gamma_2$ and B is stable under $K \times K$ action on G given by equation 7.*

Proof. Let U' be a relatively compact open neighborhood of e in G. Let $U = KU'K$. Then U is relatively compact and hence it intersects atmost finitely many conjugacy classes C_γ. Since the map $G \to G//K$ is proper and the conjugacy class C_γ is closed, the set $KC_\gamma K$ is closed in G. Since U is K-stable, $KC_\gamma K \cap U$ is non-empty if and only if $C_\gamma \cap U$ is non-empty. Hence, the set $E = \bigcup_{\gamma \neq e} [KC_\gamma K] \cap U$, being a finite union of closed sets, is a $K \times K$-stable closed subset of U. By Lemma 4.3, the identity element e does not belong to E. Choose an open set $V \subseteq U$ containing e such that $E \cap V = \emptyset$. Let $B = KVK \cap K^c$, where K^c is the complement of K in G. It can be seen that B satisfies the desired property.   □

Now we give the proof of Theorem 1.2.

Proof. By hypothesis of Theorem 1.2, there exists a finite subset S of $\widehat{G}_s$ such that

$$m(\pi, \Gamma_1) = m(\pi, \Gamma_2) \ \ \forall \ \pi \notin S.$$

Let $f \in C_c^\infty(G//K)$. Since f is bi-K-invariant, $\chi_\pi(f) = 0$ if $\pi \notin \widehat{G}_s$. Using the Selberg trace formula for f , we get :

$$(8) \qquad \sum_{\pi \in S} t_\pi \chi_\pi(f) = \sum_{[\gamma] \in [\Gamma_1]_G \cup [\Gamma_2]_G} (a(\gamma, \Gamma_1) - a(\gamma, \Gamma_2)) \, O_\gamma(f)$$

Let $\phi = \sum_{\alpha \in S} t_\pi \phi_\pi$. By using proposition 4.1, we get :

$$\int_G f(g) \, \phi(g) \, d\mu(g) = \sum_{[\gamma] \in [\Gamma_1]_G \cup [\Gamma_2]_G} (a(\gamma, \Gamma_1) - a(\gamma, \Gamma_2)) \, O_\gamma(f).$$

Let B be as in the proof of Proposition 4.2. The term on right hand side in above equation vanishes for every function f in $C_c^\infty(G//K)$ which is supported on B. Hence for such functions f,

$$\int_G f(g) \, \phi(g) \, d\mu(g) = 0.$$

By Lemma 4.1, the functions f separate points on B. Hence ϕ must vanish on the open subset B of G. Since ϕ is analytic, it vanishes on all of G. By the linear independence of functions ϕ_π (Proposition 4.1), we conclude that

$$m(\pi, \Gamma_1) = m(\pi, \Gamma_2) \quad \forall \, \pi \in \widehat{G}_s.$$

$\square$

5. **Proof of Theorem 1.3**

We now proceed to derive Theorem 1.3 from Theorem 1.2. We follow the notation given in the introduction. Let π be an irreducible, unitary, spherical representation of G. Let e_π be a K-fixed vector of unit length in π. The associated spherical function ϕ_π is an eigenfunction of $D(G/K)$ with eigencharacter λ_π:

$$D(\phi_\pi) = \lambda_\pi(D)\phi_\pi \quad D \in D(G/K).$$

The main observation is the following proposition.

Proposition 5.1. *Let Γ be a torsion-free uniform lattice in G. Let π be an irreducible, unitary spherical representation of G. Then*

$$m(\pi, \Gamma) = dim \, (V(\lambda_\pi, \Gamma)).$$

In particular, $V(\lambda_\pi, \Gamma)$ is finite dimensional.

Conversely, if λ is a character of $D(G/K)$ and the dimension of $V(\lambda_\pi, \Gamma)$ is positive, then $\lambda = \lambda_\pi$ for some spherical representation π of G.

Remark 4. When $G = PSL(2, \mathbb{R})$ this is the duality theorem proved by Gelfand, Graev and Pyatetskii-Shapiro ([GGP, p.50]), relating the spectrum of the Laplace-Beltrami on the upper half plane and the multiplicities of spherical representations of $PSL(2, \mathbb{R})$ occurring in $L^2(\Gamma \backslash PSL(2, \mathbb{R}))$. We follow their proof. The above fact is probably well known to the experts but we have included a proof for sake of completeness. The proof also indicates that Theorems 1.2 and Theorem 1.3 are equivalent.

Proof. Let $\mathfrak{g}$ be the complexification of Lie algebra of G consisting of left invariant vector fields on G. Let $\mathfrak{U}(\mathfrak{g})^K$ be the K-invariant subspace of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ under the right action of K on $\mathfrak{U}(\mathfrak{g})$. We consider the right action of G on itself. This gives raise to a surjective map from $\mathfrak{U}(\mathfrak{g})^K$ to $D(G/K)$ ([GV, page 52, Proposition 1.7.5]). Hence it can be seen that $D(e_\pi)$ is a K-fixed vector for each $D \in D(G/K)$. Since the dimension of the space of K-fixed vectors of π is one, it follows that e_π is an eigenvector of $D(G/K)$ with respect to the eigencharacter λ_π i.e. it lies in the eigenspace in $V(\lambda_\pi, \Gamma)$. Therefore, we conclude that $m(\pi, \Gamma) \leq \dim (V(\lambda_\pi, \Gamma))$.

Conversely let $f \in C^\infty(X)$ be an eigenvector of some character λ of $D(G/K)$. Since

$$L^2(\Gamma \backslash G) = \bigoplus_{\pi \in \widehat{G}} m(\pi, \Gamma) \ \pi$$

we write

(9)
$$f = \sum_{\pi \in \widehat{G}} a_\pi \ v_\pi$$

such that $v_\pi \in \pi$ is a vector of unit length. Let W be the space of K-invariants of $L^2(\Gamma \backslash G)$. Let P_W be the orthogonal projection of $L^2(\Gamma \backslash G)$ onto W. Since f is right invariant under K, $P_W(f) = f$. Hence we get :

$$f = \sum_{\pi \in \widehat{G}} a_\pi \ P_W(v_\pi)$$

The algebra $D(G/K)$ is generated by essentially self-adjoint differential operators. Hence, if the character λ_π is distinct from λ, there exists an essentially self-adjoint $D \in D(G/K)$ such that $\lambda_\pi(D) \neq \lambda(D)$. Hence the eigenvectors v_π and f are orthogonal. If π is not a spherical representation, $P_W(v_\pi) = 0$. Hence the indexing set in equation (9) is restricted to those irreducible unitary spherical representations with character λ_π equal to λ.

Since the associated spherical functions to inequivalent representations are linearly independent, the characters are distinct. Hence we conclude that there is an unique irreducible unitary spherical representation π of G such that $\lambda = \lambda_\pi$. Hence,

$$m(\pi, \Gamma) = \dim \left(V(\lambda_\pi, \Gamma) \right).$$

$\square$

Now we give the proof of Theorem 1.3. Let T be a finite subset of characters of $D(G/K)$ such that

$$\dim \left(V(\lambda, \Gamma_1) \right) = \dim \left(V(\lambda, \Gamma_2) \right)$$

for all characters $\lambda \notin T$. By above Proposition 5.1, we get that :

$$m(\pi, \Gamma_1) = m(\pi, \Gamma_2)$$

for all but finitely many irreducible, unitary spherical representations of G. Hence using Theorem 1.2 and Proposition 5.1, we get a proof of Theorem 1.3.

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