ON COMPARISON THEOREMS FOR ELLIPTIC INEQUALITIES

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1. Introduction

Suppose that Ω is a non-empty open subset of $\mathbb{R}^n$, $n \ge 2$. Let us denote: $\Omega_{R_0,R_1} = \{x \in \Omega : R_0 < |x| < R_1\}$ and $\Gamma_{R_0,R_1} = \{x \in \partial\Omega : R_0 < |x| < R_1\}$. By B_r^x we mean the open ball in $\mathbb{R}^n$ of radius $r > 0$ and center at a point x. Also put $S_r^x = \partial B_r^x$. In the case of $x = 0$, we write B_r and S_r instead of B_r^0 and S_r^0, respectively.

Consider the inequality

$$\operatorname{div} A(x, Du) \ge F(x, u, Du) \quad \text{in } \Omega_{R_0,R_1},\ 0 \le R_0 < R_1 \le \infty, \tag{1.1}$$

where $D = (\partial/\partial x_1, \dots, \partial/\partial x_n)$ is the gradient operator and $A : \Omega_{R_0,R_1} \times \mathbb{R}^n \to \mathbb{R}^n$ is s measurable function such that

$$C_1|\xi|^p \le \xi A(x, \xi), \quad |A(x, \xi)| \le C_2|\xi|^{p-1}$$

with some constants $C_1 > 0$, $C_2 > 0$, and $p > 1$ for almost all $x \in \Omega_{R_0,R_1}$ and for all $\xi \in \mathbb{R}^n$. We say that u is a solution of (1.1) if $u \in W_p^1(\Omega_{R_0,r}) \cap L_\infty(\Omega_{R_0,r})$, $A(x, Du) \in L_{p/(p-1)}(\Omega_{R_0,r})$, and $F(x, u, Du) \in L_{p/(p-1)}(\Omega_{R_0,r})$ for any real number $r \in (R_0, R_1)$ and, moreover,

$$-\int_{\Omega_{R_0,R_1}} A(x, Du) D\varphi\, dx \ge \int_{\Omega_{R_0,R_1}} F(x, u, Du)\varphi\, dx$$

for any non-negative function $\varphi \in C_0^\infty(\Omega_{R_0,R_1})$ [4]. In so doing, the condition

$$u|_{\Gamma_{R_0,R_1}} = 0 \tag{1.2}$$

means that $\varphi u \in \overset{\circ}{W}{}_p^1(\Omega_{R_0,R_1})$ for any $\varphi \in C_0^\infty(B_{R_0,R_1})$, where $B_{R_0,R_1} = \{x \in \mathbb{R}^n : R_0 < |x| < R_1\}$. In particular, if $\Omega = \mathbb{R}^n$, then (1.2) is fulfilled for all $u \in W_{p,loc}^1(B_{R_0,R_1})$.

Throughout this paper, we assume that $S_r \cap \Omega \ne \emptyset$ for any $r \in (R_0, R_1)$. Let u be a solution of (1.1), (1.2). Put

$$M(r; u) = \operatorname*{ess\,sup}_{S_r \cap \Omega} u, \quad r \in (R_0, R_1), \tag{1.3}$$

where the restriction of u to $S_r \cap \Omega$ is understood in the sense of the trace and the ess sup in the right-hand side of (1.3) is with respect to $(n-1)$-dimensional Lebesgue measure on S_r. We also assume that the right-hand side of inequality (1.1) satisfies the following condition: there exist a real number $\sigma > 1$ and locally bounded measurable functions $f : [R_0, R_1) \times (0, \infty) \to [0, \infty)$ and $b : [R_0, R_1) \to [0, \infty)$ such that

$$f(r, t - 0) = f(r, t) \quad \text{for all } R_0 < r < R_1,\ t > 0,$$
$$f(r, t_1) \ge f(r, t_2) \quad \text{for all } R_0 < r < R_1,\ t_1 \ge t_2 > 0$$

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and, moreover,

$$F(x,t,\xi) \ge \sup_{r \in (|x|/\sigma, \sigma|x|) \cap (R_0,R_1)} f(r,t) - |\xi|^{p-1} \inf_{r \in (|x|/\sigma, \sigma|x|) \cap (R_0,R_1)} b(r) \tag{1.4}$$

for almost all $x \in \Omega_{R_0,R_1}$ and for all $t \in (0,\infty)$ and $\xi \in \mathbb{R}^n$.

The questions studied in this article were earlier investigated by a number of authors [1]–[3], [5]–[10]. Our aim is to estimate the function $M(\cdot;u)$ by a solution of an ordinary differential equation, which contains the radial p-Laplace operator with the lowest terms.

2. Main results

Theorem 2.1. *Let u be a non-negative solution of problem (1.1), (1.2) such that $M(\cdot;u)$ is a non-decreasing function on the interval (R_0,R_1) with*

$$M(R_0+0;u) > 0. \tag{2.1}$$

Then for all real numbers $a > p-2$ and $k > 0$ there exist constants $\alpha > 0$ and $\beta > 0$ depending only on n, p, a, k, σ, C_1, and C_2 such that the Cauchy problem

$$\frac{1}{r^{1+a}} \frac{d}{dr} \left(r^{1+a} \left| \frac{dm}{dr} \right|^{p-2} \frac{dm}{dr} \right) + k b(r) \left| \frac{dm}{dr} \right|^{p-2} \frac{dm}{dr} = \alpha f(r, \beta m), \tag{2.2}$$

$$m(R_0) = M(R_0+0;u), \quad m'(R_0) = 0, \tag{2.3}$$

has a solution on $[R_0,R_1)$ satisfying the estimate

$$M(r;u) \ge m(r) > 0$$

for any $r \in (R_0,R_1)$.

Theorem 2.2. *Under the hypotheses of Theorem 2.1, for all real numbers $a > p-2$ and $k > 0$ there exist constants $\alpha > 0$ and $\beta > 0$ depending only on n, p, a, k, σ, C_1, and C_2 such that*

$$\begin{aligned} &M(r;u) - M(R_0+0;u) \\ &\quad \ge \int_{R_0}^r dt \left(\frac{\alpha}{t^{1+a}} \int_{R_0}^t \xi^{1+a} e^{-k \int_\xi^t b(\zeta)\, d\zeta} f(\xi, \beta M(\xi;u))\, d\xi \right)^{1/(p-1)} \end{aligned} \tag{2.4}$$

for any $r \in (R_0,R_1)$.

Example 2.1. Consider the inequality

$$\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i} \ge c(x,u)$$

for a linear uniformly elliptic operator with locally bounded measurable coefficients. Setting $p = 2$ and

$$F(x,t,\xi) = c(x,t) - \sum_{i=1}^n b_i(x) \xi_i,$$

one can show that relation (1.4) is filfilled if b and f are non-negative functions such that

$$b(r) \ge \sup_{x \in \Omega_{r/\sigma, r\sigma} \cap \Omega_{R_0,R_1}} \sum_{i=1}^n |b_i(x)| \quad \text{for all } r \in (R_0,R_1)$$

and

$$f(r,t) \le \inf_{x \in \Omega_{r/\sigma, r\sigma} \cap \Omega_{R_0,R_1}} c(x,t) \quad \text{for all } r \in (R_0,R_1),\ t \in (0,\infty).$$

In this case, equation (2.2) takes the form

$$\frac{d^2 m}{dr^2} + \left(\frac{1+a}{r} + kb(r) \right) \frac{dm}{dr} = \alpha f(r, \beta m). \tag{2.5}$$

Putting $a = n-2$ and $k = 1$, we obviously obtain the radial part of the operator $\triangle + b(|x|)D|x|D$ in the left-hand side of (2.5).

Proof of Theorem 2.1. Assume that Theorem 2.2 is already proved. Let us construct a sequence of maps $m_i : [R_0, R_1) \to (0, \infty)$ by setting $m_0(r) = M(R_0 + 0; u)$ and

$$m_i(r) = M(R_0+0;u) + \int_{R_0}^r dt \left(\frac{\alpha}{t^{1+a}} \int_{R_0}^t \xi^{1+a} e^{-k \int_\xi^t b(\zeta)\, d\zeta} f(\xi, \beta m_{i-1}(\xi))\, d\xi \right)^{1/(p-1)}$$

$i = 1, 2, \ldots$. We have $M(r;u) \ge m_i(r) \ge m_{i-1}(r)$ for all $r \in (R_0, R_1)$, $i = 1, 2, \ldots$. Therefore, there exists a map $m : [R_0, R_1) \to (0, \infty)$ such that m_i tends to m everywhere on the interval $[R_0, R_1)$ as $i \to \infty$.

It is obvious that $M(r;u) \ge m(r)$ for all $r \in (R_0, R_1)$. In addition, the following integral equation is valid:

$$m(r) = M(R_0+0;u) + \int_{R_0}^r dt \left(\frac{\alpha}{t^{1+a}} \int_{R_0}^t \xi^{1+a} e^{-k \int_\xi^t b(\zeta)\, d\zeta} f(\xi, \beta m(\xi))\, d\xi \right)^{1/(p-1)}$$

Thus, to complete the proof it remains to verify by direct differentiation that m is a solution of problem (2.2), (2.3). ☐

3. Proof of Theorem 2.2

From now on we assume that $a > p-2$ and $k > 0$ are some fixed real numbers and $u \ge 0$ is a solution of problem (1.1), (1.2) such that $M(\cdot; u)$ is a non-decreasing function on the interval (R_0, R_1) satisfying condition (2.1). Without loss of generality it can also be assumed that

$$\inf_{(R_0,R_1)} b > 0;$$

otherwise we prove (2.4) with b replaced by $b+\delta$, where δ is a positive real number, and let δ tend to zero afterwards.

From the maximum principle, it follows that

$$M(r-0;u) = M(r;u), \quad r \in (R_0, R_1), \tag{3.1}$$

(see Corollary 4.1, Section 4).

Lemma 3.1. *Let $0 < \beta < 1$, $R_0 < r_0 < r_1 < R_1$, and $\sigma^2 r_0 \ge r_1$. If $\beta^{1/2} M(r_1;u) \le M(r_0;u)$, then*

$$M(r_1;u) - M(r_0;u) \ge \gamma_1 \min\left\{ (r_1 - r_0)^{p/(p-1)}, \frac{r_1 - r_0}{\lambda^{1/(p-1)}} \right\} f^{1/(p-1)}(s, \beta M(r_1;u))$$

for all $s \in [r_1/\sigma, \sigma r_0] \cap (R_0, R_1)$, where

$$\lambda = \inf_{[r_1/\sigma, \sigma r_0] \cap (R_0,R_1)} b$$

and the constant $\gamma_1 > 0$ depends only on n, p, C_1, C_2, and β.

The proof of Lemma 3.1 is given in Section 4.

Corollary 3.1. *Suppose that $0 < \beta < 1$, $R_0 < r_0 < r_1 < R_1$, $\sigma r_0 \ge r_1$ and, moreover, $\beta^{1/2} M(r_1; u) \le M(r_0; u)$. Then*

$$M(r_1;u)-M(r_0;u) \ge \gamma_2 (r_1 - r_0) \left(\int_{\rho_0}^{\rho_1} e^{-k \int_\xi^{\rho_1} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)} \tag{3.2}$$

for all real numbers $R_0 < \rho_0 < \rho_1 < R_1$ satisfying the inequalities $r_1/\sigma \le \rho_0$, $\rho_1 \le r_1$, and $\rho_1 - \rho_0 \le r_1 - r_0$, where the constant $\gamma_2 > 0$ depends only on n, p, k, C_1, C_2, and β.

Proof. We have

$$\begin{aligned} \int_{\rho_0}^{\rho_1} e^{-k \int_\xi^{\rho_1} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi &\le f(\xi_*, \beta M(\xi_*; u)) \int_{\rho_0}^{\rho_1} e^{-k \int_\xi^{\rho_1} b(\zeta)\, d\zeta}\, d\xi \\ &\le f(\xi_*, \beta M(\xi_*; u)) \int_{\rho_0}^{\rho_1} e^{-k\lambda(\rho_1 - \xi)}\, d\xi \end{aligned}$$

for some $\xi_* \in (\rho_0, \rho_1)$, where

$$\lambda = \inf_{(\rho_0, \rho_1)} b.$$

Since

$$\int_{\rho_0}^{\rho_1} e^{-k\lambda(\rho_1 - \xi)}\, d\xi = \frac{1 - e^{-k\lambda(\rho_1 - \rho_0)}}{k\lambda} \le \min \left\{ \rho_1 - \rho_0, \frac{1}{k\lambda} \right\},$$

this implies the estimate

$$\begin{aligned} &(r_1 - r_0) \left(\int_{\rho_0}^{\rho_1} e^{-k \int_\xi^{\rho_1} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)} \\ &\quad \le \min \left\{ (r_1 - r_0)^{p/(p-1)}, \frac{r_1 - r_0}{(k\lambda)^{1/(p-1)}} \right\} f^{1/(p-1)}(\xi_*, \beta M(\xi_*; u)), \end{aligned}$$

whence in accordance with Lemma 3.1 we obtain (3.2). □

Corollary 3.2. *Let the conditions of Corollary 3.1 be fulfilled, then*

$$\begin{aligned} &M(r_1; u) - M(r_0; u) \\ &\quad \ge \gamma_3 (r_1 - r_0) \left(\frac{r_0 - \rho_1}{\rho_1 - \rho_0} \int_{\rho_0}^{\rho_1} e^{-k \int_\xi^{r_0} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)} \end{aligned}$$

for all real numbers $R_0 < \rho_0 < \rho_1 < R_1$ satisfying the inequalities $r_1/\sigma \le \rho_0$, $\rho_1 < r_0$, and $r_0 - \rho_1 \le r_1 - r_0$, where the constant $\gamma_3 > 0$ depends only on n, p, k, C_1, C_2, and β.

Proof. There exists $\xi_* \in (\rho_0, \rho_1)$ such that

$$\begin{aligned} \int_{\rho_0}^{\rho_1} e^{-k \int_\xi^{r_0} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi &\le f(\xi_*, \beta M(\xi_*; u)) \int_{\rho_0}^{\rho_1} e^{-k \int_\xi^{r_0} b(\zeta)\, d\zeta}\, d\xi \\ &\le f(\xi_*, \beta M(\xi_*; u)) \int_{\rho_0}^{\rho_1} e^{-k\lambda(r_0 - \xi)}\, d\xi, \end{aligned} \tag{3.3}$$

where

$$\lambda = \inf_{(\rho_0, r_0)} b.$$

We have

$$\int_{\rho_0}^{\rho_1} e^{-k\lambda(r_0-\xi)}\,d\xi = \frac{e^{-k\lambda(r_0-\rho_1)} - e^{-k\lambda(r_0-\rho_0)}}{k\lambda} \le (\rho_1-\rho_0)e^{-k\lambda(r_0-\rho_1)}.$$

In addition,

$$\begin{aligned} e^{-k\lambda(r_0-\rho_1)} &\le \frac{1-e^{-k\lambda(r_0-\rho_1)}}{k\lambda(r_0-\rho_1)} \\ &\le \frac{1}{r_0-\rho_1}\min\left\{r_0-\rho_1, \frac{1}{k\lambda}\right\} \\ &\le \frac{1}{r_0-\rho_1}\min\left\{r_1-r_0, \frac{1}{k\lambda}\right\}. \end{aligned}$$

Hence, we obtain

$$\int_{\rho_0}^{\rho_1} e^{-k\lambda(r_0-\xi)}\,d\xi \le \frac{\rho_1-\rho_0}{r_0-\rho_1}\min\left\{r_1-r_0, \frac{1}{k\lambda}\right\}.$$

The last formula and (3.3) imply the estimate

$$\begin{aligned} &(r_1-r_0)\left(\frac{r_0-\rho_1}{\rho_1-\rho_0}\int_{\rho_0}^{\rho_1} e^{-k\int_\xi^{r_0} b(\zeta)\,d\zeta} f(\xi,\beta M(\xi;u))\,d\xi\right)^{1/(p-1)} \\ &\quad\le \min\left\{(r_1-r_0)^{p/(p-1)}, \frac{r_1-r_0}{(k\lambda)^{1/(p-1)}}\right\} f^{1/(p-1)}(\xi_*,\beta M(\xi_*;u)). \end{aligned}$$

Thus, to complete the proof it remains to use Lemma 3.1. □

Lemma 3.2. *Suppose that $0<\beta<1$, $R_0<r_0<r_1<R_1$, and $\beta^{1/2}M(r_1;u)\le M(r_0;u)$, then*

$$\begin{aligned} &M(r_1;u)-M(r_0;u) \\ &\quad\ge \gamma_4\int_{r_0}^{r_1} dt\left(\frac{1}{t^{1+a}}\int_{r_0}^{t} \xi^{1+a} e^{-k\int_\xi^t b(\zeta)\,d\zeta} f(\xi,\beta M(\xi;u))\,d\xi\right)^{1/(p-1)}, \end{aligned} \tag{3.4}$$

where the constant $\gamma_4>0$ depends only on n, p, a, k, σ, C_1, C_2, and β.

Proof. In the case of $\sigma r_0\ge r_1$, taking $\xi_*\in(r_0,r_1)$ such that

$$\operatorname*{ess\,sup}_{\xi\in(r_0,r_1)} f(\xi,\beta M(\xi;u)) \le 2f(\xi_*,\beta M(\xi_*;u)),$$

we have

$$\begin{aligned} &\int_{r_0}^{r_1} dt\left(\frac{1}{t^{1+a}}\int_{r_0}^{t}\xi^{1+a}e^{-k\int_\xi^t b(\zeta)\,d\zeta} f(\xi,\beta M(\xi;u))\,d\xi\right)^{1/(p-1)} \\ &\quad\le 2^{1/(p-1)} f^{1/(p-1)}(\xi_*,\beta M(\xi_*;u))\int_{r_0}^{r_1} dt\left(\frac{1}{t^{1+a}}\int_{r_0}^{t}\xi^{1+a}e^{-k\int_\xi^t b(\zeta)\,d\zeta}\,d\xi\right)^{1/(p-1)} \\ &\quad\le 2^{1/(p-1)} f^{1/(p-1)}(\xi_*,\beta M(\xi_*;u))\int_{r_0}^{r_1} dt\left(\int_{r_0}^{t} e^{-k\lambda(t-\xi)}\,d\xi\right)^{1/(p-1)}, \end{aligned}$$

where

$$\lambda = \inf_{(r_0,r_1)} b.$$

It presents no special problems to verify that

$$\int_{r_0}^{r_1} dt \left(\int_{r_0}^{t} e^{-k\lambda(t-\xi)}\, d\xi \right)^{1/(p-1)} = \int_{r_0}^{r_1} dt \left(\frac{1 - e^{-k\lambda(t-r_0)}}{k\lambda} \right)^{1/(p-1)} \le (r_1 - r_0) \left(\min \left\{ r_1 - r_0, \frac{1}{k\lambda} \right\} \right)^{1/(p-1)};$$

therefore,

$$\int_{r_0}^{r_1} dt \left(\frac{1}{t^{1+a}} \int_{r_0}^{t} \xi^{1+a} e^{-k \int_\xi^t b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)} \le \gamma_5 \min \left\{ (r_1 - r_0)^{p/(p-1)}, \frac{r_1 - r_0}{\lambda^{1/(p-1)}} \right\} f^{1/(p-1)}(\xi_*, \beta M(\xi_*; u)),$$

where the constant $\gamma_5 > 0$ depends only on p and k, whence estimate (3.4) immediately follows according to Lemma 3.1.

Now, let $\sigma r_0 < r_1$ and N be the maximal integer such that $\sigma^N r_0 < r_1$. We put $\rho_i = \sigma^i r_0$, $i = 0, \ldots, N$, and $\rho_{N+1} = r_1$. It can be seen that

$$\begin{aligned} &\int_{\rho_{i-1}}^{\rho_i} dt \left(\frac{1}{t^{1+a}} \int_{r_0}^{t} \xi^{1+a} e^{-k \int_\xi^t b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)} \\ &\quad \le 2^{1/(p-1)} \int_{\rho_{i-1}}^{\rho_i} dt \left(\frac{1}{t^{1+a}} \int_{\rho_{i-1}}^{t} \xi^{1+a} e^{-k \int_\xi^t b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)} \\ &\qquad + 2^{1/(p-1)} \int_{\rho_{i-1}}^{\rho_i} dt \left(\frac{1}{t^{1+a}} \int_{r_0}^{\rho_{i-1}} \xi^{1+a} e^{-k \int_\xi^t b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)} \end{aligned} \tag{3.5}$$

for all $i = 2, \ldots, N+1$. Repeating the previous arguments, we obtain

$$\begin{aligned} &M(\rho_i; u) - M(\rho_{i-1}; u) \\ &\quad \ge \gamma_6 \int_{\rho_{i-1}}^{\rho_i} dt \left(\frac{1}{t^{1+a}} \int_{\rho_{i-1}}^{t} \xi^{1+a} e^{-k \int_\xi^t b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)} \end{aligned} \tag{3.6}$$

for all $i = 2, \ldots, N+1$, where the constant $\gamma_6 > 0$ depends only on n, p, k, C_1, C_2, and β. Analogously,

$$\begin{aligned} &M(\rho_1; u) - M(r_0; u) \\ &\quad \ge \gamma_6 \int_{r_0}^{\rho_1} dt \left(\frac{1}{t^{1+a}} \int_{r_0}^{t} \xi^{1+a} e^{-k \int_\xi^t b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)}. \end{aligned} \tag{3.7}$$

Further, in the case of $p \ge 2$, we have

$$\begin{aligned} &\left(\int_{r_0}^{\rho_{i-1}} \xi^{1+a} e^{-k \int_\xi^t b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)} \\ &\quad \le \sum_{j=2}^{i} \left(\int_{\rho_{j-2}}^{\rho_{j-1}} \xi^{1+a} e^{-k \int_\xi^t b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)} \\ &\quad \le \sum_{j=2}^{i} \left(\int_{\rho_{j-2}}^{\rho_{j-1}} \xi^{1+a} e^{-k \int_\xi^{\rho_{j-1}} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)} \end{aligned}$$

for all $t \in (\rho_{i-1}, \rho_i)$, $i = 2, \dots, N+1$. In particular,

$$\sum_{i=2}^{N+1} \int_{\rho_{i-1}}^{\rho_i} dt \left(\frac{1}{t^{1+a}} \int_{r_0}^{\rho_{i-1}} \xi^{1+a} e^{-k \int_\xi^t b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)}$$
$$\le \sum_{i=2}^{N+1} \int_{\rho_{i-1}}^{\rho_i} \frac{dt}{\rho_{i-1}^{(1+a)/(p-1)}} \left(\int_{r_0}^{\rho_{i-1}} \xi^{1+a} e^{-k \int_\xi^t b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)}$$
$$\le \sum_{i=2}^{N+1} \frac{\rho_i - \rho_{i-1}}{\rho_{i-1}^{(1+a)/(p-1)}} \sum_{j=2}^{i} \left(\int_{\rho_{j-2}}^{\rho_{j-1}} \xi^{1+a} e^{-k \int_\xi^{\rho_{j-1}} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)},$$

whence in accordance with the evident inequalities

$$\int_{\rho_{j-2}}^{\rho_{j-1}} \xi^{1+a} e^{-k \int_\xi^{\rho_{j-1}} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi$$
$$\le \rho_{j-1}^{1+a} \int_{\rho_{j-2}}^{\rho_{j-1}} e^{-k \int_\xi^{\rho_{j-1}} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi$$

and $\rho_i - \rho_{i-1} \le \sigma \rho_{i-1}$, $2 \le j \le i \le N+1$, we obtain

$$\sum_{i=2}^{N+1} \int_{\rho_{i-1}}^{\rho_i} dt \left(\frac{1}{t^{1+a}} \int_{r_0}^{\rho_{i-1}} \xi^{1+a} e^{-k \int_\xi^t b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)}$$
$$\le \sigma \sum_{i=2}^{N+1} \sum_{j=2}^{i} \left(\frac{\rho_{j-1}}{\rho_{i-1}} \right)^{(a-p+2)/(p-1)}$$
$$\times \rho_{j-1} \left(\int_{\rho_{j-2}}^{\rho_{j-1}} e^{-k \int_\xi^{\rho_{j-1}} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)}$$
$$= \sigma \sum_{j=2}^{N+1} \sum_{i=j}^{N+1} \left(\frac{\rho_{j-1}}{\rho_{i-1}} \right)^{(a-p+2)/(p-1)}$$
$$\times \rho_{j-1} \left(\int_{\rho_{j-2}}^{\rho_{j-1}} e^{-k \int_\xi^{\rho_{j-1}} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)}. \tag{3.8}$$

From the definition of the real numbers ρ_i, $i = 0, \dots, N+1$, it follows that

$$\sigma \sum_{i=j}^{N+1} \left(\frac{\rho_{j-1}}{\rho_{i-1}} \right)^{(a-p+2)/(p-1)} = \sum_{i=j}^{N+1} \sigma^{-(a-p+2)(i-j)/(p-1)+1} \le \gamma_7$$

for all $j = 2, \dots, N+1$, where the constant $\gamma_7 > 0$ depends only on p, a, and σ. Consequently, relation (3.8) implies the estimate

$$\sum_{i=2}^{N+1} \int_{\rho_{i-1}}^{\rho_i} dt \left(\frac{1}{t^{1+a}} \int_{r_0}^{\rho_{i-1}} \xi^{1+a} e^{-k \int_\xi^t b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)}$$
$$\le \gamma_7 \sum_{j=2}^{N+1} \rho_{j-1} \left(\int_{\rho_{j-2}}^{\rho_{j-1}} e^{-k \int_\xi^{\rho_{j-1}} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)}. \tag{3.9}$$

In so doing, Corollary 3.1 enable us to assert that

$$\begin{aligned}
&M(\rho_{j-1}; u) - M(\rho_{j-2}; u) \\
&\quad \ge \gamma_2 (\rho_{j-1} - \rho_{j-2}) \left(\int_{\rho_{j-2}}^{\rho_{j-1}} e^{-k \int_\xi^{\rho_{j-1}} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)} \\
&\quad = \gamma_2 \left(1 - \frac{1}{\sigma} \right) \rho_{j-1} \left(\int_{\rho_{j-2}}^{\rho_{j-1}} e^{-k \int_\xi^{\rho_{j-1}} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)}
\end{aligned}$$

for all $j = 2, \ldots, N+1$. Thus,

$$\begin{aligned}
&M(\rho_N; u) - M(r_0; u) \\
&\quad \ge \gamma_8 \sum_{i=2}^{N+1} \int_{\rho_{i-1}}^{\rho_i} dt \left(\frac{1}{t^{1+a}} \int_{r_0}^{\rho_{i-1}} \xi^{1+a} e^{-k \int_\xi^t b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)}, \quad (3.10)
\end{aligned}$$

where the constant $\gamma_8 > 0$ depends only on n, p, a, k, σ, C_1, C_2, and β.

Now, assume that $1 < p < 2$. Since $a > p - 2$, there exists a real number $\delta > 0$ satisfying the condition $a - p + 2 - \delta > 0$. In particular, we have $1 + a - \delta > 0$. It is obvious that

$$\begin{aligned}
&\int_{r_0}^{\rho_{i-1}} \xi^{1+a} e^{-k \int_\xi^t b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \\
&\quad = \sum_{j=2}^{i} \int_{\rho_{j-2}}^{\rho_{j-1}} \xi^{1+a} e^{-k \int_\xi^t b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \\
&\quad \le \sum_{j=2}^{i} \int_{\rho_{j-2}}^{\rho_{j-1}} \xi^{1+a} e^{-k \int_\xi^{\rho_{j-1}} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi
\end{aligned}$$

for all $t \in (\rho_{i-1}, \rho_i)$, $i = 2, \ldots, N+1$. Combining this with the estimates

$$\begin{aligned}
&\int_{\rho_{j-2}}^{\rho_{j-1}} \xi^{1+a} e^{-k \int_\xi^{\rho_{j-1}} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \\
&\quad \le \rho_{j-1}^{\delta} \int_{\rho_{j-2}}^{\rho_{j-1}} \xi^{1+a-\delta} e^{-k \int_\xi^{\rho_{j-1}} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi, \ j = 2, \ldots, N+1,
\end{aligned}$$

we obtain

$$\begin{aligned}
&\int_{r_0}^{\rho_{i-1}} \xi^{1+a} e^{-k \int_\xi^t b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \\
&\quad \le \sum_{j=2}^{i} \rho_{j-1}^{\delta} \int_{\rho_{j-2}}^{\rho_{j-1}} \xi^{1+a-\delta} e^{-k \int_\xi^{\rho_{j-1}} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi
\end{aligned}$$

for all $t \in (\rho_{i-1}, \rho_i)$, $i = 2, \ldots, N+1$, whence in accordance with the Hölder inequality it follows that

$$\begin{aligned}
&\int_{r_0}^{\rho_{i-1}} \xi^{1+a} e^{-k \int_\xi^t b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \le \left(\sum_{j=2}^{i} \rho_{j-1}^{\delta/(2-p)} \right)^{2-p} \\
&\quad \times \left(\sum_{j=2}^{i} \left(\int_{\rho_{j-2}}^{\rho_{j-1}} \xi^{1+a-\delta} e^{-k \int_\xi^{\rho_{j-1}} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)} \right)^{p-1}
\end{aligned}$$

for all $t \in (\rho_{i-1}, \rho_i)$, $i = 2, \dots, N+1$. At the same time,

$$\sum_{j=2}^{i} \rho_{j-1}^{\delta/(2-p)} = \rho_{i-1}^{\delta/(2-p)} \sum_{j=2}^{i} \sigma^{-\delta(i-j)/(2-p)} \le \frac{\rho_{i-1}^{\delta/(2-p)}}{1 - \sigma^{-\delta/(2-p)}}, \ i = 2, \dots, N+1.$$

Consequently, we have

$$\left(\int_{r_0}^{\rho_{i-1}} \xi^{1+a} e^{-k \int_\xi^t b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)}$$
$$\le \gamma_9 \rho_{i-1}^{\delta/(p-1)} \sum_{j=2}^{i} \left(\int_{\rho_{j-2}}^{\rho_{j-1}} \xi^{1+a-\delta} e^{-k \int_\xi^{\rho_{j-1}} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)}$$

for all $t \in (\rho_{i-1}, \rho_i)$, $i = 2, \dots, N+1$, where the constant $\gamma_9 > 0$ depends only on δ, p, a, and σ. This immediately implies the estimate

$$\sum_{i=2}^{N+1} \int_{\rho_{i-1}}^{\rho_i} dt \left(\frac{1}{t^{1+a}} \int_{r_0}^{\rho_{i-1}} \xi^{1+a} e^{-k \int_\xi^t b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)}$$
$$\le \sum_{i=2}^{N+1} \int_{\rho_{i-1}}^{\rho_i} \frac{dt}{\rho_{i-1}^{(1+a)/(p-1)}} \left(\int_{r_0}^{\rho_{i-1}} \xi^{1+a} e^{-k \int_\xi^t b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)}$$
$$\le \gamma_9 \sum_{i=2}^{N+1} \frac{\rho_i - \rho_{i-1}}{\rho_{i-1}^{(1+a-\delta)/(p-1)}}$$
$$\times \sum_{j=2}^{i} \left(\int_{\rho_{j-2}}^{\rho_{j-1}} \xi^{1+a-\delta} e^{-k \int_\xi^{\rho_{j-1}} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)}.$$

Therefore, taking into account the fact that

$$\int_{\rho_{j-2}}^{\rho_{j-1}} \xi^{1+a-\delta} e^{-k \int_\xi^{\rho_{j-1}} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi$$
$$\le \rho_{j-1}^{1+a-\delta} \int_{\rho_{j-2}}^{\rho_{j-1}} e^{-k \int_\xi^{\rho_{j-1}} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi$$

and $\rho_i - \rho_{i-1} \le \sigma \rho_{i-1}$, $2 \le j \le i \le N+1$, we obtain

$$\sum_{i=2}^{N+1} \int_{\rho_{i-1}}^{\rho_i} dt \left(\frac{1}{t^{1+a}} \int_{r_0}^{\rho_{i-1}} \xi^{1+a} e^{-k \int_\xi^t b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)}$$
$$\le \gamma_9 \sigma \sum_{i=2}^{N+1} \sum_{j=2}^{i} \left(\frac{\rho_{j-1}}{\rho_{i-1}} \right)^{(a-p+2-\delta)/(p-1)}$$
$$\times \rho_{j-1} \left(\int_{\rho_{j-2}}^{\rho_{j-1}} e^{-k \int_\xi^{\rho_{j-1}} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)}$$
$$= \gamma_9 \sigma \sum_{j=2}^{N+1} \sum_{i=j}^{N+1} \left(\frac{\rho_{j-1}}{\rho_{i-1}} \right)^{(a-p+2-\delta)/(p-1)}$$
$$\times \rho_{j-1} \left(\int_{\rho_{j-2}}^{\rho_{j-1}} e^{-k \int_\xi^{\rho_{j-1}} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)}.$$

Since

$$\sum_{i=j}^{N+1} \left(\frac{\rho_{j-1}}{\rho_{i-1}} \right)^{(a-p+2-\delta)/(p-1)} = \sum_{i=j}^{N+1} \sigma^{-(a-p+2-\delta)(i-j)/(p-1)} \le \gamma_{10}$$

for all $j = 2, \dots, N+1$, where the constant $\gamma_{10} > 0$ depends only on δ, p, a, and σ, this again implies inequality (3.9), whence we immediately derive (3.10).

From (3.5), (3.6), and (3.10), it follows that

$$\begin{aligned} & M(r_1; u) - M(r_0; u) \\ & \quad \ge \gamma_{11} \sum_{i=2}^{N+1} \int_{\rho_{i-1}}^{\rho_i} dt \left(\frac{1}{t^{1+a}} \int_{r_0}^{t} \xi^{1+a} e^{-k \int_\xi^t b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)} \\ & \quad = \gamma_{11} \int_{\rho_1}^{r_1} dt \left(\frac{1}{t^{1+a}} \int_{r_0}^{t} \xi^{1+a} e^{-k \int_\xi^t b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)}, \end{aligned}$$

where the constant $\gamma_{11} > 0$ depends only on n, p, a, k, σ, C_1, C_2, and β. Thus, to complete the proof it remains to combine the last formula with (3.7). □

Lemma 3.3. *In the hypotheses of Lemma 3.2, let $\sigma^{1/2} r_0 \le r_1$. Then*

$$\begin{aligned} & M(r_1; u) - M(r_0; u) \\ & \quad \ge \gamma_{12} r_1^{-(a-p+2)/(p-1)} \left(\int_{r_0}^{r_1} \xi^{1+a} e^{-k \int_\xi^{r_1} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)}, \end{aligned}$$

where the constant $\gamma_{12} > 0$ depends only on n, p, a, k, σ, C_1, C_2, and β.

Proof. Using Corollary 3.1, one can show that

$$\begin{aligned} & M(r_1; u) - M(\sigma^{-1/2} r_1; u) \\ & \quad \ge \gamma_2 (1 - \sigma^{-1/2}) r_1 \left(\int_{\sigma^{-1/2} r_1}^{r_1} e^{-k \int_\xi^{r_1} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)} \\ & \quad \ge \gamma_2 (1 - \sigma^{-1/2}) r_1^{-(a-p+2)/(p-1)} \\ & \qquad \times \left(\int_{\sigma^{-1/2} r_1}^{r_1} \xi^{1+a} e^{-k \int_\xi^{r_1} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)}. \end{aligned}$$

Combining this with the inequality

$$\begin{aligned} & M(r_1; u) - M(r_0; u) \\ & \quad \ge \gamma_4 \int_{\sigma^{-1/2} r_1}^{r_1} dt \left(\frac{1}{t^{1+a}} \int_{r_0}^{\sigma^{-1/2} r_1} \xi^{1+a} e^{-k \int_\xi^{r_1} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)} \\ & \quad \ge \gamma_4 (1 - \sigma^{-1/2}) r_1^{-(a-p+2)/(p-1)} \\ & \qquad \times \left(\int_{r_0}^{\sigma^{-1/2} r_1} \xi^{1+a} e^{-k \int_\xi^{r_1} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)}, \end{aligned}$$

which follows from Lemma 3.2, we complete the proof. □

Further in this section, we assume that

$$\beta = \left(\min \left\{ \frac{1}{4^{p/(p-1)+2} \sigma^{1/2}}, \frac{(1 - \sigma^{-1/2})(a-p+2)}{8^{p/(p-1)+1}(p-1)} \right\} \right)^2$$

and

$$\alpha = \left(\min \left\{ \gamma_2 \beta^{1/2}, \frac{\gamma_2}{4^{p/(p-1)+1} \sigma^{1/2}}, \frac{\gamma_3}{4^{p/(p-1)}}, \frac{\gamma_{12}(a-p+2)}{4^{p/(p-1)}(p-1)}, \frac{\gamma_4}{2^{p/(p-1)}} \right\} \right)^{p-1}.$$

Lemma 3.4. *Suppose that* $M(r_0; u) \le \beta^{1/2} M(r_1; u) \le M(r_0 + 0; u)$ *and*

$$M(r_0; u) \ge \int_{R_0}^{r_0} dt \left(\frac{\alpha}{t^{1+a}} \int_{R_0}^{t} \xi^{1+a} e^{-k \int_\xi^t b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)} \tag{3.11}$$

for some real numbers $R_0 < r_0 < r_1 < R_1$*. If* $\sigma^{1/2} r_0 \le r_1$*, then*

$$\begin{aligned} &M(r_1; u) - M(r_0; u) \\ &\quad \ge 2^{-p/(p-1)} (1 - \sigma^{-1/2}) \beta^{-1/2} r_0^{-(a-p+2)/(p-1)} \\ &\qquad \times \left(\alpha \int_{R_0}^{r_0} \xi^{1+a} e^{-k \int_\xi^{r_0} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)}. \end{aligned} \tag{3.12}$$

Proof. At first, let

$$\begin{aligned} &\frac{1}{2} \int_{R_0}^{r_0} \xi^{1+a} e^{-k \int_\xi^{r_0} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \\ &\qquad \le \int_{r_*}^{r_0} \xi^{1+a} e^{-k \int_\xi^{r_0} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi, \end{aligned} \tag{3.13}$$

where $r_* = \max\{R_0, r_0/\sigma^{1/2}\}$. By Corollary 3.1, we obtain

$$\begin{aligned} &M(\sigma^{1/2} r_0; u) - M(r_0 + 0; u) \\ &\quad \ge \gamma_2 (\sigma^{1/2} - 1) r_0 \left(\int_{r_*}^{r_0} e^{-k \int_\xi^{r_0} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)} \\ &\quad \ge \gamma_2 2^{-1/(p-1)} (\sigma^{1/2} - 1) r_0^{-(a-p+2)/(p-1)} \\ &\qquad \times \left(\int_{R_0}^{r_0} \xi^{1+a} e^{-k \int_\xi^{r_0} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)}. \end{aligned}$$

The last relation immediately implies (3.12).

Now, assume that (3.13) is not valid. In this case, we have $r_* = r_0/\sigma^{1/2} > R_0$ and

$$\begin{aligned} &\frac{1}{2} \int_{R_0}^{r_0} \xi^{1+a} e^{-k \int_\xi^{r_0} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \\ &\qquad \le \int_{R_0}^{r_*} \xi^{1+a} e^{-k \int_\xi^{r_0} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi. \end{aligned} \tag{3.14}$$

From (3.11), it can be seen that

$$\begin{aligned} M(r_0; u) &\ge \int_{r_*}^{r_0} \frac{dt}{r_0^{(1+a)/(p-1)}} \left(\alpha \int_{R_0}^{r_*} \xi^{1+a} e^{-k \int_\xi^{r_0} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)} \\ &= (1 - \sigma^{-1/2}) r_0^{-(a-p+2)/(p-1)} \\ &\quad \times \left(\alpha \int_{R_0}^{r_*} \xi^{1+a} e^{-k \int_\xi^{r_0} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)}. \end{aligned}$$

Combining this with formula (3.14) and the inequality $M(r_1;u) - M(r_0;u) \ge (\beta^{-1/2} - 1)M(r_0;u) \ge \beta^{-1/2}M(r_0;u)/2$, we again obtain (3.12). The proof is completed. □

Lemma 3.5. *Let $R_0 < r_0 < r < R_1$, $r \le \sigma^{1/2} r_0$ and, moreover,*

$$M(\zeta;u) \ge \int_{R_0}^{\zeta} dt \left(\frac{\alpha}{t^{1+a}} \int_{R_0}^{t} \xi^{1+a} e^{-k\int_\xi^t b(\zeta)\,d\zeta} f(\xi, \beta M(\xi;u))\,d\xi \right)^{1/(p-1)}$$

for all $\zeta \in (R_0, r_0)$. If $M(r_0;u) \le \beta^{1/2} M(r;u) \le M(r_0+0;u)$, then

$$\begin{aligned} M(r;u) &- M(r_0;u) \\ &\ge \frac{2^{p/(p-1)}(p-1)}{a-p+2}(r_0^{-(a-p+2)/(p-1)} - r^{-(a-p+2)/(p-1)}) \\ &\times \left(\alpha \int_{R_0}^{r_0} \xi^{1+a} e^{-k\int_\xi^{r_0} b(\zeta)\,d\zeta} f(\xi, \beta M(\xi;u))\,d\xi \right)^{1/(p-1)}. \end{aligned} \tag{3.15}$$

Proof. We put $r_1 = \max\{R_0, r_0 - \sigma^{-1/2}(r - r_0)/2\}$. By Corollary 3.1,

$$M(r;u) - M(r_0+0;u) \ge \gamma_2 (r - r_0) \left(\int_{r_1}^{r_0} e^{-k\int_\xi^{r_0} b(\zeta)\,d\zeta} f(\xi, \beta M(\xi;u))\,d\xi \right)^{1/(p-1)}.$$

Combining this with the inequality

$$\frac{p-1}{a-p+2}(r_0^{-(a-p+2)/(p-1)} - r^{-(a-p+2)/(p-1)}) \le r_0^{-(1+a)/(p-1)}(r - r_0), \tag{3.16}$$

we have

$$\begin{aligned} M(r;u) &- M(r_0+0;u) \\ &\ge \frac{4^{p/(p-1)}(p-1)}{a-p+2}(r_0^{-(a-p+2)/(p-1)} - r^{-(a-p+2)/(p-1)}) \\ &\times \left(\alpha \int_{r_1}^{r_0} \xi^{1+a} e^{-k\int_\xi^{r_0} b(\zeta)\,d\zeta} f(\xi, \beta M(\xi;u))\,d\xi \right)^{1/(p-1)}. \end{aligned} \tag{3.17}$$

The proof of Lemma 3.5 is by induction over the positive integer N defined as follows: $N = 1$ if $r_1 = R_0$; otherwise N is the minimal positive integer such that $M(R_0+0;u) \ge \beta^{N/2} M(r_1;u)$.

Consider the case of $N = 1$. If $r_1 - R_0 \le r_0 - r_1$, then $r_0 \le \sigma^{1/2} R_0$. Hence, repeating the arguments given in the proof of (3.17) with r_1 replaced by R_0, we obviously obtain (3.15). Let $r_1 - R_0 > r_0 - r_1$. For $r_1 \le \sigma^{1/2} R_0$, taking into account Corollary 3.1, we have

$$M(r_1;u) - M(R_0+0;u) \ge \gamma_2 (r_1 - R_0) \left(\int_{R_0}^{r_1} e^{-k\int_\xi^{r_1} b(\zeta)\,d\zeta} f(\xi, \beta M(\xi;u))\,d\xi \right)^{1/(p-1)}.$$

The last formula, bound (3.16), and the relation $r_1 - R_0 > r_0 - r_1 = \sigma^{-1/2}(r - r_0)/2$ enable us to assert that

$$\begin{aligned} M(r_1;u) &- M(R_0+0;u) \\ &\ge \frac{\gamma_2(p-1)}{2\sigma^{1/2}(a-p+2)}(r_0^{-(a-p+2)/(p-1)} - r^{-(a-p+2)/(p-1)}) \\ &\times \left(\int_{R_0}^{r_1} \xi^{1+a} e^{-k\int_\xi^{r_1} b(\zeta)\,d\zeta} f(\xi, \beta M(\xi;u))\,d\xi \right)^{1/(p-1)}. \end{aligned}$$

In so doing,

$$M(r;u) - M(r_0;u) \ge (\beta^{-1/2} - 1)M(r_0;u) \ge M(r_1;u); \tag{3.18}$$

therefore, we obtain

$$\begin{aligned} &M(r;u) - M(r_0;u) \\ &\quad \ge \frac{\gamma_2(p-1)}{2\sigma^{1/2}(a-p+2)} (r_0^{-(a-p+2)/(p-1)} - r^{-(a-p+2)/(p-1)}) \\ &\quad\quad \times \left(\int_{R_0}^{r_1} \xi^{1+a} e^{-k\int_\xi^{r_1} b(\zeta)\,d\zeta} f(\xi, \beta M(\xi;u))\,d\xi \right)^{1/(p-1)}, \end{aligned}$$

whence it can be seen that

$$\begin{aligned} &M(r;u) - M(r_0;u) \\ &\quad \ge \frac{4^{p/(p-1)}(p-1)}{a-p+2} (r_0^{-(a-p+2)/(p-1)} - r^{-(a-p+2)/(p-1)}) \\ &\quad\quad \times \left(\alpha \int_{R_0}^{r_1} \xi^{1+a} e^{-k\int_\xi^{r_0} b(\zeta)\,d\zeta} f(\xi, \beta M(\xi;u))\,d\xi \right)^{1/(p-1)}. \end{aligned} \tag{3.19}$$

On the other hand, if of $r_1 > \sigma^{1/2} R_0$, then in accordance with Lemma 3.3 we have

$$\begin{aligned} &M(r_1;u) - M(R_0+0;u) \\ &\quad \ge \gamma_{12} r_1^{-(a-p+2)/(p-1)} \left(\int_{R_0}^{r_1} \xi^{1+a} e^{-k\int_\xi^{r_1} b(\zeta)\,d\zeta} f(\xi, \beta M(\xi;u))\,d\xi \right)^{1/(p-1)} \\ &\quad \ge \gamma_{12} (r_0^{-(a-p+2)/(p-1)} - r^{-(a-p+2)/(p-1)}) \\ &\quad\quad \times \left(\int_{R_0}^{r_1} \xi^{1+a} e^{-k\int_\xi^{r_1} b(\zeta)\,d\zeta} f(\xi, \beta M(\xi;u))\,d\xi \right)^{1/(p-1)}. \end{aligned}$$

By (3.18), this implies the estimate

$$\begin{aligned} &M(r;u) - M(r_0;u) \\ &\quad \ge \gamma_{12} (r_0^{-(a-p+2)/(p-1)} - r^{-(a-p+2)/(p-1)}) \\ &\quad\quad \times \left(\int_{R_0}^{r_1} \xi^{1+a} e^{-k\int_\xi^{r_1} b(\zeta)\,d\zeta} f(\xi, \beta M(\xi;u))\,d\xi \right)^{1/(p-1)}, \end{aligned}$$

whence (3.19) follows again. Finally, summing (3.17) and (3.19), we derive (3.15).

Assume further that Lemma 3.5 is proved for all $N \le N_0$, where N_0 is a positive integer. We shall prove the lemma for $N = N_0 + 1$.

Let us construct the finite sequence of real numbers $R_0 = r_l < \ldots < r_2 < r_1$. The real number r_1 is defined in the beginning of the proof. If r_i is already known, then we put

$$r_{i+1} = \inf\{\xi \in (R_0, r_i) : M(\xi;u) > \beta^{1/2} M(r_i;u)\}. \tag{3.20}$$

In the case of $r_{i+1} = R_0$, we set $l = i+1$ and stop.

From (3.1), it can be seen that $\{\xi \in (R_0, r_i) : M(\xi;u) > \beta^{1/2} M(r_i;u)\} \ne \emptyset$ for all $i = 1, \ldots, l-1$. Thus, the right-hand side of (3.20) is well-defined. Also note that $l \ge 3$ as $N \ge 2$.

By Ξ we mean the set of integers $\nu \in \{2, \dots, l-1\}$ safisfying the conditions $r_{i-1} \le \sigma^{1/2} r_i$, $r_{i-1} - r_i \le 2^{-i+1}(r_0 - r_1)$, and

$$\int_{R_0}^{r_i} \xi^{1+a} e^{-k \int_\xi^{r_0} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \ge \frac{1}{2} \int_{R_0}^{r_{i-1}} \xi^{1+a} e^{-k \int_\xi^{r_0} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi$$

for all $i \in \{2, \dots, \nu\}$. We put $j = \max \Xi$ if the set Ξ is not empty and $j = 1$, otherwise.

As indicated above, to prove the lemma it is sufficient to establish the validity of estimate (3.19). It presents no special problems to verify that at least one of the following propositions is valid:

(1) $\sigma^{1/2} r_{j+1} < r_j$ and

$$\int_{r_{j+1}}^{r_j} \xi^{1+a} e^{-k \int_\xi^{r_0} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \ge \frac{1}{2} \int_{R_0}^{r_j} \xi^{1+a} e^{-k \int_\xi^{r_0} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi; \qquad (3.21)$$

(2) $\sigma^{1/2} r_{j+1} \ge r_j$ and $r_j - r_{j+1} > 2^{-j}(r_0 - r_1)$;

(3) $r_j - r_{j+1} \le 2^{-j}(r_0 - r_1)$ and, moreover, relation (3.21) holds;

(4) $\sigma^{1/2} r_{j+1} < r_j$ and

$$\int_{R_0}^{r_{j+1}} \xi^{1+a} e^{-k \int_\xi^{r_0} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \ge \frac{1}{2} \int_{R_0}^{r_j} \xi^{1+a} e^{-k \int_\xi^{r_0} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi.$$

In case (1), Lemma 3.3 implies the estimate

$$M(r_j; u) - M(r_{j+1} + 0; u) \ge \gamma_{12} r_j^{-(a-p+2)/(p-1)} \left(\int_{r_{j+1}}^{r_j} \xi^{1+a} e^{-k \int_\xi^{r_j} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)}$$

from which, by the inequalities

$$M(r; u) - M(r_0; u) \ge \frac{1}{2} \beta^{-1/2} M(r_0; u) \ge \frac{1}{2} \beta^{-j/2} M(r_j; u) \qquad (3.22)$$

and

$$\int_{r_{j+1}}^{r_j} \xi^{1+a} e^{-k \int_\xi^{r_j} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \ge \int_{r_{j+1}}^{r_j} \xi^{1+a} e^{-k \int_\xi^{r_0} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \ge \frac{1}{2^j} \int_{R_0}^{r_1} \xi^{1+a} e^{-k \int_\xi^{r_0} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi, \qquad (3.23)$$

we obtain

$$M(r; u) - M(r_0; u) \ge \gamma_{12} 2^{-j/(p-1)-1} \beta^{-j/2} \left(r_0^{-(a-p+2)/(p-1)} - r^{-(a-p+2)/(p-1)}\right) \times \left(\int_{R_0}^{r_1} \xi^{1+a} e^{-k \int_\xi^{r_0} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi; u))\, d\xi \right)^{1/(p-1)}.$$

The last formula immediately implies (3.19).

Let proposition (2) be valid. If (3.21) holds, then

$$M(r_j;u)-M(r_{j+1}+0;u) \ge \gamma_2(r_j-r_{j+1}) \left(\int_{r_{j+1}}^{r_j} e^{-k\int_\xi^{r_j} b(\zeta)\,d\zeta} f(\xi,\beta M(\xi;u))\,d\xi \right)^{1/(p-1)}$$

by Corollary 3.1. Therefore, taking into account (3.16) and the fact that $r_j - r_{j+1} > 2^{-j}(r_0 - r_1) = 2^{-j-1}\sigma^{-1/2}(r - r_0)$, we have

$$\begin{aligned} & M(r_j;u) - M(r_{j+1}+0;u) \\ & \quad \ge \frac{\gamma_2(p-1)}{2^{j+1}\sigma^{1/2}(a-p+2)} (r_0^{-(a-p+2)/(p-1)} - r^{-(a-p+2)/(p-1)}) \\ & \quad \times \left(\int_{r_{j+1}}^{r_j} \xi^{1+a} e^{-k\int_\xi^{r_j} b(\zeta)\,d\zeta} f(\xi,\beta M(\xi;u))\,d\xi \right)^{1/(p-1)}. \end{aligned}$$

Combining this with inequalities (3.22) and (3.23), one can show that

$$\begin{aligned} & M(r;u) - M(r_0;u) \\ & \quad \ge \frac{\gamma_2(p-1)}{2^{jp/(p-1)+2}\beta^{j/2}\sigma^{1/2}(a-p+2)} (r_0^{-(a-p+2)/(p-1)} - r^{-(a-p+2)/(p-1)}) \\ & \quad \times \left(\int_{R_0}^{r_1} \xi^{1+a} e^{-k\int_\xi^{r_0} b(\zeta)\,d\zeta} f(\xi,\beta M(\xi;u))\,d\xi \right)^{1/(p-1)}, \end{aligned}$$

whence (3.19) follows. On the other hand, if (3.21) is not fulfilled, then $r_{j+1} > R_0$ and

$$\begin{aligned} & \int_{R_0}^{r_{j+1}} \xi^{1+a} e^{-k\int_\xi^{r_{j+1}} b(\zeta)\,d\zeta} f(\xi,\beta M(\xi;u))\,d\xi \\ & \quad \ge \int_{R_0}^{r_{j+1}} \xi^{1+a} e^{-k\int_\xi^{r_0} b(\zeta)\,d\zeta} f(\xi,\beta M(\xi;u))\,d\xi \\ & \quad \ge \frac{1}{2} \int_{R_0}^{r_j} \xi^{1+a} e^{-k\int_\xi^{r_0} b(\zeta)\,d\zeta} f(\xi,\beta M(\xi;u))\,d\xi \\ & \quad \ge \frac{1}{2^j} \int_{R_0}^{r_1} \xi^{1+a} e^{-k\int_\xi^{r_0} b(\zeta)\,d\zeta} f(\xi,\beta M(\xi;u))\,d\xi. \end{aligned} \tag{3.24}$$

By the induction hypothesis, we have

$$\begin{aligned} & M(r_j;u) - M(r_{j+1};u) \\ & \quad \ge \frac{2^{p/(p-1)}(p-1)}{a-p+2} (r_{j+1}^{-(a-p+2)/(p-1)} - r_j^{-(a-p+2)/(p-1)}) \\ & \quad \times \left(\alpha \int_{R_0}^{r_{j+1}} \xi^{1+a} e^{-k\int_\xi^{r_{j+1}} b(\zeta)\,d\zeta} f(\xi,\beta M(\xi;u))\,d\xi \right)^{1/(p-1)}; \end{aligned}$$

therefore,

$$\begin{aligned} & M(r_j;u) - M(r_{j+1};u) \\ & \quad \ge \frac{2^{p/(p-1)}(p-1)}{2^{j/(p-1)}(a-p+2)} (r_{j+1}^{-(a-p+2)/(p-1)} - r_j^{-(a-p+2)/(p-1)}) \\ & \quad \times \left(\alpha \int_{R_0}^{r_1} \xi^{1+a} e^{-k\int_\xi^{r_0} b(\zeta)\,d\zeta} f(\xi,\beta M(\xi;u))\,d\xi \right)^{1/(p-1)}. \end{aligned}$$

Combining this with (3.22) and the relation

$$r_{j+1}^{-(a-p+2)/(p-1)} - r_j^{-(a-p+2)/(p-1)} \ge \frac{a-p+2}{p-1}(r_j - r_{j+1}) r_j^{-(a+1)/(p-1)}$$
$$\ge \frac{a-p+2}{2^j(p-1)}(r_0 - r_1) r_0^{-(a+1)/(p-1)}$$
$$= \frac{a-p+2}{2^{j+1}\sigma^{1/2}(p-1)}(r - r_0) r_0^{-(a+1)/(p-1)}$$
$$\ge 2^{-j-1}\sigma^{-1/2}(r_0^{-(a-p+2)/(p-1)} - r^{-(a-p+2)/(p-1)}),$$

one can establish the validity of the estimate

$$M(r;u) - M(r_0;u)$$
$$\ge \frac{2^{p/(p-1)}(p-1)}{2^{jp/(p-1)+2}\beta^{j/2}\sigma^{1/2}(a-p+2)}(r_0^{-(a-p+2)/(p-1)} - r^{-(a-p+2)/(p-1)})$$
$$\times \left(\alpha \int_{R_0}^{r_1} \xi^{1+a} e^{-k\int_\xi^{r_0} b(\zeta)\,d\zeta} f(\xi, \beta M(\xi;u))\,d\xi \right)^{1/(p-1)}$$

from which (3.19) follows again.

Now, let proposition (3) be valid. It is obvious that

$$r_1 - r_{j+1} = \sum_{i=1}^{j}(r_i - r_{i+1}) \le \sum_{i=1}^{j} 2^{-i}(r_0 - r_1) \le r_0 - r_1.$$

In particular, $\sigma^{1/2} r_{j+1} \ge r_0$. Consequently, Corollary 3.2 implies the inequality

$$M(r;u) - M(r_0+0;u)$$
$$\ge \gamma_3(r - r_0)\left(\frac{r_0 - r_j}{r_j - r_{j+1}} \int_{r_{j+1}}^{r_j} e^{-k\int_\xi^{r_0} b(\zeta)\,d\zeta} f(\xi, \beta M(\xi;u))\,d\xi \right)^{1/(p-1)},$$

whence in accordance with (3.16), (3.23) and the fact that $r_j - r_{j+1} \le 2^{-j}(r_0 - r_1) \le 2^{-j}(r_0 - r_j)$ we obtain

$$M(r;u) - M(r_0+0;u)$$
$$\ge \frac{\gamma_3(p-1)}{a-p+2}(r_0^{-(a-p+2)/(p-1)} - r^{-(a-p+2)/(p-1)})$$
$$\times \left(\int_{R_0}^{r_1} \xi^{1+a} e^{-k\int_\xi^{r_0} b(\zeta)\,d\zeta} f(\xi, \beta M(\xi;u))\,d\xi \right)^{1/(p-1)}.$$

This obviously implies (3.19).

Finally, let proposition (4) be valid. If $r_{j+1} = R_0$, then the right-hand side of (3.19) is equal to zero; therefore, estimate (3.19) is trivial. Thus, one can assume that $r_{j+1} > R_0$. In this case, we have $M(r_{j+1};u) \le \beta^{1/2} M(r_j;u) \le M(r_{j+1}+0;u)$ and Lemma 3.4 allows us to assert that

$$M(r_j;u) - M(r_{j+1};u)$$
$$\ge 2^{-p/(p-1)}(1 - \sigma^{-1/2})\beta^{-1/2} r_{j+1}^{-(a-p+2)/(p-1)}$$
$$\times \left(\alpha \int_{R_0}^{r_{j+1}} \xi^{1+a} e^{-k\int_\xi^{r_{j+1}} b(\zeta)\,d\zeta} f(\xi, \beta M(\xi;u))\,d\xi \right)^{1/(p-1)}.$$

Combining the last relation with (3.22) and (3.24), we obtain

$$\begin{aligned} & M(r;u) - M(r_0;u) \\ & \quad \geq 2^{-(p+j)/(p-1)-1}(1-\sigma^{-1/2})\beta^{-(j+1)/2} r_{j+1}^{-(a-p+2)/(p-1)} \\ & \quad \times \left(\alpha \int_{R_0}^{r_1} \xi^{1+a} e^{-k \int_\xi^{r_0} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi;u))\, d\xi \right)^{1/(p-1)}, \end{aligned}$$

whence (3.19) follows at once. The proof is completed. □

Proof of Theorem 2.2. The proof is by induction over the minimal positive integer N such that $M(R_0+0;u) \geq \beta^{N/2} M(r;u)$. If $N = 1$, then (2.4) follows from Lemma 3.2. Assume that Theorem 2.2 is already proved for all $N \leq N_0$, where N_0 is some positive integer. Let us prove it for $N = N_0 + 1$. Put

$$r_0 = \inf\{\xi \in (R_0, r) : M(\xi;u) > \beta^{1/2} M(r;u)\}.$$

We have $R_0 < r_0 < r$ and, moreover, $M(r_0;u) \leq \beta^{1/2} M(r;u) \leq M(r_0+0;u)$.

By the induction hypothesis,

$$\begin{aligned} & M(r_0;u) - M(R_0+0;u) \\ & \quad \geq \int_{R_0}^{r_0} dt \left(\frac{\alpha}{t^{1+a}} \int_{R_0}^{t} \xi^{1+a} e^{-k \int_\xi^{t} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi;u))\, d\xi \right)^{1/(p-1)}. \end{aligned} \tag{3.25}$$

At the same time, it can be shown that

$$\begin{aligned} & M(r;u) - M(r_0;u) \\ & \quad \geq \int_{r_0}^{r} dt \left(\frac{\alpha}{t^{1+a}} \int_{R_0}^{t} \xi^{1+a} e^{-k \int_\xi^{t} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi;u))\, d\xi \right)^{1/(p-1)}. \end{aligned} \tag{3.26}$$

Really, the right-hand side of the last expression satisfies the inequality

$$\begin{aligned} & \int_{r_0}^{r} dt \left(\frac{\alpha}{t^{1+a}} \int_{R_0}^{t} \xi^{1+a} e^{-k \int_\xi^{t} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi;u))\, d\xi \right)^{1/(p-1)} \\ & \quad \leq 2^{1/(p-1)} \int_{r_0}^{r} dt \left(\frac{\alpha}{t^{1+a}} \int_{r_0}^{t} \xi^{1+a} e^{-k \int_\xi^{t} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi;u))\, d\xi \right)^{1/(p-1)} \\ & \quad + 2^{1/(p-1)} \int_{r_0}^{r} dt \left(\frac{\alpha}{t^{1+a}} \int_{R_0}^{r_0} \xi^{1+a} e^{-k \int_\xi^{t} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi;u))\, d\xi \right)^{1/(p-1)}. \end{aligned}$$

Thus, formula (3.26) will be proved if we succeed in proving the estimates

$$\begin{aligned} & M(r;u) - M(r_0+0;u) \\ & \quad \geq 2^{p/(p-1)} \int_{r_0}^{r} dt \left(\frac{\alpha}{t^{1+a}} \int_{r_0}^{t} \xi^{1+a} e^{-k \int_\xi^{t} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi;u))\, d\xi \right)^{1/(p-1)} \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} M(r;u) &- M(r_0;u) \\ &\ge \frac{2^{p/(p-1)}(p-1)}{a-p+2}(r_0^{-(a-p+2)/(p-1)} - r^{-(a-p+2)/(p-1)}) \\ &\times \left(\alpha \int_{R_0}^{r_0} \xi^{1+a} e^{-k \int_\xi^{r_0} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi;u))\, d\xi \right)^{1/(p-1)} \\ &\ge 2^{p/(p-1)} \int_{r_0}^{r} dt \left(\frac{\alpha}{t^{1+a}} \int_{R_0}^{r_0} \xi^{1+a} e^{-k \int_\xi^{t} b(\zeta)\, d\zeta} f(\xi, \beta M(\xi;u))\, d\xi \right)^{1/(p-1)}. \end{aligned} \tag{3.28}$$

Estimate (3.27) is a consequence of Lemma 3.2, whereas (3.28) can be obtained by Lemma 3.4 if $\sigma^{1/2} r_0 \le r$ or by Lemma 3.5 if $\sigma^{1/2} r_0 > r$.

To compete the proof it remains to sum inequalities (3.25) and (3.26). □

4. Proof of Lemma 3.1

As in the previous section, we assume that u is a non-negative solution of problem (1.1), (1.2) and, moreover, $M(\cdot;u)$ is a non-decreasing function on the interval (R_0, R_1) satisfying condition (2.1).

Lemma 4.1. *There is a symmetric $n \times n$-matrix $\|a_{ij}\|$ with measurable coefficients such that the function $A = (A_1, \ldots, A_n)$ on the left in (1.1) can be written as follows:*

$$A_i(x,\xi) = \sum_{j=1}^{n} a_{ij}(x,\xi) |\xi|^{p-2} \xi_j, \quad i = 1, \ldots, n,$$

for almost all $x \in \Omega_{R_0,R_1}$ and for all $\xi \in \mathbb{R}^n$. In so doing, we have

$$\lambda_1 |\zeta|^2 \le \sum_{i,j=1}^{n} a_{ij}(x,\xi) \zeta_i \zeta_j \le \lambda_2 |\zeta|^2 \tag{4.1}$$

for almost all $x \in \Omega_{R_0,R_1}$ and for all $\xi, \zeta \in \mathbb{R}^n$, where the constants $\lambda_1 > 0$ and $\lambda_2 > 0$ depend only on C_1 and C_2.

The proof is given in [3, Lemma 4.1].

We put

$$q(x,\xi) = \left(\sum_{i,j=1}^{n} q_{ij}(x) \xi_i \xi_j \right)^{(p-2)/2},$$

where

$$q_{ij}(x) = \begin{cases} a_{ij}(x, Du), & x \in \Omega_{R_0,R_1}, \\ (\lambda_1 + \lambda_2)\delta_{ij}/2, & x \in \mathbb{R}^n \setminus \Omega_{R_0,R_1}. \end{cases}$$

Also let

$$Q_i(x,\xi) = h(x) q(x,\xi) \sum_{j=1}^{n} q_{ij}(x) \xi_j, \quad i = 1, \ldots, n, \tag{4.2}$$

where $h(x) = |Du|^{p-2}/q(x, Du)$ for all $x \in \Omega_{R_0,R_1}$ such that $Du(x) \ne 0$ and $h(x) = (\lambda_1^{(2-p)/2} + \lambda_2^{(2-p)/2})/2$ for all other $x \in \mathbb{R}^n$. In the case of $\xi = 0$, we assume that the right-hand side of (4.2) is equal to zero.

Relation (4.1) implies the inequalities

$$\min\{\lambda_1^{(2-p)/2}, \lambda_2^{(2-p)/2}\} \le h(x) \le \max\{\lambda_1^{(2-p)/2}, \lambda_2^{(2-p)/2}\} \tag{4.3}$$

and

$$\lambda_1|\xi|^2 \le \sum_{i,j=1}^{n} q_{ij}(x)\xi_i\xi_j \le \lambda_2|\xi|^2 \tag{4.4}$$

for almost all $x \in \mathbb{R}^n$ and for all $\xi \in \mathbb{R}^n$.

From Lemma 4.1, it follows that

$$\operatorname{div} Q(x, Du) \ge F(x, u, Du) \quad \text{in } \Omega_{R_0,R_1}, \tag{4.5}$$

where $Q = (Q_1, \ldots, Q_n)$. In addition, we obtain

$$(Q(x, \xi) - Q(x, \zeta))(\xi - \zeta) > 0$$

for almost all $x \in \mathbb{R}^n$ and for all $\xi, \zeta \in \mathbb{R}^n$, $\xi \ne \zeta$.

Let ω_1 and ω_2 be open subsets of $\mathbb{R}^n$ and $v \in W^1_{p,loc}(\omega_1 \cap \omega_2)$. We say that

$$v|_{\omega_2 \cap \partial\omega_1} = 0$$

if $\varphi v \in \overset{\circ}{W}{}^1_p(\omega_1 \cap \omega_2)$ for any $\varphi \in C_0^\infty(\omega_2)$. Analogously,

$$v|_{\omega_2 \cap \partial\omega_1} \le 0$$

if $\varphi \max\{v, 0\} \in \overset{\circ}{W}{}^1_p(\omega_1 \cap \omega_2)$ for any $\varphi \in C_0^\infty(\omega_2)$.

Lemma 4.2. *Suppose that $v \in W^1_p(\omega_1 \cap \omega_2)$ is a solution of the problem*

$$\operatorname{div} Q(x, Dv) \ge g(x) \quad \text{in } \omega_1 \cap \omega_2, \quad v|_{\omega_2 \cap \partial\omega_1} \le 0,$$

where ω_1 and ω_2 are bounded open subsets of $\mathbb{R}^n$ and $g \in L_{p/(p-1)}(\omega_1 \cap \omega_2)$ is some function. We denote: $\omega_0 = \{x \in \omega_1 \cap \omega_2 : v(x) > 0\}$,

$$v_0(x) = \begin{cases} v(x), & x \in \omega_0, \\ 0, & x \in \omega_2 \setminus \omega_0 \end{cases}$$

and

$$g_0(x) = \begin{cases} g(x), & x \in \omega_0, \\ 0, & x \in \omega_2 \setminus \omega_0. \end{cases}$$

Then

$$\operatorname{div} Q(x, Dv_0) \ge g_0(x) \quad \text{in } \omega_2.$$

Lemma 4.3. *For every non-negative function $w \in W^1_p(B^y_r) \cap L_\infty(B^y_r)$, $r > 0$, $y \in \mathbb{R}^n$, there exists a function $\psi \in \overset{\circ}{W}{}^1_p(B^y_r) \cap L_\infty(B^y_r)$ such that $0 \le \psi \le 1$ almost everywhere on B^y_r, $\psi = 1$ almost everywhere on $B^y_{r/2}$ and, moreover,*

$$\operatorname*{ess\,sup}_{B^y_r} w^{p-1} \ge -Cr^{p-n} \int_{B^y_r} Q(x, Dw) D\psi \, dx,$$

where the constant $C > 0$ depends only on n, p, C_1, and C_2.

The proof of Lemmas 4.2 and 4.3 is given in [3, Lemmas 4.2 and 4.4].

Proposition 4.1 (maximum principle)**.** *Suppose that* $v \in W_p^1(\omega) \cap L_\infty(\omega)$*, where* ω *is an open bounded subset of* $\mathbb{R}^n$ *with an infinitely smooth boundary and, moreover,*

$$\operatorname{div} Q(x, Dv) + H(x)|Dv|^{p-1} \geq 0 \tag{4.6}$$

in ω *for some funtion* $H \in L_\infty(\omega)$*. Then*

$$\operatorname{ess\,sup} v|_{\partial\omega} = \operatorname*{ess\,sup}_{\omega} v, \tag{4.7}$$

where the restriction of v *to* $\partial\omega$ *is understood in the sense of the trace and the* ess sup *in the left-hand side of* (4.7) *is with respect to* $(n-1)$*-dimensional Lebesgue measure on* $\partial\omega$*.*

Proposition 4.2. *Let* $v \in W_p^1((0,l)^n)$*,* $l > 0$*. If* $\operatorname{mes}\{x \in (0,l)^n : v(x) = 0\} \geq l^n/2$*, then*

$$\int_{(0,l)^n} |v|^p \, dx \leq C l^p \int_{(0,l)^n} |Dv|^p \, dx,$$

where the constant $C > 0$ *depends only on* n *and* p*.*

Proposition 4.3 (Moser's inequality)**.** *Assume that* $v \in W_p^1(B_r^y) \cap L_\infty(B_r^y)$ *is a non-negative solution of inequality* (4.6) *in the ball* B_r^y*,* $r > 0$*,* $y \in \mathbb{R}^n$*, where* $H \in L_\infty(B_r^y)$ *satisfies the condition*

$$r \operatorname*{ess\,sup}_{B_r^y} |H| \leq 1. \tag{4.8}$$

Then

$$\operatorname*{ess\,sup}_{B_{r/2}^y} v^p \leq C r^{-n} \int_{B_r^y} v^p \, dx,$$

where the constant $C > 0$ *depends only on* n*,* p*,* C_1*, and* C_2*.*

We omit the proof of Propositions 4.1–4.3 as it is pretty standard (see [4], [8]).

Corollary 4.1. *For all* $r \in (R_0, R_1)$

$$M(r; u) = \operatorname*{ess\,sup}_{\Omega_{R_0,r}} u. \tag{4.9}$$

Proof. Without loss of generality it can be assumed that $R_0 > 0$; otherwise we pass in (4.9) to the limit as $R_0 \to +0$. Take some $r \in (R_0, R_1)$. By (1.4) and (4.5), the function u satisfies the inequality

$$\operatorname{div} Q(x, Du) + b(|x|)|Du|^{p-1} \geq 0 \quad \text{in } \Omega_{R_0,r}.$$

Let us put

$$u_0(x) = \begin{cases} u(x), & x \in \omega_0, \\ 0, & x \in B_{R_0,r} \setminus \omega_0, \end{cases}$$

where $\omega_0 = \{x \in \Omega_{R_0,r} : u(x) > 0\}$ and $B_{R_0,r} = \{x \in \mathbb{R}^n : R_0 < |x| < r\}$. Lemma 4.2 obviously implies that

$$\operatorname{div} Q(x, Du_0) + b(|x|)|Du_0|^{p-1} \geq 0 \quad \text{in } B_{R_0,r}.$$

Thus,

$$\operatorname{ess\,sup} u_0|_{\partial B_{R_0,r}} = \operatorname*{ess\,sup}_{B_{R_0,r}} u_0$$

according to Proposition 4.1. To complete the proof it remains to notice that

$$M(r; u) = \operatorname{ess\,sup} u_0|_{\partial B_{R_0,r}}$$

and

$$\operatorname*{ess\,sup}_{B_{R_0,r}} u = \operatorname*{ess\,sup}_{B_{R_0,r}} u_0.$$

□

Lemma 4.4. *Let the hypotheses of Proposition 4.3 be fulfilled, then for any $\varepsilon > 0$ there exists a real number $\delta > 0$ depending only on n, p, C_1, C_2, and ε such that the relation* $\operatorname{mes}\{x \in B_r^y : v(x) > 0\} \le \delta r^n$ *implies the estimate*

$$\operatorname*{ess\,sup}_{B_{r/8}^y} v \le \varepsilon \operatorname*{ess\,sup}_{B_r^y} v.$$

Proof. In the case of $\operatorname{ess\,sup}_{B_r^y} v = 0$, Lemma 4.4 is trivial. Without loss of generality it can be assumed that $\operatorname{ess\,sup}_{B_r^y} v = 1$; otherwise we replace the function v by $v / \operatorname{ess\,sup}_{B_r^y} v$. Also it can be assumed that $r = 1$ and $y = 0$; otherwise we use the change of variables.

Take a non-negative function $\eta \in C_0^\infty(B_1)$ such that $\eta|_{B_{1/2}} = 1$. From (4.6), we have

$$-\int_{B_1} Q(x, Dv) D(\eta^p v)\, dx + \int_{B_1} H(x) |Dv|^{p-1} \eta^p v\, dx \ge 0$$

or, in other words,

$$\int_{B_1} \eta^p Q(x, Dv) Dv\, dx \le \int_{B_1} H(x) |Dv|^{p-1} \eta^p v\, dx - p \int_{B_1} \eta^{p-1} v Q(x, Dv) D\eta\, dx.$$

Using Young's inequality, one can show that

$$\int_{B_1} |H(x)| |Dv|^{p-1} \eta^p v\, dx \le \mu \int_{B_1} \eta^p |Dv|^p\, dx + \mu_* \int_{B_1} \eta^p v^p\, dx$$

and

$$\int_{B_1} \eta^{p-1} v Q(x, Dv) D\eta\, dx \le \mu \int_{B_1} \eta^p |Q(x, Dv)|^{p/(p-1)}\, dx + \mu_* \int_{B_1} |D\eta|^p v^p\, dx$$

for all real numbers $\mu > 0$, where the constant $\mu_* > 0$ depends only on p and μ. On the other hand, in accordance with (4.3) and (4.4) there are constants $\varkappa > 0$ and $\varkappa_* > 0$ depending only on n, p, C_1, and C_2 such that

$$\varkappa \int_{B_1} \eta^p |Dv|^p\, dx \le \int_{B_1} \eta^p Q(x, Dv) Dv\, dx$$

and

$$\int_{B_1} \eta^p |Q(x, Dv)|^{p/(p-1)}\, dx \le \varkappa_* \int_{B_1} \eta^p |Dv|^p\, dx.$$

Hence, choosing sufficiently small $\mu > 0$, we obtain the estimate

$$\int_{B_{1/2}} |Dv|^p\, dx \le \int_{B_1} \eta^p |Dv|^p\, dx \le \tau \int_{B_1} v^p\, dx,$$

where the constant $\tau > 0$ depends only on n, p, C_1, and C_2.

Let $\operatorname{mes}\{x \in B_1 : v(x) > 0\} \le \delta$ for some $\delta > 0$ such that $\delta < 8^{-n} n^{-n/2}$. There exists a finit family of the disjoint open cubs J_i, $i = 1, 2, \ldots, N$, with the edge length equal to $2\delta^{1/n}$ such that $B_{1/4} \subset \bigcup_{i=1}^N \overline{J}_i \subset B_{1/2}$. According to Proposition 4.2, we have

$$\int_{J_i} v^p\, dx \le \zeta \delta^{p/n} \int_{J_i} |Dv|^p\, dx, \quad i = 1, 2, \ldots, N,$$

where the constant $\zeta > 0$ depends only on n and p; therefore,

$$\int_{B_{1/4}} v^p\,dx \le \zeta\delta^{p/n} \int_{B_{1/2}} |Dv|^p\,dx.$$

At the same time, from Moser's inequality, it follows that

$$\operatorname*{ess\,sup}_{B_{1/8}} v^p \le \theta \int_{B_{1/4}} v^p\,dx,$$

where the constant $\theta > 0$ depends only on n, p, C_1, and C_2. Thus, we obtain

$$\operatorname*{ess\,sup}_{B_{1/8}} v^p \le \tau\zeta\theta\delta^{p/n} \int_{B_1} v^p\,dx \le \tau\zeta\theta\delta^{p/n} \operatorname{mes} B_1.$$

To complete the proof it remains to take the real number $\delta > 0$ satisfying the condition $\tau\zeta\theta\delta^{p/n} \operatorname{mes} B_1 \le \varepsilon^p$. □

Lemma 4.5. *Let the hypotheses of Proposition 4.3 be fulfilled, then*

$$\int_{B^y_{r/2}} |H(x)||Dv|^{p-1}\,dx \le Cr^{n-p} \operatorname*{ess\,sup}_{B^y_r} v^{p-1}, \tag{4.10}$$

where the constant $C > 0$ depends only on n, p, C_1, and C_2.

Proof. Applying Lemma 4.3 with $w = v^{p/(p-1)}$, we have

$$\operatorname*{ess\,sup}_{B^y_r} v^p \ge -\mu r^{p-n} \int_{B^y_r} Q(x, Dv) D\psi v\,dx \tag{4.11}$$

for some non-negative function $\psi \in \overset{\circ}{W}{}^1_p(B^y_r) \cap L_\infty(B^y_r)$ such that $0 \le \psi \le 1$ almost everywhere on B^y_r and $\psi = 1$ almost everywhere on $B^y_{r/2}$, where the constant $\mu > 0$ depends only on n, p, C_1, and C_2. Put

$$I = \int_{B^y_r} |H(x)||Dv|^{p-1}\psi\,dx.$$

By the Hölder inequality,

$$\begin{aligned} I &\le \operatorname*{ess\,sup}_{B^y_r} |H| \int_{B^y_r} |Dv|^{p-1}\psi\,dx \\ &\le \operatorname*{ess\,sup}_{B^y_r} |H| \left(\int_{B^y_r} \psi\,dx \right)^{1/p} \left(\int_{B^y_r} |Dv|^p\psi\,dx \right)^{(p-1)/p}; \end{aligned}$$

therefore, taking into account condition (4.8), we obtain

$$I \le r^{n/p-1} (\operatorname{mes} B_1)^{1/p} \left(\int_{B^y_r} |Dv|^p\psi\,dx \right)^{(p-1)/p}. \tag{4.12}$$

At the same time, relations (4.3) and (4.4) imply the estimate

$$\varkappa \int_{B^y_r} |Dv|^p\psi\,dx \le \int_{B^y_r} Q(x, Dv) Dv\psi\,dx, \tag{4.13}$$

where the constant $\varkappa > 0$ depends only on p, C_1, and C_2. Since the function v is a non-negative solution of inequality (4.6) in B^y_r, we have

$$-\int_{B^y_r} Q(x, Dv) D(\psi v)\,dx + \int_{B^y_r} H(x)|Dv|^{p-1}\psi v\,dx \ge 0.$$

Consequently,

$$\begin{aligned}\int_{B_r^y} Q(x,Dv)Dv\psi\,dx &= \int_{B_r^y} Q(x,Dv)D(\psi v)\,dx - \int_{B_r^y} Q(x,Dv)D\psi v\,dx \\ &\le \int_{B_r^y} H(x)|Dv|^{p-1}\psi v\,dx - \int_{B_r^y} Q(x,Dv)D\psi v\,dx.\end{aligned}$$

The last relation and (4.11) allow us to assert that

$$\int_{B_r^y} Q(x,Dv)Dv\psi\,dx \le I \operatorname*{ess\,sup}_{B_r^y} v + \frac{r^{n-p}}{\mu} \operatorname*{ess\,sup}_{B_r^y} v^p.$$

Combining this with (4.12) and (4.13), we obtain

$$I \le \zeta r^{n/p-1} \left(I \operatorname*{ess\,sup}_{B_r^y} v + r^{n-p} \operatorname*{ess\,sup}_{B_r^y} v^p \right)^{(p-1)/p}, \tag{4.14}$$

where the constant $\zeta > 0$ depends only on n, p, C_1, and C_2.

In the case of

$$I \operatorname*{ess\,sup}_{B_r^y} v \ge r^{n-p} \operatorname*{ess\,sup}_{B_r^y} v^p, \tag{4.15}$$

formula (4.14) enables one to establish the validity of the inequality

$$I \le 2^{(p-1)/p} \zeta r^{n/p-1} \left(I \operatorname*{ess\,sup}_{B_r^y} v \right)^{(p-1)/p}$$

or, in other words,

$$I \le 2^{p-1} \zeta^p r^{n-p} \operatorname*{ess\,sup}_{B_r^y} v^{p-1},$$

whence (4.10) immediately follows. On the other hand, if (4.15) does not hold, then in accordance with (4.14) we have

$$I \le 2^{(p-1)/p} \zeta r^{n-p} \operatorname*{ess\,sup}_{B_r^y} v^{p-1}.$$

This also implies (4.10). The lemma is completely proved. □

Proof of Lemma 3.1. Let $s \in [r_1/\sigma, \sigma r_0] \cap (R_0, R_1)$, $r = \min\{1/\lambda, (r_1 - r_0)/4\}$ and, moreover, N be the maximal integer such that $Nr < (r_1 - r_0)/2$. Put $\rho_i = r_0 + ir$, $i = 1, \ldots, N$. For each $i \in \{1, \ldots, N\}$ we take a point $y_i \in S_{\rho_i} \cap \Omega$ satisfying the condition

$$\lim_{\varepsilon \to +0} \operatorname*{ess\,sup}_{B_\varepsilon^{y_i} \cap \Omega} u \ge M(\rho_i; u). \tag{4.16}$$

Assume that

$$\operatorname*{ess\,sup}_{B_r^{y_i} \cap \Omega} u \ge (2 - \beta^{1/2}) M(\rho_i; u) \tag{4.17}$$

for some $i \in \{1, \ldots, N\}$. We denote $\omega_i = \{x \in B_r^{y_i} \cap \Omega : u(x) > \beta^{1/2} M(\rho_i; u)\}$ and

$$v_i(x) = \begin{cases} u(x) - \beta^{1/2} M(\rho_i; u), & x \in \omega_i, \\ 0, & x \in B_r^{y_i} \setminus \omega_i. \end{cases}$$

From Lemma 4.2 and relation (4.5), it follows that

$$\operatorname{div} Q(x, Dv_i) \ge F(x, u, Dv_i) \chi_{\omega_i}(x) \quad \text{in } B_r^{y_i}, \tag{4.18}$$

where χ_{ω_i} is the characteristic function of ω_i. Therefore, in accordance with (1.4) we obtain

$$\operatorname{div} Q(x, Dv_i) + \lambda |Dv_i|^{p-1} \ge f(s, \beta^{1/2} M(\rho_i; u)) \chi_{\omega_i}(x) \quad \text{in } B_r^{y_i}. \tag{4.19}$$

Since $\lambda r \le 1$, Lemma 4.5 implies the inequality

$$\mu r^{n-p} \operatorname*{ess\,sup}_{B_r^{y_i}} v_i^{p-1} \ge \lambda \int_{B_{r/2}^{y_i}} |Dv_i|^{p-1}\, dx. \tag{4.20}$$

where the constant $\mu > 0$ depends only on n, p, C_1, and C_2. At the same time, by Lemma 4.3, there exists a function $\psi_i \in \overset{\circ}{W}{}_p^1(B_{r/2}^{y_i}) \cap L_\infty(B_{r/2}^{y_i})$ such that $0 \le \psi_i \le 1$ almost everywhere on $B_{r/2}^{y_i}$, $\psi_i = 1$ almost everywhere on $B_{r/4}^{y_i}$ and, moreover,

$$\operatorname*{ess\,sup}_{B_{r/2}^{y_i}} v_i^{p-1} \ge -\nu r^{p-n} \int_{B_{r/2}^{y_i}} Q(x, Dv_i) D\psi_i\, dx, \tag{4.21}$$

where the constant $\nu > 0$ depends only on n, p, C_1, and C_2. According to (4.19), we have

$$-\int_{B_{r/2}^{y_i}} Q(x, Dv_i) D\psi_i\, dx + \lambda \int_{B_{r/2}^{y_i}} |Dv_i|^{p-1} \psi_i\, dx \ge f(s, \beta^{1/2} M(\rho_i; u)) \int_{\omega_i \cap B_{r/2}^{y_i}} \psi_i\, dx.$$

Combining the last relation with (4.20) and (4.21), one can show that

$$\operatorname*{ess\,sup}_{B_r^{y_i}} v_i^{p-1} \ge \varkappa r^{p-n} \operatorname{mes}(\omega_i \cap B_{r/4}^{y_i}) f(s, \beta^{1/2} M(\rho_i; u)), \tag{4.22}$$

where the constant $\varkappa > 0$ depends only on n, p, C_1, and C_2. By condition (4.16),

$$\lim_{\varepsilon \to +0} \operatorname*{ess\,sup}_{B_\varepsilon^{y_i}} v_i \ge (1 - \beta^{1/2}) M(\rho_i; u) > 0.$$

Since $M(\rho_i; u) \ge M(r_0; u) \ge \beta^{1/2} M(r_1; u)$ and

$$M(r_1; u) = \operatorname*{ess\,sup}_{\Omega_{R_0, r_1}} u \ge \operatorname*{ess\,sup}_{B_r^{y_i}} v_i,$$

this implies the estimate

$$\lim_{\varepsilon \to +0} \operatorname*{ess\,sup}_{B_\varepsilon^{y_i}} v_i \ge (1 - \beta^{1/2}) \beta^{1/2} \operatorname*{ess\,sup}_{B_r^{y_i}} v_i.$$

Therefore, by Lemma 4.4, there exists a real number $\delta > 0$ depending only on n, p, C_1, C_2, and β such that $\operatorname{mes}(\omega_i \cap B_{r/4}^{y_i}) \ge \delta r^n$. At the same time, taking into account (4.17), we obtain

$$\operatorname*{ess\,sup}_{B_r^{y_i} \cap \Omega} u - M(\rho_i; u) \ge (1 - \beta^{1/2}) M(\rho_i; u) \ge (1 - \beta^{1/2}) \beta^{1/2} \operatorname*{ess\,sup}_{B_r^{y_i}} v_i.$$

Thus, formula (4.22) enable us to assert that

$$\operatorname*{ess\,sup}_{B_r^{y_i} \cap \Omega} u - M(\rho_i; u) \ge \tau r^{p/(p-1)} f^{1/(p-1)}(s, \beta^{1/2} M(\rho_i; u)), \tag{4.23}$$

where the constant $\tau > 0$ depends only on n, p, C_1, C_2, and β. Now, assume that (4.17) is not valid, then there exists a real number $\zeta > 0$ for which

$$\operatorname*{ess\,sup}_{B_r^{y_i} \cap \Omega} u + \zeta < (2 - \beta^{1/2}) M(\rho_i; u). \tag{4.24}$$

We denote $\omega_i = \{x \in B_r^{y_i} \cap \Omega : u(x) > 2M(\rho_i; u) - \operatorname{ess\,sup}_{B_r^{y_i} \cap \Omega} u - \zeta\}$ and

$$v_i(x) = \begin{cases} u(x) - 2M(\rho_i; u) + \operatorname{ess\,sup}_{B_r^{y_i} \cap \Omega} u + \zeta, & x \in \omega_i \\ 0, & x \in B_r^{y_i} \setminus \omega_i. \end{cases}$$

As above, the function v_i satisfies inequality (4.18). From (4.24), it follows that $u(x) > \beta^{1/2} M(\rho_i; u)$ for all $x \in \omega_i$. Hence, in accordance with (1.4) the function v_i also satisfies inequality (4.19). Consequently, repeating the previous arguments, we again obtain (4.22). Further, it presents no special problems to verify that

$$\operatorname*{ess\,sup}_{B_r^{y_i}} v_i = 2 \left(\operatorname*{ess\,sup}_{B_r^{y_i} \cap \Omega} u - M(\rho_i; u) \right) + \zeta.$$

In addition, formula (4.16) implies the estimate

$$\lim_{\varepsilon \to +0} \operatorname*{ess\,sup}_{B_\varepsilon^{y_i}} v_i \ge \operatorname*{ess\,sup}_{B_r^{y_i} \cap \Omega} u - M(\rho_i; u) + \zeta > 0.$$

Therefore, we have

$$\lim_{\varepsilon \to +0} \operatorname*{ess\,sup}_{B_\varepsilon^{y_i}} v_i \ge \frac{1}{2} \operatorname*{ess\,sup}_{B_r^{y_i}} v_i > 0$$

and Lemma 4.4 enables us to assert that $\operatorname{mes}(\omega_i \cap B_{r/4}^{y_i}) \ge \delta r^n$, where the constant $\delta > 0$ depends only on n, p, C_1, and C_2. Thus,

$$2 \left(\operatorname*{ess\,sup}_{B_r^{y_i} \cap \Omega} u - M(\rho_i; u) \right) + \zeta \ge (\varkappa \delta)^{1/(p-1)} r^{p/(p-1)} f^{1/(p-1)}(s, \beta^{1/2} M(\rho_i; u))$$

by inequality (4.22). Finally, passing to the limit in the last expression as $\zeta \to +0$, we derive (4.23) once more.

Since

$$M(r_1; u) - M(r_0; u) \ge \operatorname*{ess\,sup}_{B_r^{y_1} \cap \Omega} u - M(\rho_1; u)$$

and $M(\rho_1; u) \ge M(r_0; u) \ge \beta^{1/2} M(r_1; u)$, relation (4.23) with $i = 1$ proves the lemma in the case of $r = (r_1 - r_0)/4$. For $r = 1/\lambda$, using (4.23), we obtain

$$\sum_{i=1}^{N} \left(\operatorname*{ess\,sup}_{B_r^{y_i} \cap \Omega} u - M(\rho_i; u) \right) \ge \tau N r \lambda^{-1/(p-1)} f^{1/(p-1)}(s, \beta M(r_1; u)),$$

whence in accordance with the inequalities $Nr \ge (r_1 - r_0)/4$,

$$M(\rho_{i+1}; u) \ge \operatorname*{ess\,sup}_{B_r^{y_i} \cap \Omega} u, \quad i = 1, \ldots, N-1,$$

and

$$M(r_1; u) \ge \operatorname*{ess\,sup}_{B_r^{y_N} \cap \Omega} u$$

it follows that

$$M(r_1; u) - M(\rho_1; u) \ge \frac{1}{4} \tau (r_1 - r_0) \lambda^{-1/(p-1)} f^{1/(p-1)}(s, \beta M(r_1; u)).$$

Lemma 3.1 is completely proved. □

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