A NOTE ON SCHRÖDINGER EQUATION WITH LINEAR POTENTIAL AND HITTING TIMES

GERARDO HERNÁNDEZ-DEL-VALLE

ABSTRACT. In this note we derive a solution to the Schrödinger-type backward equation which satisfies a necessary boundary condition used in hitting-time problems [as described in Hernández-del-Valle (2010a)]. We do so by using an idea introduced by Bluman and Shtelen (1996) which is worked out in Hernández-del-Valle (2010b). This example is interesting since it is independent of the parameter λ, namely:

$$\kappa(s,x)=\frac{x}{\sqrt{2\pi s}}\exp\left\{-\frac{(x+\int_0^s f'(u))^2}{2s}\right\}.$$

and suggest a procedure to generating more vanishing solutions and $x=0$.

1. The Example.

In Hernández-del-Valle (2010b) the author finds solutions to a Schrödinger-type backward equation which satisfy some necessary boundary condition used in hitting-time problems [see Hernandez-del-Valle (2010a)]. Namely, the PDE of interest is

$$-\frac{\partial w}{\partial t}(t,x)+xf''(t)w(t,x)=\frac{1}{2}\frac{\partial^2 w}{\partial x^2}(t,x) \tag{1}$$

which alternatively is related to the following expectation:

$$w(t,x)=\tilde{\mathbb{E}}\left[\exp\left\{-\int_t^s f''(u)\tilde{X}_u du\right\}\Big|\tilde{X}_t=x\right]$$

where process $\tilde{X}$ is the so-called three-dimensional Bessel bridge and f corresponds to a moving boundary. Furthermore it was shown in Hernandez-del-Valle (2010a) that w should satisfy the following inequality

$$0\le w(t,x)\le h(s-t,x)\qquad \forall\, 0\le t<s, x\ge 0, \tag{2}$$

where h is the so-called derived heat source solution defined as

$$h(s,x):=\frac{x}{\sqrt{2\pi s^3}}\exp\left\{-\frac{x^2}{2s}\right\}.$$

(Which is also the density of the first hitting time of one-dimensional standard Brownian motion to the fixed boundary x.) Thus, for all time t it follows that w should satisfy the following boundary condition:

$$\lim_{x\to 0} w(t,x) = 0. \tag{3}$$

Yet, for the specific problem of finding the density of hitting a moving boundary we only need the previous inequality (2) and boundary condition (3) to hold at $t = 0$:

$$w(0,x) = \tilde{\mathbb{E}}\left[\exp\left\{-\int_0^s f''(u)\tilde{X}_u du\right\}\Big|\tilde{X}_0 = x\right]. \tag{4}$$

This is accomplished by using an idea introduced in Bluman and Shtelen (1996) which relates the following PDE

$$-\frac{\partial u}{\partial t}(t,x) + V_1(t,x)u(t,x) = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x)$$

and its adjoint

$$\frac{\partial \Phi}{\partial t}(t,x) + V_1(t,x)\Phi(t,x) = \frac{1}{2}\frac{\partial^2 \Phi}{\partial x^2}(t,x) \tag{5}$$

with the following backward equation

$$-\frac{\partial w}{\partial t}(t,x) + V_2(t,x)w(t,x) = \frac{1}{2}\frac{\partial^2 w}{\partial x^2}(t,x)$$

where

$$V_2(t,x) = V_1(t,x) - \frac{\partial^2}{\partial x^2}\log\Phi.$$

It is done so through the following expression:

$$w(t,x) = \frac{1}{\Phi(t,x)}\left[\int_0^x u(t,\xi)\Phi(t,\xi)d\xi + B_2(t)\right] \tag{6}$$

with $B_2(t)$ satisfying the condition

$$\frac{dB_2}{dt} = \frac{1}{2}\left(\frac{\partial \Phi}{\partial x}(t,0)u(t,0) - \Phi(t,0)\frac{\partial u}{\partial x}(t,0)\right). \tag{7}$$

Thus at $t = 0$, see equation (4),

$$w(0,x) = \frac{1}{\Phi(0,x)}\left[\int_0^x u(0,\xi)\Phi(0,\xi)d\xi\right]$$

and hence

$$\lim_{x\to 0} w(0,x) = 0$$

as long as

$$\lim_{x\to 0} \Phi(0,x) \neq 0.$$

Of course, in general we will not be solving the same PDE unless

$$\frac{\partial^2}{\partial x^2}\log\Phi = 0, \tag{8}$$

in which case $V_2(t,x) = V_1(t,x)$, that is the form of the PDE is preserved. The reader may check [or consult at Hernandez-del-Valle (2010a) and (2010b)] that given $V_1(t,x) = xf''(t)$ a solution to (5) which also satisfies condition (8) is for instance:

$$\begin{aligned}\Phi(t,x)\\ = \exp\left\{\frac{1}{2}\int_0^t (f'(u))^2du - xf'(t) - \frac{1}{2}\lambda^2 t - i\lambda\left(x - \int_0^t f'(u)du\right)\right\}\end{aligned}$$

where $i=\sqrt{-1}$ and λ is some scalar.

In the remainder of this note we derive a solution to (1), which is independent of λ, and also satisfies boundary condition (3) at $t=0$. We do so by using Bluman and Shtelen's representation, equations (6) and (7), and a solution to (5) given by:

$$u(t,x) = \exp\left\{\frac{1}{2}\int_t^s (f'(u))^2du + xf'(t)\right\} \tag{9}$$

$$\times\exp\left\{-\frac{1}{2}\lambda^2(s-t) + i\lambda\left(x + \int_t^s f'(u)du\right)\right\}. \tag{10}$$

[The reader may consult Hernández-del-Valle (2007).] It follows that:

$$\Phi(t,x)u(t,x) = \exp\left\{\frac{1}{2}\int_0^s (f'(u))^2du - \frac{1}{2}\lambda^2 s + i\lambda\int_0^s f'(u)du\right\}$$

and

$$\begin{aligned}\frac{dB_2}{dt}(t) &= \frac{1}{2}\Phi u\left(\frac{\Phi_x}{\Phi} - \frac{u_x}{u}\right)\\ &= -(f'(t)+i\lambda)u\Phi.\end{aligned}$$

Hence, B_2 might be written in the two following ways

$$B_2(t) = -\left(\int_0^t f'(u)du + i\lambda t\right)u\Phi \tag{11}$$

or

$$B_2'(t) = \left(\int_t^s f'(u)du + i\lambda(s-t)\right)u\Phi. \tag{12}$$

From (6) w satisfies

$$\begin{aligned} w(t,x) &= \frac{1}{\Phi}\left[\int_0^x u\Phi dy + B_2(t)\right] \\ &= x\frac{u\Phi}{\Phi} + \frac{B_2(t)}{\Phi}, \end{aligned}$$

and hence for B_2 as in (11)

$$w = \left(\left\{x - \int_0^t f'(u)du\right\} - i\lambda t\right) u, \tag{13}$$

and for B_2' as in (12)

$$w = \left(\left\{x + \int_t^s f'(u)du\right\} + i\lambda(s-t)\right) u. \tag{14}$$

Finally, recall the following Fourier representations:

$$\begin{aligned} k(t,x) &:= \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\lambda^2 t + i\lambda x} d\lambda \end{aligned} \tag{15}$$

$$\begin{aligned} h(t,x) &:= \frac{x}{\sqrt{2\pi t^3}} e^{-\frac{x^2}{2t}} \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} (-i\lambda) e^{-\frac{1}{2}\lambda^2 t + i\lambda x} d\lambda \end{aligned} \tag{16}$$

also known as the source and *derived* source heat equations respectively.

After contour integration with respect to λ, (13) becomes:

$$\begin{aligned} w(t,x) &= \exp\left\{\frac{1}{2}\int_t^s (f'(u))^2 du + x f'(t)\right\} \\ &\quad \times \Bigg\{ \left(x - \int_0^t f'(u)du\right) k\left(s-t, x + \int_t^s f'(u)du\right) \\ &\quad + t \cdot \frac{(x + \int_t^s f'(u)du)}{(s-t)} k\left(s-t, x + \int_t^s f'(u)du\right) \Bigg\} \end{aligned}$$

and for B_2' or equation (14)

$$\begin{aligned} w(t,x) &= \exp\left\{\frac{1}{2}\int_t^s (f'(u))^2 du + xf'(t)\right\} \\ &\times\left\{ \left(x+\int_t^s f'(u)du\right) k\left(s-t, x+\int_t^s f'(u)du\right)\right. \\ &\quad \left. - \left(x+\int_t^s f'(u)du\right) k\left(s-t, x+\int_t^s f'(u)du\right)\right\} \\ &\equiv 0. \end{aligned}$$

The second solution is identically zero, and the first evaluated at $t = 0$ is

$$w(0,x) = \exp\left\{\frac{1}{2}\int_0^s (f'(u))^2 du + xf'(0)\right\} x \cdot k\left(s, x+\int_0^s f'(u)du\right)$$

This alternatively implies that an approximation to the first hitting time density is given by:

$$\frac{x}{\sqrt{2\pi s}} \exp\left\{-\frac{(x+\int_0^s f'(u)du)^2}{2s}\right\}.$$

1.1. **More vanishing solutions at $x = 0$ and $t = 0$.** Observe that if u is as in (9) which alternatively solves (5) then

$$u'(t,x) = \Gamma(\lambda, s)u(t,x)$$

is also a solution to (5) for an arbitrary function Γ. For instance, suppose that $\Gamma(\lambda, s) = (-i\lambda)$ then equation (13) becomes:

$$\begin{aligned} w' &= \left(\left\{x - \int_0^t f'(u)du\right\} - i\lambda t\right)(-i\lambda)u \\ &= \left(\left\{x - \int_0^t f'(u)du\right\}(-i\lambda) + (-i\lambda)^2 t\right) u. \end{aligned}$$

After contour integration and observing that

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} (-i\lambda)^2 e^{-\frac{1}{2}\lambda^2 t + i\lambda x} d\lambda = \left(\frac{x^2}{t^2} - \frac{1}{t}\right) k(t,x)$$

it follows that the new solution w' is given by

$$\begin{aligned} &w'(t,x) \\ &= \exp\left\{\frac{1}{2}\int_t^s (f'(u))^2du + xf'(t)\right\} \\ &\times\Bigg\{ \left(x - \int_0^t f'(u)du\right) h\left(s-t, x+\int_t^s f'(u)du\right) \\ &+t\cdot\left[\frac{\left(x+\int_t^s f'(u)du\right)^2}{(s-t)^2} - \frac{1}{(s-t)}\right] k\left(s-t, x+\int_t^s f'(u)du\right)\Bigg\} \end{aligned}$$

or

$$w'(0,x) = \exp\left\{\frac{1}{2}\int_0^s (f'(u))^2du + xf'(0)\right\} x\cdot h\left(s, x+\int_0^s f'(u)du\right)$$

and h is as in (16).

References

[1] Bluman, G. and V. Shtelen (1996). New Classes of Schrödinger equations equivalent to the free particle equation through non-local transformations. *J. Phys. A: Math Gen.* **29** 4473–4480.

[2] Hernández-del-Valle, G. (2007). On Schrödinger's equation, 3-dimensional Bessel Bridges, and Passage Time Problems. *submitted.*

[3] Hernández-del-Valle, G. (2010a). On Schrödinger's equation, 3-dimensional Bessel Bridges, and Passage Time Problems (with heat polynomials). *submitted.*

[4] Hernández-del-Valle, G. (2010b). Schrödinger equation with linear potential and hitting times. *submitted.*