

Remarks on Scattering Properties of the Solution to a Nonlinear Schrödinger Equation with Combined Power-Type Nonlinearities

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Abstract

In this paper, we consider the Cauchy problem of Nonlinear Schrödinger equation

$$\begin{cases} iu_t + \Delta u = \lambda_1 |u|^{p_1} u + \lambda_2 |u|^{p_2} u, & t \in \mathbb{R}, \quad x \in \mathbb{R}^N \\ u(0, x) = \varphi(x), & x \in \mathbb{R}^N, \end{cases}$$

where $N \geq 3$, $0 < p_1 < p_2 < \frac{4}{N-2}$, λ_1 and λ_2 are real constants. Using the methods in [1] and analyzing the interaction between the nonlinearity $\lambda_1 |u|^{p_1} u$ and $\lambda_2 |u|^{p_2} u$, we not only partly solve the open problems of Terence Tao, Monica Visan and Xiaoyi Zhang's [15] but also obtain other scattering properties of the solutions.

Keywords: Nonlinear Schrödinger equation; Global existence; Scattering property.

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1 Introduction

In this paper, we consider the following Cauchy problem

$$\begin{cases} iu_t + \Delta u = \lambda_1 |u|^{p_1} u + \lambda_2 |u|^{p_2} u, & t \in \mathbb{R}, \quad x \in \mathbb{R}^N \\ u(0, x) = \varphi(x), & x \in \mathbb{R}^N, \end{cases} \quad (1.1)$$

where $N \geq 3$, $0 < p_1 < p_2 < \frac{4}{N-2}$, λ_1 and λ_2 are real constants. We expect that the nonlinearities in (1.1) become negligible and $u(t)$ behaves like a solution of linear Schrödinger equation as $t \rightarrow +\infty$ or $t \rightarrow -\infty$. The scattering theory formalizes this

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kind of property. In convenience, we take the same basic notions of scattering theory as those in [1] below.

Let I be an interval containing 0, Duhamel's formula implies that u is a solution of (1.1) on I if and only if u satisfies

$$u(t) = \mathcal{J}(t)\varphi - i \int_0^t \mathcal{J}(t-s)\lambda_1|u(s)|^{p_1}u(s)ds - i \int_0^t \mathcal{J}(t-s)\lambda_2|u(s)|^{p_2}u(s)ds \quad (1.2)$$

for all $t \in I$, where $\mathcal{J}(t) = e^{it\Delta}$ is the one parameter group generated by the free Schrödinger equation. Let X be a Banach space – X can be Σ , $H^1(\mathbb{R}^N)$ or $L^2(\mathbb{R}^N)$ in this paper. Here the pseudoconformal space

$$\Sigma := \{f \in H^1(\mathbb{R}^N); |x|f \in L^2(\mathbb{R}^N)\} \quad \text{with norm } \|f\|_\Sigma = \|f\|_{H_x^1} + \|xf\|_{L_x^2}. \quad (1.3)$$

Assume that the solution $u_\varphi(t, x)$ is defined for all $t \geq 0$ with initial value $\varphi \in X$. We say that u_+ is the scattering state of φ at $+\infty$ if the limit

$$u_+ = \lim_{t \rightarrow +\infty} \mathcal{J}(-t)u_\varphi(t) \quad (1.4)$$

exists in X . Similarly, we say that u_- is the scattering state of φ at $-\infty$ if the limit

$$u_- = \lim_{t \rightarrow -\infty} \mathcal{J}(-t)u_\varphi(t) \quad (1.5)$$

exists in X .

Set

$$\mathcal{R}_+ = \{\varphi \in \Sigma : T_{\max} = +\infty \text{ and } u_+ = \lim_{t \rightarrow +\infty} \mathcal{J}(-t)u_\varphi(t) \text{ exists}\}, \quad (1.6)$$

$$\mathcal{R}_- = \{\varphi \in \Sigma : T_{\min} = +\infty \text{ and } u_- = \lim_{t \rightarrow -\infty} \mathcal{J}(-t)u_\varphi(t) \text{ exists}\}. \quad (1.7)$$

For $\varphi \in \mathcal{R}_\pm$, we define the operators

$$U_\pm(\varphi) = \lim_{t \rightarrow \pm\infty} \mathcal{J}(-t)u_\varphi(t), \quad (1.8)$$

where the limit holds in Σ . Set

$$\mathcal{U}_\pm = U_\pm(\mathcal{R}_\pm). \quad (1.9)$$

If the mappings U_\pm are injective, we can define the wave operators

$$\Omega_\pm = U_\pm^{-1} : \mathcal{U}_\pm \rightarrow \mathcal{R}_\pm. \quad (1.10)$$

And we also introduce the sets

$$\mathcal{O}_\pm = U_\pm(\mathcal{R}_+ \cap \mathcal{R}_-). \quad (1.11)$$

Denote the scattering operator \mathbf{S} by

$$\mathbf{S} = U_+ \Omega_- : \mathcal{O}_- \rightarrow \mathcal{O}_+. \quad (1.12)$$

Since $\mathcal{J}(-t)z = \overline{\mathcal{J}(t)\bar{z}}$, we have

$$\mathcal{J}(-t)u_\varphi(t) = \overline{\mathcal{J}(t)u_\varphi(t)} = \overline{\mathcal{J}(t)u_\varphi(-t)}.$$

Consequently, it is easy to see that

$$\mathcal{R}_- = \overline{\mathcal{R}_+} = \{\varphi \in \Sigma : \bar{\varphi} \in \mathcal{R}_+\}, \quad (1.13)$$

$$\mathcal{U}_- = \overline{\mathcal{U}_+} = \{\psi \in \Sigma : \bar{\psi} \in \mathcal{U}_+\}, \quad (1.14)$$

$$\mathcal{O}_- = \overline{\mathcal{O}_+} = \{\psi \in \Sigma : \bar{\psi} \in \mathcal{O}_+\}, \quad (1.15)$$

$$U_- \varphi = \overline{U_+ \bar{\varphi}}, \quad \Omega_- \varphi = \overline{\Omega_+ \bar{\varphi}} \quad \text{for every } \varphi \in \mathcal{R}_-. \quad (1.16)$$

Now we will give a review of some results about the scattering theory of nonlinear Schrödinger equation. About the topic of scattering theory, there are many results on the Cauchy problem of Schrödinger equation

$$\begin{cases} iu_t + \Delta u = \lambda |u|^p u, & t \in \mathbb{R}, \quad x \in \mathbb{R}^N \\ u(0, x) = \varphi(x), & x \in \mathbb{R}^N. \end{cases} \quad (1.17)$$

Different scattering theories had been constructed in many papers (see [1, 2, 3, 4, 5, 11, 16, 17]). A low energy scattering theory exists in Σ if $\lambda > 0$ and $p > \frac{4}{N+2}$. If $\lambda > 0$ and $\frac{2}{N} < p < \frac{4}{N+2}$, then every solution with initial value $\varphi \in \Sigma$ has a scattering state in $L^2(\mathbb{R}^N)$. However, if $\lambda > 0$ and $p \leq \frac{2}{N}$, then there are no nontrivial solution of (1.17) has scattering states, even for $L^2(\mathbb{R}^N)$ topology. For the case of $\lambda < 0$, there is no low energy scattering if $p < \frac{4}{N+2}$. If $\frac{4}{N+2} < p < \frac{4}{N}$, then a low energy scattering theory exists in Σ . However, if $\lambda < 0$ and $p \geq \frac{4}{N}$, then some solutions will blow up in finite time, some solutions with small initial data in $H^1(\mathbb{R}^N)$ are global and bounded in $H^1(\mathbb{R}^N)$ (see [3, 6, 14, 18] and the references therein).

Recently, in [15], Tao et al. studied the scattering properties of (1.1) with large initial data in the energy space $H^1(\mathbb{R}^N)$ and in Σ . Their results are the following theorems

Theorem 1.3 in [15] (Energy Space Scattering) *Assume that u is the unique solution to (1.1) with $\frac{4}{N} \leq p_1 < p_2 \leq \frac{4}{N-2}$ and initial value $\varphi \in H^1(\mathbb{R}^N)$. Suppose that there exists a unique global solution v to the defocusing L^2 -critical NLS*

$$\begin{cases} iv_t + \Delta v = |v|^{\frac{4}{N}} v, & t \in \mathbb{R}, \quad x \in \mathbb{R}^N \\ v(0, x) = v_0(x) \in H^1(\mathbb{R}^N), & x \in \mathbb{R}^N \end{cases} \quad (1.18)$$

and satisfies that

$$\|v\|_{L_{t,x}^{\frac{2(N+2)}{N}}(\mathbb{R} \times \mathbb{R}^N)} \leq C(\|v_0\|_{L_x^2}).$$

Then there exists unique $u_{\pm} \in H^1(\mathbb{R}^N)$ such that

$$\|u(t) - e^{it\Delta}u_{\pm}\|_{H^1} \rightarrow 0 \quad \text{as } t \rightarrow \pm\infty$$

in each of the following two cases:

(1) $\lambda_1 > 0, \lambda_2 > 0$;

(2) $\lambda_1 < 0, \lambda_2 > 0$ with the small mass condition $M \leq c(\|\nabla\varphi\|_2)$ for some suitably small quantity $c(\|\nabla\varphi\|_2) > 0$ depending only on $\|\nabla\varphi\|_2$.

Theorem 1.8 in [15] (Pseudoconformal Space Scattering) *Assume that $\lambda_1 > 0, \lambda_2 > 0, \alpha(N) < p_1 < p_2 \leq \frac{4}{N-2}$ with $\alpha(N)$ is the Strauss exponent $\alpha(N) := \frac{2-N+\sqrt{N^2+12N+4}}{2N}$, u is the unique global solution of (1.1) with $\varphi \in \Sigma$. Then there exists unique scattering states $u_{\pm} \in \Sigma$ such that*

$$\|e^{-it\Delta}u(t) - u_{\pm}\|_{\Sigma} \rightarrow 0 \quad \text{as } t \rightarrow \pm\infty.$$

However, just like they summarized in Table 1 of [15], there are little results about the scattering theory of (1.1) in the following cases:

Case (i) $\lambda_2 < 0, \lambda_1 \in \mathbb{R}, 0 < p_1 < p_2 < \frac{4}{N}$.

Case (ii) $\lambda_2 > 0, \lambda_1 < 0, 0 < p_1 < \frac{4}{N}$.

Our aim is to give some results on the scattering theory of (1.1) in the two cases above. To do this, we need some observations. First, noticing the nonlinearities in (1.1) and that in (1.17) are power types, it is natural to use the methods in [1] to deal with (1.1). On the other hand, if one of λ_1 and λ_2 is positive and another is negative, then one of the nonlinearities is defocusing and another is focusing, hence we need to analyze the interaction between the nonlinearity $\lambda_1|u|^{p_1}u$ and $\lambda_2|u|^{p_2}u$, which is the complications of this problem. Under some suitable assumptions, we obtain some new scattering properties of the solution of (1.1) and partly solve the open problems of Terence Tao, Monica Visan and Xiaoyi Zhang's [15]. However, we cannot deal with the case $p_2 = \frac{4}{N-2}$ by the technical difficult.

Our main results are the following four theorems. Theorem 1 and Theorem 2 with their proofs are similar to Theorem 7.5.7 and Theorem 7.5.9 in [1].

Theorem 1. *Assume that $\frac{4}{N+2} < p_1 < p_2 < \frac{4}{N-2}$. Then*

(i) *The sets \mathcal{R}_{\pm} and \mathcal{U}_{\pm} are open subsets of Σ with $0 \in \mathcal{R}_{\pm}$ and $0 \in \mathcal{U}_{\pm}$.*

(ii) *The operators $U_{\pm} : \mathcal{R}_{\pm} \rightarrow \mathcal{U}_{\pm}$ and $\Omega_{\pm} : \mathcal{U}_{\pm} \rightarrow \mathcal{R}_{\pm}$ are all bicontinuous bijections for the Σ topology.*

(iii) *The sets \mathcal{O}_{\pm} are open subsets of Σ with $0 \in \mathcal{O}_{\pm}$, and the scattering operator \mathbf{S} is a bicontinuous bijection $\mathcal{O}_{-} \rightarrow \mathcal{O}_{+}$ for the Σ topology.*

As an immediate consequence of Theorem 1, the following corollary gives the scattering property of the solution to (1.1) in the case of (i).

Corollary 1. *Assume that $\lambda_1 \in \mathbb{R}, \lambda_2 < 0$ and $\frac{4}{N+2} < p_1 < p_2 < \frac{4}{N}$. Then a low energy scattering theory exists in Σ .*

We have further results about the wave operators Ω_{\pm} which can be read as

Theorem 2. *Assume that $\lambda_1 \in \mathbb{R}$, $\lambda_2 < 0$, $\frac{4}{N+2} < p_1 < p_2 < \frac{4}{N}$. Then $\mathcal{U}_{\pm} = \Sigma$. Hence the wave operators Ω_{\pm} are bicontinuous bijections $\Sigma \rightarrow \mathcal{R}_{\pm}$.*

Next we will consider the scattering property of the solution to (1.1) in the case of (ii) $\lambda_2 > 0$, $\lambda_1 < 0$ and $0 < p_1 < \frac{4}{N}$.

Theorem 3. (No Scattering Results) *Assume that $u(t, x)$ is the nontrivial solution of (1.1) with initial value $\varphi \in \Sigma$. Then $\mathcal{J}(-t)u(t)$ does not have any strong limit in $L^2(\mathbb{R}^N)$ if (i) $0 < p_1 < p_2 \leq \frac{2}{N}$ or (ii) $0 < p_1 \leq \frac{2}{N} < p_2 < \frac{4}{N-2}$ with $N \geq 6$.*

Remark 1.1. We would like to compare Theorem 3 with Theorem 7.5.4 in [1]. In fact, Theorem 7.5.4 in [1] only gives some results on (1.17) with $\lambda \geq 0$. If we write (1.1) as

$$\begin{cases} iu_t + \Delta u = -\nu|u|^{p_1}u + \mu|u|^{p_2}u, & t \in \mathbb{R}, \quad x \in \mathbb{R}^N \\ u(0, x) = \varphi(x), & x \in \mathbb{R}^N \end{cases} \quad (1.19)$$

with $\nu = |\lambda_1|$ and $\mu = \lambda_2 > 0$. It is a natural way to consider the roles of the nonlinearities $-\nu|u|^{p_1}u$ and $\mu|u|^{p_2}u$. The results of Theorem 3 show that: If $p_1 < \frac{2}{N}$, the role of $-\nu|u|^{p_1}u$ is prominent. Hence we can look the nonlinearity $\lambda_2|u|^{p_2}u$ as a disturbance.

Theorem 4. (Scattering in $L^2(\mathbb{R}^N)$) *Assume that $u(t, x)$ is the nontrivial solution of (1.1) with $\lambda_2 > 0$, $\lambda_1 < 0$ and initial value $\varphi \in \Sigma$. Then there exist $u_{\pm} \in L^2(\mathbb{R}^N)$ such that*

$$\mathcal{J}(-t)u(t) \rightarrow u_{\pm} \quad \text{in } L^2(\mathbb{R}^N) \quad \text{as } t \rightarrow \pm\infty \quad (1.20)$$

if

$$(i) \quad \frac{2}{N} < p_1 < \frac{4}{N} \leq p_2 < \frac{4}{N-2}$$

or

$$(ii) \quad \frac{2}{N} < p_1 < p_2 < \frac{4}{N} \quad \text{and}$$

$$\frac{C_g \lambda_2 N (p_2 - p_1)}{(p_2 + 2)(4 - N p_1)^{\frac{4 - N p_1}{N(p_2 - p_1)}}} \left(\frac{4 - N p_2}{|\lambda_1| \varepsilon_1} \right)^{\frac{4 - N p_2}{N(p_2 - p_1)}} \|\varphi(x)\|_{L^2}^{\frac{4}{N}} < \frac{1}{2} \quad (1.21)$$

with

$$\varepsilon_1 < \min \left\{ \frac{(N p_1 - 2)}{2}, \frac{(p_2 + 1)(N p_1 - 2) + (N p_2 - 2)}{(N + 2)p_2} \right\}. \quad (1.22)$$

Here C_g is the best constant in Gagliardo-Nirenberg's inequality

$$\int_{\mathbb{R}^N} |f|^{\frac{4}{N}+2} dx \leq C_g \left(\int_{\mathbb{R}^N} |\nabla f|^2 dx \right) \left(\int_{\mathbb{R}^N} |f|^2 dx \right)^{\frac{4}{N}}. \quad (1.23)$$

Remark 1.2. We would like to compare Theorem 4 with Remark 7.5.5 (ii) in [1]. In fact, Remark 7.5.5 (ii) in [1] only gives some results on (1.17) with $\lambda < 0$. Similar to Remark 1.1, we can write (1.1) as (1.19), we also need to consider the interaction between $\mu|u|^{p_2}u$ and $-nu|u|^{p_1}u$. Theorem 4 shows that: If $\frac{2}{N} < p_1 < p_2 < \frac{4}{N-2}$, the role of nonlinearity $\lambda_2|u|^{p_2}u$ overwhelm that of $\lambda_1|u|^{p_1}u$. Hence we can look the nonlinearity $\lambda_1|u|^{p_1}u$ as a disturbance.

This paper is organized as follows: In Section 2, we will give some preliminaries. In Section 3, we give two lemmas and prove Theorem 1 and Theorem 2. In Section 4, we will prove Theorem 3 and Theorem 4. In the last of this paper, we will give some discussions on the scattering theory of (1.1).

2 Preliminaries

Similar to Section 7.5 of [1], we will study (1.1) by using pseudoconformal transformation. We also use the conventional notations in [1] below.

For $(s, y) \in \mathbb{R} \times \mathbb{R}^N$, let

$$s = \frac{t}{1-t}, \quad y = \frac{x}{1-t}, \quad \text{or equivalently, } t = \frac{s}{1+s}, \quad x = \frac{y}{1+s}. \quad (2.1)$$

For the function u defined on $(a, b) \times \mathbb{R}^N$ ($0 \leq a < b < +\infty$ are given), set

$$v(t, x) = (1-t)^{-\frac{N}{2}} u\left(\frac{t}{1-t}, \frac{x}{1-t}\right) e^{-i\frac{|x|^2}{4(1-t)}} = (1+s)^{\frac{N}{2}} u(s, y) e^{-i\frac{|x|^2}{4(1+s)}} \quad (2.2)$$

for $x \in \mathbb{R}^N$ and $\frac{a}{1+a} < t < \frac{b}{1+b}$. Obviously, if u is defined on $(0, +\infty)$, then v is defined on $(0, 1)$. And $u \in C([a, b], \Sigma)$ if and only if $v \in C([\frac{a}{1+a}, \frac{b}{1+b}], \Sigma)$. And it is easy to verify the following identities

$$\|\nabla v(t)\|_{L^2}^2 = \frac{1}{4} \|(y + 2i(1+s)\nabla)u(s)\|_{L^2}^2, \quad (2.3)$$

$$\|\nabla u(s)\|_{L^2}^2 = \frac{1}{4} \|(x - 2i(1-t)\nabla)v(t)\|_{L^2}^2, \quad (2.4)$$

$$\|v(t)\|_{L^{p_1+2}}^{p_1+2} = (1+s)^{\frac{Np_1}{2}} \|u(s)\|_{L^{p_1+2}}^{p_1+2}, \quad (2.5)$$

$$\|v(t)\|_{L^{p_2+2}}^{p_2+2} = (1+s)^{\frac{Np_2}{2}} \|u(s)\|_{L^{p_2+2}}^{p_2+2}. \quad (2.6)$$

Consider the Cauchy problem

$$\begin{cases} iv_t + \Delta v &= \lambda_1(1-t)^{\frac{Np_1-4}{2}} |v|^{p_1}v + \lambda_2(1-t)^{\frac{Np_2-4}{2}} |v|^{p_2}v \\ &:= \lambda_1 h_1(t) |v|^{p_1}v + \lambda_2 h_2(t) |v|^{p_2}v, \quad t > 0, \quad x \in \mathbb{R}^N \\ v(0, x) &= \psi(x), \quad x \in \mathbb{R}^N. \end{cases} \quad (2.7)$$

(2.7) equals to the following integral equation

$$\begin{aligned} v(t) &= \mathcal{J}(t)\psi - i \int_0^t \mathcal{J}(t-s)\lambda_1 h_1(s)|v(s)|^{p_1} v(s) ds \\ &\quad - i \int_0^t \mathcal{J}(t-s)\lambda_2 h_2(s)|v(s)|^{p_2} v(s) ds. \end{aligned} \quad (2.8)$$

Set

$$\begin{aligned} E_1(t) &= \frac{1}{2} \|\nabla v(t)\|_2^2 + \frac{\lambda_1(1-t)^{\frac{Np_1-4}{2}}}{p_1+2} \|v(t)\|_{L^{p_1+2}}^{p_1+2} + \frac{\lambda_2(1-t)^{\frac{Np_2-4}{2}}}{p_2+2} \|v(t)\|_{L^{p_2+2}}^{p_2+2}, \\ E_2(t) &= (1-t)^{\frac{4-Np_1}{2}} E_1(t) \\ &= \frac{(1-t)^{\frac{4-Np_1}{2}}}{2} \|\nabla v(t)\|_2^2 + \frac{\lambda_1}{p_1+2} \|v(t)\|_{L^{p_1+2}}^{p_1+2} + \frac{\lambda_2(1-t)^{\frac{N(p_2-p_1)}{2}}}{p_2+2} \|v(t)\|_{L^{p_2+2}}^{p_2+2}, \\ E_3(t) &= \frac{1}{8} \|(x-2i(1-t)\nabla)v(t)\|_2^2 + \frac{\lambda_1(1-t)^{\frac{Np_1}{2}}}{p_1+2} \|v(t)\|_{L^{p_1+2}}^{p_1+2} + \frac{\lambda_2(1-t)^{\frac{Np_2}{2}}}{p_2+2} \|v(t)\|_{L^{p_2+2}}^{p_2+2}. \end{aligned}$$

After some elementary computations, we get

$$\begin{aligned} \frac{d}{dt} E_1(t) &= \frac{\lambda_1(4-Np_1)(1-t)^{\frac{Np_1-6}{2}}}{2(p_1+2)} \|v(t)\|_{L^{p_1+2}}^{p_1+2} \\ &\quad + \frac{\lambda_2(4-Np_2)(1-t)^{\frac{Np_2-6}{2}}}{2(p_2+2)} \|v(t)\|_{L^{p_2+2}}^{p_2+2}, \end{aligned} \quad (2.9)$$

$$\begin{aligned} \frac{d}{dt} E_2(t) &= \frac{Np_1-4}{4} (1-t)^{\frac{2-Np_1}{2}} \|\nabla v(t)\|_2^2 \\ &\quad + \frac{\lambda_2 N(p_1-p_2)}{2(p_2+2)} (1-t)^{\frac{N(p_2-p_1)-2}{2}} \|v(t)\|_{L^{p_2+2}}^{p_2+2}, \end{aligned} \quad (2.10)$$

$$\frac{d}{dt} E_3(t) = 0. \quad (2.11)$$

Our results in this paper are based on the following observation, its proof is similar to that of Proposition 7.5.1 in [1], we omit the details here.

Proposition 2.1. *Assume that $u \in C([0, +\infty), \Sigma)$ is the solution of (1.1) and $v \in C([0, 1), \Sigma)$ is the corresponding solution of (2.7) defined by (2.2). Then $\mathcal{J}(-s)u(s)$ has a strong limit in Σ (respectively, in $L^2(\mathbb{R}^N)$) as $s \rightarrow +\infty$ if and only if $v(t)$ has a strong limit in Σ (respectively, in $L^2(\mathbb{R}^N)$) as $t \rightarrow 1$, and in that case*

$$\lim_{s \rightarrow +\infty} \mathcal{J}(-s)u(s) = e^{i\frac{|x|^2}{4}} \mathcal{J}(-1)v(1) \quad \text{in } \Sigma \text{ (respectively, in } L^2(\mathbb{R}^N)). \quad (2.12)$$

Now we discuss the existence, uniqueness and the continuous dependence of the solution on the initial value of (2.8).

Denote $2^* = \frac{2N}{N-2}$ and let

$$1 - \frac{2}{r_i} = \frac{p_i}{2^*}, \quad \frac{2}{q_i} = N\left(\frac{1}{2} - \frac{1}{r_i}\right), \quad \theta_i = \frac{4}{4 - p_i(N-2)}, \quad i = 1, 2. \quad (2.13)$$

Obviously, if q'_i is the conjugate of $q_i \in [1, +\infty)$ given by $\frac{1}{q_i} + \frac{1}{q'_i} = 1$, then

$$\frac{1}{q'_i} = \frac{1}{\theta_i} + \frac{1}{q_i}. \quad (2.14)$$

The following lemma deals with the existence and uniqueness of the solution of (2.8).

Lemma 2.2. *Assume that $0 < p_1 < p_2 < \frac{4}{N-2}$. Then for every $\psi \in H^1(\mathbb{R}^N)$, there exists a unique, maximal solution of equation (2.8) $v \in C((-T_{\min}, T_{\max}), H^1(\mathbb{R}^N)) \cap W_{loc}^{1,\theta_1}((-T_{\min}, T_{\max}), H^{-1}(\mathbb{R}^N)) \cap W_{loc}^{1,\theta_2}((-T_{\min}, T_{\max}), H^{-1}(\mathbb{R}^N))$ with $T_{\max}, T_{\min} > 0$. Here v is the maximal solution of (2.8) means that if $T_{\max} < +\infty$ (or $T_{\min} < +\infty$), then*

$$\lim_{t \rightarrow T_{\max}} \|v(t, \cdot)\|_{H^1} = +\infty \quad (\text{or} \quad \lim_{t \rightarrow -T_{\min}} \|v(t, \cdot)\|_{H^1} = +\infty).$$

Moreover, v satisfies

(i) If $T_{\max} < +\infty$, then

$$\liminf_{t \rightarrow T_{\max}} \{ \|v(t)\|_{H^1}^{p_1} \|h_1(t)\|_{L^{\theta_1}(t, T_{\max})} + \|v(t)\|_{H^1}^{p_2} \|h_2(t)\|_{L^{\theta_2}(t, T_{\max})} \} > 0.$$

(ii) If $T_{\min} < +\infty$, then

$$\liminf_{t \rightarrow -T_{\min}} \{ \|v(t)\|_{H^1}^{p_1} \|h_1(t)\|_{L^{\theta_1}(-T_{\min}, t)} + \|v(t)\|_{H^1}^{p_2} \|h_2(t)\|_{L^{\theta_2}(-T_{\min}, t)} \} > 0.$$

(iii) $v \in L_{loc}^{q_1}((-T_{\min}, T_{\max}), W^{1,r_1}(\mathbb{R}^N)) \cap L_{loc}^{q_2}((-T_{\min}, T_{\max}), W^{1,r_2}(\mathbb{R}^N))$.

(iv) There exists $\delta > 0$, depending only on N, p_1, p_2, θ_1 and θ_2 , satisfies that: If

$$\|\psi\|_{H^1}^{p_1\theta_1} \int_{-\tau}^{\tau} |h_1(s)|^{\theta_1} ds + \|\psi\|_{H^1}^{p_2\theta_2} \int_{-\tau}^{\tau} |h_2(s)|^{\theta_2} ds \leq \delta,$$

then $[-\tau, \tau] \subset (-T_{\min}, T_{\max})$ and $\|v\|_{L^{q_1}((-\tau, \tau), W^{1,r_1})} + \|v\|_{L^{q_2}((-\tau, \tau), W^{1,r_2})} \leq K\|\psi\|_{H^1}$, where K depends only on $N, p_1, p_2, q_1, q_2, \theta_1$ and θ_2 . Furthermore, if ψ' is another initial value satisfies the above condition and v' is the corresponding solution of (2.8), then $\|v - v'\|_{L^\infty((-\tau, \tau), L^2)} \leq K\|\psi - \psi'\|_{L^2}$.

(v) If $|x|\psi \in L^2(\mathbb{R}^N)$, then $|x|v \in C((-T_{\min}, T_{\max}), L^2(\mathbb{R}^N))$.

Proof: The proof is similar to that of Proposition 4.11.1 in [1]. Roughly, if we replace $h(t)|u|^\alpha u$ by $h_1(t)|u|^{p_1}u + h_2(t)|u|^{p_2}u$, we can obtain the similar results. We omit the details here. \square

The following lemma deals with the continuous dependence of the solution on the initial value.

Lemma 2.3. *Assume that v be the solution of (2.8) given by Lemma 2.2. Then*

(i) The mappings $\psi \rightarrow T_{\max}$ and $\psi \rightarrow T_{\min}$ are lower semicontinuous $H^1(\mathbb{R}^N) \rightarrow (0, +\infty]$.

(ii) Suppose that v_n is the solution of (2.8) with initial value ψ_n satisfying $\psi_n \rightarrow \psi$ as $n \rightarrow \infty$. Then $v_n \rightarrow v$ in $C([-T_1, T_2], H^1(\mathbb{R}^N))$ for any interval $[-T_1, T_2] \subset (-T_{\min}, T_{\max})$. Furthermore, $|x|v_n \rightarrow |x|v$ in $C([-T_1, T_2], L^2(\mathbb{R}^N))$ if $|x|\psi_n \rightarrow |x|\psi$ in $L^2(\mathbb{R}^N)$.

Proof: The proof is similar to that of Proposition 4.11.2 in [1]. Roughly, if we replace $h(t)|u|^\alpha u$ by $h_1(t)|u|^{p_1}u + h_2(t)|u|^{p_2}u$, we can obtain the similar results. We omit the details here. \square

3 Pseudoconformal Space Scattering

By the results of Lemma 2.2 and Lemma 2.3, we can obtain a proposition as follows

Proposition 3.1. Assume that $\frac{4}{N+2} < p_1 < p_2 < \frac{4}{N-2}$. Then for every $t_0 \in \mathbb{R}$ and $\psi \in \Sigma$, there exist $T_m(t_0, \varphi) < t_0 < T_M(t_0, \psi)$ and a unique maximal solution $v \in C((T_m, T_M), \Sigma)$ of equation (2.7). And the solution v satisfies the following properties:

(i) If $T_M = 1$, then

$$\liminf_{t \rightarrow 1} \left\{ \left((1-t)^{\frac{(N+2)p_1-4}{4p_1}} + (1-t)^{\frac{(N+2)p_2-4}{4p_2}} \right) \|v(t)\|_{H^1} \right\} > 0.$$

(ii) v depends continuously on ψ in the sense of the mapping $\psi \rightarrow T_M$ is lower semicontinuous $\Sigma \rightarrow (0, +\infty]$ and the mapping $\psi \rightarrow T_m$ is upper semicontinuous $\Sigma \rightarrow [-\infty, 0)$. Let v_n be the solution of (2.7) with initial value ψ_n . If $\psi_n \rightarrow \psi$ in Σ as $n \rightarrow \infty$ and if $[T_1, T_2] \in (T_m, T_M)$, then $v_n \rightarrow v$ in $C([T_1, T_2], \Sigma)$.

Proof: Set

$$f_1(t) = \begin{cases} \lambda_1(1-t)^{\frac{Np_1-4}{2}}, & \text{if } -\infty < t < 1 \\ \lambda_1, & \text{if } t \geq 1, \end{cases} \quad (3.1)$$

and

$$f_2(t) = \begin{cases} \lambda_2(1-t)^{\frac{Np_2-4}{2}}, & \text{if } -\infty < t < 1 \\ \lambda_2, & \text{if } t \geq 1, \end{cases} \quad (3.2)$$

Applying Lemma 2.2 and Lemma 2.3 with $h_1(t) = f_1(t - t_0)$ and $h_2(t) = f_2(t - t_0)$, we can get the results of Proposition 3.1. \square

We will use Proposition 2.1 and Proposition 3.1 to prove Theorem 1 and Theorem 2.

The proof of Theorem 1: The proof is similar to that of Theorem 7.5.7 in [1], we omit the details here. \square

The proof of Theorem 2: The proof is similar to that of Theorem 7.5.9 in [1], we omit the details here. \square

4 The Proofs of Theorem 3 and Theorem 4

In this section, we are devoted to prove Theorem 3 and Theorem 4.

The proof of Theorem 3: We only give the proof of it for the case of $t \rightarrow +\infty$. The proof of the case of $t \rightarrow -\infty$ is similar. Assume that

$$\mathcal{J}(-t)u(t) \rightarrow u_+ \quad \text{in } L^2(\mathbb{R}^N) \quad \text{as } t \rightarrow +\infty$$

by contradiction. Consequently,

$$\|u_+\|_{L^2} = \|u(t)\|_{L^2} = \|\varphi\|_{L^2} > 0. \quad (4.1)$$

By the results of Proposition 2.1, we have

$$v(t) \rightarrow w \quad \text{in } L^2(\mathbb{R}^N) \quad \text{as } t \rightarrow 1,$$

where

$$w = \mathcal{J}(1)(e^{-i\frac{|x|^2}{4}}u_+) \neq 0.$$

Noticing that $p_1 + 1 < p_2 + 1 \leq 2$ under the assumptions of ours, we have

$$\begin{aligned} |v(t)|^{p_1}v(t) &\rightarrow |w|^{p_1}w \neq 0 \quad \text{in } L^{\frac{2}{p_1+1}}(\mathbb{R}^N), \\ |v(t)|^{p_2}v(t) &\rightarrow |w|^{p_2}w \neq 0 \quad \text{in } L^{\frac{2}{p_2+1}}(\mathbb{R}^N) \end{aligned}$$

as $t \rightarrow 1$. Let $\theta \in \mathcal{D}(\mathbb{R}^N)$ be the function satisfying

$$\langle i|w|^{p_1}w, \theta \rangle = 1. \quad (4.2)$$

Using (2.7), we have

$$\begin{aligned} \frac{d}{dt} \langle v(t), \theta \rangle &= \langle i\Delta v, \theta \rangle + \lambda_1(1-t)^{\frac{Np_1-4}{2}} \langle i|v|^{p_1}v, \theta \rangle + \lambda_2(1-t)^{\frac{Np_2-4}{2}} \langle i|v|^{p_2}v, \theta \rangle \\ &= \langle iv, \Delta\theta \rangle + \lambda_1(1-t)^{\frac{Np_1-4}{2}} \langle i|v|^{p_1}v, \theta \rangle + \lambda_2(1-t)^{\frac{Np_2-4}{2}} \langle i|v|^{p_2}v, \theta \rangle. \end{aligned}$$

Noticing that v is bounded in $L^2(\mathbb{R}^N)$ and (4.2), we can get

$$\begin{aligned} \left| \frac{d}{dt} \langle v(t), \theta \rangle \right| &\geq \frac{1}{2}|\lambda_1|(1-t)^{\frac{Np_1-4}{2}} - C(1-t)^{\frac{Np_2-4}{2}} - C \\ &\geq \frac{1}{4}|\lambda_1|(1-t)^{\frac{Np_1-4}{2}} - C \end{aligned} \quad (4.3)$$

if t is closed to 1 enough. However, (4.3) implies that $|\langle v(t), \theta \rangle| \rightarrow +\infty$ as $t \rightarrow 1$ because $\frac{Np_1-4}{2} \leq -1$, which is absurd. \square

Before the proof of Theorem 4, we will prove a lemma as follows.

Lemma 4.1. *Assume that $v(t, x)$ is the solution of (2.7) with $\lambda_1 < 0$, $\lambda_2 > 0$. Then*

$$\|v(t)\|_{L^2} \leq C. \quad (4.4)$$

Moreover, if $\frac{2}{N} < p_1 < \frac{4}{N} \leq p_2 < \frac{4}{N-2}$, then

$$\|\nabla v(t)\|_{L^2}^2 \leq C(1-t)^{\frac{Np_1-4}{2}}, \quad (4.5)$$

$$\|v(t)\|_{L^{p_1+2}}^{p_1+2} \leq C, \quad (4.6)$$

$$\|v(t)\|_{L^{p_2+2}}^{p_2+2} \leq C(1-t)^{\frac{N(p_1-p_2)}{4}}. \quad (4.7)$$

If $\frac{2}{N} < p_1 < p_2 < \frac{4}{N}$ and (1.21) is true with (1.22), then

$$\|\nabla v(t)\|_{L^2}^2 \leq C(1-t)^{\frac{(Np_1-4)}{2(1-\varepsilon_1)}}, \quad (4.8)$$

$$\|v(t)\|_{L^{p_1+2}}^{p_1+2} \leq C(1-t)^{\frac{\varepsilon_1(Np_1-4)}{2(1-\varepsilon_1)}}, \quad (4.9)$$

$$\|v(t)\|_{L^{p_2+2}}^{p_2+2} \leq C(1-t)^{\frac{N(p_1-p_2)+(Np_2-4)\varepsilon_1}{2(1-\varepsilon_1)}}. \quad (4.10)$$

Proof: Noticing that

$$\frac{d}{dt}\|v(t)\|_{L^2} = 0,$$

we can obtain

$$\|v(t)\|_{L^2} \leq C.$$

Multiplying the first equation of (2.7) by $2\bar{v}_t$, integrating it on $[0, t] \times \mathbb{R}^N$ and taking the real part of the resulting expression, we have

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v(t)|^2 dx + \frac{|\lambda_1|(1-t)^{\frac{Np_1-4}{2}}}{p_1+2} \int_{\mathbb{R}^N} |v(t)|^{p_1+2} dx - \frac{\lambda_2(1-t)^{\frac{Np_2-4}{2}}}{p_2+2} \int_{\mathbb{R}^N} |v(t)|^{p_2+2} dx \\ &= \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \psi|^2 dx + \frac{|\lambda_1|}{p_1+2} \int_{\mathbb{R}^N} |\psi|^{p_1+2} dx - \frac{\lambda_2}{p_2+2} \int_{\mathbb{R}^N} |\psi|^{p_2+2} dx \\ & \quad + \frac{|\lambda_1|(4-Np_1)}{2(p_1+2)} \int_0^t (1-s)^{\frac{Np_1-4}{2}-1} \int_{\mathbb{R}^N} |v(s)|^{p_1+2} dx ds \\ & \quad + \frac{\lambda_2(Np_2-4)}{2(p_2+2)} \int_0^t (1-s)^{\frac{Np_2-4}{2}-1} \int_{\mathbb{R}^N} |v(s)|^{p_2+2} dx ds. \end{aligned} \quad (4.11)$$

If $\|\nabla v(t)\|_{L^2}^2 \leq C$ for all $0 < t < 1$, then (4.5)-(4.10) are true by Gagliardo-Nirenberg's inequality.

Without loss of generality, we only need to prove (4.5)-(4.10) under the condition

$$\lim_{t \rightarrow 1^-} \|\nabla v(t)\|_{L^2}^2 = +\infty. \quad (4.12)$$

That is, there exists a $t_0 \in (0, 1)$ such that

$$\frac{1}{4} \|\nabla v(t)\|_{L^2}^2 \geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \psi|^2 dx + \frac{|\lambda_1|}{p_1+2} \int_{\mathbb{R}^N} |\psi|^{p_1+2} dx + \frac{\lambda_2}{p_2+2} \int_{\mathbb{R}^N} |\psi|^{p_2+2} dx \quad (4.13)$$

for $t_0 \leq t < 1$. We will prove (4.5)-(4.10) in two cases.

Case (i) $\frac{2}{N} < p_1 < \frac{4}{N} \leq p_2 < \frac{4}{N-2}$.

Using the Sobolev's inequality

$$\|v(t)\|_{L^{p_2+2}}^{p_2+2} \leq C\|\nabla v(t)\|_{L^2}^2 + C\|v(t)\|_{L^2}^2,$$

noticing that $Np_2 \geq 4$ and (4.4), from (4.11), we have

$$\begin{aligned} & c \int_{\mathbb{R}^N} |\nabla v(t)|^2 dx + \frac{|\lambda_1|(1-t)^{\frac{Np_1-4}{2}}}{p_1+2} \int_{\mathbb{R}^N} |v(t)|^{p_1+2} dx \\ & \leq \frac{|\lambda_1|(4-Np_1)}{2(p_1+2)} \int_{t_0}^t (1-s)^{\frac{Np_1-4}{2}-1} \int_{\mathbb{R}^N} |v(s)|^{p_1+2} dx ds \\ & \quad + \frac{c(4-Np_1)}{2} \int_{t_0}^t (1-s)^{-1} \int_{\mathbb{R}^N} |\nabla v(s)|^2 dx ds + C(1-t)^{\frac{Np_2-4}{2}}. \end{aligned} \quad (4.14)$$

Letting

$$\eta(t) = \frac{\int_{t_0}^t \frac{|\lambda_1|}{2(p_1+2)} (1-s)^{\frac{Np_1-4}{2}} \int_{\mathbb{R}^N} |v(s)|^{p_1+2} dx ds + c \int_{\mathbb{R}^N} |\nabla v(s)|^2 dx}{(1-s)} ds,$$

(4.14) implies that

$$(1-t)\eta'(t) \leq \frac{4-Np_1}{2}\eta(t) + C(1-t)^{\frac{Np_2-4}{2}}.$$

Applying Gronwall's lemma, we obtain

$$\eta(t) \leq C_1(1-t)^{\frac{(Np_1-4)}{2}} + C_2(1-t)^{\frac{(Np_2-4)+(Np_1-4)}{2}} \leq C(1-t)^{\frac{(Np_1-4)}{2}}.$$

Therefore, we have

$$\|v(t)\|_{L^{p_1+2}}^{p_1+2} \leq C, \quad \|\nabla v(t)\|_{L^2}^2 \leq C(1-t)^{\frac{Np_1-4}{2}}. \quad (4.15)$$

Case (ii) $\frac{2}{N} < p_1 < p_2 < \frac{4}{N}$.

Noticing that $\lambda_2 > 0$, from (4.11), we can get

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v(t)|^2 dx + \frac{|\lambda_1|(1-t)^{\frac{Np_1-4}{2}}}{p_1+2} \int_{\mathbb{R}^N} |v(t)|^{p_1+2} dx \\ & \leq \frac{|\lambda_1|(4-Np_1)}{2(p_1+2)} \int_0^t (1-s)^{\frac{Np_1-4}{2}-1} \int_{\mathbb{R}^N} |v(s)|^{p_1+2} dx ds \\ & \quad + \frac{\lambda_2(1-t)^{\frac{Np_2-4}{2}}}{p_2+2} \int_{\mathbb{R}^N} |v(t)|^{p_2+2} dx + C \\ & \leq \frac{|\lambda_1|(4-Np_1)}{2(p_1+2)} \int_0^t (1-s)^{\frac{Np_1-4}{2}-1} \int_{\mathbb{R}^N} |v(s)|^{p_1+2} dx ds \\ & \quad + \frac{\varepsilon_1|\lambda_1|(1-t)^{\frac{Np_1-4}{2}}}{(p_1+2)} \int_{\mathbb{R}^N} |v(t)|^{p_1+2} dx \\ & \quad + \frac{\lambda_2 N(p_2-p_1)}{(p_2+2)(4-Np_1)^{\frac{4-Np_1}{N(p_2-p_1)}}} \left(\frac{4-Np_1}{|\lambda_1|\varepsilon_1} \right)^{\frac{4-Np_2}{N(p_2-p_1)}} \int_{\mathbb{R}^N} |v(t)|^{\frac{4}{N}+2} dx + C. \end{aligned} \quad (4.16)$$

Using Gagliardo-Nirenberg's inequality, we have

$$\int_{\mathbb{R}^N} |v(t)|^{\frac{4}{N}+2} dx \leq C_g \left(\int_{\mathbb{R}^N} |\nabla v(t)|^2 dx \right) \left(\int_{\mathbb{R}^N} |v(t)|^2 dx \right)^{\frac{4}{N}}. \quad (4.17)$$

Since $\|v(t)\|_{L^2} = \|\psi(x)\|_{L^2} = \|\varphi(x)\|_{L^2}$, if (4.21) is true, then (4.16) and (4.17) imply that

$$\begin{aligned} & \frac{(1-\varepsilon_1)|\lambda_1|(1-t)^{\frac{Np_1-4}{2}}}{(p_1+2)} \int_{\mathbb{R}^N} |v(t)|^{p_1+2} dx \\ & \leq c \int_{\mathbb{R}^N} |\nabla v(t)|^2 dx + \frac{(1-\varepsilon_1)|\lambda_1|(1-t)^{\frac{Np_1-4}{2}}}{(p_1+2)} \int_{\mathbb{R}^N} |v(t)|^{p_1+2} dx \\ & \leq \frac{|\lambda_1|(4-Np_1)}{2(p_1+2)} \int_{t_0}^t (1-s)^{\frac{Np_1-4}{2}-1} \int_{\mathbb{R}^N} |v(s)|^{p_1+2} dx ds. \end{aligned} \quad (4.18)$$

Letting

$$\chi(t) = \frac{|\lambda_1|}{(p_1+2)} \int_{t_0}^t (1-t)^{\frac{Np_1-4}{2}-1} \int_{\mathbb{R}^N} |v(t)|^{p_1+2} dx,$$

from (4.18), we can obtain

$$(1-t)\chi'(t) \leq \frac{(4-Np_1)}{2(1-\varepsilon_1)} \chi(t). \quad (4.19)$$

Applying Gronwall's lemma, from (4.19), we have

$$\chi(t) \leq C(1-t)^{\frac{(Np_1-4)}{2(1-\varepsilon_1)}}.$$

Consequently,

$$\int_{\mathbb{R}^N} |v(t)|^{p_1+2} dx \leq C(1-t)^{\frac{\varepsilon_1(Np_1-4)}{2(1-\varepsilon_1)}}. \quad (4.20)$$

From (4.18) and (4.20), we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla v(t)|^2 dx & \leq \frac{|\lambda_1|(4-Np_1)}{2(p_1+2)} \int_{t_0}^t (1-s)^{\frac{Np_1-4}{2}-1} \int_{\mathbb{R}^N} |v(s)|^{p_1+2} dx ds \\ & \leq C(1-t)^{\frac{Np_1-4}{2(1-\varepsilon_1)}}. \end{aligned} \quad (4.21)$$

(4.20) and (4.21) mean that (4.8) and (4.9) are true if $\frac{2}{N} < p_1 < p_2 < \frac{4}{N}$ under the conditions of the lemma.

If $\lambda_2 > 0$, we have

$$\begin{aligned} \frac{d}{dt} E_2(t) & = \frac{\lambda_2 N(p_1-p_2)}{2(p_2+2)} (1-t)^{\frac{N(p_2-p_1)-2}{2}} \|v(t)\|_{L^{p_2+2}}^{p_2+2} \\ & \quad + \frac{Np_1-4}{4} (1-t)^{\frac{2-Np_1}{2}} \|\nabla v(t)\|_{L^2}^2 \leq 0. \end{aligned} \quad (4.22)$$

(4.22) implies that

$$E_2(0) \geq E(t) = \frac{(1-t)^{\frac{4-Np_1}{2}}}{2} \|\nabla v(t)\|_2^2 + \frac{\lambda_1}{p_1+2} \|v(t)\|_{L^{p_1+2}}^{p_1+2} + \frac{\lambda_2(1-t)^{\frac{N(p_2-p_1)}{2}}}{p_2+2} \|v(t)\|_{L^{p_2+2}}^{p_2+2}. \quad (4.23)$$

If $\frac{2}{N} < p_1 < \frac{4}{N} \leq p_2 < \frac{4}{N-2}$, then (4.4)–(4.5) and (4.23) mean that

$$\|v(t)\|_{L^{p_2+2}}^{p_2+2} \leq C(1-t)^{\frac{N(p_1-p_2)}{2}}$$

And (4.8)–(4.9) and (4.23) mean that

$$\|v(t)\|_{L^{p_2+2}}^{p_2+2} \leq C(1-t)^{\frac{N(p_1-p_2)+\varepsilon_1(Np_2-4)}{2(1-\varepsilon_1)}}$$

if $\frac{2}{N} < p_1 < p_2 < \frac{4}{N}$. \square

The proof of Theorem 4. By the results of Proposition 2.1, we only need to prove that there exists a $w \in L^2(\mathbb{R}^N)$ satisfying

$$v(t) \rightarrow w \quad \text{in } L^2(\mathbb{R}^N) \quad \text{as } t \rightarrow 1.$$

By the embedding

$$L^{\frac{p_1+2}{p_1+1}}(\mathbb{R}^N) \hookrightarrow H^{-1}(\mathbb{R}^N) \hookrightarrow H^{-2}(\mathbb{R}^N), \quad L^{\frac{p_2+2}{p_2+1}}(\mathbb{R}^N) \hookrightarrow H^{-1}(\mathbb{R}^N) \hookrightarrow H^{-2}(\mathbb{R}^N)$$

and equation (1.1), we can get

$$\begin{aligned} \|v_t\|_{H^{-2}} &\leq \|\Delta v\|_{H^{-2}} + C(1-t)^{\frac{Np_1-4}{2}} \| |v|^{p_1} v \|_{H^{-2}} + C(1-t)^{\frac{Np_2-4}{2}} \| |v|^{p_2} v \|_{H^{-2}} \\ &\leq C\|v\|_{L^2} + C(1-t)^{\frac{Np_1-4}{2}} \|v\|_{L^{p_1+2}}^{p_1+1} + C(1-t)^{\frac{Np_2-4}{2}} \|v\|_{L^{p_2+2}}^{p_2+1}. \end{aligned}$$

From (4.4)–(4.10) and the inequality above, we obtain

$$\|v_t\|_{H^{-2}} \leq C + C(1-t)^{\frac{Np_1-4}{2}} + C(1-t)^{\frac{(Np_2-4)+(Np_1-4)(p_2+1)}{2(p_2+2)}} \quad (4.24)$$

if $\frac{2}{N} < p_1 < \frac{4}{N} \leq p_2 < \frac{4}{N-2}$ and

$$\|v_t\|_{H^{-2}} \leq C + C(1-t)^{\frac{(Np_1-4)(p_1+2-\varepsilon_1)}{2(1-\varepsilon_1)(p_1+2)}} + C(1-t)^{\frac{[N(p_1-p_2)(p_2+1)+(p_2+2-\varepsilon_1)(Np_2-4)]}{2(1-\varepsilon_1)(p_2+2)}} \quad (4.25)$$

if $\frac{2}{N} < p_1 \leq p_2 < \frac{4}{N}$. From (4.24), we can see that

$$v_t \in L^1((0, 1), H^{-2}(\mathbb{R}^N))$$

if $\frac{2}{N} < p_1 < \frac{4}{N} \leq p_2 < \frac{4}{N-2}$. Choosing

$$\varepsilon_1 < \min\left\{\frac{(Np_1-2)}{2}, \frac{(p_2+1)(Np_1-2) + (Np_2-2)}{(N+2)p_2}\right\},$$

from (4.25), we can verify that

$$v_t \in L^1((0, 1), H^{-2}(\mathbb{R}^N))$$

if $\frac{2}{N} < p_1 \leq p_2 < \frac{4}{N}$. Therefore, there exists $w \in H^{-2}(\mathbb{R}^N)$ satisfying $v(t) \rightarrow w$ in $H^{-2}(\mathbb{R}^N)$ as $t \rightarrow 1$. From (4.4), we know that $w \in L^2(\mathbb{R}^N)$ and

$$v(t) \rightharpoonup w \quad \text{in } L^2(\mathbb{R}^N) \text{ as } t \rightarrow 1. \quad (4.26)$$

For any $\psi \in H^1(\mathbb{R}^N)$ and $0 \leq t \leq \tau < 1$, we can get

$$\begin{aligned} (v(\tau) - v(t), \psi)_{L^2} &= \int_t^\tau \langle v_t, \psi \rangle_{H^{-1}, H^1} ds \\ &= \int_t^\tau (i\nabla v, \nabla \psi)_{L^2} ds + \int_t^\tau (1-s)^{\frac{Np_1-4}{2}} \langle i\lambda_1 |v|^{p_1} v, \psi \rangle_{L^{\frac{p_1+2}{p_1+1}}, L^{p_1+2}} ds \\ &\quad + \int_t^\tau (1-s)^{\frac{Np_2-4}{2}} \langle i\lambda_1 |v|^{p_2} v, \psi \rangle_{L^{\frac{p_2+2}{p_2+1}}, L^{p_2+2}} ds. \end{aligned}$$

Consequently,

$$\begin{aligned} |(v(\tau) - v(t), \psi)_{L^2}| &= C \|\nabla \psi\|_{L^2} \int_t^\tau \|\nabla v\|_{L^2} ds + C \|\psi\|_{L^{p_1+2}} \int_t^\tau (1-s)^{\frac{Np_1-4}{2}} \|v\|_{L^{p_1+2}}^{p_1+1} ds \\ &\quad + C \|\psi\|_{L^{p_2+2}} \int_t^\tau (1-s)^{\frac{Np_2-4}{2}} \|v\|_{L^{p_2+2}}^{p_2+1} ds \\ &\leq C \|\nabla \psi\|_{L^2} \int_t^\tau (1-s)^{\frac{Np_1-4}{4(1-\varepsilon_1)}} ds + C \|\psi\|_{L^{p_1+2}} \int_t^\tau (1-s)^{\frac{(Np_1-4)(p_1+2-\varepsilon_1)}{2(1-\varepsilon_1)(p_1+2)}} ds \\ &\quad + C \|\psi\|_{L^{p_2+2}} \int_t^\tau (1-s)^{\frac{[N(p_1-p_2)(p_2+1)+(p_2+2-\varepsilon_1)(Np_2-4)]}{2(1-\varepsilon_1)(p_2+2)}} ds. \end{aligned}$$

Letting $\tau \rightarrow 1$ and using (4.26), we have

$$\begin{aligned} |(w - v(t), \psi)_{L^2}| &\leq C \|\nabla \psi\|_{L^2} \int_t^1 (1-s)^{\frac{Np_1-4}{4(1-\varepsilon_1)}} ds + C \|\psi\|_{L^{p_1+2}} \int_t^1 (1-s)^{\frac{(Np_1-4)(p_1+2-\varepsilon_1)}{2(1-\varepsilon_1)(p_1+2)}} ds \\ &\quad + C \|\psi\|_{L^{p_2+2}} \int_t^1 (1-s)^{\frac{[N(p_1-p_2)(p_2+1)+(p_2+2-\varepsilon_1)(Np_2-4)]}{2(1-\varepsilon_1)(p_2+2)}} ds. \end{aligned}$$

Especially, if $\psi = v(t)$, we can get

$$\begin{aligned} |(w - v(t), v(t))_{L^2}| &\leq C \|\nabla v(t)\|_{L^2} \int_t^1 (1-s)^{\frac{Np_1-4}{4(1-\varepsilon_1)}} ds + C \|v(t)\|_{L^{p_1+2}} \int_t^1 (1-s)^{\frac{(Np_1-4)(p_1+2-\varepsilon_1)}{2(1-\varepsilon_1)(p_1+2)}} ds \\ &\quad + C \|v(t)\|_{L^{p_2+2}} \int_t^1 (1-s)^{\frac{[N(p_1-p_2)(p_2+1)+(p_2+2-\varepsilon_1)(Np_2-4)]}{2(1-\varepsilon_1)(p_2+2)}} ds. \end{aligned}$$

Noticing (4.4)–(4.7), we obtain

$$\begin{aligned} &|(w - v(t), v(t))_{L^2}| \\ &\leq C(1-t)^{\frac{(Np_1-2)-2\varepsilon_1}{2(1-\varepsilon_1)}} + C(1-t)^{\frac{(Np_1-2)-2\varepsilon_1}{2(1-\varepsilon_1)}} + C(1-t)^{\frac{(Np_1-2)-2\varepsilon_1}{2(1-\varepsilon_1)}} \\ &\leq C(1-t)^\theta \leq C. \end{aligned} \quad (4.27)$$

Here

$$\theta = \frac{(Np_1 - 2) - 2\varepsilon_1}{2(1 - \varepsilon_1)} \geq 0.$$

Now using (4.26) and (4.27), we have

$$\|v(t) - w\|_{L^2}^2 = -(w - v(t), v(t))_{L^2} + (w - v(t), w)_{L^2} \rightarrow 0$$

as $t \rightarrow 1$. □

5 Discussions

The proofs of Theorem 1 and Theorem 2 are still true for the case of $\lambda_1 > 0$, $\lambda_2 > 0$ and $\frac{4}{N+2} < p_1 < p_2 < \frac{4}{N-2}$. In fact, we have

Theorem 5. *Assume that $\lambda_1 > 0$, $\lambda_2 > 0$ and $\frac{4}{N+2} < p_1 < p_2 < \frac{4}{N-2}$. Then a low energy scattering theory exists in Σ .*

Since there is little result for the case of $\lambda_1 > 0$, $\lambda_2 > 0$, $\frac{4}{N+2} < p_1 < \alpha(N)$ in [15], Theorem 5 extends the ranges of p_1 and p_2 to $\frac{4}{N+2} < p_1 < p_2 < \frac{4}{N-2}$.

By (4.22) and (4.23), the conclusions of Lemma 4.1 are also true in the case of $\lambda_1 > 0$, $\lambda_2 > 0$ and $\frac{2}{N} < p_1 < p_2 < \frac{4}{N-2}$. Similar to the proof of Theorem 4, we have

Theorem 6. *Assume that $\lambda_1 > 0$, $\lambda_2 > 0$ and $\frac{2}{N} < p_1 < p_2 < \frac{4}{N-2}$. Then a low energy scattering theory exists in $L^2(\mathbb{R}^N)$.*

Since there is also little result for the case of $\lambda_1 > 0$, $\lambda_2 > 0$, $\frac{2}{N} < p_1 < \frac{4}{N+2}$ in [15], Theorem 6 establishes the scattering theory of (1.1) with $\lambda_1 > 0$, $\lambda_2 > 0$, $\frac{2}{N} < p_1 < \frac{4}{N+2}$ and $p_1 < p_2 < \frac{4}{N-2}$. Since the nonlinearities in Theorem 5 and Theorem 6 are all defocusing, hence their roles are positive to each other.

We suspect that there exist initial data $u_0(x)$ of arbitrary small Σ -norm such that the solution u of (1.1) doesn't possess a scattering state in Σ (or even in $L^2(\mathbb{R}^2)$) if $p_1 < \frac{4}{N+2}$, $\lambda_1 \in \mathbb{R}$ and $\lambda_2 < 0$. However, we cannot obtain such $u_0(x)$ in this paper. The difficulty is the failure of the equation in (1.1) to be scale invariant. We also suspect that the nontrivial solution of (1.1) doesn't possess any scattering state in $L^2(\mathbb{R}^N)$ if $0 < p_1 \leq \frac{2}{N} < p_2 < \frac{4}{N-2}$ when $N = 3, 4, 5$.

The methods in this paper and those of [15] can be used to deal with the following Cauchy problem

$$\begin{cases} iu_t + \Delta u = \sum_{i=1}^m \lambda_i |u|^{p_i} u, & x \in \mathbb{R}^N, \quad t \in \mathbb{R} \\ u(0, x) = \varphi(x), & x \in \mathbb{R}^N, \end{cases} \quad (5.1)$$

where $N \geq 3$, $0 < p_1 < p_2 < \dots < p_m < \frac{4}{N-2}$, λ_i , $i = 1, 2, \dots, m$ are real constants. In many cases, whether the solution of (5.1) possess a scattering state or not are essentially depended on the nonlinearities $\lambda_1 |u|^{p_1} u$ and $\lambda_m |u|^{p_m} u$, because $\lambda_i |u|^{p_i} u$, $i = 2, \dots, (m-1)$ can be controlled by $\lambda_1 |u|^{p_1} u$ and $\lambda_m |u|^{p_m} u$ if one use Young's inequality.

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