Simple closed curves, word length, and nilpotent quotients of free groups

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Abstract

We consider the fundamental group π of a surface of finite type equipped with the infinite generating set consisting of all simple closed curves. We show that every nilpotent quotient of π has finite diameter with respect to the word metric given by this set. This is in contrast with a result of Danny Calegari that shows that π has infinite diameter with respect to this set. Furthermore, we give a general criterion for a finitely generated group equipped with a generating set to have this property.

1 Introduction

A surface of finite type is a surface whose fundamental group is finitely generated. Given such a surface there is no canonical choice of generating set. If one wishes to define a suitably canonical generating set of a geometric nature then it becomes necessary to consider infinite generating sets. One such set is the set of all elements whose conjugacy class can be represented by a simple closed curve. These are in some sense the simplest elements of the fundamental group, and are thus a natural choice for a generating set.

Benson Farb posed the question whether the fundamental group, endowed with the word metric given by this set, has finite diameter. This question was answered negatively by Danny Calegari [1]. In this paper our goal is to investigate the same question for some quotients of the fundamental group. In contrast with Calegari's result, we find the following.

Theorem 1.1. *Let Σ be a surface of finite type, $\pi = \pi_1(\Sigma)$, and $\mathcal{S} \subset \pi$ be any generating set containing at least one element in each conjugacy class that is represented by a nonseparating simple closed curve. Let $\rho : \pi \to N$ be a homomorphism into any nilpotent group. Then $\rho(\pi)$ has finite diameter in the word metric with respect to the set $\rho(\mathcal{S})$.*

Note that in surfaces of genus > 1, π has many nilpotent quotients of every degree of nilpotency. Furthermore, it is residually nilpotent, that is for every $x \in \pi$ there is some nilpotent quotient $q : \pi \to N$ such that $q(x) \neq 1$.

We say that a group G is *nilpotent-bounded with respect to the set S* if any nilpotent quotient of G has finite diameter with respect to the word metric given by the image of S. As part of the proof we prove the following more general result.

Theorem 1.2. *Let G be a finitely generated group, and let $S \subset G$ be a generating set such that $G/[G,G]$ has finite diameter with respect to the word metric given by S. Then G is nilpotent-bounded with respect to S.*

Using Theorem 1.2, it is possible to find smaller generating sets for which π is nilpotent bounded. We give one such set here, but it is relatively simple to find many of them. In order to do so, we need a simple corollary.

Corollary 1.3. *Let G be a finitely generate group. Let $H = H_1(G,\mathbb{Z}) \cong G/[G,G]$. Suppose that $H \cong H_1 \oplus \ldots \oplus H_k$, and that for each $i = 1,\ldots,k$ we are given a set $S_i \subset \Sigma$ whose projection to H is contained in H_i and generates H_i with finite diameter. Then G is nilpotent bounded with respect to $S_1 \cup \ldots \cup S_k$.*

An example of an application of Corollary 1.3 is the following. Let Σ be an orientable of genus $g > 1$. It is common to choose a generating set for $\pi = \pi_1(\Sigma)$ of the form $S' = \{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ where all of the above are represented by simple closed curves, the geometric intersection number of α_i and β_i is one and they can be realized disjointly from all the other curves. Let $\Gamma_i =< \alpha_i, \beta_i >$. The group Γ_i is the fundamental group of an embedded torus. Let $H = H_1(\Sigma)$, and H_i be the projection to H of Γ_i. Then $H = H_1 \oplus \ldots \oplus H_g$. Thus, if we let $\mathcal{S}$ be the any set containing at least one representative in each conjugacy class of a simple closed curves that lies in one of the g embedded tori described above, then π is nilpotent bounded with respect to $\mathcal{S}$.

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2 Nilpotent Groups and Lower Central Series.

Given a group Γ, we define a decreasing sequence of subgroups of Γ called *the lower central series of* Γ by the following rule:

$$\Gamma_0 = \Gamma,\ \Gamma_{n+1} = [\Gamma, \Gamma_n].$$

A group is nilpotent if $\Gamma_n = \langle 1 \rangle$ for some n. A group is called n-step nilpotent if $L_n = 1$, and $L_{n-1} \neq 1$. For every n, the group $L_n := \Gamma/\Gamma_n$ is a nilpotent group. These groups have the property that any nilpotent quotient of G factors through one of the projections $\Gamma \to L_n$.

Let $A_n := \Gamma_n/\Gamma_{n+1}$. It is a standard fact that $A_n = Z(L_n)$, the center of L_n. Furthermore, if Γ is finitely generated then A_n is also finitely generated. Given a generating set S of Γ, the group A_n is generated by the images of elements of the form $[a_1, \dots, a_n]$ where $a_1, \dots, a_n \in S$ and $[a_1, \dots, a_n]$ denotes a generalized commutator, i.e:

$$[a_1, \dots, a_n] = [\dots [a_1, a_2], a_3], \dots, a_n]$$

In the course of the proof, we require the following technical lemma about generalized commutators in nilpotent groups.

Lemma 2.1. *Let Γ be any group, $n, k \in \mathbb{N}$, and $a_1, \dots, a_n \in \Gamma$. Then:*

$$[a_1, \dots, a_n]^k \equiv_{n+1} ([a_1^k, \dots, a_n])$$

where $\equiv_i$ is understood as having equal images in L_i

Proof. First, recall that $A_n = Z(L_{n+1})$. Let $x \in \Gamma_{n-1}$ and $y \in \Gamma$. Note that $[x,y] \in \Gamma_n$. Thus we have that:

$$[x^k, y] \equiv_{n+1} x^k y x^{-k} y^{-1} \equiv_{n+1} x^k y [x,y]^k y^{-1} x^{-k} \equiv_{n+1} [x,y]^k.$$

The last equality stems from the fact that $[x,y]^k$ is central in L_{n+1} and thus is invariant under conjugation. Note that this proves the claim for the case $n = 1$. We now proceed by induction.
By the case $n = 1$ we have that:

$$[a_1, \dots, a_n]^k \equiv_{n+1} [[a_1, \dots, a_{n-1}], a_n]^k \equiv_{n+1} [[a_1, \dots, a_{n-1}]^k, a_n].$$

By induction, we can write:

$$[a_1, \dots, a_{n-1}]^k \equiv_{n+1} [[a_1, \dots, a_{n-2}]^k, a_{n-1}] \gamma_n,$$

where $\gamma_n \in \Gamma_n$. Since the image of Γ_n is central in L_{n+1} we have that :

$$[[a_1, \dots, a_{n-1}]^k \gamma_n^{-1}, a_n] \equiv_{n+1} [a_1, \dots, a_{n-1}]^k, a_n].$$

Proceeding similarly we get the claim of the lemma. □

3 Proof of the Main Theorems

Lemma 3.1. *Let $n \in \mathbb{N}$ and let $e_1, \ldots, e_{2n}$ be the standard basis for $\mathbb{Z}^{2n}$. Then the set $\mathcal{S} = Sp_{2n}(\mathbb{Z}) \cdot e_1$ generates $\mathbb{Z}^{2n}$ with finite diameter.*

Proof. We prove this fact first for $n = 1$. In this case $Sp_{2n}(\mathbb{Z}) = SL_2(\mathbb{Z})$. Given a vector $v = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \in \mathbb{Z}^2$ such that $\gcd(a,b) = 1$, there exist $x, y \in \mathbb{Z}$ such that $ax + by = 1$. In this case

$$A = \begin{pmatrix} a & -y \\ b & x \end{pmatrix} \in SL_2(\mathbb{Z})$$

and $A \cdot e_1 = v$ and thus $v \in \mathcal{S}$. For a general vector $v = \begin{pmatrix} a \\ b \end{pmatrix}$ notice that

$$v = \begin{pmatrix} a-1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ b-1 \end{pmatrix}$$

and that $\gcd(1, a-1) = \gcd(1, b-1) = 1$, and thus $v \in \mathcal{S} + \mathcal{S}$.

Now consider the case $n > 1$. In this case we have that $D < Sp_{2n}(\mathbb{Z})$ where $D \cong \prod_{i=1}^{n} SL_2(\mathbb{Z})$ is the group of matrices containing n copies of $SL_2(\mathbb{Z})$ along the diagonal, and zeroes in all other entries. Notice further that $\widehat{e} = e_1 + e_3 + \ldots + e_{2n-1} \in \mathcal{S}$. Given $\left(\begin{smallmatrix} a_i \\ b_i \end{smallmatrix}\right)_{i=1}^{n} \in \mathbb{Z}^{2n}$, by the $n = 1$ case there are $2n$ matrices $A_1, \ldots, A_n, B_1, \ldots B_n \in SL_2(\mathbb{Z})$ such that:

$$A_i \cdot e_1 = \begin{pmatrix} a_i - 1 \\ 1 \end{pmatrix}, \; B_i \cdot e_1 = \begin{pmatrix} 1 \\ b_i - 1 \end{pmatrix}.$$

Let $A = \mathrm{diag}(A_1, \ldots, A_n)$, $B = diag(B_1, \ldots, B_n)$ and . Then

$$v = A \cdot \widehat{e} + B \cdot \widehat{e}.$$

Thus $\mathbb{Z}^{2n}$ is generated by $\mathcal{S}$ with finite diameter. □

Lemma 3.2. *Let Γ be a finitely generated group, and let $n \in \mathbb{N}$. Suppose that $\mathcal{S} \subset \Gamma$ generates Γ and generates L_n with finite diameter. Then $\mathcal{S}$ generates L_{n+1} with finite diameter.*

Proof. By assumption, there exists a N_0 such that for any $w \in \Gamma$ there exist $s_1, \ldots s_m \in \mathcal{S}$ (with $m < N_0$) such that

$$(s_1 \ldots s_m)^{-1} w \in \Gamma_n.$$

Thus, it is enough to show that the image of $\mathcal{S}$ in L_{n+1} generates A_n with finite diameter. The group A_n is a finitely generated abelian group which is generated by elements of the form $[s_1, \ldots, s_n]$ where $s_1, \ldots s_n \in \mathcal{S}$. Choose p such generators: $\gamma_1, \ldots, \gamma_p$. Consider $\gamma_1 =$

$[s_1,\ldots,s_n]$. Given any $k\in\mathbb{N}$, by Lemma 2.1, we have that $\gamma_1^k \equiv_{n+1} [s_1^k,\ldots,s_n]$. Further, note that there exist elements $\sigma_1,\ldots,\sigma_m\in\mathcal{S}$ with $m<N_0$ and an element $\gamma\in\Gamma_n$ such that

$$s_1^k=\sigma_1\ldots\sigma_m\gamma$$

The elements $\sigma_1,\ldots,\sigma_m,\gamma$ depend on γ_1 and k, but their number does not. Now, we have:

$$\gamma_1^k\equiv_{n+1}[\sigma_1\ldots\sigma_m\gamma,\ldots,s_n]\equiv_{n+1}[\sigma_1\ldots\sigma_m,\ldots,s_n],$$

where the last equality stems from the centrality of Γ_n. The last expression is a word in the elements of $\mathcal{S}$, whose length is bounded from above by a number that does not depend on k. This fact is true not just for γ_1, but for $\gamma_2,\ldots,\gamma_p$. Since the group A_n is abelian, and every element in it can be written as a product of powers of $\gamma_1,\ldots,\gamma_p$, we get that A_n is generated by $\mathcal{S}$ with finite diameter, as required. □

Proof of Theorem 1.2. Theorem 1.2 is a direct consequence of Lemma 3.2 and induction. □

Proof of Theorem 1.1. Let $H=H_1(S,\mathbb{Z})$. There exists a simple closed curve in π that is mapped to e_1 under this mapping. The mapping class group acts on H, and it is well known that this action induces a surjective homomorphism onto $Sp_{2g}(\mathbb{Z})$ ([3]). Furthermore, the non-separating simple closed curves form a single mapping class group orbit. Thus, by Lemma 3.1 and Theorem 1.2, π is nilpotent-bounded with respect to $\mathcal{S}$. □

Proof of Corollary 1.3 This is a direct result of Theorem 1.2, and the fact that any element of $x\in H$ can be written as $x=h_1+\ldots+h_k$ with $h_i\in H_i$.

4 Further Questions.

The contrast between the result in this paper and Calegari's result that π has infinite diameter with respect to $\mathcal{S}$ gives rise to several questions.

Question 1. Recall that $L_n=\pi/\pi_n$. By Theorem 1.1, L_n has finite diameter with respect to $\mathcal{S}$. Call this diameter d_n. The sequence $\{d_n\}_{n=1}^{\infty}$ is nondecreasing. Is this sequence bounded? If so, by what value. If not, what is its asymptotic growth rate?

Question 2. The lower central series is but one of the important series of nested subgroups of π. Another such series is the derived series, whose elements are quotiens of surjections onto solvable groups. This sequence is defined by:

$$\Gamma^{(0)}=\Gamma,\Gamma^{(n+1)}=[\Gamma^{(n)},\Gamma^{(n)}]$$

Is the conclusion of Theorem 1.1 if we replace the word nilpotent with the word solvable?

References

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