# $\mathbb{T}^2$-COBORDISM OF QUASITORIC 4-MANIFOLDS

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ABSTRACT. We show the $\mathbb{T}^2$-cobordism group of the category of 4-dimensional quasitoric manifolds is generated by the $\mathbb{T}^2$-cobordism class of $\mathbb{C}P^2$. The main tool is the theory of quasitoric manifolds.

## 1. Introduction

Cobordism was explicitly introduced by Lev Pontryagin in geometric work on manifolds. In the early 1950's René Thom [Tho] showed that cobordism groups could be computed by results of homotopy theory. Thom showed that the cobordism class of $G$-manifolds for a Lie group $G$ are in one to one correspondence with the elements of the homotopy group of the Thom space of the group $G \subseteq O(n)$. We consider the following category: the objects are all quasitoric manifolds and morphisms are torus equivariant maps between quasitoric manifolds. We compute the $\mathbb{T}^2$-cobordism group of 4-dimensional manifolds in this category. We show the $\mathbb{T}^2$-cobordism group of the category of 4-dimensional quasitoric manifolds is generated by the $\mathbb{T}^2$-cobordism class of $\mathbb{C}P^2$. The main tool is the theory of quasitoric manifolds.

Quasitoric manifolds and small covers were introduced by Davis and Januskiewicz in [DJ]. A manifold with quasitoric (small cover) boundary is a manifold with boundary where the boundary is a disjoint union of some quasitoric manifolds (respectively small covers).

We give the brief definition of some manifolds with quasitoric and small cover boundary in a constructive way in section 3. There is a natural torus action on these manifolds with quasitoric boundary having a simple convex polytope as the orbit space. The fixed point set of the torus action on the manifold with quasitoric boundary corresponds to the disjoint union of closed intervals of positive length. Interestingly, we show that such a manifold with quasitoric boundary could be viewed as the quotient space of a quasitoric manifold corresponding to a certain circle action on it. This is done in the subsection 3.3.

In section 4 we show these manifolds with quasitoric boundary are orientable and compute their Euler characteristic.

In the subsection 5.2 we show the $\mathbb{T}^2$-cobordism group of 4-dimensional quasitoric manifolds is generated by the $\mathbb{T}^2$-cobordism class of the complex projective space $\mathbb{C}P^2$, see theorem 5.4. Following [OR] we discuss the classification of 4-dimensional quasitoric manifolds in subsection 5.1. This classification is needed to prove the theorem 5.4.

## 2. Edge-Simple Polytopes

An $n$-dimensional simple convex polytope is a convex polytope where exactly $n$ bounding hyperplanes meet at each vertex. The codimension one faces of a convex polytope are called facets. We introduce a particular type of polytope, which we call an edge-simple polytope.

**Definition 2.1.** *An $n$-dimensional convex polytope $P$ is called an $n$-dimensional edge-simple polytope if each edge of $P$ is the intersection of exactly $(n-1)$ facets of $P$.*

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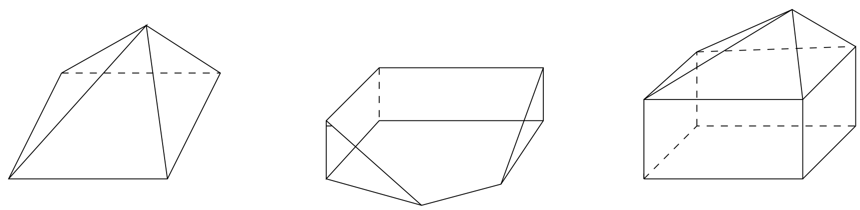
**Example 2.1.** (1) *An $n$-dimensional simple convex polytope is an $n$-dimensional edge-simple polytope.*

(2) *The following convex polytopes are edge-simple polytopes of dimension* 3.



(3) *The dual polytope of a* 3*-dimensional simple convex polytope is a* 3*-dimensional edge-simple polytope. This result is not true for higher dimensional polytopes, that is if $P$ is a simple convex polytope of dimension $n \geq 4$ the dual polytope of $P$ may not be an edge-simple polytope. For example the dual of the* 4*-dimensional standard cube in $\mathbb{R}^4$ is not an edge-simple polytope.*

**Proposition 2.2.** ***(a)*** *If $P$ is a* 2*-dimensional simple convex polytope then the suspension $SP$ on $P$ is an edge-simple polytope and $SP$ is not a simple convex polytope.*

***(b)*** *If $P$ is an $n$-dimensional simple convex polytope then the cone $CP$ on $P$ is an $(n+1)$-dimensional edge-simple polytope.*

*Proof.* ***(a)*** Let $P$ be a 2-dimensional simple polytope with $m$ vertices $\{v_i \colon i \in I = \{1, 2, \ldots, m\}\}$ and $m$ edges $\{e_i \colon i \in I\}$. Let $a$ and $b$ be the other two vertices of $SP$. Then facets of $SP$ are the cone $(Ce_i)_x$ on $e_i$ at $x = a, b$. Edges of $SP$ are $\{xv_i \colon x = a, b$ and $i \in I\} \cup \{e_i : i \in I\}$. The edge $xv_i$ is the intersection of $(Ce_{i_1})_x$ and $(Ce_{i_2})_x$ if $v_i = e_{i_1} \cap e_{i_2}$ for $x = a, b$ and $e_i = (Ce_i)_a \cap (Ce_i)_b$. Hence $SP$ is an edge-simple polytope. If $v$ is a vertex of the polytope $P$, $v$ is the intersection of 4 facets of $SP$. So $SP$ is not a simple convex polytope.

***(b)*** Let $P$ be an $n$-dimensional simple convex polytope in $\mathbb{R}^n \times 0 \subseteq \mathbb{R}^{n+1}$ with $m$ facets $\{F_i \colon i \in I = \{1, 2, \ldots, m\}\}$ and $k$ vertices $\{v_1, v_2, \ldots, v_k\}$. Assume that the cone are taken at a fixed point $a$ in $\mathbb{R}^{n+1} - \mathbb{R}^n$ lying above the centroid of $P$. Then facets of $CP$ are $\{(CF_i) \colon i = 1, 2, \ldots, m\} \cup \{P\}$. Edges of $CP$ are $\{av_i = C(\{v_i\}) \colon i = 1, 2, \ldots, k\} \cup \{e_l : e_l$ is an edge of $P\}$. Since $P$ is a simple convex polytope, each vertex $v_i$ of $P$ is the intersection of exactly $n$ facets of $P$, namely $\{v_i\} = \cap_{j=1}^{n} F_{i_j}$ and each edge $e_l$ is the intersection of unique collection of $(n-1)$ facets $\{F_{l_1}, \ldots, F_{l_{n-1}}\}$. Then $C\{v_i\} = \cap_{j=1}^{n} CF_{i_j}$ and $e_l = P \cap CF_{l_1} \cap CF_{l_2} \cap \ldots \cap CF_{l_{n-1}}$. That is $C\{v_i\}$ and $\{e_l\}$ are the intersection of exactly $n$ facets of $CP$. Hence $CP$ is an $(n+1)$-dimensional edge-simple polytope. □

Cut off a neighborhood of each vertex $v_i, i = 1, 2, \ldots, k$ of an $n$-dimensional edge-simple polytope $P \subset \mathbb{R}^n$ by an affine hyperplane $H_i, i = 1, 2, \ldots, k$ in $\mathbb{R}^n$ such that $H_i \cap H_j \cap P$ are empty sets for $i \neq j$. Then the remaining subset of the convex polytope $P$ is a simple convex polytope of dimension $n$, denote it by $Q_P$. Suppose $P_{H_i} = P \cap H_i = H_i \cap Q_P$ for $i = 1, 2, \ldots k$. Then $P_{H_i}$ is a facet of $Q_P$ called the facet corresponding to the vertex $v_i$ for each $i = 1, \ldots, k$. Since each vertex of $P_{H_i}$ is an interior point of an edge of $P$ and $P$ is an edge-simple polytope, $P_{H_i}$ is an $(n-1)$-dimensional simple convex polytope for each $i = 1, 2, \ldots, k$.

**Lemma 2.3.** *Let $F$ be a codimension $l < n$ face of $P$. Then $F$ is the intersection of unique set of $l$ facets of $P$.*

*Proof.* The intersection $F \cap Q_P$ is a codimension $l$ face of $Q_P$ not contained in $\cup_{i=0}^{k} \{P_{H_i}\}$. Since $Q_P$ is a simple convex polytope, $F \cap Q_P = \cap_{j=1}^{l} F'_{i_j}$ for some facets $\{F'_{i_1}, \ldots, F'_{i_1}\}$ of

$Q_P$. Let $F_{i_j}$ be the unique facet of $P$ such that $F'_{i_j} \subseteq F_{i_j}$. Then $F = \cap_1^l F_{i_j}$. Hence each face of $P$ of codimension $l < n$ is the intersection of unique set of $l$ facets of $P$. □

**Remark 2.4.** *If $v_i$ is the intersection of facets $\{F_{i_1}, \ldots, F_{i_l}\}$ of $P$ for some positive integer $l$, the facets of $P_{H_i}$ are $\{P_{H_i} \cap F_{i_1}, \ldots, P_{H_i} \cap F_{i_l}\}$.*

## 3. Construction of Manifolds with Boundary

Let $P$ be an edge-simple polytope of dimension $n$ with $m$ facets $F_1, \ldots, F_m$ and $k$ vertices $v_1, \ldots, v_k$. Let $e$ be an edge of $P$. Then $e$ is the intersection of unique collection of $(n-1)$ facets $\{F_{i_j} : j = 1, \ldots, (n-1)\}$. Suppose $\mathcal{F}(P) = \{F_1, \ldots, F_m\}$.

**Definition 3.1.** *The functions $\lambda\colon \mathcal{F}(P) \to \mathbb{Z}^{n-1}$ and $\lambda^s\colon \mathcal{F}(P) \to \mathbb{F}_2^{n-1}$ are called the istropy function and $\mathbb{F}_2$-istropy function respectively of the edge-simple polytope $P$ if the set of vectors $\{\lambda(F_{i_1}), \ldots, \lambda(F_{i_{n-1}})\}$ and $\{\lambda^s(F_{i_1}), \ldots, \lambda^s(F_{i_{n-1}})\}$ form a basis of $\mathbb{Z}^{n-1}$ and $\mathbb{F}_2^{n-1}$ respectively whenever the intersection of the facets $\{F_{i_1}, \ldots, F_{i_{n-1}}\}$ is an edge of $P$.*

*The vectors $\lambda_i := \lambda(F_i)$ and $\lambda_i^s := \lambda^s(F_i)$ are called istropy vectors and $\mathbb{F}_2$-istropy vectors respectively.*

We define some istropy functions of the edge-simple polytopes $I^3$ and $P_0$ in examples 3.3 and 3.4 respectively.

**Remark 3.1.** *It may not possible to define an istropy function on the set of facets of all edge-simple polytopes. For example there does not exist an istropy function of the standard $n$-simplex $\triangle^n$ for each $n \geq 3$.*

3.1. **Manifolds with Quasitoric Boundary.** Let $F$ be a face of $P$ of codimension $l < n$. Then $F$ is the intersection of a unique collection of $l$ facets $F_{i_1}, F_{i_2}, \ldots, F_{i_l}$ of $P$. Let $\mathbb{T}_F$ be the torus subgroup of $\mathbb{T}^{n-1}$ corresponding to the submodule generated by $\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_l}$ in $\mathbb{Z}^{n-1}$. Assume $\mathbb{T}_v = \mathbb{T}^n$ for each vertex $v$ of $P$. We define an equivalence relation $\sim$ on the product $\mathbb{T}^{n-1} \times P$ as follows.

$$(t, p) \sim (u, q) \text{ if and only if } p = q \text{ and } tu^{-1} \in \mathbb{T}_F \tag{3.1}$$

where $F$ is the unique face of $P$ containing $p$ in its relative interior. We denote the quotient space $(\mathbb{T}^{n-1} \times P)/\sim$ by $X(P, \lambda)$. The space $X(P, \lambda)$ is not a manifold except when $P$ is a 2-dimensional polytope. If $P$ is 2-dimensional polytope the space $X(P, \lambda)$ is homeomorphic to the 3-dimensional sphere.

But whenever $n > 2$ we can construct a manifold with boundary from the space $X(P, \lambda)$. We restrict the equivalence relation $\sim$ on the product $(\mathbb{T}^{n-1} \times Q_P)$ where $Q_P \subset P$ is a simple polytope as constructed in section 2 corresponding to the edge-simple polytope $P$. Let $W(Q_P, \lambda) = (\mathbb{T}^{n-1} \times Q_P)/\sim \subset X(P, \lambda)$ be the quotient space. The natural action of $\mathbb{T}^{n-1}$ on $W(Q_P, \lambda)$ is induced by the group operation in $\mathbb{T}^{n-1}$.

**Theorem 3.2.** *The space $W(Q_P, \lambda)$ is a manifold with boundary. The boundary is a disjoint union of quasitoric manifolds.*

For each edge $e$ of $P$, $e' = e \cap Q_P$ is an edge of the simple convex polytope $Q_P$. Let $U_{e'}$ be the open subset of $Q_P$ obtained by deleting all facets of $Q_P$ that does not contain $e'$ as an edge. Then the set $U_{e'}$ is diffeomorphic to $I^0 \times \mathbb{R}_{\geqslant 0}^{n-1}$ where $I^0$ is the open interval $(0, 1)$ in $\mathbb{R}$. The facets of $I^0 \times \mathbb{R}_{\geqslant 0}^{n-1}$ are $I^0 \times \{x_1 = 0\}, \ldots, I^0 \times \{x_{n-1} = 0\}$ where $\{x_j = 0,\ j = 1, 2, \ldots, n-1\}$ are the coordinate hyperplanes in $\mathbb{R}^{n-1}$. Let $F'_{i_1}, \ldots, F'_{i_{n-1}}$ be the facets of $Q_P$ such that $\cap_{j=1}^{n-1} F'_{i_j} = e'$. Suppose the diffeomorphism $\phi\colon U_{e'} \to I^0 \times \mathbb{R}_{\geqslant 0}^{n-1}$ sends $F'_{i_j} \cap U_{e'}$ to $I^0 \times \{x_j = 0\}$ for all $j = 1, 2, \ldots, n-1$. Define an isotropy function $\lambda_e$

on the set of all facets of $I^0 \times \mathbb{R}^{n-1}_{\geqslant 0}$ by $\lambda_e(I^0 \times \{x_j = 0\}) = \lambda_{i_j}$ for all $j = 1, 2, \ldots, n-1$. We define an equivalence relation $\sim_e$ on $(\mathbb{T}^{n-1} \times I^0 \times \mathbb{R}^{n-1}_{\geqslant 0})$ as follows.

$$(t, b, x) \sim_e (u, c, y) \text{ if and only if } (b, x) = (c, y) \text{ and } tu^{-1} \in \mathbb{T}_{\phi(F)}. \tag{3.2}$$

where $\phi(F)$ is the unique face of $I^0 \times \mathbb{R}^{n-1}_{\geqslant 0}$ containing $(b, x)$ in its relative interior, for a unique face $F$ of $U_{e'}$ and $\mathbb{T}_{\phi(F)} = \mathbb{T}_F$. So for each $a \in I^0$ the restriction of $\lambda_e$ on $\{(\{a\} \times \{x_j = 0\}) : j = 1, 2, \ldots, n-1\}$ define a characteristic function (see definition 5.1) on the set of facets of $\{a\} \times \mathbb{R}^{n-1}_{\geqslant 0}$. From the constructive definition of quasitoric manifold given in [DJ] it is clear that the quotient space $\{a\} \times (\mathbb{T}^{n-1} \times \mathbb{R}^{n-1}_{\geqslant 0})/\sim_e$ is diffeomorphic to $\{a\} \times \mathbb{R}^{2(n-1)}$. Hence the quotient space

$$(\mathbb{T}^{n-1} \times I^0 \times \mathbb{R}^{n-1}_{\geqslant 0})/\sim_e = I^0 \times (\mathbb{T}^{n-1} \times \mathbb{R}^{n-1}_{\geqslant 0})/\sim_e \cong I^0 \times \mathbb{R}^{2(n-1)}.$$

Since the quotient maps $\pi : (\mathbb{T}^{n-1} \times U_{e'}) \to (\mathbb{T}^{n-1} \times U_{e'})/\sim$ and $\pi_e : (\mathbb{T}^{n-1} \times I^0 \times \mathbb{R}^{n-1}_{\geqslant 0}) \to (\mathbb{T}^{n-1} \times I^0 \times \mathbb{R}^{n-1}_{\geqslant 0})/\sim_e$ are open maps and $\phi$ is a diffeomorphism, the following commutative diagram ensure that the lower horizontal map $\phi_e$ is a homeomorphism.

$$\begin{array}{ccccc} (\mathbb{T}^{n-1} \times U_{e'}) & \xrightarrow{id \times \phi} & (\mathbb{T}^{n-1} \times I^0 \times \mathbb{R}^{n-1}_{\geqslant 0}) & & \\ \pi \downarrow & & \pi_e \downarrow & & \\ (\mathbb{T}^{n-1} \times U_{e'})/\sim & \xrightarrow{\phi_e} & (\mathbb{T}^{n-1} \times I^0 \times \mathbb{R}^{n-1}_{\geqslant 0})/\sim_e & \xrightarrow{\cong} & I^0 \times \mathbb{R}^{2(n-1)} \end{array} \tag{3.3}$$

Let $v'_1$ and $v'_2$ be the vertices of the edge $e'$ of $Q_P$. Suppose $H_1 \cap e' = \{v'_1\}$ and $H_2 \cap e' = \{v'_2\}$, where $H_1$ and $H_2$ are affine hyperplanes as considered in section 2 corresponding to the vertices $v_1$ and $v_2$ of $e$ respectively. Let $U_{v'_1}$ and $U_{v'_2}$ be the open subset of $Q_P$ obtained by deleting all facets of $Q_P$ not containing $v'_1$ and $v'_2$ respectively. Hence there exist diffeomorphism $\phi^1 : U_{v'_1} \to [0, 1) \times \mathbb{R}^{n-1}_{\geqslant 0}$ and $\phi^2 : U_{v'_2} \to [0, 1) \times \mathbb{R}^{n-1}_{\geqslant 0}$ satisfying the same property as the map $\phi$. We get the following commutative diagram and homeomorphisms $\phi^j_e$ for $j = 1, 2$.

$$\begin{array}{ccccc} (\mathbb{T}^{n-1} \times U_{v'_j}) & \xrightarrow{id \times \phi^j} & (\mathbb{T}^{n-1} \times [0, 1) \times \mathbb{R}^{n-1}_{\geqslant 0}) & & \\ \pi \downarrow & & \pi_e \downarrow & & \\ (\mathbb{T}^{n-1} \times U_{v'_j})/\sim & \xrightarrow{\phi^j_e} & (\mathbb{T}^{n-1} \times [0, 1) \times \mathbb{R}^{n-1}_{\geqslant 0})/\sim_e & \xrightarrow{\cong} & [0, 1) \times \mathbb{R}^{2(n-1)} \end{array} \tag{3.4}$$

Hence each point of $(\mathbb{T}^{n-1} \times Q_P)/\sim$ has a neighborhood homeomorphic to an open subset of $[0, 1) \times \mathbb{R}^{2(n-1)}$. So $W(Q_P, \lambda)$ is a manifold with boundary. From the above discussion the interior of $W(Q_P, \lambda)$ is

$$\cup_{e'} (\mathbb{T}^{n-1} \times U_{e'})/\sim = W(Q_P, \lambda) \smallsetminus \{(\mathbb{T}^{n-1} \times \sqcup_{i=1}^k P_{H_i})/\sim\}$$

and the boundary is $\sqcup_{i=1}^k \{(\mathbb{T}^{n-1} \times P_{H_i})/\sim\}$. Let $F(H)_{i_j}$ be a facet of $P_{H_i}$. So there exists a unique facet $F_j$ of $P$ such that $F(H)_{i_j} = F_j \cap Q_P \cap H_i$. The restriction of the function $\lambda$ on the set of all facets of $P_{H_i}$ ( namely $\lambda(F(H)_{i_j}) = \lambda_j$ ) give a characteristic function of a quasitoric manifold over $P_{H_i}$. Hence restricting the equivalence relation $\sim$ on $(\mathbb{T}^{n-1} \times P_{H_i})$ we get that the quotient space $W_i = (\mathbb{T}^{n-1} \times P_{H_i})/\sim$ is a quasitoric manifold over $P_{H_i}$. Hence the boundary $\partial W(Q_P, \lambda) = \sqcup_{i=1}^k W_i$, where $W_i$ is a quasitoric manifold.

**Example 3.3.** *An isotropy function of the standard cube $I^3$ is described in the following figure 1. Here simple convex polytopes $P_{H_1}, \ldots, P_{H_8}$ are triangles. The restriction of the isotropy function on $P_{H_i}$ gives that the space $(\mathbb{T}^2 \times P_{H_i})/\sim$ is the complex projective* *space $\mathbb{C}P^2$ for each $i \in \{1, \dots, 8\}$. Hence the disjoint union $\sqcup_{i=1}^{8} \mathbb{C}P^2$ is the boundary of $(\mathbb{T}^2 \times Q_{I^3})/\sim$.*

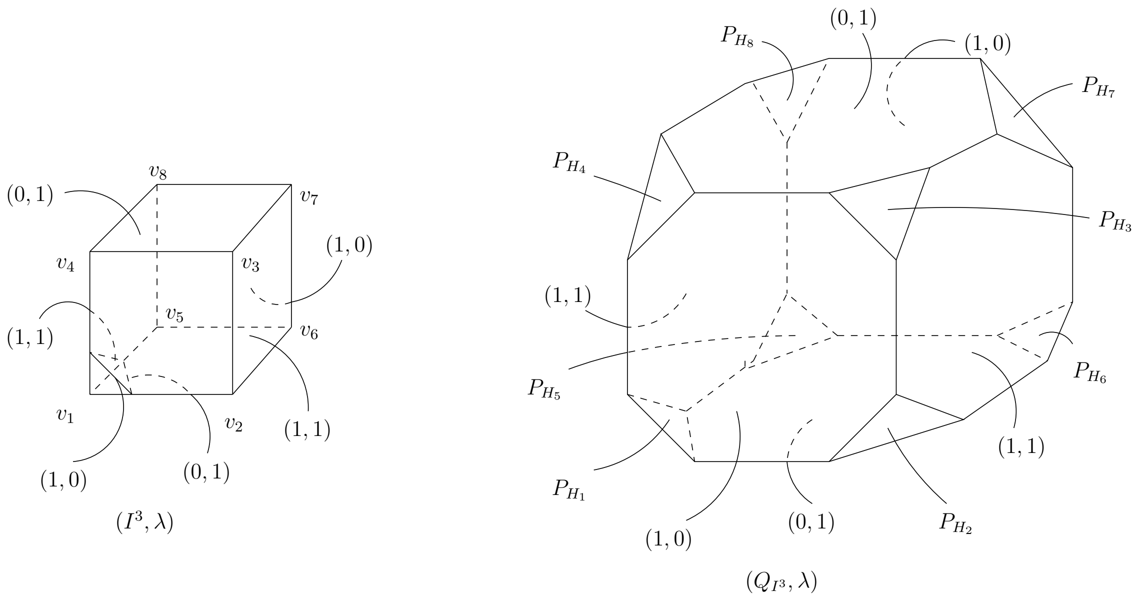



FIGURE 1. An isotropy function $\lambda$ of the edge-simple polytope $I^3$

**Example 3.4.** *In the following figure 2 we define an isotropy function of the edge-simple polytope $P_0$. Here simple convex polytopes $P_{H_1}, P_{H_2}, P_{H_3}, P_{H_4}$ are triangles and the simple convex polytope $P_{H_5}$ is a rectangle. The restriction of the isotropy function on $P_{H_i}$ gives that the space $(\mathbb{T}^2 \times P_{H_i})/\sim$ is $\mathbb{C}P^2$ for each $i \in \{1, 2, 3, 4\}$ and $(\mathbb{T}^2 \times P_{H_5})/\sim$ is $\mathbb{C}P^1 \times \mathbb{C}P^1$. Hence the space $\sqcup_{i=1}^{4} \mathbb{C}P^2 \sqcup (\mathbb{C}P^1 \times \mathbb{C}P^1)$ is the boundary of $(\mathbb{T}^2 \times Q_{P_0})/\sim$.*

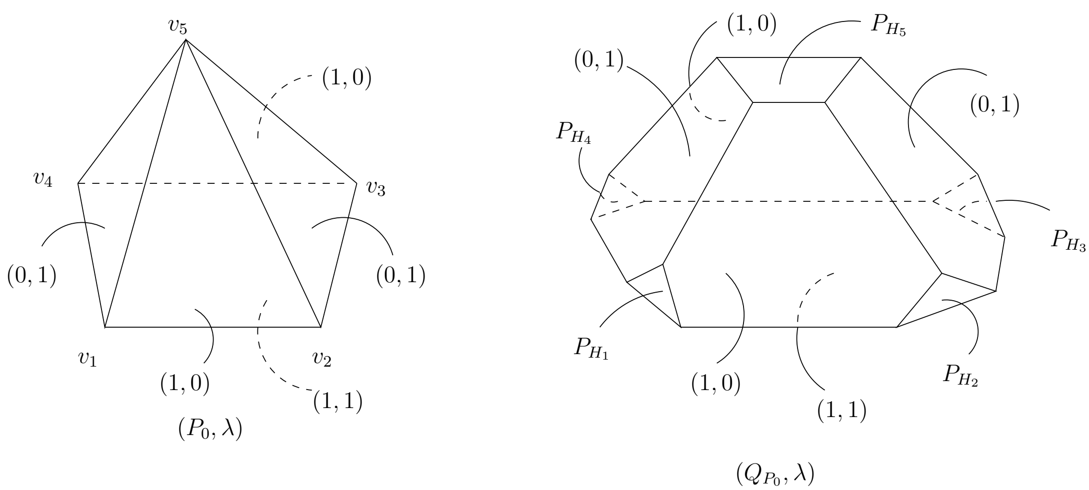



FIGURE 2. An isotropy function $\lambda$ of the edge-simple polytope $P_0$

3.2. **Manifolds with small cover boundary.** We assign each face $F$ to the subgroup $G_F$ of $\mathbb{F}_2^{n-1}$ determined by the vectors $\lambda_{i_1}^s, \dots, \lambda_{i_l}^s$ where $F$ is the intersection of the facets $F_{i_1}, \dots, F_{i_l}$. Let $\sim_s$ be an equivalence relation on $(\mathbb{F}_2^{n-1} \times P)$ defined by the following.

$$(t, p) \sim_s (u, q) \text{ if and only if } p = q \text{ and } t - u \in G_F \tag{3.5}$$

where $F$ is the unique face of $P$ containing $p$ in its relative interior. The quotient space $(\mathbb{F}_2^{n-1} \times Q_P)/\sim_s \subset (\mathbb{F}_2^{n-1} \times P)/\sim_s$, denoted by $S(Q_P, \lambda^s)$, is a manifold with boundary. This can be shown by the same arguments given in the subsection 3.1. The boundary of this manifold is $\{(\mathbb{F}_2^{n-1} \times \sqcup_{i=1}^k P_{H_i})/\sim_s\} = \sqcup_{i=1}^k \{(\mathbb{F}_2^{n-1} \times P_{H_i})/\sim_s\}$. Clearly the restriction of the $\mathbb{F}_2$-isotropy function $\lambda^s$ on the set of all facets of $P_{H_i}$ gives the characteristic function of a small cover over $P_{H_i}$. So $(\mathbb{F}_2^{n-1} \times P_{H_i})/\sim_s$ is a small cover for each $i = 0, \ldots, k$. Hence $S(Q_P, \lambda^s)$ is a manifold with small cover boundary.

3.3. **Some observations.** The set of all facets of the simple convex polytope $Q_P$ are $\mathcal{F}(Q_P) = \{P_{H_j} : j = 1, 2, \ldots, k\} \cup \{F_i' : i = 1, 2, \ldots, m\}$, where $F_i' = F_i \cap Q_P$ for a unique facets $F_i$ of $P$. We define the function $\eta \colon \mathcal{F}(Q_P) \to \mathbb{Z}^n$ as follows.

$$\eta(F) = \begin{cases} (0, \ldots, 0, 1) \in \mathbb{Z}^n & \text{if } F = P_{H_j} \text{ and } j \in \{1, \ldots, k\} \\ \lambda_i \in \mathbb{Z}^{n-1} \times \{0\} \subset \mathbb{Z}^n & \text{if } F = F_i \text{ and } i \in \{1, 2, \ldots, m\} \end{cases} \tag{3.6}$$

So the function $\eta$ satisfies the condition for the characteristic function (see definition 5.1) of a quasitoric manifold over the $n$-dimensional simple convex polytope $Q_P$. Hence from the characteristic pair $(Q_P, \eta)$ we can construct the quasitoric manifold $M(Q_P, \eta)$ over $Q_P$. There is a natural $\mathbb{T}^n$ action on $M(Q_P, \eta)$. Let $\mathbb{T}_H$ be the circle subgroup of $\mathbb{T}^n$ determined by the submodule $\{0\} \times \{0\} \times \ldots \times \{0\} \times \mathbb{Z}$ of $\mathbb{Z}^n$. Hence $W(Q_P, \lambda)$ is the orbit space of the circle $\mathbb{T}_H$ action on $M(Q_P, \eta)$. The quotient map $\phi_H \colon M(Q_P, \eta) \to W(Q_P, \lambda)$ is not a fiber bundle map.

**Remark 3.5.** *The manifold $S(Q_p, \lambda_s)$ with small cover boundary constructed in subsection 3.2 is the orbit space of $\mathbb{Z}_2$ action on a small cover.*

## 4. Orientability of $W(Q_P, \lambda)$

Suppose $W = W(Q_P, \lambda)$. The boundary $\partial W$ has a collar neighborhood in $W$. Hence by the proposition 2.22 of [Hat] we get $H_i(W, \partial W) = \widetilde{H}_i(W/\partial W)$ for all $i$. We show the space $W/\partial W$ has a $CW$-structure. Realize $Q_P$ as a simple convex polytope in $\mathbb{R}^n$ and choose a linear functional $\phi : \mathbb{R}^n \to \mathbb{R}$ which distinguishes the vertices of $Q_P$, as in the proof of Theorem 3.1 in [DJ]. The vertices are linearly ordered according to ascending value of $\phi$. We make the 1-skeleton of $Q_P$ into a directed graph by orienting each edge such that $\phi$ increases along edges. For each vertex $v$ of $Q_P$ define its index, $ind(v)$, as the number of incident edges that point towards $v$. Suppose $\mathcal{V}(Q_P)$ is the set of all vertices and $\mathcal{E}(Q_P)$ is the set of edges of $Q_P$. For each $j \in \{1, 2, \ldots, n\}$, let

$$I_j = \{(v, e_v) \in \mathcal{V}(Q_P) \times \mathcal{E}(Q_P) : ind(v) = j \textit{ and } e_v \textit{ is the incident edge that points}$$
$$\textit{towards } v \textit{ such that } e_v = e \cap Q_P \textit{ for an edge } e \textit{ of } P\}.$$

Suppose $(v, e_v) \in I_j$. Let $F_v$ be the unique face of $Q_P$ containing $e_v$ such that $ind(v)$ is the dimension of $F_v$. Let $U_{e_v}$ be the open subset of $F_v$ obtain by deleting all faces of $F_v$ not containing the edge $e_v$. The restriction of the equivalence relation $\sim$ on $(\mathbb{T}^{n-1} \times U_{e_v})$ gives that the quotient space $(\mathbb{T}^{n-1} \times U_{e_v})/\sim$ is homeomorphic to the open disk $B^{2j-1}$. Hence the quotient space $(W/\partial W)$ has a $CW$-complex structure with odd dimensional cells and one zero dimensional cell only. The number of $(2j-1)$-dimensional cell is $|I_j|$, the cardinality of $I_j$ for $j = 1, 2, \ldots, n$. So we get the following theorem.

**Theorem 4.1.** $H_i(W, \partial W) = \begin{cases} \bigoplus_{|I_j|} \mathbb{Z} & \textit{if } i = 2j-1 \textit{ and } j \in \{1, \ldots, n\} \\ \mathbb{Z} & \textit{if } i = 0 \\ 0 & \textit{otherwise} \end{cases}$

When $j = n$ the cardinality of $I_j$ is one. So $H_{2n-1}(W, \partial W) = \mathbb{Z}$. Hence $W$ is an oriented manifold with boundary. In [DJ] the authors showed that the odd dimensional homology of quasitoric manifolds are zero. So $H_{2i-1}(\partial W) = 0$ for all $i$. Hence we get the following exact sequences for the collared pair $(W, \partial W)$.

(4.1)

$$\begin{array}{ccccccc}
0 \to H_{2n-1}(W) & \xrightarrow{j_*} & H_{2n-1}(W, \partial W) & \xrightarrow{\partial} & H_{2n-2}(\partial W) & \xrightarrow{i_*} & H_{2n-2}(W) \to 0 \\
\vdots & & \vdots & & \vdots & & \vdots \\
0 \to H_3(W) & \xrightarrow{j_*} & H_3(W, \partial W) & \xrightarrow{\partial} & H_2(\partial W) & \xrightarrow{i_*} & H_2(W) \to 0 \\
0 \to H_1(W) & \xrightarrow{j_*} & H_1(W, \partial W) & \xrightarrow{\partial} & H_0(\partial W) & \xrightarrow{i_*} & H_0(W) \twoheadrightarrow \mathbb{Z}
\end{array}$$

Where $\mathbb{Z} \cong H_0(W, \partial W)$. Let $(h_{i_0}, \ldots, h_{i_{n-1}})$ be the *h-vector* of $P_{H_i}$, for $i = 1, 2, \ldots, k$. The definition of *h-vector* of simple convex polytope is given in [DJ]. Hence the Euler characteristic of the manifold $W$ with quasitoric boundary is $\Sigma_{i=1}^{k} \Sigma_{j=0}^{n-1} h_{i_j} - \Sigma_{j=1}^{n-1} |I_j|$.

## 5. Torus Cobordism of Quasitoric Manifolds

### 5.1. Classification of 4-dimensional quasitoric manifolds.

**Definition 5.1.** *Let $Q$ be an $n$-dimensional simple convex polytope and $\mathcal{F}(Q)$ be the set of all facets of $Q$. A map $\eta : \mathcal{F}(Q) \to \mathbb{Z}^n$ is called a characteristic function if the span of $\eta(F_{j_1}), \ldots, \eta(F_{j_l})$ is a $l$-dimensional direct summand of $\mathbb{Z}^n$ whenever the intersection of the facets $F_{j_1}, \ldots, F_{j_l}$ is nonempty. The vectors $\eta_j = \eta(F_j)$ are called characteristic vectors and the pair $(Q, \eta)$ is called a characteristic pair.*

In [DJ] the authors show that we can construct a quasitoric manifold from the pair $(Q, \eta)$ and given a quasitoric manifold we can define a characteristic pair. There is a bijective correspondence between quasitoric manifolds and characteristic pairs modulo the sign of characteristic vectors.

**Example 5.1.** *Let $Q$ be a triangle $\triangle^2$ in $\mathbb{R}^2$. The possible characteristic functions are indicated by the following figures 3. The quasitoric manifold corresponding to the first*

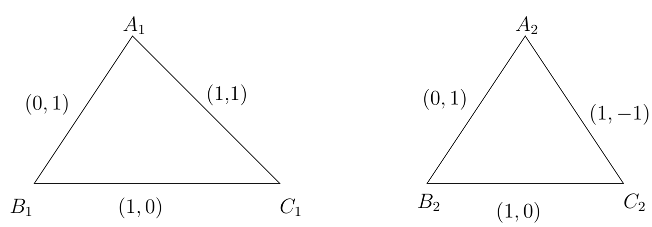



Figure 3. The characteristic functions corresponding to a triangle.

*characteristic pair is $\mathbb{C}P^2$ with the usual $\mathbb{T}^2$ action and standard orientation. The second correspond to the same $\mathbb{T}^2$ action with the reverse orientation on $\mathbb{C}P^2$, we denote it by $\overline{\mathbb{C}P^2}$.*

**Example 5.2.** *Suppose that $Q$ is combinatorially a square in $\mathbb{R}^2$. In this case there are many possible characteristic functions. Some examples are given by the figure 4.*

*The first characteristic pairs may construct an infinite family of 4-dimensional quasitoric manifolds, denote them by $M_k^4$ for each $k \in \mathbb{Z}$. The manifolds $\{M_k^4 : k \in \mathbb{Z}\}$ are equivariantly distinct. Let $L(k)$ be the complex line bundle over $\mathbb{C}P^1$ with the first Chern class $k$. The associated projective bundle is the Hirzebruch surface $\mathbb{P}(L(k) \oplus L(k))$. In* [Oda]

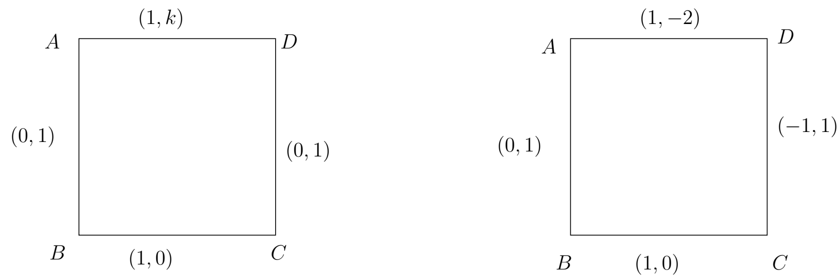



FIGURE 4. Some characteristic functions corresponding to a square.

*the author shows that with the natural action of $\mathbb{T}^2$ on $\mathbb{P}(L(k) \oplus L(k))$ it is equivariantly homeomorphic to $M_k^4$ for each $k$.*

*On the other hand the second combinatorial model gives the quasitoric manifold $\mathbb{C}P^2 \# \mathbb{C}P^2$, the equivariant connected sum of $\mathbb{C}P^2$.*

**Remark 5.3.** *Orlik and Raymond ( [OR], p. 553) show that any 4-dimensional quasitoric manifold $M^4$ over 2-dimensional simple convex polytope is an equivariant connected sum of some copies of $\mathbb{C}P^2$, $\overline{\mathbb{C}P^2}$ and $M_k^4$ for some $k \in \mathbb{Z}$.*

5.2. **$\mathbb{T}^2$-cobordism of quasitoric manifolds.** Let $\mathfrak{C}$ be the following category: the objects are all quasitoric manifolds and morphisms are torus equivariant maps between quasitoric manifolds. We are considering torus cobordism in this category only. Quasitoric manifolds are orientable manifolds, see [DJ].

**Definition 5.2.** *Two $2n$-dimensional quasitoric manifolds $M_1$ and $M_2$ are said to be $\mathbb{T}^n$-cobordant if there exist an oriented $\mathbb{T}^n$ manifold $W$ with boundary $\partial W$ such that $\partial W$ is $\mathbb{T}^n$ equivariantly homeomorphic to $M_1 \sqcup (-M_2)$ under an orientation preserving homeomorphism. Here $-M_2$ represent the reverse orientation of $M_2$.*

We denote the $\mathbb{T}^n$-cobordism class of quasitoric $2n$-manifold $M$ by $[M]$.

**Definition 5.3.** *The $n$-th torus cobordism group is the group of all cobordism classes of $2n$-dimensional quasitoric manifolds with the operation of disjoint union. We denote this group by $CG_n$.*

Let $M \to Q$ be a 4-dimensional quasitoric manifold over the 2-dimensional simple convex polytope $Q$ with the characteristic function $\eta : \mathcal{F}(Q) \to \mathbb{Z}^2$. Suppose the number of facets of $Q$ is $m$. We construct an oriented $\mathbb{T}^2$ manifold $W$ with boundary $\partial W$, where $\partial W$ is equvariantly homeomorphic to $M \sqcup \sqcup_{k_1} \mathbb{C}P^2 \sqcup \sqcup_{k_2} \overline{\mathbb{C}P^2}$ for some integer $k_1, k_2$. To show this we construct a 3-dimensional edge-simple polytope $P_{\mathcal{E}}$ such that $P_{\mathcal{E}}$ has exactly one vertex $O$ which is the intersection of $m$ facets with $P_{\mathcal{E}} \cap H_O = Q$ and other vertices of $P_{\mathcal{E}}$ are intersection of 3 facets. We define an isotropy function $\lambda$, extending the characteristic function $\eta$ of $M$, from the set of facets of $P_{\mathcal{E}}$ to $\mathbb{Z}^2$. Then $W(Q_{P_{\mathcal{E}}}, \lambda)$ is the required oriented $\mathbb{T}^2$ manifold with quasitoric boundary. To compute the group $CG_2$ we use the induction on the number of facets of 2-dimensional simple convex polytope in $\mathbb{R}^2$. We made explicit calculation for 4-dimensional quasitoric manifold on rectangle.

Let $ABCD$ be a rectangle ( see figure 5 ) belongs to $\{(x, y, z) \in \mathbb{R}^3_{\geq 0} : x + y + z = 1\}$. Let $\eta : \{AB, BC, CD, DA\} \to \mathbb{Z}^2$ be the characteristic function for a quasitoric manifold $M$ over $ABCD$ such that the characteristic vectors are

$$\eta(AB) = \eta_1, \ \eta(BC) = \eta_2, \ \eta(CD) = \eta_3 \text{ and } \eta(DA) = \eta_4.$$

We may assume that $\eta_1 = (0, 1)$ and $\eta_2 = (1, 0)$. From the classification results given in subsection 5.1, it is enough to consider the following cases only.

$$\eta_3 = (0,1) \text{ and } \eta_4 = (1,0) \tag{5.1}$$

$$\eta_3 = (0,1) \text{ and } \eta_4 = (1,k),\ k = 1 \text{ or } -1 \tag{5.2}$$

$$\eta_3 = (0,1) \text{ and } \eta_4 = (1,k),\ k \in \mathbb{Z} - \{-1,0,1\} \tag{5.3}$$

$$\eta_3 = (-1,1) \text{ and } \eta_4 = (1,-2) \tag{5.4}$$

**For the case** 5.1: In this case the edge-simple polytope $\widetilde{P}_1$, given in figure 5, is the required edge-simple polytope. The isotropy vectors of $\widetilde{P}_1$ are given by

$$\lambda(OGH) = \eta_1,\ \lambda(OHI) = \eta_2,\ \lambda(OIJ) = \eta_3,\ \lambda(OGJ) = \eta_4 \text{ and } \lambda(GHIJ) = \eta_1 + \eta_2.$$

So we get an oriented $\mathbb{T}^2$ manifold $W(Q_{\widetilde{P}_1}, \lambda)$ with quasitoric boundary where the boundary is the quasitoric manifold $M \sqcup \sqcup_{k_1} \mathbb{C}P^2 \sqcup \sqcup_{k_2} \overline{\mathbb{C}P^2}$ for some integers $k_1, k_2$. Since $[\overline{\mathbb{C}P^2}] = -[\mathbb{C}P^2]$, $[M] = k_3[\mathbb{C}P^2]$ for some integer $k_3$.

**For the case** 5.2: In this case $|det(\eta_2, \eta_4)| = 1$. Let $O$ be the origin of $\mathbb{R}^3$. Let $C_Q$ be the open cone on rectangle $ABCD$ at the origin $O$. Let $G, H, I, J$ be points on extended $OA, OB, OC, OD$ respectively. Let $E$ and $F$ be two points in the interior of the open cones on $AB$ and $CD$ at $O$ respectively such that $|OG| < |OE|$, $|OH| < |OE|$ and $|OI| < |OF|$, $|OJ| < |OF|$. Then the convex polytope $P_1 \subset C_Q$ on the set of vertices $\{O, G, E, H, I, F, J\}$ is an edge-simple polytope (see figure 5) of dimension 3. Define a function, denote by $\lambda$, on the set of facets of $P_1$ by

$$\begin{aligned} &\lambda(OGEH) = \eta_1,\ \lambda(OHI) = \eta_2,\ \lambda(OJFI) = \eta_3,\ \lambda(OJG) = \eta_4, \\ &\lambda(HIFE) = \eta_4 \text{ and } \lambda(GJFE) = \eta_2. \end{aligned} \tag{5.5}$$

Hence $\lambda$ is an isotropy function on the edge-simple polytope $P_1$. The boundary of the oriented $\mathbb{T}^2$ manifold $W(Q_{P_1}, \lambda)$ is the quasitoric manifold $M \sqcup \sqcup_{k_1} \mathbb{C}P^2 \sqcup \sqcup_{k_2} \overline{\mathbb{C}P^2}$ for some integers $k_1, k_2$. Hence $[M] = k_3[\mathbb{C}P^2]$ for some integer $k_3$.

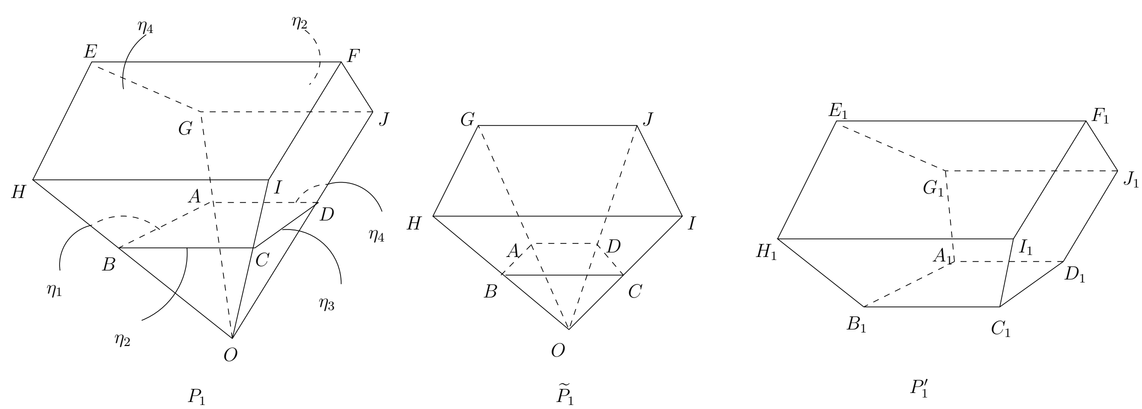



FIGURE 5. The edge-simple polytope $P_1, \widetilde{P}_1$ and the convex polytope $P'_1$ respectively.

**For the case** 5.3: Suppose $det(\eta_2, \eta_4) = k > 1$. Define a function $\lambda^{(1)}$ on the set of facets of $P_1$ except $GEFJ$ by

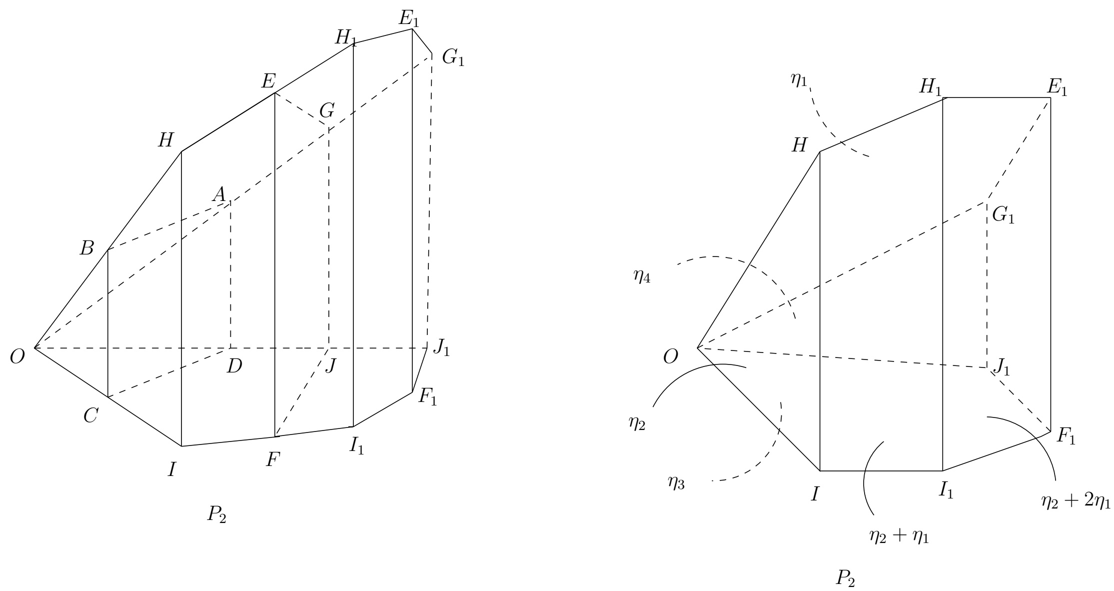



FIGURE 6. The edge-simple polytope $P_2$ with the function $\lambda^{(2)}$.

$$\lambda^{(1)}(OGEH) = \eta_1,\ \lambda^{(1)}(OHI) = \eta_2,\ \lambda^{(1)}(OIFJ) = \eta_3,\ \lambda^{(1)}(OGJ) = \eta_4, \\ \text{and } \lambda^{(1)}(EHIF) = \eta_2 + \eta_1. \tag{5.6}$$

So the function $\lambda^{(1)}$ satisfies the condition of an isotropy function of the edge-simple polytope $P_1$ along each edge except the edges of the rectangle $GEFJ$. The restriction of the function $\lambda^{(1)}$ on the edges $GE, EF, FJ, GJ$ of the rectangle $GEFJ$ gives the following equations,

$$|det[\lambda^{(1)}(GE), \lambda^{(1)}(EF)]| = 1,\ |det[\lambda^{(1)}(EF), \lambda^{(1)}(FJ)]| = 1, \\ |det[\lambda^{(1)}(FJ), \lambda^{(1)}(GJ)]| = 1,\ |det[\lambda^{(1)}(GJ), \lambda^{(1)}(GE)]| = 1 \\ \text{and } det[\lambda^{(1)}(EF), \lambda^{(1)}(GJ)] = k - 1 < k. \tag{5.7}$$

Let $P_1'$ be a 3-dimensional convex polytope as in the figure 5. Identifying the facet $GEFJ$ of $P_1$ and $A_1B_1C_1D_1$ of $P_1'$ through a suitable diffeomorphism of manifold with corners such that the vertices $G, E, F, J$ maps to the vertices $A_1, B_1, C_1, D_1$ respectively, we can form a new convex polytope $P_2$, see figure 6. After the identification following holds.

(1) The facet of $P_1$ containing $GE$ and the facet of $P_1'$ containing $A_1B_1$ make the facet $OHH_1E_1G_1$ of $P_2$.
(2) The facet of $P_1$ containing $EF$ and the facet of $P_1'$ containing $B_1C_1$ make the facet $HH_1I_1I$ of $P_2$.
(3) The facet of $P_1$ containing $FJ$ and the facet of $P_1'$ containing $C_1D_1$ make the facet $OII_1F_1J_1$ of $P_2$.
(4) The facet of $P_1$ containing $JG$ and the facet of $P_1'$ containing $D_1A_1$ make the facet $OJ_1G_1$ of $P_2$.

The polytope $P_2$ is an edge-simple polytope. We define a function $\lambda^{(2)}$ on the set of facets of $P_2$ except $G_1E_1F_1J_1$ by

$$\begin{aligned}(5.8)\qquad &\lambda^{(2)}(OHH_1E_1G_1) = \eta_1,\ \lambda^{(2)}(OIH) = \eta_2,\ \lambda^{(2)}(OII_1F_1J_1) = \eta_3,\\ &\lambda^{(2)}(OJ_1G_1) = \eta_4,\ \lambda^{(2)}(HH_1I_1I) = \eta_2 + \eta_1\\ &\text{and } \lambda^{(2)}(H_1I_1F_1E_1) = \eta_2 + 2\eta_1.\end{aligned}$$

So the function $\lambda^{(2)}$ satisfies the condition of an isotropy function of the edge-simple polytope $P_2$ along each edge except the edges of the rectangle $G_1E_1F_1J_1$. The restriction of the function $\lambda^{(2)}$ on the edges namely $G_1E_1, E_1F_1, F_1J_1, G_1J_1$ of the rectangle $G_1E_1F_1J_1$ gives the following equations,

$$\begin{aligned}(5.9)\qquad &|det[\lambda^2(G_1E_1), \lambda^2(E_1F_1)]| = 1,\ |det[\lambda^2(E_1F_1), \lambda^2(F_1J_1)]| = 1,\\ &|det[\lambda^2(F_1J_1), \lambda^2(G_1J_1)]| = 1,\ |det[\lambda^2(G_1J_1), \lambda^2(G_1E_1)]| = 1\\ &\text{and } det[\lambda^2(E_1F_1), \lambda^2(G_1J_1)] = k-2 < k-1.\end{aligned}$$

Proceeding in this way, at $k$-th step we construct an edge-simple polytope $P_k$ with the function $\lambda^{(k)}$, extending the function $\lambda^{(k-1)}$, on the set of facets of $P_k$ such that

$$\begin{aligned}(5.10)\qquad &\lambda^{(k)}(H_{k-2}H_{k-1}I_{k-1}I_{k-2}) = \eta_2 + (k-1)\eta_1 = \lambda^{(k-1)}(H_{k-2}I_{k-2}F_{k-2}E_{k-2}),\\ &\lambda^{(k)}(OG_{k-1}J_{k-1}) = \eta_4 = \lambda^{(k-1)}(OG_{k-2}J_{k-2}),\\ &\lambda^{(k)}(H_{k-1}I_{k-1}F_{k-1}E_{k-1}) = \eta_4 \text{ and } \lambda^{(k)}(G_{k-1}E_{k-1}F_{k-1}J_{k-1}) = \eta_2 + (k-1)\eta_1.\end{aligned}$$

Observe that the function $\lambda := \lambda^{(k)}$ is an isotropy function of the edge-simple polytope $P_k$. So we get an oriented $\mathbb{T}^2$-manifold with boundary $W(Q_{P_k}, \lambda)$ where the boundary is the quasitoric manifold $M \sqcup \sqcup_{k_1} \mathbb{C}P^2 \sqcup \sqcup_{k_2} \overline{\mathbb{C}P^2}$ for some integers $k_1, k_2$. Hence $[M] = k_3[\mathbb{C}P^2]$ for some integer $k_3$.

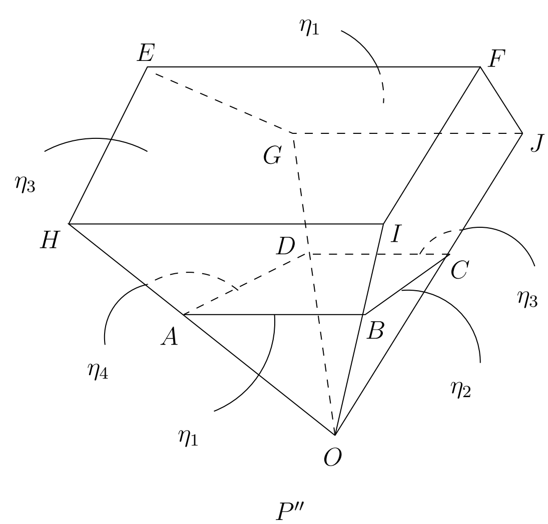



FIGURE 7. The edge-simple polytope $P''$ and an isotropy function $\lambda$ associated to the case 5.4.

**For the case** 5.4: In this case $|det[\eta_1, \eta_3]| = 1$. Following case 5.2, we can construct an edge simple polytope $P''$ and an isotropy function $\lambda$ over this edge-simple polytope, see figure 7. Hence we can construct an oriented $\mathbb{T}^2$ manifold with quasitoric boundary $W(Q_{P''}, \lambda)$ having the desired property.

Now consider the case of a quasitoric manifold $M$ over a convex 2-polytope $P$ with $m$ facets, where $m > 4$. By the classification result of 4-dimensional quasitoric manifold which is discussed in subsection 5.1, $M$ is one of the following equivariant connected sum.

$$(5.11)\qquad M = N_1 \# \mathbb{C}P^2$$

$$M = N_2 \# \overline{\mathbb{C}P^2} \tag{5.12}$$

$$M = N_3 \# M_k^4 \tag{5.13}$$

The quasitoric manifolds $N_1, N_2$ and $N_3$ are associated to the 2-polytopes $Q_1, Q_2$ and $Q_3$ respectively. The number of facets of $Q_1, Q_2$ and $Q_3$ are $m-1$, $m-1$ and $m-2$ respectively. The quasitoric manifold $M_k^4$ is defined in subsection 5.1.

Suppose for a quasitoric manifold $N$ over a convex 2-polytope $Q$ we have constructed a 3-dimensional edge-simple polytope $P_{\mathcal{E}}$ such that

(1) $P_{\mathcal{E}}$ has exactly one vertex $O$ with $P_{\mathcal{E}} \cap H_O = Q$, where $H_O$ is an affine hyperplane corresponding to the vertex $O$ as we considered in section 2,
(2) all other vertices of $P_{\mathcal{E}}$ are intersection of 3 facets,
(3) there exists an isotropy function $\lambda$, extending the characteristic function $\eta$ of $N$, from the set of facets of $P_{\mathcal{E}}$ to $\mathbb{Z}^2$.

**Definition 5.4.** *We call the pair $(P_{\mathcal{E}}, \lambda)$ an isotropy pair associated to the quasitoric manifold $N$.*

We have already constructed an isotropy pair associated to $N$ over a convex 2-polytope $Q$ with $|\mathcal{F}(Q)| = 4$. Now we construct an isotropy pair associated to $M$ for the cases 5.11, 5.12 and 5.13. We use the induction on $m$, the cardinality of the set of facets of 2-polytope $Q$. Let for any quasitoric manifold $N$ over a convex 2-polytope $Q$ with $|\mathcal{F}(Q)| = j < m$, we have constructed an isotropy pair associated to $N$.

**For the case** 5.11: In this case $N_1 \# \mathbb{C}P^2$ is a quasitoric manifold over the 2-polytope $Q_1' = Q_1 \# A_1B_1C_1$. Here the triangle $A_1B_1C_1$ is the orbit space associated to $\mathbb{C}P^2$ with the characteristic function given in the figure 3. We may assume that the characteristic vectors of facets meeting at $x \in Q_1$ are $(1,0)$ and $(0,1)$ as given in the figure 8. Suppose the connected sum of $N_1$ and $\mathbb{C}P^2$ take place at the fixed points corresponding to the vertices $x$ of $Q_1$ and $B_1$ of $A_1B_1C_1$, see figure 8.

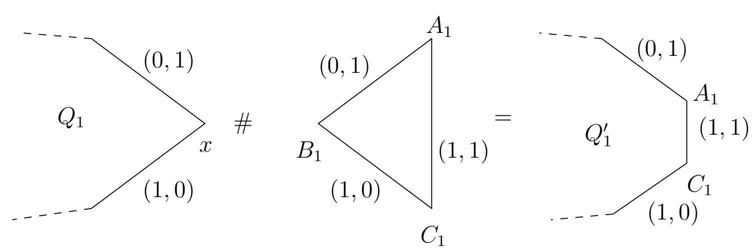



FIGURE 8. Connected sum of $Q_1$ and the triangle $A_1B_1C_1$.

Let $P_{\mathcal{E}_1}$ be the 3-dimensional edge-simple polytope associated to the quasitoric manifold $N_1$. Let $\lambda' : \mathcal{F}(P_{\mathcal{E}_1}) \to \mathbb{Z}^2$ be an isotropy function such that the oriented $\mathbb{T}^2$ manifold with boundary $W(Q_{P_{\mathcal{E}_1}}, \lambda')$ has the boundary $N_1 \sqcup \sqcup_{k_1} \mathbb{C}P^2 \sqcup \sqcup_{k_2} \overline{\mathbb{C}P^2}$ for some integers $k_1, k_2$.

Let $OC_3$ be the edge of $P_{\mathcal{E}_1}$ containing the vertex $x$ of $Q_1$ in its relative interior. Let $A_3$ and $B_3$ be two points belongs to the relative interior of the edges $e_1$ and $e_2$ respectively, see figure 9. Let $H'_{A_3B_3}$ be the closed half space of an affine hyperplane $H_{A_3B_3}$ such that

(1) the plane $H_{A_3B_3}$ passes through the points $A_3, B_3$ and $O$,
(2) the point $C_3$ does not belongs to $H'_{A_3B_3}$.

Let

$$P'_{\mathcal{E}_1} = P_{\mathcal{E}_1} \cap H'_{A_3B_3}. \tag{5.14}$$

So $P'_{\mathcal{E}_1}$ is an edge-simple polytope and $F_{A_3B_3} = P'_{\mathcal{E}_1} \cap H_{A_3B_3}$ is a facet of $P'_{\mathcal{E}_1}$. Let $F_{x_1}, F_{x_2}$ and $F_{x_3}$ be the facets of $P_{\mathcal{E}_1}$ meeting at $C_3$. So $\lambda'(F_{x_1}) = (1,0)$ and $\lambda'(F_{x_2}) = (0,1)$. Since $det[\lambda'(F_{x_1}), \lambda'(F_{x_2})] = 1$, we do the following.

Let $D_3, E_3, G_3$ and $H_3$ be the points in the relative interior of the edges $e_1 \cap P'_{\mathcal{E}_1}, OA_3$, $OB_3$ and $e_2 \cap P'_{\mathcal{E}_1}$ respectively, see figure 9. Let $H'_{D_3E_3}$ and $H'_{G_3H_3}$ be closed half space of affine hyperplanes $H_{D_3E_3}$ and $H_{G_3H_3}$ respectively satisfying the following

(1) $D_3, E_3 \in H_{D_3E_3}$, $A_3 \notin H'_{D_3E_3}$ and $B_3 \in H'_{D_3E_3}$,
(2) $G_3, H_3 \in H_{G_3H_3}$, $A_3 \in H'_{G_3H_3}$ and $B_3 \notin H'_{G_3H_3}$,
(3) the intersection $A_3B_3 \cap H'_{D_3E_3} \cap H'_{G_3H_3}$ is empty.

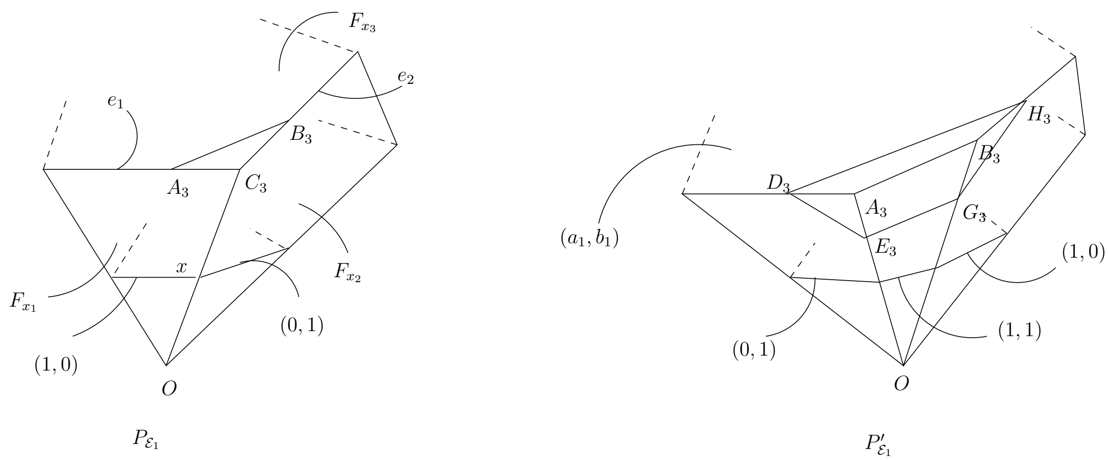



FIGURE 9. The edge-simple polytope $P_{\mathcal{E}_1}$ and $P'_{\mathcal{E}_1}$.

Let

$$P_{\mathcal{E}} = P'_{\mathcal{E}_1} \cap H'_{D_3E_3} \cap H'_{G_3H_3}. \tag{5.15}$$

So the polytope $P_{\mathcal{E}}$ is a 3-dimensional edge-simple polytope in $\mathbb{R}^3$. Let

$$\begin{aligned} &F'_{x_1} = P_{\mathcal{E}} \cap F_{x_1},\ F'_{x_2} = P_{\mathcal{E}} \cap F_{x_2},\ F'_{x_3} = P_{\mathcal{E}} \cap F_{x_3}, \\ &F'_{A_3B_3} = P_{\mathcal{E}} \cap H_{A_3B_3},\ F'_{D_3E_3} = P_{\mathcal{E}} \cap H_{D_3E_3} \text{ and } F'_{G_3H_3} = P_{\mathcal{E}} \cap H_{G_3H_3}. \end{aligned} \tag{5.16}$$

The facets of $P_{\mathcal{E}}$ are

$$\mathcal{F}(P_{\mathcal{E}}) = \{\mathcal{F}(P_{\mathcal{E}_1}) - \{F_{x_1}, F_{x_2}, F_{x_3}\}\} \cup \{F'_{x_1}, F'_{x_2}, F'_{x_3}, F'_{A_3B_3}, F'_{D_3E_3}, F'_{G_3H_3}\}. \tag{5.17}$$

Define a function $\lambda : \mathcal{F}(P_{\mathcal{E}}) \to \mathbb{Z}^2$ as follows,

$$\lambda(F) = \begin{cases} \lambda'(F) & \text{if } F \in \{\mathcal{F}(P_{\mathcal{E}_1}) - \{F_{x_1}, F_{x_2}, F_{x_3}\}\} \\ \lambda'(F_{x_i}) & \text{if } F = F'_{x_i},\ i = 1,2,3 \\ (1,1) & \text{if } F = F_{A_3B_3} \\ \lambda'(F_{x_2}) & \text{if } F = F_{D_3E_3} \\ \lambda'(F_{x_1}) & \text{if } F = F_{G_3H_3} \end{cases} \tag{5.18}$$

Observe that $\lambda$ is an isotropy function on $P_{\mathcal{E}}$ such that the restriction of $\lambda$ on the set of facets of $Q'_1 = P_{\mathcal{E}} \cap H_O$ is the characteristic function for $M$ over $Q'_1$. Hence we get an oriented $\mathbb{T}^2$ manifold with boundary $W(Q_{P_{\mathcal{E}}}, \lambda)$ where the boundary is the quasitoric manifold $M \sqcup \sqcup_{k_1} \mathbb{C}P^2 \sqcup \sqcup_{k_2} \overline{\mathbb{C}P^2}$ for some integers $k_1, k_2$.

**For the case** 5.12: In this case the construction of an isotropy pair associated to $M = N_2 \# \overline{\mathbb{C}P^2}$ is similar to the case 5.11.

**For the case** 5.13: In this case the construction of edge-simple polytope and an isotropy function is almost similar to the case 5.11 with some exceptions. The manifold

$M = N_3 \# M^4_k$ is a quasitoric manifold over the 2-polytope $Q_3' = Q_3 \# ABCD$. Here the rectangle $ABCD$ is the orbit space associated to $M^4_k$ with a characteristic function given in the figure 4. Assume that the characteristic vectors of facets meeting at $z$ are $(1,0)$ and $(0,1)$. Suppose the connected sum of $N_3$ and $M^4_k$ take place at the fixed points corresponding to the vertex $z$ of $Q_3$ and the vertex $B$ of $ABCD$, see figure 10.

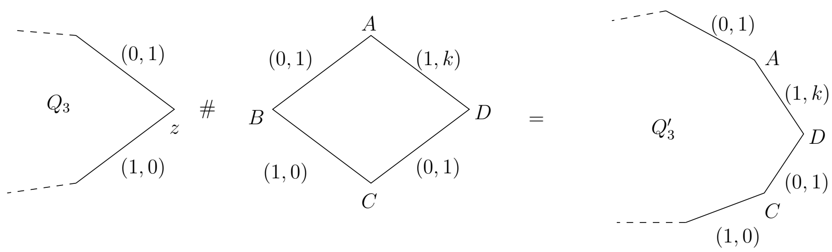



FIGURE 10. Connected sum of $Q_3$ and the rectangle $ABCD$.

Let $P_{\mathcal{E}_3}$ be the 3-dimensional edge-simple polytope associated to the quasitoric manifold $N_3$. Let $\lambda' : \mathcal{F}(P_{\mathcal{E}_3}) \to \mathbb{Z}^2$ be an isotropy function such that the oriented $\mathbb{T}^2$ manifold with boundary $W(Q_{P_{\mathcal{E}_3}}, \lambda')$ has the boundary $N_3 \sqcup \sqcup_{k_1} \mathbb{C}P^2 \sqcup \sqcup_{k_2} \overline{\mathbb{C}P^2}$ for some integers $k_1, k_2$.

Let $OC_5$ be the edge of $P_{\mathcal{E}_3}$ containing the vertex $z$ of $Q_3$ in its relative interior. Let $A_5$ and $D_5$ be two points that belong to the relative interior of the edges $e_5$ and $e_6$ respectively, see figure 11. Let $B_5$ be a point belongs to the relative interior of the triangle $A_5C_5D_5 \subset F_{z_2}$.

Let $H'_{A_5B_5}$ and $H'_{B_5D_5}$ be the closed half spaces of affine hyperplanes $H_{A_5B_5}$ and $H_{B_5D_5}$ respectively such that

(1) the points $O, A_5, B_5$ belong to $H_{A_5B_5}$ and the points $O, B_5, D_5$ belong to $H_{B_5D_5}$,
(2) the point $C_5$ does not belongs to $H'_{A_5B_5}$ and $H'_{B_5D_5}$.

Let

$$P'_{\mathcal{E}_3} = P_{\mathcal{E}_3} \cap H'_{A_5B_5} \cap H'_{B_5D_5}. \tag{5.19}$$

So $P'_{\mathcal{E}_3}$ is a 3-dimensional edge-simple polytope in $\mathbb{R}^3$. Let the facets $F_{z_1}, F_{z_2}$ and $F_{z_3}$ of $P_{\mathcal{E}_3}$ meet at $C_5$, see figure 11. So $\lambda'(F_{z_1}) = (0,1)$ and $\lambda'(F_{z_3}) = (1,0)$. Let

$$\begin{aligned} &F'_{z_1} = P'_{\mathcal{E}_3} \cap F_{z_1},\; F'_{z_2} = P'_{\mathcal{E}_3} \cap F_{z_2},\; F'_{z_3} = P'_{\mathcal{E}_3} \cap F_{z_3},\\ &F'_{A_5B_5} = P'_{\mathcal{E}_3} \cap H_{A_5B_5} \text{ and } F'_{B_5D_5} = P'_{\mathcal{E}_3} \cap H_{B_5D_5}. \end{aligned} \tag{5.20}$$

The facets of $P'_{\mathcal{E}_3}$ are

$$\mathcal{F}(P'_{\mathcal{E}_3}) = \{\mathcal{F}(P_{\mathcal{E}_3}) - \{F_{z_1}, F_{z_2}, F_{z_3}\}\} \cup \{F'_{z_1}, F'_{z_2}, F'_{z_3}, F'_{A_5B_5}, F'_{B_5D_5}\}. \tag{5.21}$$

Define a function $\overline{\lambda} : \mathcal{F}(P'_{\mathcal{E}_3}) \to \mathbb{Z}^2$ as follows,

$$\overline{\lambda}(F) = \begin{cases} \lambda'(F) & \text{if } F \in \{\mathcal{F}(P_{\mathcal{E}_3}) - \{F_{z_1}, F_{z_2}, F_{z_3}\}\} \\ \lambda'(F_{z_i}) & \text{if } F = F'_{z_i},\; i = 1,2,3 \\ (0,1) & \text{if } F = F'_{A_5B_5} \\ (1,k) & \text{if } F = F'_{B_5D_5} \end{cases} \tag{5.22}$$

So the function $\overline{\lambda}$ is an isotropy function of $P'_{\mathcal{E}_3}$ if and only if $|det[\lambda(F_{z_2}), (1,k)]| = 1$. If $|det[\lambda(F_{z_2}), (1,k)]| \neq 1$, then we do the following.

Let $E_5, F_5, I_5$ be points in the relative interior of the edges $e_5 \cap P'_{\mathcal{E}_3}, OA_3$ and $e_6 \cap P'_{\mathcal{E}_3}$ of $P'_{\mathcal{E}_3}$ respectively as given in the figure 11. Let $H_C$ be an affine hyperplane in $\mathbb{R}^3$ passing through the points $\{E_5, F_5, I_5\}$. Clearly the points $G_5 = OB_5 \cap H_C$ and $H_5 = OD_5 \cap H_C$ belong to the relative interior of $OB_5$ and $OD_5$ respectively. Let $H'_C$ be the closed half space of $H_C$ such that the points $\{A_5, B_5, D_5\}$ does not belong to $H'_C$.

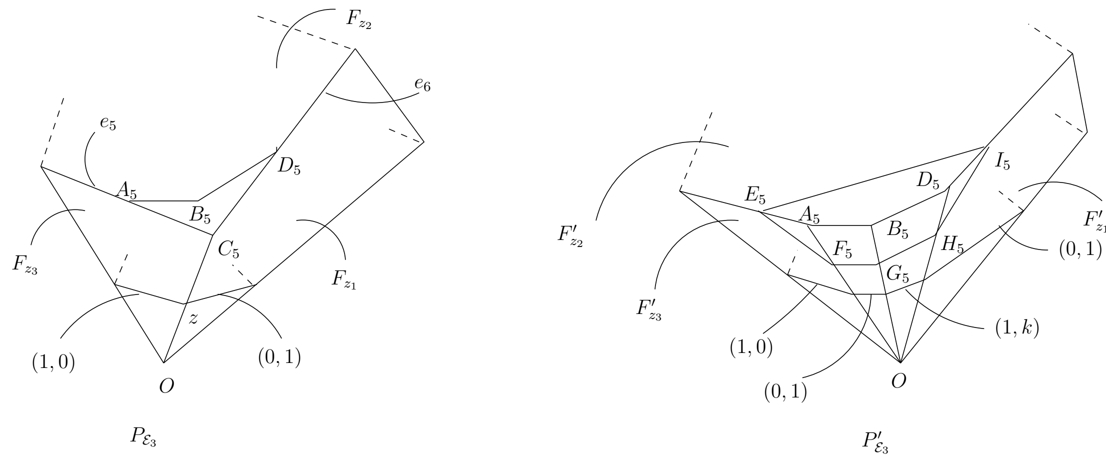



FIGURE 11. The edge-simple polytope $P_{\mathcal{E}_3}$ and $P'_{\mathcal{E}_3}$.

Let

$$P''_{\mathcal{E}_3} = P'_{\mathcal{E}_3} \cap H'_C. \tag{5.23}$$

So $P''_{\mathcal{E}_3}$ is a 3-dimensional edge-simple polytope. Let

$$\begin{aligned} &F''_{z_1} = P''_{\mathcal{E}_3} \cap F'_{z_1},\ F''_{z_2} = P''_{\mathcal{E}_3} \cap F'_{z_2},\ F''_{z_3} = P'_{\mathcal{E}_3} \cap F'_{z_3}, \\ &F''_{A_5B_5} = P''_{\mathcal{E}_3} \cap H'_{A_5B_5},\ F''_{B_5D_5} = P''_{\mathcal{E}_3} \cap H'_{B_5D_5} \text{ and } F_C = P''_{\mathcal{E}_3} \cap H_C. \end{aligned} \tag{5.24}$$

The facets of $P''_{\mathcal{E}_3}$ are

$$\mathcal{F}(P''_{\mathcal{E}_3}) = \{\mathcal{F}(P_{\mathcal{E}_3}) - \{F_{z_1}, F_{z_2}, F_{z_3}\}\} \cup \{F''_{z_1}, F''_{z_2}, F''_{z_3}, F''_{A_5B_5}, F''_{B_5D_5}, F_C\}. \tag{5.25}$$

The restriction $\eta_C$ of $\overline{\lambda}$ on the set of facets of $F_C$ is given in the figure 12. Note that $\eta_C$ is the characteristic function of a quasitoric manifold $N_C$ over $F_C$. Since the number of facets of $F_C$ is 5, by the case 5.11 or 5.12 we can construct an edge-simple polytope $P_{\mathcal{E}_C}$ and an isotropy function $\lambda_C : \mathcal{F}(P_{\mathcal{E}_C}) \to \mathbb{Z}^2$ of $P_{\mathcal{E}_C}$ such that

(1) there exists a unique vertex $O_C$ of $P_{\mathcal{E}_C}$ with $P_{\mathcal{E}_C} \cap H_{O_C} = \widehat{F}_C \cong F_C$ see figure 12, where $H_{O_C}$ is an affine hyperplane corresponding to the vertex $O_C$ as we considered in section 2,
(2) all other vertices of $P_{\mathcal{E}_C}$ are intersection of 3 facets,
(3) the restriction of $\lambda_C$ on $\widehat{F}_C$ is the characteristic function $\eta_C$ of $N_C$.

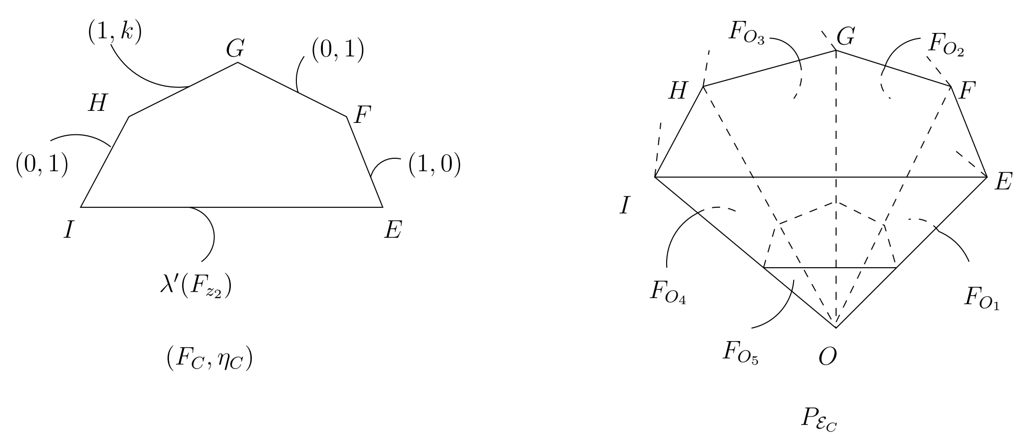



FIGURE 12. The polytope $F_C$ and the edge-simple polytope $P_{\mathcal{E}_C}$.

Let $H'_{O_C}$ be the closed half space of $H_{O_C}$ not containing $O_C \in P_{\mathcal{E}_C}$. Let

$$P'_{\mathcal{E}_C} = P_{\mathcal{E}_C} \cap H'_{O_C} \text{ and } F'_{O_i} = F_{O_i} \cap P'_{\mathcal{E}_C} \text{ for } i = 1, \ldots, 5. \tag{5.26}$$

Now we construct an edge-simple polytope $P_{\mathcal{E}}$ by identifying the facets $F_C$ of $P''_{\mathcal{E}_3}$ and $\widehat{F}_C$ of $P'_{\mathcal{E}_C}$ via a suitable diffeomorphism. We can define a diffeomorphism $f$ from $P'_{\mathcal{E}_C}$ onto its image in $\mathbb{R}^3$ such that following holds,

(1) $P''_{\mathcal{E}_3} \cap f(P'_{\mathcal{E}_C}) = F_C$,
(2) $P''_{\mathcal{E}_3} \cup f(P'_{\mathcal{E}_C}) = P_{\mathcal{E}}$ is an edge-simple polytope,
(3) $F''_{z_3} \cup f(F'_{O_1})$, $F''_{A_5B_5} \cup f(F'_{O_2})$, $F''_{B_5D_5} \cup f(F'_{O_3})$, $F'_{z_2} \cup f(F'_{O_4})$ and $F'_{z_3} \cup f(F'_{O_5})$ are facets of $P_{\mathcal{E}}$ containing $E_5F_5$, $F_5G_5$, $G_5H_5$, $H_5I_5$ and $I_5E_5$ respectively.

Let $\overline{\mathcal{F}}(P_{\mathcal{E}_C}) = \{f(F) : F \in \{\mathcal{F}(P_{\mathcal{E}_C}) - \{F_{O_i} : i = 1, \ldots, 5\}\}\}$, $F_{OZ_1} = F''_{z_3} \cup f(F'_{O_1})$, $F_{OZ_2} = F''_{A_5B_5} \cup f(F'_{O_2})$, $F_{OZ_3} = F''_{B_5D_5} \cup f(F'_{O_3})$, $F_{OZ_4} = F'_{z_2} \cup f(F'_{O_4})$ and $F_{OZ_5} = F'_{z_3} \cup f(F'_{O_5})$.

Hence the facets of $P_{\mathcal{E}}$ are

$$\mathcal{F}(P_{\mathcal{E}}) = \{\mathcal{F}(P_{\mathcal{E}_3}) - \{F_{z_1}, F_{z_2}, F_{z_3}\}\} \cup \{F_{OZ_i} : i = 1, \ldots, 5\} \cup \{\overline{\mathcal{F}}(P_{\mathcal{E}_C})\}. \tag{5.27}$$

Define a function $\lambda : \mathcal{F}(P_{\mathcal{E}}) \to \mathbb{Z}^2$ as follows,

$$\lambda(F) = \begin{cases} \lambda'(F) & \text{if } F \in \{\mathcal{F}(P_{\mathcal{E}_3}) - \{F_{z_1}, F_{z_2}, F_{z_3}\}\} \\ (1,0) & \text{if } F = F_{OZ_1} \\ (0,1) & \text{if } F \in \{F_{OZ_2}, F_{OZ_4}\} \\ (1,k) & \text{if } F = F_{OZ_3} \\ \lambda'(F_{z_2}) & \text{if } F = F_{OZ_5} \\ \lambda_C(\overline{F}) & \text{if } F = f(\overline{F}) \in \overline{\mathcal{F}}(P_{\mathcal{E}_C}) \end{cases} \tag{5.28}$$

Observe that $\lambda$ is an isotropy function on $P_{\mathcal{E}}$ such that the restriction of $\lambda$ on the set of facets of $Q'_3 = P_{\mathcal{E}} \cap H_O$ is the characteristic function for $M$ over $Q'_3$. Hence we get an oriented $\mathbb{T}^2$ manifold with boundary $W(Q_{P_{\mathcal{E}}}, \lambda)$ where the boundary is the quasitoric manifold $M \sqcup \sqcup_{k_1} \mathbb{C}P^2 \sqcup \sqcup_{k_2} \overline{\mathbb{C}P^2}$ for some integers $k_1, k_2$. Hence we have proved the following theorem.

**Theorem 5.4.** *The oriented torus cobordism group $CG_2$ is an infinite cyclic group generated by $\mathbb{T}^2$-cobordism class of complex projective space $\mathbb{C}P^2$.*

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