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Contractions of Filippov algebras

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Abstract

We introduce in this paper the contractions $\mathfrak{G}_c$ of n-Lie (or Filippov) algebras $\mathfrak{G}$ and show that they have a semidirect structure as their $n = 2$ Lie algebra counterparts. As an example, we compute the non-trivial contractions of the simple A_{n+1} Filippov algebras. By using the İnönü-Wigner and the generalized Weimar-Woods contractions of ordinary Lie algebras, we compare (in the $\mathfrak{G} = A_{n+1}$ simple case) the Lie algebras $\mathrm{Lie}\,\mathfrak{G}_c$ (the Lie algebra of inner endomorphisms of $\mathfrak{G}_c$) with certain contractions $(\mathrm{Lie}\,\mathfrak{G})_{IW}$ and $(\mathrm{Lie}\,\mathfrak{G})_{W-W}$ of the Lie algebra $\mathrm{Lie}\,\mathfrak{G}$ associated with $\mathfrak{G}$.

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1 Introduction

In 1985, Filippov [1, 2] initiated the study of certain linear algebras (called n-Lie algebras by him) endowed with a completely antisymmetric bracket with n entries that satisfies a characteristic identity, the Filippov identity (FI). These *n-Lie* or *Filippov algebras* (FA) $\mathfrak{G}$ reduce for $n = 2$ to ordinary Lie algebras $\mathfrak{g}$.

The properties of Filippov algebras [1] have been studied further in parallel with those of the Lie algebras, specially by Kasymov [3, 4] and Ling [5] (see [6] for a review). It has been shown, for instance, that it is possible to define solvable ideals, simple and semisimple Filippov algebras, etc. Semisimple algebras satisfy a Cartan-like criterion [4] and, as in the Lie algebra case, they are given by the direct sums of simple ones. One result, however, in which FAs differ significantly from their $n = 2$ Lie algebra counterparts is that for each $n > 2$ there is only one complex simple finite Filippov algebra [1, 5], which is $(n + 1)$-dimensional. The real Euclidean simple n-Lie algebras A_{n+1}, which are constructed on Euclidean $(n + 1)$-dimensional vector spaces, are thus the only $(n > 2)$-Lie (Filippov) algebra generalizations of the simple $so(3)$ Lie algebra. Similarly, the simple pseudoeuclidean ones may be considered as $n > 2$ generalizations of $so(1, 2)$.

Other properties of FAs, such as deformations (or *e.g.*, central extensions) may be studied. As in the general and Lie algebra cases [7, 8], deformations are associated with FA cohomology. The Filippov algebra cohomology suitable for deformations of Filippov algebras was given in [9] in the context of Nambu-Poisson algebras (see further [10, 11, 12]); the FA cohomology generalizes the Lie algebra cohomology complexes (see also [6]). The FA cohomology is not completely straightforward. For instance, for $n > 3$ it turns out that the p-cochains are mappings $\alpha^p : \wedge^{n-1}\mathfrak{G} \otimes \overset{p}{\cdots} \otimes \wedge^{n-1}\mathfrak{G} \wedge \mathfrak{G} \to \mathbb{R}$ (*e.g.* in the cohomology suitable for central extensions of FAs), rather than $\alpha^p : \wedge^p \mathfrak{g} \to \mathbb{R}$ as they would be for Lie algebras $\mathfrak{g}$. Thus, it is convenient to label the p-cochains by the number p of arguments $\mathcal{X} \in \wedge^{n-1}\mathfrak{G}$ that they contain rather than by the number of elements of $\mathfrak{G}$ itself (the $\mathcal{X}$s were called *fundamental objects* in [12]). It has been proved recently [12] that there is a Whitehead lemma for Filippov algebras: semisimple FAs do not have non-trivial central extensions and are moreover rigid *i.e.*, they do not admit non-trivial deformations. As a result, the Whitehead lemma holds true for all n-Lie semisimple FAs, $n \geq 2$.

Besides the above finite-dimensional simple FAs there are also infinite-dimensional simple ones (see [13]), as those defined by the n-bracket bracket given by the Jacobian of functions. This bracket, which satisfies [1, 2] the FI and therefore determines an infinite-dimensional FA, had actually been considered long before by Nambu [14]. He studied specially the $n = 3$ case, as a generalization of the two-entries Poisson bracket, in an attempt to introducing a new type of dynamics beyond the standard Hamilton-Poisson one; the Nambu bracket satisfies additionally Leibniz's rule. Nambu did not write the FI that is satisfied by his bracket; this was done later by Sahoo and Valsakumar [15] who considerd it as a consistency condition for the time evolution of Nambu mechanics, as reflected by the derivation property that is expressed by the FI. The general $n > 3$ case was studied in detail by Takhtajan [16], leading to an n-ary generalization of the Poisson structures that he called *Nambu-Poisson structures.* This sparkled an extensive analysis of various issues related with them, including the notoriously difficult problem of the quantization of Nambu-Poisson mechanics that also had been discussed by Nambu himself [14] (and which, in our view, does not admit a completely satisfactory solution, see [17, 6]). In the last few years, FAs have reappeared in physics in another context, namely in the Bagger-Lambert-Gustavsson model [18, 19, 20], originally proposed as a candidate for the low-energy effective action of a system of coincident membranes in M-theory. These and other physical aspects of FAs are reviewed in [6], to which we refer for further information and references.

In this paper, however, we address a mathematical problem: the İnönü-Wigner type contractions of Filippov algebras. These are introduced and discussed in generality here. As is well known, all Filippov algebras $\mathfrak{G}$ have an associated *Lie* algebra $\mathrm{Lie}\,\mathfrak{G}$, the algebra of the inner derivations of $\mathfrak{G}$. Thus, a natural question to ask is whether there is any relation between the Lie algebra $\mathrm{Lie}\,\mathfrak{G}_c$ associated with some contraction $\mathfrak{G}_c$ of a FA $\mathfrak{G}$ and a (İnönü-Wigner [21] (IW) or a generalized Weimar-Woods [22] (W-W)) contraction of the Lie algebra $\mathrm{Lie}\,\mathfrak{G}_c$ associated with the contracted FA $\mathfrak{G}_c$. Clearly $\mathrm{Lie}\,\mathfrak{G}_c \neq (\mathrm{Lie}\,\mathfrak{G})_c$ in general, but it is still possible to compare the structure of $(\mathrm{Lie}\,\mathfrak{G})_c$ and $\mathrm{Lie}\,\mathfrak{G}_c$ for a given $\mathfrak{G}$. We shall use the simple A_{n+1} FAs to illustrate this point.

The plan of the paper is as follows. Sec. 2 briefly describes the FA structure, including the fundamental objects $\mathcal{X}$ and the simple finite-dimensional FAs. Sec. 3 contains the description of the Lie algebra associated with a FA and, in particular, considers the case of $\mathrm{Lie}\,A_{n+1} = so(n+1)$. Sec. 4 is devoted to the description of contractions $\mathfrak{G}_c$ of arbitrary FAs $\mathfrak{G}$, starting with the simplest $n = 3$ case. Sec. 4.2.1 describes the structure of the Lie algebra $\mathrm{Lie}\,\mathfrak{G}_c$ associated with a given contraction $\mathfrak{G}_c$; Sec. 4.2.2 considers the non-trivial contractions $(A_{n+1})_c$ of the simple A_{n+1} FAs and gives the structure of their associated $\mathrm{Lie}\,(A_{n+1})_c$ Lie algebras. Sec. 5 discusses the relation between $\mathrm{Lie}\,\mathfrak{G}_c$ and $(\mathrm{Lie}\,\mathfrak{G})_c$. To this end, we find the IW and W-W contractions of $\mathrm{Lie}\,\mathfrak{G}$ for the simple FAs A_{n+1} that follow the patterns suggested by the structure $\mathrm{Lie}\,\mathfrak{G}_c$, and then compare the results with it. Finally, Sec. 6 contains some conclusions.

All FAs considered below are real and finite-dimensional.

2 n-Lie or Filippov algebras

A *Filippov algebra* (FA) [1, 3] or *n-Lie algebra* $\mathfrak{G}$ (see also [4, 2, 5] and *e.g.* [6] for a review and further references) is a vector space endowed with a n-linear fully skewsymmetric map $[\ ,\ ,\overset{n}{\cdots},\ ,\ ] : \mathfrak{G} \times \overset{n}{\cdots} \times \mathfrak{G} \to \mathfrak{G}$ such that the *Filippov identity* (FI),

$$[X_{l_1}, \ldots, X_{l_{n-1}}, [Y_{k_1}, \ldots, Y_{k_n}]] = [[X_{l_1}, \ldots, X_{l_{n-1}}, Y_{k_1}], Y_{k_2}, \ldots, Y_{k_n}] \tag{1}$$

$$+[Y_{k_1},[X_{l_1},\ldots,X_{l_{n-1}},Y_{k_2}],Y_{k_3},\ldots,Y_{k_n}]+\ldots+[Y_{k_1},\ldots,Y_{k_{n-1}},[X_{l_1},\ldots,X_{l_{n-1}},Y_{k_n}]]\ ,$$

or, equivalently [23, 20, 6]

$$[[X_{[k_1},X_{k_2},\ldots,X_{k_n}],X_{l_1}],\ldots,X_{l_{n-1}}]=0\ , \tag{2}$$

is satisfied. Both the vector space and the FA structure will be denoted by the same symbol $\mathfrak{G}$; its meaning will be clear from the context. For $n=2$ the FI becomes the Jacobi identity (JI) and the Filippov algebra $\mathfrak{G}$ is an ordinary Lie algebra $\mathfrak{g}$.

2.1 Structure constants of n-Lie algebras

Chosen a basis $\{X_l\}$ of $\mathfrak{G}$, the FA bracket may be defined by the n-Lie algebra structure constants,

$$[X_{l_1},\ldots,X_{l_n}]=f_{l_1\ldots l_n}{}^k X_k \quad , \quad l,k=1,\ldots,\mathrm{dim}\mathfrak{G}\ . \tag{3}$$

The $f_{l_1\ldots l_n}{}^k$ are fully skewsymmetric in the $l_1\ldots l_n$ indices and satisfy the condition

$$f_{k_1\ldots k_n}{}^l f_{l_1\ldots l_{n-1}l}{}^k=\sum_{i=1}^n f_{l_1\ldots l_{n-1}k_i}{}^l f_{k_1\ldots k_{i-1}lk_{i+1}\ldots k_n}{}^k\ , \tag{4}$$

which expresses the FI (1) in terms of the structure constants of $\mathfrak{G}$. The form (2) of the FI leads in coordinates to the expression[1]

$$f_{[k_1\ldots k_n}{}^l f_{l_1]l_2\ldots l_{n-1}l}{}^k=0 \quad . \tag{5}$$

2.2 Fundamental objects of a FA and their properties

In a n-Lie algebra $\mathfrak{G}$ it is convenient to introduce objects $\mathcal{X}=(X_1,\ldots,X_{n-1})$, $X_i\in\mathfrak{G}$, antisymmetric in its $(n-1)$-arguments, $\mathcal{X}\in\wedge^{n-1}(\mathfrak{G})$; they define inner derivations of the FA through the adjoint action. This is defined by

$$ad_{\mathcal{X}}:Z\mapsto ad_{\mathcal{X}}Z\equiv\mathcal{X}\cdot Z:=[X_1,\ldots,X_{n-1},Z]\ ,\quad \forall Z\in\mathfrak{G} \quad . \tag{6}$$

In terms of $ad_{\mathcal{X}}=ad_{(X_1,\ldots,X_{n-1})}$, the FI is written as

$$ad_{\mathcal{X}}[Y_{l_1},\ldots,Y_{l_n}]=\sum_{i=1}^n[Y_{l_1},\ldots,ad_{\mathcal{X}}Y_{l_i},\ldots,Y_{l_n}]\ ,\quad l=1,\ldots\dim\mathfrak{G}\ , \tag{7}$$

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[1] Eqs. (2), (5) are to be compared with the *generalized Jacobi identity (GJI)*

$$[X_{[l_1},\ldots,X_{l_{n-1}},[X_{k_1},\ldots,X_{k_n]}]]=0 \quad , \quad C_{[k_1\ldots k_n}{}^l C_{l_1l_2\ldots l_{n-1}]l}{}^k=0 \quad ,$$

n even, which is the characteristic identity that satisfies another n-ary generalization of Lie algebras, the *generalized* or *higher order Lie algebras* [24, 25, 26], which will not be considered here (see [6] for a parallel analysis of Filippov and higher order Lie algebras and their associated n-ary Poisson structures).

which expresses that $ad_{\mathcal{X}} \in \mathrm{End}\,\mathfrak{G}$ is an inner derivation of the FA n-bracket. For convenience, we refer to the $\mathcal{X} \in \wedge^{n-1}\mathfrak{G}$ as the *fundamental objects* of the n-Lie algebra $\mathfrak{G}$. Since $ad : \wedge^{n-1}\mathfrak{G} \to \mathrm{End}\,\mathfrak{G}$ may have a non-trivial kernel, the correspondence between fundamental objects and inner derivations, $\mathcal{X}_{a_1\dots a_{n-1}} \mapsto ad_{\mathcal{X}_{a_1\dots a_{n-1}}}$, is not injective in general: $\mathcal{X} \in \ker ad$ when $ad_{\mathcal{X}}$ is the trivial endomorphism of $\mathfrak{G}$ and, for instance, $\ker ad = \wedge^{n-1}\mathfrak{G}$ and ad is trivial if $\mathfrak{G}$ is abelian.

The coordinates of the $(\dim\mathfrak{G} \times \dim\mathfrak{G})$-dimensional matrix $ad_{(X_{l_1},\dots,X_{l_{n-1}})} \equiv ad_{\mathcal{X}_{l_1\dots l_{n-1}}} \in \mathrm{End}\,\mathfrak{G}$ are given by

$$ad_{(X_{l_1},\dots,X_{l_{n-1}})}{}^{l}{}_{k} = f_{l_1\dots l_{n-1}k}{}^{l} \quad , \quad ad_{(X_{l_1},\dots,X_{l_{n-1}})}X_k = [X_{l_1},\dots,X_{l_{n-1}},X_k] = f_{l_1\dots l_{n-1}k}{}^{l}X_l \ . \tag{8}$$

Then, in terms of the structure constants of the FA, the FI (7) takes the form

$$f_{l_1\dots l_n}{}^{l}ad_{(X_{k_1},\dots,X_{k_{n-1}})}X_l = (-1)^{n-i}\sum_{i=1}^{n} f_{k_1\dots k_{n-1}\,l_i}{}^{l}ad_{(Y_{l_1},\dots,Y_{l_{i-1}},Y_{l_{i+1}},\dots,Y_{l_n})}X_l \ . \tag{9}$$

Given two fundamental objects $\mathcal{X}$, $\mathcal{Y}$ their composition $\mathcal{X}\cdot\mathcal{Y} \in \wedge^{n-1}\mathfrak{G}$ is the fundamental object given by the formal sum [9]

$$\begin{aligned}\mathcal{X}\cdot\mathcal{Y} &:= \sum_{i=1}^{n-1}(Y_1,\dots,ad_{\mathcal{X}}Y_i,\dots,Y_{n-1}) \\ &= \sum_{i=1}^{n-1}(Y_1,\dots,[X_1,\dots,X_{n-1},Y_i],\dots,Y_{n-1}) \ ,\end{aligned} \tag{10}$$

which is the natural extension on $\mathcal{Y} \in \wedge^{n-1}\mathfrak{G}$ of the action of the adjoint derivative $ad_{\mathcal{X}}$ on $\mathfrak{G}$; thus, eq. (10) may be rewritten as

$$\mathcal{X}\cdot\mathcal{Y} = ad_{\mathcal{X}}\mathcal{Y} \quad . \tag{11}$$

The composition of fundamental objects is not associative. In fact, due to the FI, the dot product of fundamental objects $\mathcal{X}$ of a n-Lie algebra $\mathfrak{G}$ satisfies the relation[2]

$$\mathcal{X}\cdot(\mathcal{Y}\cdot\mathcal{Z}) - \mathcal{Y}\cdot(\mathcal{X}\cdot\mathcal{Z}) = (\mathcal{X}\cdot\mathcal{Y})\cdot\mathcal{Z} \qquad \forall\mathcal{X},\ \mathcal{Y},\ \mathcal{Z} \in \wedge^{n-1}\mathfrak{G} \quad , \tag{12}$$

and, as a result,

$$\begin{aligned}&\mathcal{X}\cdot(\mathcal{Y}\cdot Z) - \mathcal{Y}\cdot(\mathcal{X}\cdot Z) = (\mathcal{X}\cdot\mathcal{Y})\cdot Z \qquad \text{or, equivalently,} \\ &\quad ad_{\mathcal{X}}ad_{\mathcal{Y}}\,Z - ad_{\mathcal{Y}}ad_{\mathcal{X}}\,Z = ad_{\mathcal{X}\cdot\mathcal{Y}}\,Z \quad \forall\mathcal{X},\ \mathcal{Y}\ \in \wedge^{n-1}\mathfrak{G}\ ,\ \forall Z \in \mathfrak{G} \quad .\end{aligned} \tag{13}$$

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[2] In the case of Lie algebras, $n = 2$, $\mathcal{X}$ reduces to a single element $X \in \mathfrak{g}$, $X\cdot Y = [X,Y]$ and, of course, $X\cdot(Y\cdot Z) - Y\cdot(X\cdot Z) = (X\cdot Y)\cdot Z$ is simply the Jacobi identity, $[X,[Y,Z]] - [Y,[X,Z]] = [[X,Y],Z]$.

Thus, the FI may be written as

$$[ad_{\mathcal{X}}, ad_{\mathcal{Y}}] = ad_{\mathcal{X}\cdot\mathcal{Y}} \quad ; \tag{14}$$

clearly, $ad_{(ad_{\mathcal{X}}\mathcal{Y})} = ad_{\mathcal{X}\cdot\mathcal{Y}}$. Note that although in general $\mathcal{X}\cdot\mathcal{Y} \neq -\mathcal{Y}\cdot\mathcal{X}$, eq. (13) is $\mathcal{X} \leftrightarrow \mathcal{Y}$ skewsymmetric, $ad_{\mathcal{X}\cdot\mathcal{Y}} = -ad_{\mathcal{Y}\cdot\mathcal{X}}$.

2.3 The simple Euclidean FAs

The simple, finite, $(n+1)$-dimensional n-Lie algebras constructed over $(n+1)$-dimensional vector spaces were already given in [1], and found to be the only simple ones in [5]. For the purposes of this paper it will be sufficient to consider the Euclidean n-Lie algebras, constructed over $(n+1)$-dimensional Euclidean spaces. The Euclidean FAs A_{n+1} [1] are given by eq. (3) where

$$f_{l_1 \ldots l_n}{}^k = \epsilon_{l_1 \ldots l_n}{}^k \quad ; \tag{15}$$

the pseudoeuclidean FAs are simply obtained by adding appropriate signs (it will be sufficient for our purposes here to restrict ourselves to (15) when dealing with simple FAs). Lowering the index k with Euclidean metric the structure constants are given by the fully skewsymmetric tensor of an Euclidean $(n+1)$-dimensional vector space.

It is not difficult to check that these algebras are indeed simple. Clearly, $[\mathfrak{G}, \ldots, \mathfrak{G}] \neq \{0\}$ (in fact, $[\mathfrak{G}, \ldots, \mathfrak{G}] = \mathfrak{G}$) and they do not contain any non-trivial ideal (a subspace I of a FA $\mathfrak{G}$ is an ideal [1, 5] if $[\mathfrak{G}, \overset{n-1}{\ldots}, \mathfrak{G}, I] \subset I$). Further, the FI is satisfied; we present here a short proof. For $n=3$, $\mathfrak{G} = A_4$, the four terms in the FI

$$\begin{aligned} &[X_{l_1}, X_{l_2}, [Y_{k_1}, Y_{k_2}, Y_{k_3}]] = \\ &[[X_{l_1}, X_{l_2}, Y_{k_1}], Y_{k_2}, Y_{k_3}] + [Y_{k_1}, [X_{l_1}, X_{l_2}, Y_{k_2}], Y_{k_3}] + [Y_{k_1}, Y_{k_2}, [X_{l_1}, X_{l_2}, Y_{k_3}]] \quad ; \end{aligned} \tag{16}$$

are all zero unless two k indices are equal to the two l ones, $k_1\, k_2 = l_1\, l_2$, say, in which case is obviously satisfied since it reduces to

$$[X_{l_1}, X_{l_2}, [X_{l_1}, X_{l_2}, Y_{k_3}]] = [X_{l_1}, X_{l_2}, [X_{l_1}, X_{l_2}, Y_{k_3}]] \quad , \tag{17}$$

which are the only terms that survive since $[X_{l_1}, X_{l_2}, Y_{l_3}] = \epsilon_{l_1 l_2 l_3}{}^{l_4} X_{l_4}$. This argument is easily extended to general n. Since there are $2n-1$ entries in the double n-bracket and $n+1$ elements in the basis $\{X_l\}$ of A_{n+1}, at least $n-2$ elements are necessarily repeated in the double bracket. Thus, since the separate $(n-1)$ and n entries in each part of the double bracket cannot have a repeated element due to the skewsymmetry, we see that the two parts must have at least $n-2$ equal entries, $[X_{l_1}, \ldots, X_{l_{n-2}}, X_{l_{n-1}}, [X_{k_1}, \ldots, X_{k_{n-2}}, X_{k_{n-1}}, X_{k_n}]]$ with $(l_1 \ldots l_{n-2}) = (k_1 \ldots k_{n-2})$, say. If they only share these $n-2$ entries all the $n+1$ basis elements will be present in the double bracket, and then the inner n-bracket will necessarily give rise to an element already present as one of the other $n-1$ entries in the outer bracket, giving zero. If they share $n-1$ entries *e.g.*, $k_1 \dots k_{n-1} = l_1 \dots l_{n-1}$, the only non-zero terms in the FI (3) are the two that do not mix the $k_1 \dots k_{n-1}$ indices with the $l_1 \dots l_{n-1}$ ones, which give the trivial identity

$$[X_{l_1}, \dots, X_{l_{n-1}}, [X_{l_1}, \dots, X_{l_{n-1}}, Y_{k_n}]] = [X_{l_1}, \dots, X_{l_{n-1}}, [X_{l_1}, \dots, X_{l_{n-1}}, Y_{k_n}]] \ . \tag{18}$$

Thus, the FI is satisfied by the simple $(n+1)$-dimensional FAs.

When $\mathfrak{G}$ is simple, the composition of two fundamental objects $\mathcal{X} = (X_{k_1}, \dots, X_{k_{n-1}}) = \mathcal{X}_{k_1 \dots k_{n-1}}$ and $\mathcal{Y} = (X_{j_1}, \dots, X_{j_{n-1}})$ is antisymmetric, $\mathcal{X} \cdot \mathcal{Y} = -\mathcal{Y} \cdot \mathcal{X}$. To prove this we take again into account the form (15) of the structure constants. Indeed, in

$$\begin{aligned} \mathcal{X}_{k_1 \dots k_{n-1}} \cdot \mathcal{Y}_{j_1 \dots j_{n-1}} &= \sum_{i=1}^{n-1} (X_{j_1}, \dots, [X_{k_1}, \dots, X_{k_{n-1}}, X_{j_i}], \dots, X_{j_{n-1}}) \\ &= \sum_{i=1}^{n-1} \epsilon_{k_1 \dots k_{n-1} \, j_i}{}^{l} (X_{j_1}, \dots, X_{j_{i-1}}, X_l, X_{j_{i+1}} \dots, X_{j_{n-1}}) \end{aligned} \tag{19}$$

the only nonvanishing terms will be those in which $n-2$ of the indices $k_1 \, \dots \, k_{n-1}$ are equal to $n-2$ of the indices $j_1 \, \dots \, j_{n-1}$, since there are $n+1$ basis elements and the indices j_i, l, in eq. (19) must be different from both $k_1 \, \dots \, k_{n-1}$ and $j_1 \, \dots \, \hat{j_i} \dots \, j_{n-1}$. Taking, $j_i = k_i, \ i = 1, \dots, n-2$, we see that

$$\mathcal{X}_{k_1 \dots k_{n-1}} \cdot \mathcal{Y}_{k_1 \dots k_{n-2} \, j_{n-1}} = \epsilon_{k_1 \dots k_{n-1} \, j_{n-1}}{}^{l} (X_{k_1}, \dots, X_{k_{n-2}}, X_l) = -\mathcal{Y}_{k_1 \dots k_{n-2} \, j_{n-1}} \cdot \mathcal{X}_{k_1 \dots k_{n-1}} \, , \tag{20}$$

as we wanted to prove.

3 The Lie algebra Lie $\mathfrak{G}$ associated to a n-Lie algebra $\mathfrak{G}$

The inner or adjoint derivations $ad_{\mathcal{X}} \in \mathrm{End}\,\mathfrak{G}$ associated with the fundamental objects $\mathcal{X} \in \wedge^{n-1}\mathfrak{G}$ determine an ordinary Lie algebra for the bracket in $\mathrm{End}\,\mathfrak{G}$,

$$ad_{\mathcal{X}} ad_{\mathcal{Y}} - ad_{\mathcal{Y}} ad_{\mathcal{X}} = [ad_{\mathcal{X}}, ad_{\mathcal{Y}}] = ad_{(\mathcal{X} \cdot \mathcal{Y})} \quad . \tag{21}$$

Indeed, they satisfy the JI since, using eq. (12) and that $ad_{\mathcal{X} \cdot \mathcal{Y}} = -ad_{\mathcal{Y} \cdot \mathcal{X}}$,

$$\begin{aligned} &[ad_{\mathcal{X}}, [ad_{\mathcal{Y}}, ad_{\mathcal{Z}}]] + [ad_{\mathcal{Y}}, [ad_{\mathcal{Z}}, ad_{\mathcal{X}}]] + [ad_{\mathcal{Z}}, [ad_{\mathcal{X}}, ad_{\mathcal{Y}}]] \\ &= ad_{\mathcal{X} \cdot (\mathcal{Y} \cdot \mathcal{Z})} + ad_{\mathcal{Y} \cdot (\mathcal{Z} \cdot \mathcal{X})} + ad_{\mathcal{Z} \cdot (\mathcal{X} \cdot \mathcal{Y})} = ad_{\mathcal{X} \cdot (\mathcal{Y} \cdot \mathcal{Z}) - \mathcal{Y} \cdot (\mathcal{X} \cdot \mathcal{Z}) - (\mathcal{X} \cdot \mathcal{Y}) \cdot \mathcal{Z}} = 0 \qquad . \end{aligned} \tag{22}$$

This is the Lie algebra $\mathrm{Lie}\,\mathfrak{G} \equiv \mathrm{InDer}\,\mathfrak{G} \subset \mathrm{End}\,\mathfrak{G}$ of inner derivations associated with the FA $\mathfrak{G}$. Clearly, $\dim \mathrm{Lie}\,\mathfrak{G} = \begin{pmatrix} \dim \mathfrak{G} \\ n-1 \end{pmatrix} - \dim(\ker ad)$.

If $\mathfrak{G}$ is the simple FA A_{n+1}, all derivations are inner; further, $\mathrm{Lie}\, A_{n+1} = so(n+1)$ (see *e.g.* [5, 6]) and, of course, $\begin{pmatrix} n+1 \\ n-1 \end{pmatrix} = \dim so(n+1)$.

3.1 Structure constants of Lie $\mathfrak{G}$ for a 3-Lie algebra

For $n = 3$ the coordinates of the $\dim\mathfrak{G}\times\dim\mathfrak{G}$-dimensional matrix $[X_{l_1}, X_{l_2}, \quad] \equiv ad_{(X_{l_1},X_{l_2})} \in \mathrm{End}\,\mathfrak{G}$ are given by

$$ad_{(X_{l_1},X_{l_2})}{}^{l}{}_{k} = f_{l_1l_2k}{}^{l} \quad , \quad ad_{(X_{l_1},X_{l_2})}X_k = [X_{l_1}, X_{l_2}, X_k] = f_{l_1l_2k}{}^{l}X_l \ . \tag{23}$$

Then, the form (14) of the FI for $n = 3$

$$ad_{(X_{l_1},X_{l_2})}(ad_{(Y_{k_1},Y_{k_2})}) - ad_{(Y_{k_1},Y_{k_2})}(ad_{(X_{l_1},X_{l_2})}) = ad_{([X_{l_1},X_{l_2},Y_{k_1}],Y_{k_2})+(Y_{k_1},[X_{l_1},X_{l_2},Y_{k_2}])}$$

can be written as

$$[ad_{(X_{l_1},X_{l_2})}, ad_{(Y_{k_1},Y_{k_2})}]^{l}{}_{k} = f_{l_1l_2k_1}{}^{j}f_{jk_2l}{}^{k} + f_{l_1l_2k_2}{}^{j}f_{k_1jl}{}^{k} = -f_{l_1l_2[k_1}{}^{j}f_{k_2]jl}{}^{k} \ . \tag{24}$$

This shows antisymmetry under the interchange of the indices (k_1k_2) and (l_1l_2), *i.e.*,

$$f_{l_1l_2[k_1}{}^{j}f_{k_2]jl}{}^{k} = -f_{k_1k_2[l_1}{}^{j}f_{l_2]jl}{}^{k} \ ,$$

which also follows directly from the FI $f_{[k_1k_2l_1}{}^{j}f_{l_2]lj}{}^{k} = 0$ (eq. (5)).

Using eq. (23) we can write

$$f_{l_1l_2[k_1}{}^{j}ad_{X_{k_2]},X_j)} = -f_{k_1k_2[l_1}{}^{j}ad_{X_{l_2]},X_j)} \quad , \tag{25}$$

or, equivalently,

$$(f_{l_1l_2[k_1}{}^{j}\delta^{l}_{k_2]} + f_{k_1k_2[l_1}{}^{j}\delta^{l}_{l_2]})ad_{(X_l,X_j)} = 0 \ . \tag{26}$$

Using eq. (24), the commutators of Lie $\mathfrak{G}$ can be expressed as

$$[ad_{(X_{l_1},X_{l_2})}, ad_{(Y_{k_1},Y_{k_2})}]^{l}{}_{k} = \frac{1}{2}C_{l_1l_2k_1k_2}{}^{j_1j_2}ad_{(X_{j_1},X_{j_2})}{}^{l}{}_{k} \quad , \quad C_{l_1l_2k_1k_2}{}^{j_1j_2} = f_{l_1l_2[k_1}{}^{[j_1}\delta^{j_2]}_{k_2]} \ . \tag{27}$$

However, this does not mean (see also [20]) that the above C's are the structure constants of Lie $\mathfrak{G}$. Although the *r.h.s.* of eq. (24) is $(l_1l_2) \leftrightarrow (k_1k_2)$ skewsymmetric as mandated by the *l.h.s.*, this does not necessarily imply that the constants $C_{l_1l_2k_1k_2}{}^{j_1j_2}$ in eq. (27) retain this property once the sum over (j_1j_2) is removed. One may, of course, write antisymmetric C's in eq. (27) by taking

$$C_{l_1l_2k_1k_2}{}^{j_1j_2} = \frac{1}{2}\left(f_{l_1l_2[k_1}{}^{[j_1}\delta^{j_2]}_{k_2]} - (l \leftrightarrow k)\right) \ , \tag{28}$$

but this is not sufficient to look at them as structure constants of Lie $\mathfrak{G}$ since, in general, the indices (j_1j_2) that characterize $\mathcal{X}_{j_1j_2} = (X_{j_1}, X_{j_2})$ are not suitable to label the matrices

$ad_{(X_{j_1},X_{j_2})}$. Since $\mathcal{X}_{k_1k_2} \neq \mathcal{X}_{l_1l_2} \nRightarrow ad_{\mathcal{X}_{k_1k_2}} \neq ad_{\mathcal{X}_{l_1l_2}}$ in general, the (j_1, j_2)-labelled $ad_{(X_{j_1},X_{j_2})}$ may not be a basis of Lie $\mathfrak{G}$.

The Jacobi identity is of course satisfied by the endomorphisms $ad_{(X_{s_1},X_{s_2})}$ of $\mathfrak{G}$:

$$\sum_{cycl.\,(j_1j_2),(k_1k_2),(l_1l_2)} (C_{j_1j_2k_1k_2}{}^{r_1r_2} C_{l_1l_2r_1r_2}{}^{s_1s_2})\,(ad_{(X_{s_1},X_{s_2})})^l{}_k = 0\ , \tag{29}$$

$$(C_{j_1j_2k_1k_2}{}^{s_1s_2} + C_{k_1k_2j_1j_2}{}^{r_1r_2})\,(ad_{(X_{s_1},X_{s_2})})^l{}_k = 0\ , \tag{30}$$

but the $ad_{(X_{s_1},X_{s_2})}$ cannot be removed from eqs. (29) and (30).

Nevertheless,

$$\sum_{cycl.\,(j_1j_2),(k_1k_2),(l_1l_2)} (C_{j_1j_2k_1k_2}{}^{r_1r_2} C_{l_1l_2r_1r_2}{}^{s_1s_2}) = 0 \tag{31}$$

(cf. (29)) holds if the structure constants C in eq. (27) are already skewsymmetric under the interchange $l_1l_2 \leftrightarrow k_1k_2$ *i.e.*, when

$$f_{k_1k_2[l_1}{}^{[j_1}\delta^{j_2]}_{l_2]} = -f_{l_1l_2[k_1}{}^{[j_1}\delta^{j_2]}_{k_2]}\ . \tag{32}$$

We will see below that this is the case for simple n-Lie algebras, for which *e.g.* eq. (15) holds, ad is injective and the matrices $ad_{(X_{j_1},X_{j_2})}$ define a basis of the associated Lie algebra. For instance, when $n = 3$ it is easy to see that for A_4

$$\epsilon_{k_1k_2[l_1}{}^{[j_1}\delta^{j_2]}_{l_2]} = -\epsilon_{l_1l_2[k_1}{}^{[j_1}\delta^{j_2]}_{k_2]}\ , \tag{33}$$

since for $\epsilon_{k_1k_2[l_1}{}^{[j_1}\delta^{j_2]}_{l_2]}$ to be different from zero we need that one of the indices k_1, k_2 is equal to l_1 or l_2, say $k_2 = l_2$, and then $\epsilon_{k_1l_2[l_1}{}^{[j_1}\delta^{j_2]}_{l_2]} = -\epsilon_{l_1l_2[k_1}{}^{[j_1}\delta^{j_2]}_{l_2]}$ by the antisymmetry of the elements in ϵ. Then, using the relation (33) and the FI (4) for $n = 3$, the JI in eq. (31) follows.

3.2 The general n-Lie case

Let now $\mathfrak{G}$ be a n-Lie algebra, and $ad_{(X_{k_1},\ldots,X_{k_{n-1}})}$ the inner derivations associated with the fundamental objects $\mathcal{X}_{k_1\ldots k_{n-1}}$,

$$ad_{(X_{k_1},\ldots,X_{k_{n-1}})} : Z \to [X_{k_1}, \ldots, X_{k_{n-1}}, Z] \in \mathfrak{G}\ .$$

The $ad_{\mathcal{X}}$ determine the Lie algebra Lie $\mathfrak{G}$ associated with the FA $\mathfrak{G}$. In terms of components, the commutators of the elements $ad_{\mathcal{X}} \in$ Lie $\mathfrak{G}$ can be written as:

$$[ad_{\mathcal{X}}, ad_{\mathcal{Y}}] \;=\; [ad_{(X_{k_1},\ldots,X_{k_{n-1}})}, ad_{(X_{j_1},\ldots,X_{j_{n-1}})}] = \frac{1}{2} ad_{(\mathcal{X}\cdot\mathcal{Y} - \mathcal{Y}\cdot\mathcal{X})} =$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{i=1}^{n-1}\Big(f_{k_1\dots k_{n-1}\,j_i}{}^{l}ad_{(X_{j_1},\dots,X_{j_{i-1}},X_l,X_{j_{i+1}}\dots,X_{j_{n-1}})} \\
&\qquad -f_{j_1\dots j_{n-1}\,k_i}{}^{l}ad_{(X_{k_1},\dots,X_{k_{i-1}},X_l,X_{k_{i+1}}\dots,X_{k_{n-1}})}\Big) \\
&\equiv \frac{1}{(n-1)!}C_{k_1\dots k_{n-1}\,j_1\dots j_{n-1}}{}^{l_1\dots l_{n-1}}ad_{(X_{l_1},\dots,X_{l_{n-1}})}\ ,
\end{aligned}
\tag{34}
$$

where we have taken

$$
C_{k_1\dots k_{n-1}\,j_1\dots j_{n-1}}{}^{l_1\dots l_{n-1}} = \frac{1}{2(n-2)!}\Big(f_{k_1\dots k_{n-1}\,[j_1}{}^{[l_1}\delta^{l_2}_{j_2}\dots\delta^{l_{n-1}]}_{j_{n-1}]} - (k\leftrightarrow j)\Big)\quad , \tag{35}
$$

so that they are antisymmetric under the permutation of the indices $(k_1,\dots,k_{n-1})$ and $(j_1,\dots,j_{n-1})$.

The Jacobi identity for Lie $\mathfrak{G}$ (cf. eq. (29)) reads

$$
\sum_{cycl.\ j,k,l} C_{j_1\dots j_{n-1}\,k_1\dots k_{n-1}}{}^{h_1\dots h_{n-1}}C_{l_1\dots l_{n-1}\,h_1\dots h_{n-1}}{}^{i_1\dots i_{n-1}}ad_{(X_{i_1},\dots,X_{i_{n-1}})} = 0\ . \tag{36}
$$

As in the $n=3$ case, it is possible to remove the $ad_{\mathcal{X}}$ above when $\{ad_{(X_{i_1},\dots,X_{i_{n-1}})}\}$ is a basis of Lie $\mathfrak{G}$, *i.e.*, when ad is injective. This is the case for the simple FAs, for which the terms $f_{k_1\dots k_{n-1}\,[j_1}{}^{[l_1}\delta^{l_2}_{j_2}\dots\delta^{l_{n-1}]}_{j_{n-1}]}$ are skewsymmetric under the interchange $(k_1,\dots,k_{n-1})\leftrightarrow(j_1,\dots,j_{n-1})$. The proof is familiar by now (see Sec. 2.3): the only non-vanishing structure constants of Lie $\mathfrak{G}$ for a simple FA are of the form $C_{k_1\dots k_{n-1}\,j_1\dots j_{n-1}}{}^{l_1\dots l_{n-1}}$ with $n-2$ of the indices $k_1\dots k_{n-1}$ equal to $n-2$ of the indices $j_1\dots j_{n-1}$. Taking again $k_i=j_i$, $i=1,\dots,n-2$, it follows that

$$
\begin{aligned}
C_{k_1\dots k_{n-2}\,k_{n-1}\,k_1\dots k_{n-2}\,j_{n-1}}{}^{l_1\dots l_{n-1}} &= \frac{1}{(n-2)!2}\Big(-\epsilon_{k_1\dots k_{n-1}\,[j_{n-1}}{}^{[l_1}\delta^{l_2}_{k_2}\dots\delta^{l_{n-2}}_{k_{n-2}}\delta^{l_{n-1}]}_{k_1]} + (k\leftrightarrow j)\Big) = \\
\frac{1}{(n-2)!}\epsilon_{k_1\dots k_{n-2}\,j_{n-1}\,[k_{n-1}}{}^{[l_1}\delta^{l_2}_{k_2}\dots\delta^{l_{n-2}}_{k_{n-2}}\delta^{l_{n-1}]}_{k_1]} &= -C_{k_1\dots k_{n-2}\,j_{n-1}\,k_1\dots k_{n-2}\,k_{n-1}}{}^{l_1\dots l_{n-1}}.
\end{aligned}
\tag{37}
$$

3.3 A trivial example: Lie $A_4 = so(4)$

Since this case will be used later on, consider A_4. It is given by

$$
[X_{j_1},X_{j_2},X_{j_3}] = \epsilon_{j_1\,j_2\,j_3}{}^{j_4}X_{j_4}\ ,\quad j=1,2,3,4\,. \tag{38}
$$

Lie A_4 is given by the commutators (cf. eq. (27))

$$
\begin{aligned}
[ad_{(X_{k_1},X_{k_2})},ad_{(X_{l_1},X_{l_2})}] &= ad_{([X_{k_1},X_{k_2},X_{l_1}],X_{l_2})} + ad_{(X_{l_1},[X_{k_1},X_{k_2},X_{l_2}])} = \\
&= \epsilon_{k_1\,k_2\,l_1}{}^{l}ad_{(X_l,X_{l_2})} + \epsilon_{k_1\,k_2\,l_2}{}^{l}ad_{(X_{l_1},X_l)} = \frac{1}{2}C_{k_1\,k_2\,l_1\,l_2}{}^{j_1\,j_2}ad_{(X_{j_1},X_{j_2})}\ ,
\end{aligned}
\tag{39}
$$

where the structure constants of Lie A_4 are given by

$$C_{k_1\,k_2\,l_1\,l_2}{}^{j_1\,j_2} = -C_{l_1\,l_2\,k_1\,k_2}{}^{j_1\,j_2} = \epsilon_{k_1\,k_2\,[l_1}{}^{[j_1}\delta^{j_2]}_{l_2]}\;; \tag{40}$$

they may be non-zero only if one of the indices k_1, k_2 is equal to one of the indices l_1, l_2, as seen in Sec. 3.1.

Let the $\mathfrak{G}$ vector space be split into the space $\mathfrak{G}_0$ generated by one generator, say X_4, and the subspace $\mathfrak{V}$ generated by the remaining elements of the A_4 basis,

$$\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{V} \quad , \quad \mathfrak{G}_0 = \langle X_4 \rangle \quad , \quad \mathfrak{V} = \langle X_u,\ u = 1,2,3 \rangle . \tag{41}$$

This type of splitting will prove useful when considering the contractions of $\mathfrak{G}$ since $\mathfrak{G}_0$ is obviously a subalgebra of $\mathfrak{G}$. To look at Lie A_4 we split its vector space into subspaces $\langle ad_{(X_{j_1},X_{j_2})} \rangle$ according to the number of elements of $\mathfrak{V}$ that appear in the fundamental objects in the inner derivations $ad_{(X_{j_1},X_{j_2})}$ that generate each of them. Then,

$$\begin{aligned}
\mathcal{W}^{(0)} &= \langle ad_{(X_4,X_4)} \rangle = \{0\}\,, \\
\mathcal{W}^{(1)} &= \langle ad_{(X_4,X_u)} \rangle = \langle ad_{(X_4,X_1)}, ad_{(X_4,X_2)}, ad_{(X_4,X_3)} \rangle\,, \\
\mathcal{W}^{(2)} &= \langle ad_{(X_{u_1},X_{u_2})} \rangle = \langle ad_{(X_2,X_3)}, ad_{(X_3,X_1)}, ad_{(X_1,X_2)} \rangle\,,
\end{aligned} \tag{42}$$

where we have included $\mathcal{W}^{(0)}$ separately although here reduces trivially to the zero element of Lie A_4. The commutation relations of Lie A_4,

$$[ad_{\mathcal{X}^{(1)}_{4\,u_1}}, ad_{\mathcal{Y}^{(1)}_{4\,u_2}}] \equiv [ad_{(X_4,X_{u_1})}, ad_{(X_4,X_{u_2})}] = \epsilon_{4\,u_1\,u_2}{}^{u} ad_{(X_4,X_u)} \in \mathcal{W}^{(1)} \tag{43}$$

$$[ad_{\mathcal{X}^{(1)}_{4\,u_1}}, ad_{\mathcal{Y}^{(2)}_{u_1\,v_2}}] \equiv [ad_{(X_4,X_{u_1})}, ad_{(X_{u_1},X_{v_2})}] = \epsilon_{4\,u_1\,v_2}{}^{u} ad_{(X_{u_1},X_u)} \in \mathcal{W}^{(2)} \tag{44}$$

$$[ad_{\mathcal{X}^{(2)}_{u_1\,u_2}}, ad_{\mathcal{Y}^{(2)}_{u_1\,v_2}}] \equiv [ad_{(X_{u_1},X_{u_2})}, ad_{(X_{u_1},X_{v_2})}] = \epsilon_{u_1\,u_2\,v_2}{}^{4} ad_{(X_{u_1},X_4)} \in \mathcal{W}^{(1)}\ , \tag{45}$$

show that $\mathcal{W}^{(1)}$ is a $so(3)$ subalgebra. Renaming the elements of Lie A_4 as

$$\begin{aligned}
&Y_1 = ad_{(X_4,X_1)},\ Y_2 = ad_{(X_4,X_2)},\ Y_3 = ad_{(X_4,X_3)}\,, \\
&Z_1 = ad_{(X_2,X_3)},\ Z_2 = ad_{(X_3,X_1)},\ Z_3 = ad_{(X_1,X_2)}\,,
\end{aligned} \tag{46}$$

eqs. (43)-(45) can be written as

$$[Y_i, Y_j] = \epsilon_{ij}{}^{k} Y_k, \quad [Y_i, Z_j] = \epsilon_{ij}{}^{k} Z_k, \quad [Z_i, Z_j] = \epsilon_{ij}{}^{k} Y_k, \quad i,j,k = 1,2,3\ , \tag{47}$$

or, with $\tilde{Y}_i = \frac{1}{2}(Y_i + Z_i)$, $\tilde{Z}_i = \frac{1}{2}(Y_i - Z_i)$,

$$[\tilde{Y}_i, \tilde{Y}_j] = \epsilon_{ij}{}^{k} \tilde{Y}_k, \quad [\tilde{Y}_i, \tilde{Z}_j] = 0, \quad [\tilde{Z}_i, \tilde{Z}_j] = \epsilon_{ij}{}^{k} \tilde{Z}_k, \quad i,j,k = 1,2,3\ . \tag{48}$$

In fact, as is well known, Lie $A_4 = so(4) = so(3) \oplus so(3)$ (Lie A_{n+1} is simple but for $n = 3$) and $\dim \text{Lie}\, A_4 = 6$.

4 Contractions of FAs

4.1 The case of 3-Lie algebras

As is well known, the İnönü-Wigner (IW) contraction [21] of a Lie algebra $\mathfrak{g}$ is performed with respect to a subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ by rescaling the generators of the coset $\mathfrak{g}/\mathfrak{g}_0$ and then taking the contraction limit for the scaling parameter; this guarantees that the result is also a Lie algebra, $\mathfrak{g}_c$. Let $\mathfrak{g}$ be defined by

$$[X_{l_1}, X_{l_2}] = f_{l_1\,l_2}{}^{l} X_l\,, \quad X_l \in \mathfrak{g}\,, \quad l = 1, \dots \dim \mathfrak{g} \quad , \tag{49}$$

and split its underlying vector space as the sum $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{v}$,

$$\mathfrak{g}_0 = \{X_a,\, a = 1, \dots, \dim \mathfrak{g}_0\}\,, \quad \mathfrak{v} = \{X_u,\, u = \dim \mathfrak{g}_0 + 1, \dots, \dim \mathfrak{g}\}\ .$$

Then, redefining the basis of $\mathfrak{v}$ as $X'_u = \epsilon X_u$ and taking the limit $\epsilon \to 0$ the contracted algebra $\mathfrak{g}_c$ is obtained. The generators of $\mathfrak{v} = \mathfrak{g}/\mathfrak{g}_0$ become abelian in $\mathfrak{g}_c$, the preserved subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}_c$ acts on them and $\mathfrak{g}_c$ has the semidirect structure $\mathfrak{g}_c = \mathfrak{v} \niplus \mathfrak{g}_0$, where $\mathfrak{v}$ is an abelian ideal of $\mathfrak{g}_c$. Obviously, the IW contraction is dimension preserving.

To generalize the contraction procedure to $n > 2$ Filippov algebras, consider first the simplest case of a 3-Lie algebra $\mathfrak{G}$ given by

$$[X_{l_1}, X_{l_2}, X_{l_3}] = f_{l_1\,l_2\,l_3}{}^{l} X_l \quad , \quad X_l \in \mathfrak{G},\ l = 1, \dots \dim \mathfrak{G}\,. \tag{50}$$

The three-bracket satisfies the FI (eq. (7)), namely

$$ad_{\mathcal{X}}[Y_{k_1}, Y_{k_2}, Y_{k_3}] = [ad_{\mathcal{X}} Y_{k_1}, Y_{k_2}, Y_{k_3}] + [Y_{k_1}, ad_{\mathcal{X}} Y_{k_2}, Y_{k_3}] + [Y_{k_1}, Y_{k_2}, ad_{\mathcal{X}} Y_{k_3}]\ . \tag{51}$$

Let $\mathfrak{G}_0 \subset \mathfrak{G}$ be a Filippov subalgebra, $[\mathfrak{G}_0, \mathfrak{G}_0, \mathfrak{G}_0] \subset \mathfrak{G}_0$ and let us split the $\mathfrak{G}$ vector space as

$$\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{V}\,, \quad \begin{array}{l} \mathfrak{G}_0 = \langle X_a \rangle\,,\ a \in I_0 = \{1, \dots, \dim\mathfrak{G}_0\} \\ \mathfrak{V}\ = \langle X_u \rangle\,,\ u \in I_1 = \{\dim\mathfrak{G}_0 + 1, \dots, \dim\mathfrak{G}\} \quad , \end{array} \tag{52}$$

where the indices a, b, c label the elements of a basis of $\mathfrak{G}_0$, u, v, w refer to the basis of $\mathfrak{V}$ and the indices j, k, l refer to the basis of the FA $\mathfrak{G}$,

$$X_a, X_b, X_c \in \mathfrak{G}_0\ , \quad X_u, X_v, X_w \in \mathfrak{V}\ , \quad X_j, X_k, X_l \in \mathfrak{G}\,. \tag{53}$$

We now define the contraction with respect to the Filippov subalgebra $\mathfrak{G}_0$ by rescaling the basis elements of $\mathfrak{V}$, $X'_a = X_a$, $X'_u = \epsilon X_u$. The four types of brackets in the new, primed basis of $\mathfrak{G}$ are

$$[X'_{a_1}, X'_{a_2}, X'_{a_3}] = f_{a_1\,a_2\,a_3}{}^{a} X'_a \quad + \underbrace{\epsilon^{-1} f_{a_1\,a_2\,a_3}{}^{u} X'_u}_{=0\ (\mathfrak{G}_0\ \text{subalgebra})}\,, \tag{54}$$

$$
\begin{aligned}
[X'_{a_1}, X'_{a_2}, X'_{u_1}] &= \epsilon f_{a_1\,a_2\,u_1}{}^{a} X'_a + f_{a_1\,a_2\,u_1}{}^{u} X'_u \ , && (55)\\
[X'_{a_1}, X'_{u_1}, X'_{u_2}] &= \epsilon^2 f_{a_1\,u_1\,u_2}{}^{a} X'_a + \epsilon f_{a_1\,u_1\,u_2}{}^{u} X'_u \ , && (56)\\
[X'_{u_1}, X'_{u_2}, X'_{u_3}] &= \epsilon^3 f_{u_1\,u_2\,u_3}{}^{a} X'_a + \epsilon^2 f_{u_1\,u_2\,u_3}{}^{u} X'_u \ . && (57)
\end{aligned}
$$

Thus, to be able to take the $\epsilon \to 0$ contraction limit it is required that $f_{a_1\,a_2\,a_3}{}^{u} = 0$ in (54) *i.e.*, $\mathfrak{G}_0$ must be a subalgebra of $\mathfrak{G}$ as originally assumed. Then, the *contracted 3-Lie algebra* $\mathfrak{G}_c$,

$$
[X'_{l_1}, X'_{l_2}, X'_{l_3}] = f'_{l_1\,l_2\,l_3}{}^{l} X'_l \ , \tag{58}
$$

is defined by the FA structure constants

$$
f'_{l_1\,l_2\,l_3}{}^{l} = \left\{ \begin{array}{ll}
f'_{a_1\,a_2\,a_3}{}^{a} = f_{a_1\,a_2\,a_3}{}^{a} \ , & a_1,\, a_2,\, a_3,\, a \,\in\, I_0 \\
f'_{a_1\,a_2\,a_3}{}^{u} = 0 \ , & a_1,\, a_2,\, a_3 \,\in I_0\, ; \;\; u \,\in\, I_1 \\
f'_{a_1\,a_2\,u_3}{}^{a} = 0 \ , & a_1,\, a_2,\, a \,\in\, I_0\, ; \;\; u_3 \,\in I_1 \\
f'_{a_1\,a_2\,u_3}{}^{u} = f_{a_1\,a_2\,u_3}{}^{u} \ , & a_1,\, a_2 \,\in I_0\, ; \;\; u_3,\, u \,\in\, I_1 \\
f'_{a_1\,u_2\,u_3}{}^{l} = 0 \ , & a_1 \,\in\, I_0\, ; \;\; u_2,\, u_3 \,\in I_1\, ; \;\; l \in I_0 \cup I_1 \\
f'_{u_1\,u_2\,u_3}{}^{l} = 0 \ , & u_1,\, u_2,\, u_3 \,\in I_1\, ; \;\; l \in I_0 \cup I_1 \quad ,
\end{array} \right. \tag{59}
$$

since the FI is obviously satisfied (this will be shown in general in Sec. 4.2). The $\mathfrak{G}_0 \subset \mathfrak{G}$ subalgebra is preserved in the contraction process and $\dim \mathfrak{G}_c = \dim \mathfrak{G}$.

Of course, once $\mathfrak{G}_c$ has been obtained the primes may be removed throughout; we shall keep them nevertheless to indicate that we refer to the structure constants of the contracted FA. Eq. (59) shows that $\mathfrak{V}$ becomes a FA ideal in $\mathfrak{G}_c$, as it is the case for the IW contraction of Lie algebras. Hence, the general structure of the contracted 3-Lie algebra $\mathfrak{G}_c$ is

$$
[\mathfrak{G}_0, \mathfrak{G}_0, \mathfrak{G}_0] \subset \mathfrak{G}_0 \;\; \Rightarrow \mathfrak{G}_0 \text{ subalgebra} \tag{60}
$$

$$
\left. \begin{array}{l}
[\mathfrak{G}_0, \mathfrak{G}_0, \mathfrak{V}] \subset \mathfrak{V} \\
[\mathfrak{G}_0, \mathfrak{V}, \mathfrak{V}] = 0 \\
[\mathfrak{V}, \mathfrak{V}, \mathfrak{V}] = 0
\end{array} \right\} \Rightarrow \mathfrak{V} \text{ abelian ideal} \quad . \tag{61}
$$

$\mathfrak{V}$ is an ideal because $[\mathfrak{G}, \mathfrak{G}, \mathfrak{V}] \subset \mathfrak{V}$ and abelian by the last equality. As a result, *the contracted FA has the semidirect structure*[3] $\mathfrak{G}_c = \mathfrak{V} \supset\!\!\!\!\!+ \ \mathfrak{G}_0$ since $\mathfrak{G}_0 \subset \mathfrak{G}_c$ acts on the (abelian) ideal $\mathfrak{V} \subset \mathfrak{G}_c$ through the adjoint action, $ad_{\mathcal{X}_0} : \mathfrak{V} \mapsto \mathfrak{V}$, $\mathcal{X}_0 \in \wedge^2 \mathfrak{G}_0$, by the first expression in eq. (61). Of course, for $n = 2$ this reproduces the familiar semidirect structure $\mathfrak{g} = \mathfrak{v} \supset\!\!\!\!\!+ \ \mathfrak{g}_0$ of the IW contraction of Lie algebras.

---

[3] We introduce the *semidirect extension of FAs* in similarity with the Lie algebra case. Let $\mathfrak{G}$ be a n-Lie algebra, $\mathfrak{G}_0$ a subalgebra and $\mathfrak{G} = \mathfrak{V} \oplus \mathfrak{G}_0$ as a vector space. Then, $\mathfrak{G}$ is the semidirect FA extension of $\mathfrak{G}_0$ by $\mathfrak{V}$, $\mathfrak{G} = \mathfrak{V} \supset\!\!\!\!\!+ \ \mathfrak{G}_0$ if $\mathfrak{V}$ is an ideal of $\mathfrak{G}$ and $\mathfrak{G}_0$ acts on it through the (adjoint) action that results from $\mathfrak{G}_0$ being a subalgebra of $\mathfrak{G}$. For $n = 2$, this recovers the semidirect sum of Lie algebras.

4.1.1 The Lie algebra Lie $\mathfrak{G}_c$ associated with the contracted 3-Lie algebra $\mathfrak{G}_c$

Let now Lie $\mathfrak{G}_c \subset \mathrm{End}\,\mathfrak{G}_c$ be the Lie algebra associated (Sec. 3) with the contracted 3-Lie algebra $\mathfrak{G}_c$. To study its structure let us split the fundamental objects of $\mathfrak{G}_c$, $\mathcal{X}' \in \wedge^2\mathfrak{G}_c$, into three types, as suggested by the splitting of the elements of $\mathfrak{G}$ itself in eq. (52),

$$\left.\begin{array}{l} \mathcal{X}'^{(0)} = (X'_{a_1}, X'_{a_2}) \\ \mathcal{X}'^{(1)} = (X'_{a_1}, X'_{u_2}) \\ \mathcal{X}'^{(2)} = (X'_{u_1}, X'_{u_2}) \end{array}\right| \qquad X'_{a_i} \in \mathfrak{G}_0\,,\ X'_{u_i} \in \mathfrak{V}\,, \quad \mathfrak{G}_c = \mathfrak{G}_0 \oplus \mathfrak{V}\,. \tag{62}$$

From the structure of $\mathfrak{G}_c$ (see eq. (59)) it follows that

$$ad_{\mathcal{X}'^{(0)}} X_a \in \mathfrak{G}_0 \tag{63}$$

$$ad_{\mathcal{X}'^{(0)}} X_u \in \mathfrak{V} \tag{64}$$

$$ad_{\mathcal{X}'^{(1)}} X_a \in \mathfrak{V} \tag{65}$$

$$ad_{\mathcal{X}'^{(1)}} X_u = 0 \tag{66}$$

$$ad_{\mathcal{X}'^{(2)}} X_l = 0\ . \tag{67}$$

The inner derivations $ad_{(X'_{l_1}, X'_{l_2})}$ above do not determine a basis of Lie $\mathfrak{G}_c$ since no contracted FA may be simple and ad is not injective for $\mathfrak{G}_c$. In particular, by eq. (67), all the elements in $\mathcal{X}'^{(2)}$ induce the zero derivation, $\mathcal{X}'^{(2)} \in \ker ad$.

Let $\mathcal{W}'^{(r)} = \langle ad_{\mathcal{X}'^{(r)}} \rangle$, $r = 0, 1, 2$, be the vector spaces generated by the inner endomorphisms associated with the fundamental objects $\mathcal{X}'^{(r)}$ in eq. (62) ($\mathcal{W}'^{(2)}$ is actually zero by eq. (67)). Lie $\mathfrak{G}_c$ is now readily determined from the structure of $\mathfrak{G}_c$, eqs. (58), (59). The different types of Lie $\mathfrak{G}_c$ commutators are, explicitly,

$$\begin{aligned} [ad_{\mathcal{X}'^{(0)}_{a_1a_2}}, ad_{\mathcal{Y}'^{(0)}_{b_1b_2}}] &= \frac{1}{2} ad_{[(X'_{a_1},X'_{a_2})\cdot(X'_{b_1},X'_{b_2}) - (X'_{b_1},X'_{b_2})\cdot(X'_{a_1},X'_{a_2})]} \\ &= \frac{1}{2}\left(f'_{a_1\,a_2\,b_1}{}^{a} ad_{(X'_a,X'_{b_2})} + f'_{a_1\,a_2\,b_2}{}^{a} ad_{(X'_{b_1},X'_a)} - f'_{b_1\,b_2\,a_1}{}^{a} ad_{(X'_a,X'_{a_2})} - f'_{b_1\,b_2\,a_2}{}^{a} ad_{(X'_{a_1},X'_a)} \right) \\ &\in \mathcal{W}'^{(0)} \end{aligned} \tag{68}$$

$$\begin{aligned} [ad_{\mathcal{X}'^{(0)}_{a_1a_2}}, ad_{\mathcal{Y}'^{(1)}_{b_1u_2}}] &= \frac{1}{2} ad_{[(X'_{a_1},X'_{a_2})\cdot(X'_{b_1},X'_{u_2}) - (X'_{b_1},X'_{u_2})\cdot(X'_{a_1},X'_{a_2})]} \\ &= \frac{1}{2}\Bigg(f'_{a_1\,a_2\,b_1}{}^{a} ad_{(X'_a,X'_{u_2})} + \underbrace{f'_{a_1\,a_2\,b_1}{}^{u}}_{=0} ad_{(X'_u,X'_{u_2})} + \underbrace{f'_{a_1\,a_2\,u_2}{}^{a}}_{=0} ad_{(X'_{b_1},X'_a)} + f'_{a_1\,a_2\,u_2}{}^{u} ad_{(X'_{b_1},X'_u)} \\ &\quad - \underbrace{f'_{b_1\,u_2\,a_1}{}^{a}}_{=0} ad_{(X'_a,X'_{a_2})} - f'_{b_1\,u_2\,a_1}{}^{u} ad_{(X'_u,X'_{a_2})} - \underbrace{f'_{b_1\,u_2\,a_2}{}^{a}}_{=0} ad_{(X'_{a_1},X'_a)} - f'_{b_1\,u_2\,a_2}{}^{u} ad_{(X'_{a_1},X'_u)} \Bigg) \end{aligned}$$

$$\in \mathcal{W}'^{(1)} \tag{69}$$

$$\begin{aligned}
[ad_{\mathcal{X}'^{(1)}_{a_1u_1}}, ad_{\mathcal{Y}'^{(1)}_{a_2u_2}}] &= \frac{1}{2} ad_{[(X'_{a_1},X'_{u_1})\cdot(X'_{a_2},X'_{u_2})-(X'_{a_2},X'_{u_2})\cdot(X'_{a_1},X'_{u_1})]} \\
&= \frac{1}{2}\Bigg(\underbrace{f'_{a_1\,u_1\,a_2}{}^{a}}_{=0} ad_{(X'_a,X'_{u_2})} + f'_{a_1\,u_1\,a_2}{}^{u} \underbrace{ad_{(X'_u,X'_{u_2})}}_{=0} + \underbrace{f'_{a_1\,u_1\,u_2}{}^{l}}_{=0} ad_{(X'_{a_2},X'_l)} \\
&\quad - \underbrace{f'_{a_2\,u_2\,a_1}{}^{a}}_{=0} ad_{(X'_a,X'_{u_1})} - f'_{a_2\,u_2\,a_1}{}^{u} \underbrace{ad_{(X'_u,X'_{u_1})}}_{=0} - \underbrace{f'_{a_2\,u_2\,u_1}{}^{l}}_{=0} ad_{(X'_{a_1},X'_l)} \Bigg) = 0 \ ,
\end{aligned} \tag{70}$$

where the constants f' of $\mathfrak{G}_c$ are given in eq. (59).

We can see in eqs. (68)-(70) that the elements in $\mathcal{W}'^{(0)} = \langle ad_{\mathcal{X}'^{(0)}} \rangle \subset \mathrm{Lie}\,\mathfrak{G}_c$ determine a subalgebra, denoted $\mathcal{W}'^{(0)}$ as its vector space. $\mathcal{W}'^{(0)}$ is therefore the Lie algebra associated with the Filippov subalgebra $\mathfrak{G}_0 \subset \mathfrak{G}_c$, and acts on the coset $\mathcal{W}'^{(1)} = \mathrm{Lie}\,\mathfrak{G}_c / \mathcal{W}'^{(0)}$ which is an abelian ideal of $\mathrm{Lie}\,\mathfrak{G}_c$. Thus, $\mathrm{Lie}\,\mathfrak{G}_c$ has the semidirect structure $\mathrm{Lie}\,\mathfrak{G}_c = \mathcal{W}'^{(1)} \supsetplus \mathcal{W}'^{(0)}$ and $\dim \mathrm{Lie}\,\mathfrak{G}_c = \begin{pmatrix} \dim \mathfrak{G}_c \\ 2 \end{pmatrix} - \dim(\ker ad)$, where $ad : \wedge^2 \mathfrak{G}_c \to \mathrm{End}\,\mathfrak{G}_c$.

4.1.2 Example: the contractions of A_4 and their associated Lie $(A_4)_c$

The simple euclidean FA A_4 (eq. (38)) has two possible types of non-trivial subalgebras $\mathfrak{G}_0$: one-dimensional, generated by any one element of A_4, and two-dimensional, generated by any two elements of the basis of A_4. They are both abelian, $[\mathfrak{G}_0, \mathfrak{G}_0, \mathfrak{G}_0] = 0$.

- **First case: $\mathfrak{G}_0$ one-dimensional**

  Let $\mathfrak{G}_0$ be generated by X_4; the basis of $\mathfrak{V}$ is then $\{X_1, X_2, X_3\}$ and $\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{V}$.

  a) Contraction

  If $\mathfrak{G}_0 = \langle X_4 \rangle$, eqs. (60), (61) show that the contraction of A_4 with respect to $\mathfrak{G}_0$ gives rise to a four-dimensional abelian FA $(A_4)_c$. $\mathfrak{V}$ is then a subalgebra of $\mathfrak{G}_c$ acting trivially on $\mathfrak{G}_0$.

  b) Lie $(A_4)_c$

  Since $(A_4)_c$ is abelian, all $f'_{j_1\,j_2\,j_3}{}^{j_4} = 0$, $ad_{\mathcal{X}'} \in \ker ad$, $\forall \mathcal{X}' \in \wedge^2 (A_4)_c$ and Lie $(A_4)_c$ reduces to the zero derivation.

- **Second case: $\mathfrak{G}_0$ bidimensional**

  Let $\mathfrak{G}_0$ be now generated by two elements, $\{X_a,\ a=1,2\}$ say, of the basis of $\mathfrak{G}$. Thus, in $\mathfrak{G}=\mathfrak{G}_0\oplus\mathfrak{V}$, the vector space $\mathfrak{V}$ is generated by $\{X_u,\ u=3,4\}$. Clearly, $\mathfrak{G}_0$ and $\mathfrak{V}$ play in this case a similar role, and eq. (38) gives

$$\begin{aligned}
&[\mathfrak{G}_0,\mathfrak{G}_0,\mathfrak{G}_0]=0 && (71)\\
&[\mathfrak{G}_0,\mathfrak{G}_0,\mathfrak{V}]\subset\mathfrak{V} && (72)\\
&[\mathfrak{G}_0,\mathfrak{V},\mathfrak{V}]\subset\mathfrak{G}_0 && (73)\\
&[\mathfrak{V},\mathfrak{V},\mathfrak{V}]=0\,. && (74)
\end{aligned}$$

Thus, $\mathfrak{G}_0$ and $\mathfrak{V}$ play a symmetrical role, and both determine two-dimensional abelian Filippov subalgebras.

a) Contraction

The only structure constants of $(A_4)_c$ different from zero are, from eq. (59),

$$f'_{a_1a_2u_1}{}^{u_2}=\epsilon_{a_1a_2u_1}{}^{u_2}\,. \tag{75}$$

Therefore all the commutators in $(A_4)_c$ are zero except those coming from $[\mathfrak{G}_0,\mathfrak{G}_0,\mathfrak{V}]\subset\mathfrak{V}$,

$$\begin{aligned}
&[X'_{a_1},X'_{a_2},X'_{a_3}]=0 && (76)\\
&[X'_{a_1},X'_{a_2},X'_{u_1}]=\epsilon_{a_1a_2u_1}{}^{u_2}X'_{u_2} && (77)\\
&[X'_{a_1},X'_{u_1},X'_{u_2}]=0 && (78)\\
&[X'_{u_1},X'_{u_2},X'_{u_3}]=0\,, && (79)
\end{aligned}$$

i.e., except

$$[X'_1,X'_2,X'_3]=X'_4\quad,\quad[X'_1,X'_2,X'_4]=-X'_3\,. \tag{80}$$

The inner derivation associated with $\mathcal{X}'\in\wedge^2\mathfrak{G}_0$, $\mathfrak{G}_0\subset(A_4)_c$ acts on the two-dimensional abelian ideal $\mathfrak{V}\subset(A_4)_c$ as a $so(2)$ rotation.

b) Lie $(A_4)_c$

To find the associated Lie $(A_4)_c$, with $(A_4)_c$ given by eqs. (76)-(79), let us consider the vector spaces generated by the $ad_{\mathcal{X}'}$ when the $\mathcal{X}'\in\wedge^2(A_4)_c$ are labelled according to the pattern above. This leads to

$$\begin{aligned}
\mathcal{W}'^{(0)}&=\langle ad_{(X'_{a_1},X'_{a_2})}\rangle=\langle ad_{(X'_1,X'_2)}\rangle\,,\\
\mathcal{W}'^{(1)}&=\langle ad_{(X'_a,X'_u)}\rangle=\langle ad_{(X'_1,X'_3)},ad_{(X'_1,X'_4)},ad_{(X'_2,X'_3)},ad_{(X'_2,X'_4)}\rangle\,,
\end{aligned}$$

$$\mathcal{W}'^{(2)} = \langle ad_{(X'_{u_1},X'_{u_2})}\rangle = \langle ad_{(X'_3,X'_4)}\rangle = \{0\}\,. \tag{81}$$

Then, applying eqs. (68)-(70) to this case, we find that $\mathrm{Lie}\,(A_4)_c$ is given by the commutators

$$\begin{aligned}
&[ad_{\mathcal{X}'^{(0)}_{a_1a_2}}, ad_{\mathcal{Y}'^{(0)}_{b_1b_2}}] = 0 && (82)\\
&[ad_{\mathcal{X}'^{(0)}_{a_1a_2}}, ad_{\mathcal{Y}'^{(1)}_{b_1u_2}}] = \frac{1}{2}\epsilon_{a_1\,a_2\,u_2}{}^{u} ad_{(X'_{b_1},X'_u)} - \frac{1}{2}\epsilon_{b_1\,u_2\,a_1}{}^{u} ad_{(X'_u,X'_{a_2})} \\
&\qquad -\frac{1}{2}\epsilon_{b_1\,u_2\,a_2}{}^{u} ad_{(X'_{a_1},X'_u)} \quad \in \mathcal{W}'^{(1)} && (83)\\
&[ad_{\mathcal{X}'^{(1)}_{a_1u_1}}, ad_{\mathcal{Y}'^{(1)}_{a_2u_2}}] = \frac{1}{2}\epsilon_{a_1\,u_1\,a_2}{}^{u} \underbrace{ad_{(X'_u,X'_{u_2})}}_{=0} - \frac{1}{2}\epsilon_{a_2\,u_2\,a_1}{}^{u} \underbrace{ad_{(X'_u,X'_{u_1})}}_{=0} = 0 && (84)\\
&[ad_{\mathcal{X}'^{(2)}_1}, ad_{\mathcal{Y}'^{(r)}_2}] = 0\;, \quad r = 0,1,2\;. && (85)
\end{aligned}$$

The *r.h.s.* $\epsilon_{a_1\,a_2\,u_2}{}^{u} ad_{(X'_{b_1},X'_u)}$ of eq. (83) is non-zero when $b_1 = a_1$ or $b_1 = a_2$, and the *r.h.s.* of eq. (84) is always zero since $ad_{(X'_u,X'_v)} = 0$. As shown in Sec. 4.1.1, $\mathcal{W}'^{(0)} \subset \mathrm{Lie}\,\mathfrak{G}_c$ is a subalgebra, abelian in this case, that acts on the abelian ideal $\mathcal{W}'^{(1)} \subset \mathrm{Lie}\,\mathfrak{G}_c$. Thus, $\mathrm{Lie}\,(A_4)_c$ has the semidirect structure $\mathrm{Lie}\,(A_4)_c = \mathcal{W}'^{(1)} \supset\!\!\!\!\!\!+\; \mathcal{W}'^{(0)}$, and is the five-dimensional Lie algebra $(Tr_2 \oplus Tr_2) \supset\!\!\!\!\!\!+\; so(2)$ where $so(2)$ acts independently on the two bidimensional abelian subalgebras $\langle ad_{\mathcal{X}'^{(1)}_{13}}, ad_{\mathcal{X}'^{(1)}_{14}}\rangle$, $\langle ad_{\mathcal{X}'^{(1)}_{23}}, ad_{\mathcal{X}'^{(1)}_{24}}\rangle$ (translations Tr_2) of $\mathcal{W}'^{(1)}$. We check that $\dim \mathrm{Lie}\,(A_4)_c = 6-1 = 5$ since $\mathcal{X}'^{(2)}_{34} \in \ker ad$.

4.2 General case: contractions of n-Lie algebras $\mathfrak{G}$

Having discussed the $n = 3$ case it is not difficult to extend the contraction procedure to an arbitrary n-Lie algebra $\mathfrak{G}$ (eq. (3)). Let $\mathfrak{G}_0$ now be a subspace of $\mathfrak{G}$ (not yet a subalgebra) and split the vector space of $\mathfrak{G}$ as the sum

$$\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{V}\;, \quad \begin{array}{l} \{X_a\}\ \text{basis of}\ \mathfrak{G}_0\;,\ a \in I_0 = \{1,\dots,\dim\mathfrak{G}_0\} \\ \{X_u\}\ \text{basis of}\ \mathfrak{V}\;,\ u \in I_1 = \{\dim\mathfrak{G}_0 + 1,\dots,\dim\mathfrak{G}\} \end{array} \tag{86}$$

where, again, the indices a, b, c refer here to the basis of $\mathfrak{G}_0$, u, v, w to the basis of $\mathfrak{V}$ and j, k, l label the elements of the basis of the FA $\mathfrak{G}$,

$$X_a, X_b, X_c \in \mathfrak{G}_0 \subset \mathfrak{G}\;, \quad X_u, X_v, X_w \in \mathfrak{V} \subset \mathfrak{G}\;, \quad X_j, X_k, X_l \in \mathfrak{G}\;.$$

Then, an arbitrary n-Lie bracket in $\mathfrak{G}$ may be written as

$$[X_{a_1},\dots,X_{a_p},X_{u_{p+1}},\dots X_{u_n}] \;=\; f_{a_1\dots a_p\,u_{p+1}\dots u_n}{}^{l} X_l =$$

$$= \quad f_{a_1\dots a_p\, u_{p+1}\dots u_n}{}^a X_a + f_{a_1\dots a_p\, u_{p+1}\dots u_n}{}^u X_u\,. \tag{87}$$

Let us rescale the basis generators of $\mathfrak{V}$, $X_u \to X'_u \equiv \epsilon X_u$ while keeping those of $\mathfrak{G}_0$ unscaled, $X_a \to X'_a = X_a$. Then,

$$\begin{aligned} &[X'_{a_1},\dots,X'_{a_p},X'_{u_{p+1}},\dots,X'_{u_n}] = \\ &= \epsilon^{n-p}(f_{a_1\dots a_p\, u_{p+1}\dots u_n}{}^a X_a + f_{a_1\dots a_p\, u_{p+1}\dots u_n}{}^u X_u) = \\ &= \epsilon^{n-p} f_{a_1\dots a_p\, u_{p+1}\dots u_n}{}^a X'_a + \epsilon^{n-p-1} f_{a_1\dots a_p\, u_{p+1}\dots u_n}{}^u X'_u. \end{aligned} \tag{88}$$

The limit $\epsilon \to 0$ is well defined for the first term in the last equality because $n \geq p$ always, but to have a well defined limit for the second one when $n = p$ we must have $f_{a_1\dots a_n}{}^u = 0$ so that the factor ϵ^{-1} does not appear. Therefore, $\mathfrak{G}_0$ must be subalgebra of $\mathfrak{G}$: *FAs contractions* $\mathfrak{G}_c$ *have to be defined with respect to Filippov subalgebras* $\mathfrak{G}_0 \subset \mathfrak{G}$*.*

The limit $\epsilon \to 0$ defines the contraction $\mathfrak{G}_c$ of the n-Lie algebra $\mathfrak{G}$ with respect to its subalgebra $\mathfrak{G}_0$. The n-brackets of $\mathfrak{G}_c$ are given by:

$$[X'_{a_1},\dots,X'_{a_p},X'_{u_{p+1}},\dots X'_{u_n}] = f'_{a_1\dots a_p\, u_{p+1}\dots u_n}{}^l X'_l\,, \tag{89}$$

where

$$f'_{a_1\,\dots\,a_p\,u_{p+1}\dots\,u_n}{}^l = \left\{ \begin{array}{ll} \lim_{\epsilon\to 0}\, \epsilon^{n-p} f_{a_1\,\dots\,a_p\,u_{p+1}\dots\,u_n}{}^a\,, & \forall a \in I_0 \\ \lim_{\epsilon\to 0}\, \epsilon^{n-p-1} f_{a_1\,\dots\,a_p\,u_{p+1}\dots\,u_n}{}^u\,, & \forall u \in I_1 \end{array} \right. . \tag{90}$$

Therefore, the structure constants of $\mathfrak{G}_c$ are given by

$$f'_{a_1,\dots,a_p,u_{p+1}\dots,u_n}{}^l = \left\{ \begin{array}{l} \left. \begin{array}{lll} f'_{a_1\dots a_n}{}^a & = f_{a_1\dots a_n}{}^a\,, & p=n,\ \ a\in I_0, \\ f'_{a_1\dots a_n}{}^u & = 0\,, & p=n,\ \ u\in I_1, \end{array} \right\} \qquad (a) \\ \begin{array}{llll} f'_{a_1\,\dots\,a_{n-1}\,u_n}{}^a & = 0 & p=n-1,\ \ a\in I_0 & (b) \\ f'_{a_1\,\dots\,a_{n-1}\,u_n}{}^u & = f_{a_1\,\dots\,a_{n-1}\,u_n}{}^u\,, & p=n-1,\ \ u\in I_1 & (c) \\ f'_{a_1,\dots,a_p,u_{p+1}\dots,u_n}{}^l & = 0 & p<n-1,\ \ l\in I_0\cup I_1 & (d) \end{array} \end{array} \right. \tag{91}$$

Again (eq. (91a)), the Filippov subalgebra $\mathfrak{G}_0$ is preserved in the contraction. For $n = 3$, eqs. (91) reproduce eq. (59).

To see that the structure constants of (91) define indeed a n-Lie algebra $\mathfrak{G}_c$, we have to check the Filippov identity for $\mathfrak{G}_c$. As expected, this is satisfied as a consequence of the FI for the original FA $\mathfrak{G}$. Indeed, the FI for the contracted algebra,

$$[X'_1,\dots,X'_{n-1},[Y'_1,\dots,Y'_n]] = \sum_{i=1}^{n} [Y'_1 \dots Y'_{i-1},[X'_1,\dots,X'_{n-1},Y'_i],Y'_{i+1},\dots,Y'_n] \tag{92}$$

gives, in term of the primed structure constants of $\mathfrak{G}_c$

$$f'_{k_1\dots k_n}{}^i f'_{l_1\dots l_{n-1} i}{}^j = \sum_{i=1}^{n} f'_{l_1\dots l_{n-1} k_i}{}^i f'_{k_1\dots k_{i-1} i k_{i+1}\dots k_n}{}^j \quad . \tag{93}$$

The proof involves three possible cases:

1. All algebra elements in (92) belong to $\mathfrak{G}_0$.
   Then the structure constants in (93) are given by eq. (91a), and the FI holds because $\mathfrak{G}_0 \subset \mathfrak{G}_c$ is a Filippov (sub)algebra.

2. Only one element in (92) belongs to $\mathfrak{V}$, and the remaining $2n-2$ ones belong to $\mathfrak{G}_0$.
   In this case, when the index $j \in I_0$ in (93), we have the identity $0=0$. Indeed, due to (91a), (91b), all the indices in the terms $f'{}_{\_\_}{}^{j}$ must belong to I_0 to be non-zero, but then the structure constants $f'{}_{\_\_}{}^{i}$ are of the type (91b), and therefore vanish.
   When $j \in I_1$, the FI is the same for the contracted $\mathfrak{G}_c$ and the original n-Lie algebra $\mathfrak{G}$. The reason is that in this case the terms that may be non-zero in the FI (93) are the same for $\mathfrak{G}_c$ and $\mathfrak{G}$ and involve structure constants of the type $f'{}_{a_1\ldots a_{n-1}u}{}^{v} = f_{a_1\ldots a_{n-1}u}{}^{v}$ since $f'{}_{a_1\ldots a_n}{}^{u} = 0$ for both $\mathfrak{G}$ and $\mathfrak{G}_c$.

3. Two or more elements belong to $\mathfrak{V}$.
   In this case, as in the previous one, when $j \in I_0$, we have the identity $0=0$ because due to (91a), (91b), (91d) all the indices in the structure constants $f'{}_{\_\_}{}^{j}$ must be in I_0 to be non-zero, but then the other structure constants in the products are of the form (91d), and therefore are zero. When $j \in I_1$ we have again $0=0$, because in this case $f'{}_{\_\_}{}^{j}$ has to be of the form (91c) to be non-zero, and then the terms $f'{}_{\_\_}{}^{i}$ are either of the form (91b) if $i \in I_0$ or (91d) if $i \in I_1$, which vanish in both cases.

The n-brackets of the contraction $\mathfrak{G}_c$ of the FA $\mathfrak{G}$ with respect to the subalgebra $\mathfrak{G}_0$ have therefore the following general structure (see eq. (91)):

$$\begin{array}{lll}
[\mathfrak{G}_0,\ldots,\mathfrak{G}_0] & \subset \mathfrak{G}_0, & (ad_{\mathcal{X}_0}\mathfrak{G}_0 \subset \mathfrak{G}_0) \\
[\mathfrak{G}_0,\ldots,\mathfrak{G}_0,\mathfrak{V}] & \subset \mathfrak{V}, & (ad_{\mathcal{X}_0}\mathfrak{V} \subset \mathfrak{V}) \\
[\mathfrak{G}_0,\ldots,\mathfrak{G}_0,\mathfrak{V},\mathfrak{V}] & = 0, & \\
\ldots\ldots & & \\
[\mathfrak{G}_0,\mathfrak{V},\ldots,\mathfrak{V}] & = 0, & \\
[\mathfrak{V},\ldots,\mathfrak{V}] & = 0, &
\end{array} \tag{94}$$

where $\mathcal{X}_0 = (X_1,\ldots,X_{n-1})$, $X_1,\ldots,X_{n-1} \in \mathfrak{G}_0$. The elements in the coset $\mathfrak{V} = \mathfrak{G}/\mathfrak{G}_0$ become an abelian ideal in $\mathfrak{G}_c$, $[\mathfrak{G},\ldots,\mathfrak{G},\mathfrak{V}] \subset \mathfrak{V}$, $[\mathfrak{V},\ldots,\mathfrak{V}] = 0$, and the fundamental objects of $\mathfrak{G}_0 \subset \mathfrak{G}_c$ act on $\mathfrak{V}$ as derivations, $ad_{\mathcal{X}_0} : \mathfrak{V} \to \mathfrak{V}$. Thus, $\mathfrak{G}_c$ *has the FA semidirect structure* $\mathfrak{G}_c = \mathfrak{V} \supset\!\!\!\!\!+ \mathfrak{G}_0$.

4.2.1 The Lie algebra Lie $\mathfrak{G}_c$ associated with a contraction $\mathfrak{G}_c$

To describe Lie $\mathfrak{G}_c$ associated with $\mathfrak{G}_c$, it will prove again useful to split the space of fundamental objects of $\mathfrak{G}_c$ in subsets, where each subset $\mathcal{X}'^{(r)}$ is characterized by the number r of elements $X'_{u_i} \in \mathfrak{V}$ in the $\mathcal{X}'$'s that it contains. Thus,

$$\mathcal{X}'^{(0)} = (X'_{a_1},\ldots,X'_{a_{n-1}})\,, \quad X'_{a_i} \in \mathfrak{G}_0 \tag{95}$$

$$\mathcal{X}'^{(1)} = (X'_{a_1}, \dots, X'_{a_{n-2}}, X'_{u_{n-1}}) \,, \quad X'_{a_i} \in \mathfrak{G}_0 \,,\; X'_{u_i} \in \mathfrak{V} \tag{96}$$

$$\dots$$

$$\mathcal{X}'^{(r)} = (X'_{a_1}, \dots, X'_{a_{n-r-1}}, X'_{u_{n-r}}, \dots, X'_{u_{n-1}}) \,, \quad X'_{a_i} \in \mathfrak{G}_0 \,,\; X'_{u_i} \in \mathfrak{V} \tag{97}$$

$$\dots$$

$$\mathcal{X}'^{(n-1)} = (X'_{u_1}, \dots, X'_{u_{n-1}}) \,, \quad X'_{u_i} \in \mathfrak{V} \quad , \tag{98}$$

and the vector spaces generated by the inner derivations associated with the fundamental objects in $\mathcal{X}'^{(r)}$ are denoted by

$$\mathcal{W}'^{(r)} = \langle ad_{\mathcal{X}'^{(r)}} \rangle \,, \quad r = 0, \dots, n-1 \,. \tag{99}$$

Due to eqs. (91), we see that the inner derivations of Lie $\mathfrak{G}_c$ act on the elements of the contracted $\mathfrak{G}_c$ in the following way:

$$ad_{\mathcal{X}'^{(0)}} \mathfrak{G}_0 \subset \mathfrak{G}_0 \tag{100}$$

$$ad_{\mathcal{X}'^{(0)}} \mathfrak{V} \subset \mathfrak{V} \tag{101}$$

$$ad_{\mathcal{X}'^{(1)}} \mathfrak{G}_0 \subset \mathfrak{V} \tag{102}$$

$$ad_{\mathcal{X}'^{(1)}} \mathfrak{V} = 0 \tag{103}$$

$$ad_{\mathcal{X}'^{(r)}} \mathfrak{G}_c = 0 \quad \forall r \geq 2 \tag{104}$$

(eqs. (101), (102) both correspond to the second equation in (94)). Therefore $ad_{\mathcal{X}'^{(r)}} = 0$ for $r \geq 2$ *i.e.*, when $r \geq 2$ all the $\mathcal{X}'^{(r)}$ belong to $\ker ad$ and $\mathcal{W}'^{(r)} = 0$.

The composition of fundamental objects in eq. (10) and the structure constants (91) of the contracted FA $\mathfrak{G}_c$ determine the following structure for Lie $\mathfrak{G}_c$

$$\begin{aligned}
&[ad_{\mathcal{X}'^{(0)}_{a_1 \dots a_{n-1}}}, ad_{\mathcal{Y}'^{(0)}_{b_1 \dots b_{n-1}}}] = \frac{1}{2} ad_{[(X'_{a_1},\dots,X'_{a_{n-1}})\cdot(X'_{b_1},\dots,X'_{b_{n-1}}) - (X'_{b_1},\dots,X'_{b_{n-1}})\cdot(X'_{a_1},\dots,X'_{a_{n-1}})]} = \\
&\quad = \frac{1}{2} ad_{\left[\sum_{i=1}^{n-1}(X'_{b_1},\dots,[X'_{a_1},\dots,X'_{a_{n-1}},X'_{b_i}],\dots,X'_{b_{n-1}}) - (a\leftrightarrow b)\right]} = \\
&\quad \frac{1}{2}\left(\sum_{i=1}^{n-1} f_{a_1 \dots a_{n-1}\, b_i}{}^{b}\, ad_{(X'_{b_1},\dots,X'_{b_{i-1}},X'_{b},X'_{b_{i+1}}\dots,X'_{b_{n-1}})} - (a \leftrightarrow b)\right) \quad \in \mathcal{W}'^{(0)}
\end{aligned} \tag{105}$$

$$\begin{aligned}
&[ad_{\mathcal{X}'^{(0)}_{a_1 \dots a_{n-1}}}, ad_{\mathcal{Y}'^{(1)}_{b_1 \dots b_{n-2}\, v_{n-1}}}] = \\
&\quad = \frac{1}{2} ad_{[(X'_{a_1},\dots,X'_{a_{n-1}})\cdot(X'_{b_1},\dots,X'_{b_{n-2}},X'_{v_{n-1}}) - (X'_{b_1},\dots,X'_{b_{n-2}},X'_{v_{n-1}})\cdot(X'_{a_1},\dots,X'_{a_{n-1}})]} = \\
&\quad \frac{1}{2}\left(\sum_{i=1}^{n-2} f_{a_1 \dots a_{n-1}\, b_i}{}^{b}\, ad_{(X'_{b_1},\dots,X'_{b_{i-1}},X'_{b},X'_{b_{i+1}},\dots,X'_{b_{n-2}},X'_{v_{n-1}})} + f_{a_1 \dots a_{n-1}\, v_{n-1}}{}^{v}\, ad_{(X'_{b_1},\dots,X'_{b_{n-2}},X'_{v})}\right. \\
&\quad \left. - \sum_{i=1}^{n-1} f_{b_1 \dots b_{n-2}\, v_{n-1}\, a_i}{}^{v}\, ad_{(X'_{a_1},\dots,X'_{a_{i-1}},X'_{v},X'_{a_{i+1}},\dots,X'_{a_{n-1}})}\right) \quad \in \mathcal{W}'^{(1)}
\end{aligned} \tag{106}$$

$$
\begin{aligned}
&[ad_{\mathcal{X}'^{(0)}_{a_1 \dots a_{n-1}}}, ad_{\mathcal{Y}'^{(2)}_{b_1 \dots b_{n-3}\, v_{n-2}\, v_{n-1}}}] = \\
&\quad = \frac{1}{2} ad_{[(X'_{a_1},\dots,X'_{a_{n-1}})\cdot(X'_{b_1},\dots,X'_{b_{n-3}},X'_{v_{n-2}},X'_{v_{n-1}}) - (X'_{b_1},\dots,X'_{b_{n-3}},X'_{v_{n-2}},X'_{v_{n-1}})\cdot(X'_{a_1},\dots,X'_{a_{n-1}})]} \\
&\quad = \frac{1}{2}\left(\sum_{i=1}^{n-3} f_{a_1 \dots a_{n-1}\, b_i}{}^{b} ad_{(X'_{b_1},\dots,X'_{b_{i-1}},X'_b,X'_{b_{i+1}},\dots,X'_{b_{n-3}},X'_{v_{n-2}},X'_{v_{n-1}})} \right. \\
&\quad \left. + f_{a_1 \dots a_{n-1}\, v_{n-2}}{}^{v} ad_{(X'_{b_1},\dots,X'_{b_{n-3}},X'_{v_{n-1}},X'_v)} + f_{a_1 \dots a_{n-1}\, v_{n-1}}{}^{v} ad_{(X'_{b_1},\dots,X'_{b_{n-3}},X'_v,X'_{v_{n-2}})} \right) = 0 \quad (107)
\end{aligned}
$$

$$
\begin{aligned}
&[ad_{\mathcal{X}'^{(1)}_{a_1 \dots a_{n-2}\, u_{n-1}}}, ad_{\mathcal{Y}'^{(1)}_{b_1 \dots b_{n-2}\, v_{n-1}}}] = \\
&\quad = \frac{1}{2} ad_{\left[(X'_{a_1},\dots,X'_{a_{n-2}},X'_{u_{n-1}})\cdot(X'_{b_1},\dots,X'_{b_{n-2}},X'_{v_{n-1}}) - (X'_{b_1},\dots,X'_{b_{n-2}},X'_{v_{n-1}})\cdot(X'_{a_1},\dots,X'_{a_{n-2}},X'_{u_{n-1}})\right]} \\
&\quad = \frac{1}{2}\left(\sum_{i=1}^{n-2} f_{a_1 \dots a_{n-2}\, u_{n-1}\, b_i}{}^{v} ad_{(X'_{b_1},\dots,X'_{b_{i-1}},X'_v,X'_{b_{i+1}},\dots,X'_{b_{n-2}},X'_{v_{n-1}})} - (a,u \leftrightarrow b,v) \right) = 0 \quad (108)
\end{aligned}
$$

$$
[ad_{\mathcal{X}'^{(r)}}, ad_{\mathcal{Y}'^{(2)}}] = 0\,, \quad r = 1,2 \tag{109}
$$

where we have used eqs. (103), (104) and the only non-zero structure constants appearing above are the $f'_{a_1 \dots a_n}{}^{a} = f_{a_1 \dots a_n}{}^{a}$ and $f'_{a_1 \dots a_{n-1}\, u_n}{}^{u} = f_{a_1 \dots a_{n-1}\, u_n}{}^{u}$ by eq. (91). As a result, the structure of Lie $\mathfrak{G}_c$ for the n-Lie algebra $\mathfrak{G}_c$ is similar to the one found for $n = 3$. The elements $ad_{\mathcal{X}'^{(0)}}$ generate a subalgebra $\mathcal{W}'^{(0)}$ of Lie $\mathfrak{G}_c$ and the $ad_{\mathcal{X}'^{(1)}}$ an abelian ideal $\mathcal{W}'^{(1)}$. Lie $\mathfrak{G}_c$ has therefore the semidirect structure Lie $\mathfrak{G}_c = \mathcal{W}'^{(1)} \supsetplus \mathcal{W}'^{(0)}$ which, for $n = 3$, recovers the case of Sec. 4.1.1. As for any Lie algebra associated with a FA, dim Lie $\mathfrak{G}_c = \begin{pmatrix} \dim \mathfrak{G}_c \\ n-1 \end{pmatrix} - \dim(\ker ad)$ where now $ad : \wedge^{n-1}\mathfrak{G}_c \to \mathrm{End}\,\mathfrak{G}_c$.

4.2.2 Example: the contractions of A_{n+1} and their associated Lie $(A_{n+1})_c$

In this section we consider the general simple FAs $\mathfrak{G} := A_{n+1}$,

$$
[X_{l_1}, \dots, X_{l_n}] = \epsilon_{l_1 \dots l_n}{}^{l_{n+1}} X_{l_{n+1}}\ , \tag{110}
$$

generalizing the $n = 3$ results of Sec. 4.1.2. There are various subspaces that determine different non-trivial subalgebras $\mathfrak{G}_0 \subset A_{n+1}$ and corresponding vector space splittings $\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{V}$: it suffices to take $\mathfrak{G}_0$ generated by m basis elements of A_{n+1} with $m \leq (n-1)$ (if $\dim \mathfrak{G}_0 = n$, $\mathfrak{G}_0$ cannot be a subalgebra when $\mathfrak{G}$ is simple). We shall see below that only one of these splittings, when $m = n-1$, leads to a non-trivial contraction $(A_{n+1})_c$. All other $(m < n-1)$-dimensional subalgebras lead to a contraction of A_{n+1} which is an abelian $(n+1)$-dimensional n-Lie algebra.

Let then $\mathfrak{G}_0$ be generated by $n-1$ basis elements of A_{n+1} and $\mathfrak{V}$ by the remaining two,

$$\mathfrak{G}_0 = \langle X_a,\ a=1,\ldots,n-1\rangle, \quad \mathfrak{V} = \langle X_u,\ u=n,n+1\rangle. \tag{111}$$

Then, the various A_{n+1} n-brackets follow the pattern

$$\begin{aligned}
&[\mathfrak{G}_0, \overset{n}{\ldots}, \mathfrak{G}_0] = 0 && (112)\\
&[\mathfrak{G}_0, \overset{n-1}{\ldots}, \mathfrak{G}_0, \mathfrak{V}] \subset \mathfrak{V} && (113)\\
&[\mathfrak{G}_0, \overset{n-2}{\ldots}, \mathfrak{G}_0, \mathfrak{V}, \mathfrak{V}] \subset \mathfrak{G}_0 && (114)\\
&[\mathfrak{G}_0, \overset{n-3}{\ldots}, \mathfrak{G}_0, \mathfrak{V}, \mathfrak{V}, \mathfrak{V}] = 0 && (115)\\
&\ldots\\
&[\mathfrak{V}, \overset{n}{\ldots}, \mathfrak{V}] = 0\,. && (116)
\end{aligned}$$

Looking at eqs. (91) we find that the only non-zero structure constants of the contraction $(A_{n+1})_c$ of the FA A_{n+1} with respect to a $(n-1)$-dimensional subalgebra are

$$f'_{a_1\ldots a_{n-1}u_1}{}^{u_2} = \epsilon_{a_1\ldots a_{n-1}u_1}{}^{u_2}\ . \tag{117}$$

Note that any other $(m < n-1)$-dimensional $\mathfrak{G}_0$ would lead to $f'_{i_1\ldots i_n}{}^{k} = 0$. Thus, the splitting (111) is the only one leading to a non fully abelian contraction.

The contracted n-Lie algebra $(A_{n+1})_c$ is given by

$$\begin{aligned}
&[X'_{a_1},\ldots,X'_{a_n}] = 0 && (118)\\
&[X'_{a_1},\ldots,X'_{a_{n-1}},X'_{u_1}] = \epsilon_{a_1\ldots a_{n-1}u_1}{}^{u_2} X'_{u_2} && (119)\\
&[X'_{a_1},\ldots,X'_{a_{n-2}},X'_{u_1},X'_{u_2}] = 0 && (120)\\
&\ldots\\
&[X'_{u_1},\ldots,X'_{u_n}] = 0\,. && (121)
\end{aligned}$$

It has a $(n-1)$-dimensional abelian subalgebra $\mathfrak{G}_0$ acting by eq. (119) on the two-dimensional abelian ideal $\mathfrak{V}$. For $n=3$, this reproduces the contraction $(A_4)_c$ of the second case in Sec. 4.1.2.

The familiar Lie algebra case also follows in this framework. For $n=2$, $A_3 = so(3)$ and the $(n-1)$-dimensional subalgebra is of dimension one. Then, the only non-zero structure constants in eq. (117) reduce to $f'_{au}{}^{v} = \epsilon_{au}{}^{v}$, and $(A_3)_c = Tr_2 \supset\!\!\!\!\!\!+ so(2) = E_2$, the Euclidean algebra on the plane.

Let us now find Lie$\,(A_{n+1})_c$. For it, consider the adjoint maps determined by the fundamental objects of $(A_{n+1})_c$ in the subsets (95)-(98), and the corresponding vector spaces $\mathcal{W}'^{(r)}$ generated by them,

$$\mathcal{W}'^{(0)} = \langle ad_{(X'_{a_1},\ldots,X'_{a_{n-1}})}\rangle,$$

$$
\begin{aligned}
&\mathcal{W}'^{(1)} = \langle ad_{(X'_{a_1},\ldots,X'_{a_{n-2}},X'_u)} \rangle \,, \\
&\mathcal{W}'^{(2)} = \langle ad_{(X'_{a_1},\ldots,X'_{a_{n-3}},X'_{u_1},X'_{u_2})} \rangle = \{0\} \,, \\
&\ldots\ldots \\
&\mathcal{W}'^{(n-1)} = \langle ad_{(X'_{u_1},\ldots,X'_{u_{n-1}})} \rangle = \{0\} \quad , \qquad (122)
\end{aligned}
$$

where, by eq. (104), $ad_{\mathcal{X}'^{(r)}} = 0,\ r \geq 2$ so that $\mathcal{W}'^{(r)} = \{0\}$ for $r \geq 2$ (note that the non-zero commutator in eq. (114) becomes zero in $(A_{n+1})_c$, eq. (120)). Therefore, the vector space of Lie $(A_{n+1})_c$ is reduced to $\mathcal{W}'^{(0)} \oplus \mathcal{W}'^{(1)}$, of dimension $\begin{pmatrix} n-1 \\ n-1 \end{pmatrix} + 2 \begin{pmatrix} n-1 \\ n-2 \end{pmatrix} = 2n-1$.

The structure of the Lie algebra Lie $(A_{n+1})_c$ is obtained by inserting the structure constants $f'_{l_1 \ldots l_n}{}^{l_{n+1}}$ of $(A_{n+1})_c$ (as given in eqs. (91) whith $f_{l_1 \ldots l_n}{}^{l_{n+1}} = \epsilon_{l_1 \ldots l_n}{}^{l_{n+1}}$ since $\mathfrak{G} = A_{n+1}$) in eqs. (105)-(109). This leads to

$$
[ad_{\mathcal{X}'^{(0)}_{a_1 \ldots a_{n-1}}}, ad_{\mathcal{Y}'^{(0)}_{b_1 \ldots b_{n-1}}}] = 0 \qquad (123)
$$

$$
\begin{aligned}
[ad_{\mathcal{X}'^{(0)}_{a_1 \ldots a_{n-1}}}, ad_{\mathcal{Y}'^{(1)}_{b_1 \ldots b_{n-2}\, v_{n-1}}}] &= \frac{1}{2} \epsilon_{a_1 \ldots a_{n-1}\, v_{n-1}}{}^{v} ad_{(X'_{b_1},\ldots,X'_{b_{n-2}},X_v)} \\
&-\frac{1}{2} \sum_{i=1}^{n-1} \epsilon_{b_1 \ldots b_{n-2}\, v_{n-1}\, a_i}{}^{v} ad_{(X'_{a_1},\ldots,X'_{a_{i-1}},X'_v,X'_{a_{i+1}},\ldots,X'_{a_{n-1}})} \quad \in \mathcal{W}'^{(1)} \qquad (124)
\end{aligned}
$$

$$
\begin{aligned}
&[ad_{\mathcal{X}'^{(1)}_{a_1 \ldots a_{n-2}\, u_{n-1}}}, ad_{\mathcal{Y}'^{(1)}_{b_1 \ldots b_{n-2}\, v_{n-1}}}] = \\
&\quad \frac{1}{2} \left(\sum_{i=1}^{n-2} \epsilon_{a_1 \ldots a_{n-2}\, u_{n-1}\, b_i}{}^{v} \underbrace{ad_{(X'_{b_1},\ldots,X'_{b_{i-1}},X'_v,X'_{b_{i+1}},\ldots,X'_{b_{n-2}},X'_{v_{n-1}})}}_{=0} - \underbrace{[(a,\, u) \leftrightarrow (b,\, v)]}_{=0} \right) = 0 \qquad (125)
\end{aligned}
$$

The *r.h.s.* of eq. (124) may be non-zero only if $n-2$ of the a indices are equal to $n-2$ of the b indices. $\mathcal{W}'^{(0)} \subset \mathrm{Lie}\, \mathfrak{G}_c$ is an abelian one-dimensional subalgebra $so(2)$ that acts on the $2(n-1)$-dimensional abelian ideal $\mathcal{W}'^{(1)} \subset \mathrm{Lie}\, \mathfrak{G}_c$, which may be split as the sum of two $(n-1)$-dimensional abelian subalgebras $\langle ad_{(X'_{a_1},\ldots,X'_{a_{n-2}},X'_n)} \rangle \oplus \langle ad_{(X'_{b_1},\ldots,X'_{b_{n-2}},X'_{n+1})} \rangle$, where X'_n and X'_{n+1} are the basis of $\mathfrak{V}$ (eq. (111)), on which $\mathcal{W}'^{(0)}$ acts (eq. (124)) by rotating the (X'_n, X'_{n+1}) plane. Thus, Lie $(A_{n+1})_c$ has the semidirect structure Lie $(A_{n+1})_c = \mathcal{W}'^{(1)} \niplus \mathcal{W}'^{(0)}$ and is the $(2n-1)$-dimensional Lie algebra $(Tr_{n-1} \oplus Tr_{n-1}) \niplus so(2)$, where $so(2)$ rotates the two abelian subalgebras (translations Tr_{n-1}) in the abelian Lie ideal $\mathcal{W}'^{(1)}$.

We may also look here at the $n=2$ Lie algebra case. This gives $\mathrm{Lie}(A_3)_c = Tr_2 \niplus so(2)$, again E_2. This is not surprising: the centre of E_2 is trivial and, since the inner derivations of a Lie algebra $\mathfrak{g}$ are given by $\mathfrak{g}/Z(\mathfrak{g})$, we have $(A_3)_c = E_2 = \mathrm{InDer}\,(E_2) \equiv \mathrm{Lie}\,(A_3)_c$.

5 On Lie $\mathfrak{G}_c$ and the contractions of Lie $\mathfrak{G}$

In Sec. 4.2 we have studied the general structure of $\mathfrak{G}_c$ and Lie $\mathfrak{G}_c$. It is natural to ask ourselves whether there is any relation between Lie $\mathfrak{G}_c$ and some contraction $(\mathrm{Lie}\,\mathfrak{G})_c$ of the Lie algebra Lie $\mathfrak{G}$ associated with the FA $\mathfrak{G}$, or, equivalently, under which circumstances one may consider some kind of relation for the Lie algebras in the lower *r.h.s.* of the diagram

$$\begin{tikzcd}
\mathfrak{G} \arrow[rr] \arrow[d, "contr.\,limit"'] & & \mathrm{Lie}\,\mathfrak{G} \arrow[d, "contr.\,limit"] \\
\mathfrak{G}_c \arrow[r] & \mathrm{Lie}\,\mathfrak{G}_c\ ; & (\mathrm{Lie}\,\mathfrak{G})_c
\end{tikzcd} \tag{126}$$

for some contraction of Lie $\mathfrak{G}$. Note that we may not expect the closure of the diagram, because Lie $\mathfrak{G}_c$ is the algebra of derivations of $\mathfrak{G}_c$, while $(\mathrm{Lie}\,\mathfrak{G})_c$ is a contraction of an ordinary Lie algebra determined by the inner derivations of $\mathfrak{G}$ and not related with the adjoint derivations of $\mathfrak{G}_c$. Further, there is a mismatch among the dimensions of $(\mathrm{Lie}\,\mathfrak{G})_c$ and Lie $\mathfrak{G}_c$ since the inner derivations associated with the $\mathcal{X}'^{(r)} \in \wedge^{n-1}\mathfrak{G}_c$ for $r \geq 2$ are trivial by eq. (104) and then $\mathcal{W}'^{(r)} = 0$ in Lie $\mathfrak{G}_c$ for $r \geq 2$. Thus, $\dim(\mathrm{Lie}\,\mathfrak{G}) = \dim(\mathrm{Lie}\,\mathfrak{G})_c \neq \dim \mathrm{Lie}\,\mathfrak{G}_c$, and the diagram (126) does not close. However, in the case of simple FAs, the comparison of Lie $\mathfrak{G}_c$ and $(\mathrm{Lie}\,\mathfrak{G})_c$ is simpler since for $\mathfrak{G}$ ad is injective (see Secs. 3.1, 3.2) and the $ad_{\mathcal{X}_{i_1\cdots i_{n-1}}}$ derivations determine a basis of Lie A_{n+1}. We shall therefore restrict ourselves to this case, and show how Lie $\mathfrak{G}_c$ and various contractions $(\mathrm{Lie}\,\mathfrak{G})_c$, $\mathfrak{G} = A_{n+1}$, $n > 2$, may be related.

To look into the problem we first notice that, given a FA $\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{V}$ as a vector space, the splitting of Lie $\mathfrak{G}$ defined by the vector subspaces

$$\mathcal{W}^{(r)} = \langle ad_{\mathcal{X}^{(r)}} \rangle\ , \quad ad_{\mathcal{X}^{(r)}} = ad_{(X_{a_1},\ldots,X_{a_{n-r-1}},X_{u_{n-r}},\ldots,X_{u_{n-1}})}\ , \quad X_{a_i} \in \mathfrak{G}_0\ ,\ X_{u_i} \in \mathfrak{V}\ , \tag{127}$$

allows us to perform a generalized contraction of Lie $\mathfrak{G}$ in the sense of Weimar-Woods (W-W) [22]. The reason is that the splitting of Lie $\mathfrak{G} = \bigoplus \mathcal{W}^{(r)}$ does not only say that $\mathcal{W}^{(0)}$ is a subalgebra of Lie $\mathfrak{G}$; the $\mathfrak{G}_0$ Filippov subalgebra condition $f_{a_1\ldots a_n}{}^{u} = 0$ gives for the Lie $\mathfrak{G}$ commutators the structure

$$[ad_{\mathcal{X}^{(r)}}, ad_{\mathcal{Y}^{(s)}}] \in \bigoplus \langle ad_{\mathcal{Z}^{(t)}} \rangle\ , \quad t \leq r+s\ , \quad r,\ s,\ t\ =\ 0,\ldots,(n-1) \tag{128}$$

i.e.,

$$[\mathcal{W}^{(r)}, \mathcal{W}^{(s)}] \subset \bigoplus \mathcal{W}^{(t)}\ , \quad t \leq r+s \quad ,$$

(proved in the Appendix), which is precisely the general condition needed to perform a generalized contraction of Lie algebras in the sense of Weimar-Woods (W-W) [22]. This is defined as follows. Let the vector space of a Lie algebra split as $\mathfrak{g} = \oplus \mathfrak{v}_p$, $p = 0, 1, \ldots, m$. Let the subset of basis generators X of $\mathfrak{g}$ generating each subspace $\mathfrak{v}_p$ be redefined by $X \to X' = \epsilon^p X$ when $X \in \mathfrak{v}_p$. Then, a W-W Lie algebra contraction (the limit $\epsilon \to 0$) exists iff the splitting of $\mathfrak{g}$ above is such that $[\mathfrak{v}_p, \mathfrak{v}_q] \subset \oplus_s \mathfrak{v}_s$, where s runs over all the values for which $s \leq p+q$. In the present Lie $\mathfrak{G}$ case, the contracted algebra $(\mathrm{Lie}\,\mathfrak{G})_{W-W}$ is obtained by the reparametrization $ad'_{\mathcal{X}^{(r)}} = \epsilon^r ad_{\mathcal{X}^{(r)}}$ and the limit $\epsilon \to 0$.

5.1 Contractions of Lie A_4

The contractions of $A_4 = \mathfrak{G}_0 \oplus \mathfrak{V}$ with respect to its two types of non-trivial subalgebras $\mathfrak{G}_0 \subset A_4$ and their associated Lie $(A_4)_c$ algebras were given in Sec. 4.1.2. We consider here the contractions of the corresponding Lie $A_4 = \bigoplus_{r=0}^{2} \mathcal{W}^{(r)}$, where as usual r indicates the number of generators of the basis of $\mathfrak{V}$ in the elements of $\mathcal{W}^{(r)}$ as in eq. (127).

As a third case, we recall the IW contraction with respect to the subalgebra $so(3) \subset$ Lie A_4, generated by the elements in the first line in eq. (46), and corresponding to $\mathcal{W}^{(1)}$ in the splitting (42) of its vector space, $\mathcal{W}^{(1)} \oplus \mathcal{W}^{(2)}$. Since Lie A_4 is semisimple, there is another well known contraction, also mentioned in Sec. 5.1.3.

5.1.1 First case: $\mathfrak{G}_0$ one-dimensional

In this case $(A_4)_c$ is a four-dimensional abelian algebra and hence Lie $(A_4)_c$ reduces to the trivial endomorphism (Sec. 4.1.2). The IW contraction (Lie $A_4)_c$ of Lie $A_4 = so(4)$ with respect to the trivial subalgebra $\mathcal{W}^{(0)} = \langle ad_{(X_4,X_4)} \rangle = \{0\}$, associated to $\mathfrak{G}_0 = \langle X_{a_4} \rangle \subset A_4$ is obviously a six-dimensional abelian algebra; in this extreme case, $\dim\,(\text{Lie}\,A_4)_c - \dim \text{Lie}\,(A_4)_c = 6$.

The W-W contraction for the splitting (127) gives again a six-dimensional abelian algebra.

5.1.2 Second case: (Lie $A_4)_c$, $\mathfrak{G}_0$ bidimensional

Since $n = 3$ there are three types of $\mathcal{W}^{(r)}$ spaces, $r = 0, 1, 2$. Labelling the elements $ad_{(X_i,X_j)}$ as usual, the Lie A_4 commutators are given by

$$\begin{array}{ll}
[ad_{(X_{a_1},X_{a_2})}, ad_{(X_{b_1},X_{b_2})}] = 0 & \Rightarrow [\mathcal{W}^{(0)},\mathcal{W}^{(0)}] = 0 \\
[ad_{(X_{a_1},X_{a_2})}, ad_{(X_{b_1},X_{u_1})}] = \frac{1}{2}\epsilon_{a_1 a_2 u_1}{}^{u_2} ad_{(X_{b_1},X_{u_2})} & \\
\qquad -\frac{1}{2}\epsilon_{b_1 u_1 a_1}{}^{u_2} ad_{(X_{u_2},X_{a_2})} - \frac{1}{2}\epsilon_{b_1 u_1 a_2}{}^{u_2} ad_{(X_{a_1},X_{u_2})} & \Rightarrow [\mathcal{W}^{(0)},\mathcal{W}^{(1)}] \subset \mathcal{W}^{(1)} \\
\left.\begin{array}{l}
[ad_{(X_{a_1},X_{u_1})}, ad_{(X_{a_2},X_{u_2})}] = 0\,,\ a_1 \neq a_2, u_1 \neq u_2 \\
[ad_{(X_{a_1},X_{u})}, ad_{(X_{a_2},X_{u})}] = \epsilon_{a_1 u_1 a_2}{}^{v} ad_{(X_v,X_u)} \\
[ad_{(X_{a},X_{u_1})}, ad_{(X_{a},X_{u_2})}] = \epsilon_{a u_1 u_2}{}^{b} ad_{(X_a,X_b)}
\end{array}\right\} & \Rightarrow [\mathcal{W}^{(1)},\mathcal{W}^{(1)}] \subset \mathcal{W}^{(0)} \oplus \mathcal{W}^{(2)} \\
[ad_{(X_{a_1},X_{a_2})}, ad_{(X_{u_1},X_{u_2})}] = 0 & \Rightarrow [\mathcal{W}^{(0)},\mathcal{W}^{(2)}] = 0 \\
[ad_{(X_{u},X_{a_1})}, ad_{(X_{u},X_{v})}] = \epsilon_{u a_1 v}{}^{a_2} ad_{(X_u,X_{a_2})} & \Rightarrow [\mathcal{W}^{(1)},\mathcal{W}^{(2)}] \subset \mathcal{W}^{(1)} \\
[ad_{(X_{v_1},X_{v_2})}, ad_{(X_{u_1},X_{u_2})}] = 0 & \Rightarrow [\mathcal{W}^{(2)},\mathcal{W}^{(2)}] = 0 \quad .
\end{array} \tag{129}$$

- IW contraction, (Lie $A_4)_{IW}$

We contract with respect to $\mathcal{W}^{(0)}$, the one-dimensional subalgebra generated by $ad_{(X_{a_1},X_{a_2})}$. The reparametrization $ad'_{\mathcal{X}^{(0)}} = ad_{\mathcal{X}^{(0)}}$, $ad'_{\mathcal{X}^{(r)}} = \epsilon ad_{\mathcal{X}^{(r)}}$, $r = 1, 2$, and the limit $\epsilon \to 0$ gives the contracted Lie algebra $(\mathrm{Lie}\, A_4)_{IW}$

$$\begin{array}{ll}
[ad'_{(X_{a_1},X_{a_2})}, ad'_{(X_{b_1},X_{b_2})}] = 0 & \Rightarrow [\mathcal{W}^{(0)},\mathcal{W}^{(0)}] = 0 \\
[ad'_{(X_{a_1},X_{a_2})}, ad'_{(X_{b_1},X_{u_1})}] = \frac{1}{2}\epsilon_{a_1a_2u_1}{}^{u_2} ad'_{(X_{b_1},X_{u_2})} & \\
\qquad -\frac{1}{2}\epsilon_{b_1u_1a_1}{}^{u_2} ad'_{(X_{u_2},X_{a_2})} - \frac{1}{2}\epsilon_{b_1u_1a_2}{}^{u_2} ad'_{(X_{a_1},X_{u_2})} & \Rightarrow [\mathcal{W}^{(0)},\mathcal{W}^{(1)}] \subset \mathcal{W}^{(1)} \\
[ad'_{(X_{a_1},X_{u_1})}, ad'_{(X_{a_2},X_{u_2})}] = 0 & \Rightarrow [\mathcal{W}^{(1)},\mathcal{W}^{(1)}] = 0 \\
[ad'_{\mathcal{X}^{(2)}}, ad'_{\mathcal{Y}^{(r)}}] = 0 \ , \quad r = 0,1,2 & \Rightarrow [\mathcal{W}^{(2)},\mathcal{W}^{(r)}] = 0 \ , \quad r = 0,1,2 \ ,
\end{array} \tag{130}$$

where we are using the same notation $\mathcal{W}^{(r)}$ to refer now to the subspaces of the *contracted* $(\mathrm{Lie}\, A_4)_{IW}$ algebra. Thus, the contraction $(\mathrm{Lie}\, A_4)_{IW}$ contains $\mathrm{Lie}\,(A_4)_c$ as a subalgebra (see eqs. (82)-(85)), but contains an extra commuting generator $ad'_{(X_{u_1},X_{u_2})}$ that extends $\mathrm{Lie}\,(A_4)_c$ by a direct sum: $(\mathrm{Lie}\, A_4)_{IW} = (Tr_{n-1} \oplus Tr_{n-1}) \supset\!\!\!\!\!\!+\; so(2) \oplus \mathcal{W}^{(2)} = (\mathrm{Lie}\, A_4)_c \oplus \mathcal{W}^{(2)}$; dimensionally, $6 = 5 + 1$. This result also follows from eq. (47) by contracting with respect to $Z_3 \equiv ad_{\mathcal{X}_{12}}$ and with $\mathcal{W}^{(2)}$ generated by $Y_3 \equiv ad_{\mathcal{X}_{43}}$.

- W-W generalized contraction, $(\mathrm{Lie}\, A_4)_{W-W}$

This is obtained by the reparametrizations $ad'_{\mathcal{X}^{(r)}} = \epsilon^r ad_{\mathcal{X}^{(r)}}$, $r = 0,1,2$. The $\epsilon \to 0$ limit gives $(\mathrm{Lie}\, A_4)_{W-W}$ as

$$\begin{array}{ll}
[ad'_{(X_{a_1},X_{a_2})}, ad'_{(X_{b_1},X_{b_2})}] = 0 & \Rightarrow [\mathcal{W}^{(0)},\mathcal{W}^{(0)}] = 0 \\
[ad'_{(X_{a_1},X_{a_2})}, ad'_{(X_{b_1},X_{u_1})}] = \frac{1}{2}\epsilon_{a_1a_2u_1}{}^{u_2} ad'_{(X_{b_1},X_{u_2})} & \\
\qquad -\frac{1}{2}\epsilon_{b_1u_1a_1}{}^{u_2} ad'_{(X_{u_2},X_{a_2})} - \frac{1}{2}\epsilon_{b_1u_1a_2}{}^{u_2} ad'_{(X_{a_1},X_{u_2})} & \Rightarrow [\mathcal{W}^{(0)},\mathcal{W}^{(1)}] \subset \mathcal{W}^{(1)} \\
\left.\begin{array}{l}
[ad'_{(X_{a_1},X_{u_1})}, ad'_{(X_{a_2},X_{u_2})}] = 0 \ , \ a_1 \neq a_2, u_1 \neq u_2 \\
[ad'_{(X_{a_1},X_{u})}, ad'_{(X_{a_2},X_{u})}] = \epsilon_{a_1ua_2}{}^{v} ad'_{(X_v,X_u)} \\
[ad'_{(X_{a},X_{u_1})}, ad'_{(X_{a},X_{u_2})}] = 0
\end{array}\right\} & \Rightarrow [\mathcal{W}^{(1)},\mathcal{W}^{(1)}] \subset \mathcal{W}^{(2)} \\
[ad'_{\mathcal{X}^{(r)}}, ad'_{\mathcal{Y}^{(2)}}] = 0 \ , \quad r = 0,1,2 & \Rightarrow [\mathcal{W}^{(r)},\mathcal{W}^{(2)}] = 0 \ , \quad r = 0,1,2 \ .
\end{array} \tag{131}$$

This is a central extension of $\mathrm{Lie}\,(A_4)_c$ (eqs. (82)-(85)) by the one-dimensional subalgebra $\mathcal{W}^{(2)} = \langle ad'_{(X_{u_1},X_{u_2})} \rangle$. Thus, $\mathrm{Lie}\,(A_4)_c$= $(\mathrm{Lie}\, A_4)_{W-W}/\mathcal{W}^{(2)}$, and it is not a subalgebra of $(\mathrm{Lie}\, A_4)_{W-W}$.

5.1.3 Third case

Consider $\mathrm{Lie}\, A_4$ as given by the sum $\mathcal{W}^{(1)} \oplus \mathcal{W}^{(2)}$ where $\mathcal{W}^{(1)} = \langle Y_1, Y_2, Y_3 \rangle$ is a $so(3)$ subalgebra and $\mathcal{W}^{(2)} = \langle Z_1, Z_2, Z_3 \rangle$ (eq. (46)). The IW contraction with respect to the $\mathcal{W}^{(1)}$ subalgebra is the well known 6-dimensional Euclidean group E_3, $(\mathrm{Lie}\, A_4)_c = \mathcal{W}^{(2)} \supset\!\!\!\!\!\!+\; \mathcal{W}^{(1)} \equiv Tr_3 \supset\!\!\!\!\!\!+\; so(3)$.

Since $\mathrm{Lie}\, A_4 = so(3) \oplus so(3)$ is not simple, there is of course the possibility of contracting with respect to any of the $so(3)$ subalgebras in eq. (48), leading to $Tr_3 \oplus so(3)$.

5.2 Contractions of Lie A_{n+1}

In Sec. 4.2.2 we have seen that the only splitting of $\mathfrak{G}$ that leads to a non-trivial contracted Filippov algebra $(A_{n+1})_c$ requieres taking $\mathfrak{G}_0$ as an abelian subalgebra generated by $n-1$ A_{n+1} basis elements so that $\mathfrak{V}$ is generated by the remaining two, $\mathfrak{G}_0 = \langle X_a,\ a = 1, \dots, n-1\rangle$, $\quad \mathfrak{V} = \langle X_u,\ u = n, n+1\rangle$.

Labelling as in eq. (127), the commutators of Lie A_{n+1} for the different subspaces adopt the form:

$$
\begin{aligned}
[ad^{(0)}_{(X_{a_1},\dots,X_{a_{n-1}})}, ad^{(0)}_{(X_{b_1},\dots,X_{b_{n-1}})}] &= \frac{1}{2}\sum_{i=1}^{n-1} \underbrace{\epsilon_{a_1 \dots a_{n-1}\, b_i}{}^{b}}_{=0} ad_{(X_{b_1},\dots,X_{b_{i-1}},X_b,X_{b_{i+1}},\dots,X_{b_{n-1}})} \\
&-\frac{1}{2}\sum_{i=1}^{n-1} \underbrace{\epsilon_{b_1 \dots b_{n-1}\, a_i}{}^{b}}_{=0} ad_{(X_{a_1},\dots,X_{a_{i-1}},X_b,X_{a_{i+1}},\dots,X_{a_{n-1}})} = 0
\end{aligned}
\tag{132}
$$

$$
\begin{aligned}
&[ad^{(0)}_{(X_{a_1},\dots,X_{a_{n-1}})}, ad^{(1)}_{(X_{b_1},\dots,X_{b_{n-2}},X_{v_{n-1}})}] = \\
&\frac{1}{2}\sum_{i=1}^{n-2} \underbrace{\epsilon_{a_1 \dots a_{n-1}\, b_i}{}^{b}}_{=0} ad_{(X_{b_1},\dots,X_{b_{i-1}},X_b,X_{b_{i+1}},\dots,X_{b_{n-2}},X_{v_{n-1}})} \\
&+\frac{1}{2}\underbrace{\epsilon_{a_1 \dots a_{n-1}\, v_{n-1}}{}^{b}}_{=0} ad_{(X_{b_1},\dots,X_{b_{n-2}},X_b)} + \frac{1}{2}\epsilon_{a_1 \dots a_{n-1}\, v_{n-1}}{}^{v} ad_{(X_{b_1},\dots,X_{b_{n-2}},X_v)} \\
&-\frac{1}{2}\sum_{i=1}^{n-1} f_{b_1 \dots b_{n-2}\, v_{n-1}\, a_i}{}^{v} ad_{(X_{a_1},\dots,X_{a_{i-1}},X_v,X_{a_{i+1}},\dots,X_{a_{n-1}})} \qquad \in \mathcal{W}^{(1)}
\end{aligned}
\tag{133}
$$

$$
\begin{aligned}
&[ad^{(0)}_{(X_{a_1},\dots,X_{a_{n-1}})}, ad^{(2)}_{(X_{b_1},\dots,X_{b_{n-3}},X_{v_{n-2}},X_{v_{n-1}})}] = \\
&\frac{1}{2}\sum_{i=1}^{n-3} \underbrace{\epsilon_{a_1 \dots a_{n-1}\, b_i}{}^{b}}_{=0} ad_{(X_{b_1},\dots,X_{b_{i-1}},X_b,X_{b_{i+1}},\dots,X_{b_{n-3}},X_{v_{n-2}},X_{v_{n-1}})} \\
&+\frac{1}{2}\underbrace{\epsilon_{a_1 \dots a_{n-1}\, v_{n-2}}{}^{b}}_{=0} ad_{(X_{b_1},\dots,X_{b_{n-3}},X_{v_{n-1}},X_b)} + \frac{1}{2}\underbrace{\epsilon_{a_1 \dots a_{n-1}\, v_{n-2}}{}^{v} ad_{(X_{b_1},\dots,X_{b_{n-3}},X_{v_{n-1}},X_v)}}_{=0} \\
&+\frac{1}{2}\underbrace{\epsilon_{a_1 \dots a_{n-1}\, v_{n-1}}{}^{b}}_{=0} ad_{(X_{b_1},\dots,X_{b_{n-3}},X_b,X_{v_{n-2}})} + \frac{1}{2}\underbrace{\epsilon_{a_1 \dots a_{n-1}\, v_{n-1}}{}^{v} ad_{(X_{b_1},\dots,X_{b_{n-3}},X_v,X_{v_{n-2}})}}_{=0} \\
&-\frac{1}{2}\sum_{i=1}^{n-1} f_{b_1 \dots b_{n-3}\, v_{n-2}\, v_{n-1}\, a_i}{}^{b} \underbrace{ad_{(X_{a_1},\dots,X_{a_{i-1}},X_b,X_{a_{i+1}},,\dots,X_{a_{n-1}})}}_{=0}
\end{aligned}
$$

$$-\frac{1}{2}\sum_{i=1}^{n-1}\underbrace{f_{b_1\,\dots\,b_{n-3}\,v_{n-2}\,v_{n-1}\,a_i}{}^{v}}_{=0}ad_{(X_{a_1},\dots,X_{a_{i-1}},X_v,X_{a_{i+1}},\dots,X_{a_{n-1}})}\quad = 0 \tag{134}$$

$$\begin{aligned}
&[ad^{(1)}_{(X_{a_1},\dots,X_{a_{n-2}},X_{u_{n-1}})},\,ad^{(1)}_{(X_{b_1},\dots,X_{b_{n-2}},X_{v_{n-1}})}] = \\
&\quad \frac{1}{2}\left(\sum_{i=1}^{n-2}\underbrace{\epsilon_{a_1\,\dots\,a_{n-2}\,u_{n-1}\,b_i}{}^{b}}_{=0}ad_{(X_{b_1},\dots,X_{b_{i-1}},X_b,X_{b_{i+1}},\dots,X_{b_{n-2}},X_{v_{n-1}})}\right. \\
&\quad +\sum_{i=1}^{n-2}\epsilon_{a_1\,\dots\,a_{n-2}\,u_{n-1}\,b_i}{}^{v}ad_{(X_{b_1},\dots,X_{b_{i-1}},X_v,X_{b_{i+1}},\dots,X_{b_{n-2}},X_{v_{n-1}})} \\
&\quad \left.+\epsilon_{a_1\,\dots\,a_{n-2}\,u_{n-1}\,v_{n-1}}{}^{b}ad_{(X_{b_1},\dots,X_{b_{n-2}},X_b)}+\underbrace{\epsilon_{a_1\,\dots\,a_{n-3}\,u_{n-1}\,v_{n-1}}{}^{v}}_{=0}ad_{(X_{b_1},\dots,X_{b_{n-2}},X_v)}\right) \\
&\quad -[(a,\,u)\leftrightarrow(b,\,v)]\quad \in \mathcal{W}^{(0)}\oplus\mathcal{W}^{(2)}
\end{aligned} \tag{135}$$

$$\begin{aligned}
&[ad^{(1)}_{(X_{a_1},\dots,X_{a_{n-2}},X_{u_{n-1}})},\,ad^{(2)}_{(X_{b_1},\dots,X_{b_{n-3}},X_{v_{n-2}},X_{v_{n-1}})}] = \\
&\quad \frac{1}{2}\sum_{i=1}^{n-2}\underbrace{\epsilon_{a_1\,\dots\,a_{n-2}\,u_{n-1}\,b_i}{}^{b}}_{=0}ad_{(X_{b_1},\dots,X_{b_{i-1}},X_b,X_{b_{i+1}},\dots,X_{b_{n-3}},X_{v_{n-2}},X_{v_{n-1}})} \\
&\quad +\frac{1}{2}\sum_{i=1}^{n-2}\epsilon_{a_1\,\dots\,a_{n-2}\,u_{n-1}\,b_i}{}^{v}\,\underbrace{ad_{(X_{b_1},\dots,X_{b_{i-1}},X_v,X_{b_{i+1}},\dots,X_{b_{n-3}},X_{v_{n-2}},X_{v_{n-1}})}}_{=0}
\end{aligned} \tag{136}$$

$$\begin{aligned}
&\quad +\frac{1}{2}\epsilon_{a_1\,\dots\,a_{n-2}\,u_{n-1}\,v_{n-2}}{}^{b}ad_{(X_{b_1},\dots,X_{b_{n-3}},X_{v_{n-1}},X_b)}+\frac{1}{2}\underbrace{\epsilon_{a_1\,\dots\,a_{n-3}\,u_{n-1}\,v_{n-2}}{}^{v}}_{=0}ad_{(X_{b_1},\dots,X_{b_{n-3}},X_{v_{n-1}},X_v)} \\
&\quad +\frac{1}{2}\epsilon_{a_1\,\dots\,a_{n-2}\,u_{n-1}\,v_{n-1}}{}^{b}ad_{(X_{b_1},\dots,X_{b_{n-3}},X_b,X_{v_{n-2}})}+\frac{1}{2}\underbrace{\epsilon_{a_1\,\dots\,a_{n-3}\,u_{n-1}\,v_{n-1}}{}^{v}}_{=0}ad_{(X_{b_1},\dots,X_{b_{n-3}},X_v,X_{v_{n-2}})} \\
&\quad -\frac{1}{2}\sum_{i=1}^{n-2}\epsilon_{b_1\,\dots\,b_{n-3}\,v_{n-2}\,v_{n-1}\,a_i}{}^{b}ad_{(X_{a_1},\dots,X_{a_{i-1}},X_b,X_{a_{i+1}},\dots,X_{a_{n-2}},X_{u_{n-1}})} \\
&\quad -\frac{1}{2}\sum_{i=1}^{n-2}\underbrace{\epsilon_{b_1\,\dots\,b_{n-3}\,v_{n-2}\,v_{n-1}\,a_i}{}^{v}}_{=0}ad_{(X_{a_1},\dots,X_{a_{i-1}},X_v,X_{a_{i+1}},\dots,X_{a_{n-2}},X_{u_{n-1}})} \\
&\quad -\frac{1}{2}\underbrace{\epsilon_{b_1\,\dots\,b_{n-3}\,v_{n-2}\,v_{n-1}\,u_{n-1}}{}^{l}}_{=0}ad_{(X_{a_1},\dots,X_{a_{n-2}},X_l)} \\
&\quad \in \mathcal{W}^{(1)}
\end{aligned} \tag{137}$$

$$[ad^{(0)}_{(X_{a_1},\dots,X_{a_{n-1}})}, \underbrace{ad^{(3)}_{(X_{b_1},\dots,X_{b_{n-4}},X_{v_{n-3}},X_{v_{n-2}},X_{v_{n-1}})}}_{=0}] = 0 \tag{138}$$

$$\begin{aligned}
&[ad^{(r)}_{(X_{a_1},\dots,X_{a_{n-r-1}},X_{u_{n-r}},\dots,X_{u_{n-1}})}, ad^{(s)}_{(X_{b_1},\dots,X_{b_{n-s-1}},X_{v_{n-s}},\dots X_{v_{n-1}})}] = \\
&\quad = \frac{1}{2}\sum_{i=1}^{n-s-1} \epsilon_{a_1\dots a_{n-r-1}\, u_{n-r}\dots u_{n-1}\, b_i}{}^{l} ad_{(X_{b_1},\dots,X_{b_{i-1}},X_l,X_{b_{i+1}},\dots,X_{b_{n-s-1}},X_{v_{n-s}},\dots,X_{v_{n-1}})} \\
&\quad + \frac{1}{2}\sum_{i=n-s}^{n-1} \epsilon_{a_1\dots a_{n-r-1}\, u_{n-r}\dots u_{n-1}\, v_i}{}^{l} ad_{(X_{b_1},\dots,X_{b_{n-s-1}},X_{v_{n-s}},\dots,X_{v_{i-1}},X_l,X_{v_{i+1}},\dots,X_{v_{n-1}})} \\
&\quad - [(a,\, u,\, r) \leftrightarrow (b,\, v,\, s)] = 0, \qquad r+s>3\,.
\end{aligned} \tag{139}$$

where the constants $\epsilon_{l_1\dots l_n}{}^j$ are zero if they contain more than $n-1$ indices $l_i \in I_0$ (cf. (86)) or more than 2 indices $l_i \in I_1$; the inner endomorphisms $ad_{(X_{l_1},\dots,X_{l_{n-1}})}$ are zero if they contain more than two indices $l_i \in I_1$. For $n=3$, the above expressions reduce to eqs. (129).

Since $\dim \mathcal{W}^{(0)} = \begin{pmatrix} n-1 \\ n-1 \end{pmatrix} = 1\,,$ $\dim \mathcal{W}^{(1)} = 2\begin{pmatrix} n-1 \\ n-2 \end{pmatrix} = 2(n-1)\,,$ $\dim \mathcal{W}^{(2)} = \begin{pmatrix} n-1 \\ n-3 \end{pmatrix} = \frac{1}{2}(n-1)(n-2)\,,$ $\dim \mathcal{W}^{(r)} = 0,\ r>2$, we check that =$\dim \mathcal{W}^{(0)} + \dim \mathcal{W}^{(1)} + \dim \mathcal{W}^{(2)} = \begin{pmatrix} n+1 \\ 2 \end{pmatrix} = \dim \mathrm{Lie}\, A_{n+1}\,.$

5.2.1 IW contraction $(\mathrm{Lie}\, A_{n+1})_{IW}$ of Lie A_{n+1}

The contraction $(\mathrm{Lie}\, A_{n+1})_{IW}$, obtained by the reparametrization $ad'_{\mathcal{X}^{(0)}} = ad_{\mathcal{X}^{(0)}}$ $(\langle ad_{\mathcal{X}^{(0)}}\rangle = \mathrm{Lie}\,\mathfrak{G}_0)$, $ad'_{\mathcal{X}^{(r)}} = \epsilon ad_{\mathcal{X}^{(r)}}$, $r=1,\dots,n-1$, is given by

$$[ad'^{(0)}_{(X_{a_1},\dots,X_{a_{n-1}})}, ad'^{(0)}_{(X_{b_1},\dots,X_{b_{n-1}})}] = 0 \tag{140}$$

$$\begin{aligned}
&[ad'^{(0)}_{(X_{a_1},\dots,X_{a_{n-1}})}, ad'^{(1)}_{(X_{b_1},\dots,X_{b_{n-2}},X_{v_{n-1}})}] = \frac{1}{2}\epsilon_{a_1\dots a_{n-1}\, v_{n-1}}{}^{v} ad'_{(X_{b_1},\dots,X_{b_{n-2}},X_v)} \\
&\qquad -\frac{1}{2}\sum_{i=1}^{n-1} f_{b_1\dots b_{n-2}\, v_{n-1}\, a_i}{}^{v} ad'_{(X_{a_1},\dots,X_{a_{i-1}},X_v,X_{a_{i+1}},\dots,X_{a_{n-1}})} \qquad \in \mathcal{W}^{(1)}
\end{aligned} \tag{141}$$

$$[ad'^{(0)}_{(X_{a_1},\dots,X_{a_{n-1}})}, ad'^{(2)}_{(X_{b_1},\dots,X_{b_{n-3}},X_{v_{n-2}},X_{v_{n-1}})}] = 0 \tag{142}$$

$$[ad'^{(1)}_{(X_{a_1},\dots,X_{a_{n-2}},X_{u_{n-1}})}, ad'^{(1)}_{(X_{b_1},\dots,X_{b_{n-2}},X_{v_{n-1}})}] = 0 \tag{143}$$

$$[ad'^{(r)}_{(X_{a_1},\dots,X_{a_{n-r-1}},X_{u_{n-r}},\dots,X_{u_{n-1}})}, ad'^{(s)}_{(X_{b_1},\dots,X_{b_{n-s-1}},X_{v_{n-s}},\dots X_{v_{n-1}})}] = 0\ ,\ r+s>2 \tag{144}$$

(in fact, $r+s \geq 2$, see eqs. (142), (143)), and generalizes the $(\mathrm{Lie}\, A_4)_{IW}$ case of Sec. 5.1.2. We see that eqs. (140)-(144) give the Lie $(A_{n+1})_c$ algebra plus the $\begin{pmatrix} n-1 \\ n-3 \end{pmatrix}$ abelian algebra

$\mathcal{W}^{(2)}$, that is, $(\mathrm{Lie}\,A_{n+1})_{IW} = (Tr_{(n-1)} \oplus Tr_{(n-1)}) \supset\!\!\!\!+\, so(2) \oplus \mathcal{W}^{(2)}$. Further, $\dim\,(\mathrm{Lie}\,A_{n+1})_{IW} = \dim \mathcal{W}^{(1)} + \dim \mathcal{W}^{(0)} + \dim \mathcal{W}^{(2)} = \dim\,[(\mathrm{Lie}\,A_{n+1}) = so(n+1)]$. For $n = 3$ the above commutators lead to eqs. (130).

5.2.2 W-W contraction $(\mathrm{Lie}\,A_{n+1})_{W-W}$ of Lie A_{n+1}

The W-W reparametrization is now $ad'_{\mathcal{X}^{(r)}} = \epsilon^r ad_{\mathcal{X}^{(r)}}$ and in the limit $\epsilon \to 0$, eqs. (132)-(139) lead to the same n-brackets as in eqs. (140)-(144), but for (143) which is replaced by

$$\begin{aligned}
&[ad'^{(1)}_{(X_{a_1},\dots,X_{a_{n-2}},X_{u_{n-1}})}, ad'^{(1)}_{(X_{b_1},\dots,X_{b_{n-2}},X_{v_{n-1}})}] = \\
&\quad \frac{1}{2}\epsilon_{a_1 \dots a_{n-2}\, u_{n-1}\, b_i}{}^{v} ad'_{(X_{b_1},\dots,X_{b_{i-1}},X_v,X_{b_{i+1}},\dots,X_{b_{n-2}},X_{v_{n-1}})} \\
&\quad -\frac{1}{2}\epsilon_{b_1 \dots b_{n-2}\, v_{n-1}\, a_i}{}^{v} ad'_{(X_{a_1},\dots,X_{a_{i-1}},X_v,X_{a_{i+1}},\dots,X_{a_{n-2}},X_{u_{n-1}})} \in \mathcal{W}^{(2)}
\end{aligned} \qquad (145)$$

which indicates that $(\mathrm{Lie}\,A_{n+1})_{W-W}$ is a central extension of the $(2n-1)$-dimensional Lie $(A_{n+1})_c$ (see below eq. (122)) by the $\begin{pmatrix} n-1 \\ n-3 \end{pmatrix}$-dimensional abelian algebra $\mathcal{W}^{(2)} = \langle ad'_{\mathcal{X}^{(2)}} \rangle$ so that $(\mathrm{Lie}\,A_{n+1})_{W-W}/\mathcal{W}^{(2)} = \mathrm{Lie}\,(A_{n+1})_c$ as given by eqs. (123)-(125) ($\mathrm{Lie}\,(A_{n+1})_c$ is not a subalgebra of $(\mathrm{Lie}\,A_{n+1})_{W-W}$). Of course, $\begin{pmatrix} n+1 \\ 2 \end{pmatrix} - \begin{pmatrix} n-1 \\ n-3 \end{pmatrix} = \dim \mathrm{Lie}\,(A_{n+1})_c$.

6 Conclusions

We have introduced in this paper the contractions of Filippov algebras and given the non-trivial IW-type contractions of the A_{n+1} simple FAs to illustrate the procedure. As it is for the Lie algebras case, the contraction of a FA $\mathfrak{G}$ has to be done with respect to a subalgebra $\mathfrak{G}_0$ and has the semidirect FA structure $\mathfrak{G}_c = \mathfrak{V} \supset\!\!\!\!+\, \mathfrak{G}_0$, where $\mathfrak{V}$ is a FA abelian ideal of $\mathfrak{G}_c$.

We have also considered the Lie algebra $\mathrm{Lie}\,\mathfrak{G}_c$ associated with a FA contraction $\mathfrak{G}_c$, and the contractions $(\mathrm{Lie}\,\mathfrak{G})_c$ of the Lie algebra associated with the uncontracted FA $\mathfrak{G}$, and compared them in the simple $\mathfrak{G} = A_{n+1}$ case. We have seen that the IW or W-W contractions $(\mathrm{Lie}\,A_{n+1})_{IW}$, $(\mathrm{Lie}\,A_{n+1})_{W-W}$ are either a trivial or a central extension of the Lie algebra $\mathrm{Lie}\,(A_{n+1})_c$ associated with the non-trivial contraction of the simple Filippov algebra A_{n+1}.

All the examples in this paper have dealt with simple FAs. It is clear that, for semisimple FAs, a contraction that only affects the generators of a single ideal will not modify the others since they remain as spectators of the contraction process. But, already for $n = 2$, it is possible to define IW contractions of Lie algebras which have direct sum structure by using a basis that contains generators involving a combination of those of different algebras in the direct sum, be these simple ones or not. The result of a contraction of this type is a

Lie algebra that does no longer have the original direct sum structure of the uncontracted one (these contractions are sometimes called 'unconventional', 'exotic' or even 'generalized', although they are ordinary, standard IW contractions). This explains why the well known eleven dimensional, centrally extended Galilei group may be obtained by a contraction of the direct product of the Poincaré group and a $U(1)$ factor (see [27] and [28, 29] for the 'generating cohomology' properties of these contractions). Other physical examples of contractions of this type have been considered in [30], in [31] in the context of expansions of Lie algebras (a process [32, 31, 33] that is not dimension preserving in general but that includes IW contractions as a particular case) and, very recently, in [34, 35]. In our $n > 2$ case, this type of contractions may have a bearing for FAs. It is well known that it is not easy to find explicit examples of FAs beyond the semisimple ones, one of the reasons being the lack of associativity: the Filippov bracket is not constructed from associative products of its n entries. The above type of contractions, applied to direct sums of FAs, would lead to other non-trivial examples of FAs. Note that here, however, we would be dealing -as throughout this paper- with Filippov *algebras* only; for $n > 2$, there is no 'Filippov group' manifold structure and no vector fields associated with FA generators that could act on it.

Finally, one might think of applying the above contraction scheme to some physical situation. As an exercise, we have tried it on the original BLG A_4 model of (two) coincident M2 branes, but the resulting Lagrangian becomes trivial: the Chern-Simons term disappears and, further, it reduces to the free kinetic terms.

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A On the graded W-W structure of the splitting Lie $\mathfrak{G} = \mathcal{W}^{(0)} \oplus \cdots \oplus \mathcal{W}^{(n-1)}$

Let $\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{V}$ as vector space, and let $\mathcal{W}^{(r)} = \langle ad_{\mathcal{X}_{a_1\cdots a_{n-r-1}u_{n-r}\cdots u_{n-1}}} \rangle$ the Lie $\mathfrak{G}$ subspaces generated by the elements $ad_{\mathcal{X}_{a_1\cdots a_{n-r-1}u_{n-r}\cdots u_{n-1}}}$, where the superindex r indicates the number of generators $X_{u_{n-r}}, \dots, X_{u_{n-1}}$ of the basis of $\mathfrak{V}$ in the fundamental object $\mathcal{X}$ in $ad_{\mathcal{X}}$.

In terms of the structure constants of the FA $\mathfrak{G}$ and using this splitting, the Lie $\mathfrak{G}$ algebra commutators are

$$
\begin{aligned}
&[ad_{\mathcal{X}^{(r)}_{a_1\,\dots\,a_{n-r-1}\,u_{n-r}\,\cdots\,u_{n-1}}}, ad_{\mathcal{Y}^{(s)}_{b_1\,\dots\,b_{n-s-1}\,v_{n-s}\,\cdots\,v_{n-1}}}] = \\
&= \frac{1}{2} ad_{\big[(X_{a_1},\dots,X_{a_{n-r-1}},X_{u_{n-r}},\dots,X_{u_{n-1}})\cdot(X_{b_1},\dots,X_{b_{n-s-1}},X_{v_{n-s}},\dots X_{v_{n-1}}) \atop -(X_{b_1},\dots,X_{b_{n-s-1}},X_{v_{n-s}},\dots X_{v_{n-1}})\cdot(X_{a_1},\dots,X_{a_{n-r-1}},X_{u_{n-r}},\dots,X_{u_{n-1}})\big]} = \\
&= \frac{1}{2}\sum_{i=1}^{n-s-1} f_{a_1\,\dots\,a_{n-r-1}\,u_{n-r}\,\dots\,u_{n-1}\,b_i}{}^{l} ad_{(X_{b_1},\dots,X_{b_{i-1}},X_l,X_{b_{i+1}},\dots,X_{b_{n-s-1}},X_{v_{n-s}},\dots,X_{v_{n-1}})} \\
&+\frac{1}{2}\sum_{i=n-s}^{n-1} f_{a_1\,\dots\,a_{n-r-1}\,u_{n-r}\,\dots\,u_{n-1}\,v_i}{}^{l} ad_{(X_{b_1},\dots,X_{b_{n-s-1}},X_{v_{n-s}},\dots,X_{v_{i-1}},X_l,X_{v_{i+1}},\dots,X_{v_{n-1}})} \\
&-\frac{1}{2}\sum_{i=1}^{n-r-1} f_{b_1\,\dots\,b_{n-s-1}\,v_{n-s}\,\dots\,v_{n-1}\,a_i}{}^{l} ad_{(X_{a_1},\dots,X_{a_{i-1}},X_l,X_{a_{i+1}},\dots,X_{a_{n-r-1}},X_{u_{n-r}},\dots,X_{u_{n-1}})} \\
&-\frac{1}{2}\sum_{i=n-r}^{n-1} f_{b_1\,\dots\,b_{n-s-1}\,v_{n-s}\,\dots\,v_{n-1}\,u_i}{}^{l} ad_{(X_{a_1},\dots,X_{a_{n-r-1}},X_{u_{n-r}},\dots,X_{u_{i-1}},X_l,X_{u_{i+1}},\dots,X_{u_{n-1}})}\,. \qquad (146)
\end{aligned}
$$

(as mentioned, if $\mathfrak{G}$ is not simple, not all $ad_{(X_{a_1},\dots,X_{a_{n-r-1}},X_{u_{n-r}},\dots X_{u_{n-1}})}$ are independent in general). The fulfillment of the W-W condition (128) is a consequence of the dot composition of the fundamental objects (eq. (10)). Indeed, the $\mathcal{X}$'s in ad's in the *r.h.s.* contain a maximum of $r+s$ elements of the basis of $\mathfrak{V}$, except when $r=0$ (no u indices) or $s=0$ (no v indices), where the first and third summatories give a term with $ad_{\mathcal{Z}^{(t)}}$, $t=r+1$ or $t=s+1$. However, the terms with $t=r+s+1$ with r or s equal to zero are zero when $\mathfrak{G}_0$ is subalgebra, ($f_{a_1\dots a_{n-1}}{}^{u} = f_{b_1\dots b_{n-1}}{}^{u} = 0$), and then it follows that $[ad_{\mathcal{X}^{(r)}}, ad_{\mathcal{Y}^{(s)}}] \in \bigoplus\langle ad_{\mathcal{Z}^{(t)}}\rangle$, $t \leq r+s$. Therefore, when the above splitting of Lie $\mathfrak{G}$ is considered, the W-W condition

$$
[\mathcal{W}^{(r)}, \mathcal{W}^{(s)}] \subset \bigoplus \mathcal{W}^{(t)} \ , \quad t \leq r+s \qquad (147)
$$

(here, $r, s, t = 1, \dots, n-1$) is automatically fulfilled when $\mathfrak{G}_0$ is subalgebra (a condition also reflected for $r = 0 = s$).

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