Edge Growth in Graph Cubes*

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Abstract

We show that for every connected graph G of diameter ≥ 3, the graph G^3 has average degree $\geq \frac{7}{4}\delta(G)$. We also provide an example showing that this bound is best possible. This resolves a question of Hegarty [3].

1 Introduction

Throughout the paper, we only consider simple graphs. Let G be a graph. We denote by $v(G)$, $e(G)$ its number of vertices, edges respectively, and let $\delta(G)$ denote the minimum degree of G. The k^{th}*-power* of G, denoted by G^k, has vertex set $V(G)$ and edges the pair of vertices at distance at most k in G. If G is connected, the *diameter* of G is the maximum distance between a pair of vertices of G, or, equivalently, the smallest integer k so that G^k is a clique.

Consider a generating set A of a finite (multiplicative) group and suppose that $1 \in A$ and $g \in A \Rightarrow g^{-1} \in A$. Numerous important questions in Number Theory and Group Theory concern the increase in size from $|A|$ to $|A^k|$. Such problems can be phrased naturally in terms of Cayley graphs. If G is the (simple) Cayley graph generated by A, then G^k is generated by A^k and the sizes of the sets A and A^k are given by the degrees of these (regular) graphs. Thus the growth of the set A^k can be studied in terms of the number of additional edges in the graph G^k. For instance, the following result is an easy corollary of a famous theorem of Cauchy and Davenport.

Theorem 1.1 (Cauchy-Davenport)**.** *If G is a connected Cayley graph on a group of prime order with diameter $< k$ then $e(G^k) \geq ke(G)$.*

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Inspired by this connection, Hegarty considered the more general problem of how many extra edges are formed when we move from a graph G to the k^{th} power of G. Although little can be said for graphs in general, the problem is interesting for connected regular graphs with a diameter constraint. Perhaps surprisingly, even for this class of graphs, there does not exist a positive constant c so that $e(G^2) \geq (1+c)e(G)$. In contrast to this, the following holds for the third power:

Theorem 1.2 (Hegarty)**.** *There exists a positive constant c so that every connected regular graph of diameter ≥ 3 satisfies $e(G^3) \geq (1+c)e(G)$.*

Hegarty proved this for $c = 0.087$ and this was subsequently improved by Pokrovskiy [5] who showed that the same result holds with $c = \frac{1}{6}$ (Pokrovskiy also established some results for higher powers of G). These authors both raised the question of the best possible value of c. We settle this problem in the following theorem.

Theorem 1.3. *If G is a connected graph with diameter ≥ 3, then $e(G^3) \geq \frac{7}{8}\delta(G)v(G)$.*

In particular, when G is regular, this shows that c can be chosen to be $\frac{3}{4}$. To see that this is best possible, we construct a family of regular graphs defined as follows. The graph G_k is obtained from the disjoint union of the graphs $H_1, H_2, \ldots, H_5$ by adding all possible edges between vertices in H_i and H_{i+1} for $1 \leq i \leq 4$, where the graphs H_1 and H_5 are copies of K_{2k+1}, the graphs H_2 and H_4 are copies of K_{2k} minus a perfect matching, and H_3 is a single vertex. It follows that G_k is $4k$-regular with $8k+3$ vertices so $e(G_k) = \frac{1}{2}(8k+3)(4k) = 16k^2 + 6k$. Its cube G_k^3 has $4k+1$ vertices of degree $8k+2$ and $4k+2$ vertices of degree $6k+1$ so it satisfies $e(G_k^3) = \frac{1}{2}(4k+1)(8k+2) + \frac{1}{2}(4k+2)(6k+1) = 28k^2 + 16k + 2$. The family of graphs $\{G_k\}_{k\in\mathbb{N}}$ hence shows that the constant $\frac{7}{8}$ in Theorem 1.3 is best possible.

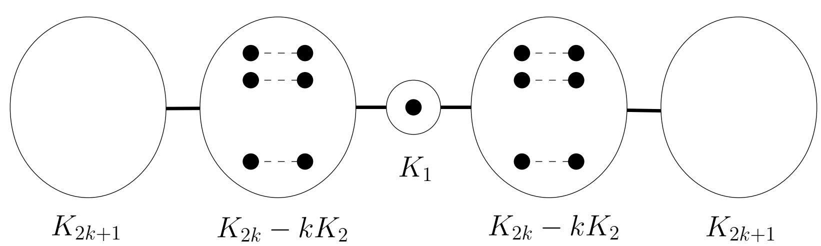

Figure 1: The graph G_k

There are a number of interesting related problems for directed graphs. Here we highlight a rather basic conjecture, which, if true, would resolve a special case of the Caccetta-Häggkvist conjecture.

Conjecture 1.4. *If D is an orientation of a simple graph and every vertex of D has indegree and outdegree equal to d then $e(D^2) \geq 2e(D)$.*

2 Proof

For a set of vertices X we let $N(X)$ denote the closed neighbourhood of X, i.e. $N(X)$ is the union of X and the set of vertices with a neighbour in X. For a nonnegative integer k we let $N^k(X)$ denote the set of vertices at distance $\leq k$ from a point in X. For a vertex v we simplify this notation by $N(v) = N(\{v\})$ and $N^k(v) = N^k(\{v\})$. Note that the degree of a v in G^3 satisfies $deg_{G^3}(v) = |N^3(v)| - 1$.

Proof of Theorem 1.3: Let G be a connected graph with minimum degree δ and diameter ≥ 3. We say that a path is *geodesic* if it is a shortest path between its endpoints. A vertex v is *doubling* if $deg_{G^3}(v) \geq 2\delta$. We let Z be the set of doubling vertices in G. We now prove a sequence of claims.

(1) If v is an internal vertex in a geodesic path of length 3, then v is doubling.

To see this, suppose that our geodesic path has vertex sequence u, v, v', u'. Now $N(u) \cap N(u') = \emptyset$ and $N(u) \cup N(u') \subseteq N^3(v)$ so v is doubling.

Now let $X_1, X_2, \ldots, X_m$ be the vertex sets of the components of $G - Z$.

(2) If v and v' both belong to the same X_i, for some $1 \leq i \leq m$, then $N^2(v) = N^2(v')$.

Since $G[X_i]$ is connected, it suffices to prove that $N^2(v) \subseteq N^2(v')$ when v, v' are adjacent. In this case, suppose that $u \in N^2(v)$. Then there is a path of length 3 from v' to u which has v as an internal vertex. By (1) this path cannot be geodesic, so there must be a path of length at most 2 from v' to u, i.e. $u \in N^2(v')$.

Next, define a relation $\sim$ on $\{X_1, \ldots, X_m\}$ by the rule that $X_i \sim X_j$ if $N(X_i) \cap N(X_j) \neq \emptyset$.

(3) If $X_i \sim X_j$, $v \in X_i$ and $v' \in X_j$, then $N^2(v) = N^2(v')$.

In light of (2), it suffices to prove this in the case that $N(v) \cap N(v') \neq \emptyset$. To see this, suppose (for a contradiction) that $u \in N^2(v) \setminus N^2(v')$. Then we have $N(u) \cap N(v') = \emptyset$ and $N(u) \cup N(v') \subseteq N^3(v)$ so v is doubling, which is contradictory.

(4) $\sim$ is an equivalence relation.

To check that $\sim$ is transitive, suppose that $X_i \sim X_j \sim X_k$ and choose $v \in X_i$ and $v' \in X_k$. It follows from (3) that $N^2(v) = N^2(v')$ but then v and v' have a common neighbour, hence $N(X_i) \cap N(X_k) \neq \emptyset$.

Let $\{Y_1, Y_2, \ldots, Y_\ell\}$ be the set of unions of equivalence classes of $\sim$.

(5) The subgraph of G^2 induced by $N(Y_i)$ is a clique for every $1 \leq i \leq \ell$.

Let $v, v' \in N(Y_i)$. If one of v, v' is in Y_i then it follows from (3) that v, v' are adjacent in G^2. In the remaining case, choose $u \in Y_i$ adjacent to v. Since $v' \in N^2(u)$ there is a path of length ≤ 3 from v to v' which has u as an internal vertex. It now follows from (1) that v and v' are distance ≤ 2 in G, so they are adjacent in G^2.

Let $y_i = |Y_i|$ for every $1 \leq i \leq \ell$.

(6) $deg_{G^3}(v) \geq \delta + y_i$ for every $v \in Y_i$.

Claim (5) shows that $N(Y_i)$ induces a clique in G^2. Since G has diameter ≥ 3 the graph G^2 is not a clique. Hence there must exist a vertex $u \in N^2(Y_i) \setminus N(Y_i)$. Now $N(u) \cap Y_i = \emptyset$ and $N(u) \cup Y_i \subseteq N^3(v)$ which gives us $deg_{G^3}(v) \geq \delta + y_i$ as desired.

Set $y = y_1 + y_2 + \ldots + y_\ell$ and set $z = |Z|$.

(7) $z \geq \delta\ell - y$

First note that $\delta \leq |N(Y_i)| = |Y_i| + |N(Y_i) \cap Z|$ so $|N(Y_i) \cap Z| \geq \delta - y_i$. Next, observe that $N(Y_i) \cap N(Y_j) = \emptyset$ whenever $i \neq j$. This gives us $z = |Z| \geq \sum_{i=1}^{\ell} |N(Y_i) \cap Z| \geq \sum_{i=1}^{\ell} (\delta - y_i) = \delta\ell - y$ as desired.

We now have the tools to complete the proof. Combining the fact that every vertex in Z has degree at least 2δ in G^3 with (6), gives us the following inequality (here we use Cauchy-Schwarz and (7) in getting to the third line)

$$\begin{aligned}
\sum_{v \in V(G)} deg_{G^3}(v) - \tfrac{7}{4}\delta v(G) &\geq 2\delta z + \sum_{i=1}^{\ell} y_i(\delta + y_i) - \tfrac{7}{4}\delta(z + y) \\
&= \tfrac{1}{4}\delta z - \tfrac{3}{4}\delta y + \sum_{i=1}^{\ell} y_i^2 \\
&\geq \tfrac{1}{4}\delta(\delta\ell - y) - \tfrac{3}{4}\delta y + \frac{y^2}{\ell} \\
&= \left(\frac{\delta\sqrt{\ell}}{2} - \frac{y}{\sqrt{\ell}} \right)^2 \\
&\geq 0.
\end{aligned}$$

This shows that G^3 has average degree $\geq \frac{7}{4}\delta$, thus completing the proof. $\square$

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