EXTENSIONS OF TORIC VARIETIES

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Abstract. In this paper, we introduce the notion of "extension" of a toric variety and study its fundamental properties. This gives rise to infinitely many toric varieties with a special property, such as being set theoretic complete intersection or arithmetically Cohen-Macaulay (Gorenstein) and having a Cohen-Macaulay tangent cone or a local ring with non-decreasing Hilbert function, from just one single example with the same property, verifying Rossi's conjecture for larger classes and extending some results appeared in literature.

1. Introduction

Toric varieties are rational algebraic varieties with special combinatorial structures making them objects on the crossroads of different areas such as algebraic statistics, dynamical systems, hypergeometric differential equations, integer programming, commutative algebra and algebraic geometry.

Affine extensions of a toric curve has been introduced for the first time by Arslan and Mete [2] and used to study Rossi's conjecture saying that Gorenstein local rings has non-decreasing Hilbert functions. Later, we have studied set-theoretic complete intersection problem for projective extensions motivated by the fact that every projective toric curve is an extension of another lying in one less dimensional projective space [14]. Our purpose here is to emphasize the nice behavior of toric varieties (of *any* dimension this time) under the operation of extensions and we hope that this approach will provide a rich source of classes for studying many other conjectures and open problems.

In the first part of the present paper we introduce extensions of toric varieties generalizing the definition given for monomial curves in [2, 14] and present results in which a minimal generating set of a toric ideal extends to its extensions by a binomial, see Propositions 2.4 and 2.12, which is not true in general by Example 2.11. In particular, if we start with a set theoretic complete intersection, arithmetically Cohen-Macaulay or Gorenstein toric variety, then we obtain *infinitely* many toric varieties having the same property, generalizing [16].

We devote the second part for the local study of extensions of toric varieties. Namely, if a toric variety has a Cohen-Macaulay tangent cone or at least its local ring has a non-decreasing Hilbert function, then we prove that its nice extensions share these properties supporting Rossi's conjecture for higher dimensional Gorenstein local rings and extending results appeared in [1, Proposition 4.1] and [2, Theorem 3.6]. Similarly, we show that if its local ring is of homogeneous type, then so are the local rings of its extensions. Local properties of toric varieties of higher dimensions have not been studied extensively, although there is a vast literature about toric curves, see [9, 12], [3, 15] and references therein. This paper might be considered as a first modest step towards the higher dimensional case.

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2. Extensions of Toric Varieties

Throughout the paper, K is an algebraically closed field of any characteristic. Let S be a subsemigroup of $\mathbb{N}^d$ generated minimally by $\mathbf{m}_1,\dots,\mathbf{m}_n$. If we set $\deg_S(x_i)=\mathbf{m}_i$, then $S-$degree of a monomial is defined by

$$\deg_S(\mathbf{x}^{\mathbf{b}})=\deg_S(x_1^{b_1}\cdots x_n^{b_n})=b_1\mathbf{m}_1+\cdots+b_n\mathbf{m}_n\in S.$$

The toric ideal of S, denoted I_S, is the prime ideal in $K[x_1,\dots,x_n]$ generated by the binomials $\mathbf{x}^{\mathbf{a}}-\mathbf{x}^{\mathbf{b}}$ with $\deg_S(\mathbf{x}^{\mathbf{a}})=\deg_S(\mathbf{x}^{\mathbf{b}})$. The set of zeroes in $\mathbb{A}^n$ is called the toric variety of S and is denoted by V_S. The projective closure of a variety V will be denoted by $\overline{V}$ as usual and we write $\overline{S}$ for the semigroup defining the toric variety $\overline{V}_S$.

Denote by $S_{\ell,\mathbf{m}}$ the affine semigroup generated by $\ell\mathbf{m}_1,\dots,\ell\mathbf{m}_n$ and $\mathbf{m}$, where ℓ is a positive integer. When $\mathbf{m}\in S$, we define $\delta(\mathbf{m})$ to be the minimum of all the sums $s_1+\cdots+s_n$ where $s_1,\dots,s_n$ are some non-negative integers such that $\mathbf{m}=s_1\mathbf{m}_1+\cdots+s_n\mathbf{m}_n$.

Definition 2.1 (Extensions)**.** With the preceding notation, we say that the affine toric variety $V_{S_{\ell,\mathbf{m}}}\subset\mathbb{A}^{n+1}$ is an *extension* of $V_S\subset\mathbb{A}^n$, if $\mathbf{m}\in S$, and ℓ is a positive integer relatively prime to a component of $\mathbf{m}$. A projective variety $\overline{E}\subset\mathbb{P}^{n+1}$ will be called an *extension* of another one $\overline{X}\subset\mathbb{P}^n$ if its affine part E is an extension of the affine part X of $\overline{X}$.

Remark 2.2.
1. Notice that $\overline{V_S}=V_{\overline{S}}$, $I_S\subset I_{S_{\ell,\mathbf{m}}}$ and $I_{\overline{S}}\subset I_{\overline{S}_{\ell,\mathbf{m}}}$.
2. The question of whether or not $I_{S_{\ell,\mathbf{m}}}$ (resp. $I_{\overline{S}_{\ell,\mathbf{m}}}$) has a minimal generating set containing a minimal generating set of I_S (resp. $I_{\overline{S}}$) is not trivial.
3. This definition generalizes the one given for monomial curves in [2, 14].
4. In [16], special extensions for which ℓ equals to a multiple of $\delta(\mathbf{m})$ has been studied without referring to them as extensions.

Example 2.3. Let $S=\mathbb{N}\{1,4,5\}$ be the semigroup defining the affine monomial (toric) curve $V_S=\{(v,v^4,v^5)\,|\,v\in K\}$. If we take $\ell=1$ and $\mathbf{m}=10$ we have an affine extension $V_{S_{1,10}}$ corresponding to the semigroup $S_{1,10}=\mathbb{N}\{1,4,5,10\}$. The projective monomial curve $V_{\overline{S}}$ in $\mathbb{P}^3$ is defined by $\overline{S}=\mathbb{N}\{(5,0),(4,1),(1,4),(0,5)\}$ and the corresponding projective extension is determined by the semigroup

$$\overline{S}_{1,10}=\mathbb{N}\{(10,0),(9,1),(6,4),(5,5),(0,10)\}.$$

Although minimal generating set of I_S extends to the toric ideal of $S_{1,10}$, as we will see below in Proposition 2.4, this is not possible for the ideal corresponding to $\overline{S}_{1,10}$ as we will illustrate in Example 2.11.

2.1. Affine Extensions.
In this section, we explore some algebro-geometric relations between a toric variety and its extensions. The results of this section generalize the results of [16].

Proposition 2.4. *If the toric variety $V_{S_{\ell,\mathbf{m}}}\subset\mathbb{A}^{n+1}$ is an extension of $V_S\subset\mathbb{A}^n$, then $I_{S_{\ell,\mathbf{m}}}=I_S+\langle F\rangle$, where $F=x_{n+1}^{\ell}-x_1^{s_1}\cdots x_n^{s_n}$. Moreover, if $\mathcal{G}$ is a reduced Gröbner basis for I_S with respect to a term order $\succ$, then $\mathcal{G}\cup\{F\}$ is a reduced Gröbner basis for $I_{S_{\ell,\mathbf{m}}}$ with respect to a term order refining $\succ$ and making x_{n+1} the biggest variable.*

Proof. First of all, $S=\mathbb{N}\{\mathbf{m}_1,\dots,\mathbf{m}_n\}$, $S_{\ell,\mathbf{m}}=\mathbb{N}T$, where the set $T=T_1\sqcup T_2$, $T_1=\{\ell\mathbf{m}_1,\dots,\ell\mathbf{m}_n\}$ and $T_2=\{\mathbf{m}\}$. We claim that $S_{\ell,\mathbf{m}}$ is the gluing of its subsemigroups $\mathbb{N}T_1$ and $\mathbb{N}T_2$. To this end we show that $\mathbb{Z}T_1\cap\mathbb{Z}T_2=\mathbb{Z}\alpha$, where $\alpha=\ell\mathbf{m}\in\mathbb{N}T_1\cap\mathbb{N}T_2$, see [14, section 3] for a similar proof.

Since $\ell\mathbf{m} = s_1\ell\mathbf{m}_1 + \cdots + s_n\ell\mathbf{m}_n$ with non-negative integers s_i, we have clearly $\mathbb{Z}T_1 \cap \mathbb{Z}T_2 \supseteq \mathbb{Z}\alpha$. Take $z\mathbf{m} = z_1\ell\mathbf{m}_1 + \cdots + z_n\ell\mathbf{m}_n \in \mathbb{Z}T_1 \cap \mathbb{Z}T_2$ and note that $z\mathbf{m} = \ell(z_1\mathbf{m}_1 + \cdots + z_n\mathbf{m}_n)$. Since ℓ is relatively prime to a component of $\mathbf{m}$ by assumption, it follows that ℓ divides z and thus $z\mathbf{m} \in \mathbb{Z}\alpha$ yielding $\mathbb{Z}T_1 \cap \mathbb{Z}T_2 \subseteq \mathbb{Z}\alpha$. By the relation between the corresponding ideals, we have $I_{S_{\ell,\mathbf{m}}} = I_S + \langle F\rangle$, since $I_{T_1} = I_S$ and $I_{T_2} = 0$.

Let $\mathcal{G} = \{F_1, \ldots, F_k\}$. Note first that $F \in I_{S_{\ell,\mathbf{m}}} - I_S$ and F_i are binomials of the form $F_i = \mathrm{LM}(F_i) - \mathrm{NLM}(F_i)$. Since $\mathrm{LM}(F_i) \in K[x_1, \ldots, x_n]$ and $\mathrm{LM}(F) = x_{n+1}^{\ell}$, it follows that $gcd(\mathrm{LM}(F_i), \mathrm{LM}(F)) = 1$, for all i. This implies that the set $\mathcal{G} \cup \{F\}$ is a Gröbner basis for $I_{S_{\ell,\mathbf{m}}}$. Now, if $\mathrm{LM}(F_i)$ does not divide $\mathrm{NLM}(F) = x_1^{s_1} \cdots x_n^{s_n}$, it follows that $\mathcal{G} \cup \{F\}$ is reduced as $\mathcal{G}$ is also. Otherwise, we can replace $\mathrm{NLM}(F)$ by a monomial with the same $S_{\ell,\mathbf{m}}$-degree and obtain a new binomial F with the required property. For if, $\mathrm{NLM}(F) = \mathrm{LM}(F_i)M$, for some monomial $M \in K[x_1, \ldots, x_n]$, we have

$$\begin{aligned}\deg_{S_{\ell,\mathbf{m}}}(\mathrm{NLM}(F)) &= \ell.[\deg_S(\mathrm{LM}(F_i)) + deg_{S_{\ell,\mathbf{m}}}(M)]\\ &= \ell.[\deg_S(\mathrm{NLM}(F_i)) + \deg_{S_{\ell,\mathbf{m}}}(M)] = \deg_{S_{\ell,\mathbf{m}}}(\mathrm{NLM}(F_i)M),\end{aligned}$$

which means that the new binomial $F = x_{n+1}^{\ell} - \mathrm{NLM}(F_i)M \in I_{S_{\ell,\mathbf{m}}}$. Since $\mathcal{G}$ is reduced and F_i are irreducible binomials, no $\mathrm{LM}(F_j)$ divides $\mathrm{NLM}(F_i)M$ and thus we are done. □

Remark 2.5. Let $\mathcal{G}$ be a reduced Gröbner basis of I. Then, in general, the set $\mathcal{G} \cup \{F\}$ may not be a reduced Gröbner basis of $I + \langle F\rangle$. This is the case, if $I = \langle wy - x^2\rangle$ and $F = y^2 - zx$, as the reduced Gröbner basis of $I + \langle F\rangle$ with respect to the reverse lexicographic ordering with $w > z > y > x$ is given by $\{wy - x^2, y^2 - zx, wzx - yx^2\}$. Therefore, it is an interesting occurrence when a reduced Gröbner basis coincide with a minimal generating set.

Corollary 2.6. *If $V_S \subset \mathbb{A}^n$ is a set theoretic complete intersection, arithmetically Cohen-Macaulay (Gorenstein), so are its extensions $V_{S_{\ell,\mathbf{m}}} \subset \mathbb{A}^{n+1}$.*

Proof. The first claim follows from Proposition 2.4 and [17]. The second part follows from Proposition 2.4 and Corollary 2.10 that will be proven below. □

2.1.1. *Minimal Free Resolutions.* Now, we present the mapping cone construction and its direct consequences which will be used later. In particular, using this construction we can determine explicit minimal free resolutions for all extensions of toric varieties. The following can be proven easily via the mapping cone $M(F)$ of the map $F : \mathfrak{C}(-\mathbf{f}) \to \mathfrak{C}$ defined by multiplication by F, see e.g. [13].

Theorem 2.7. *Let I be a multigraded ideal in a polynomial ring R over a field K and $F \in R-I$ is a multigraded polynomial of multidegree $\mathbf{f}$ which is a nonzerodivisor on R/I. Assume further that I has a minimal multigraded free resolution given by*

$$\mathfrak{C} : 0 \longrightarrow U_d \xrightarrow{A_d} U_{d-1} \xrightarrow{A_{d-1}} \cdots \xrightarrow{A_2} U_1 \xrightarrow{A_1} R \longrightarrow R/I \longrightarrow 0 .$$

Then, $\mathfrak{C}'$ is a minimal multigraded free resolution of $J := I + \langle F\rangle$ over R, where

$$\mathfrak{C}' : 0 \longrightarrow U'_{d+1} \xrightarrow{A'_{d+1}} U'_d \xrightarrow{A'_d} \cdots \xrightarrow{A'_2} U'_1 \xrightarrow{A'_1} R \longrightarrow R/J \longrightarrow 0 ,$$

$$U'_1 = U_1 \oplus R(-\mathbf{f}),\ U'_i = U_i \oplus U_{i-1}(-\mathbf{f})\ (2 \le i \le d),\ U'_{d+1} = U_d(-\mathbf{f})$$

and the matrices above are as follows

$$A'_1 = \begin{bmatrix} A_1 & F \end{bmatrix}, A'_i = \begin{bmatrix} A_i & (-1)^{i-1}F.I \\ 0 & A_{i-1} \end{bmatrix}\ (2 \le i \le d), A'_{d+1} = \begin{bmatrix} (-1)^d F.I \\ A_d \end{bmatrix}.$$

Remark 2.8. It is clear that if $F_1, \ldots, F_k$ form an R/I-sequence, then Theorem 2.7 gives a minimal free resolution of $I + \langle F_1, \ldots, F_k \rangle$ using that of I.

Remark 2.9. Notice that when $\mathfrak{C}$ is a minimal multigraded free resolution of a multigraded ideal I in $R = K[x_1, \ldots, x_n]$, $F = T_1 - T_2$ is a binomial with x_{n+1} dividing T_1, it follows that F is not in I and is a nonzerodivisor on R/I. Thus, Theorem 2.7 applies to ideals $I + \langle F \rangle$ in $R[x_{n+1}]$, whenever F is of this form. In particular, it applies to extensions of toric varieties.

Next, we list some important consequences below:

Corollary 2.10. *With $J = I + \langle F \rangle$ and the hypothesis of Thereom 2.7, we have the following:*

*(**a**)* $\operatorname{pd}(R/J) = \operatorname{pd}(R/I) + 1$.

*(**b**) R/J is Cohen-Macaulay if and only if R/I is Cohen-Macaulay. In this situation,* $\operatorname{type}(R/J) = \operatorname{type}(R/I)$ *and in particular R/J is Gorenstein if and only if R/I is Gorenstein.*

Proof. (**a**) Since the projective dimension is the length of the minimal free resolution, the assertion follows directly from Theorem 2.7.
(**b**) Since $F \notin I$, we have $\operatorname{codim}(J) = \operatorname{codim}(I)+1$. Therefore $\operatorname{pd}(R/J) = \operatorname{codim}(J)$ if and only if $\operatorname{pd}(R/I) = \operatorname{codim}(I)$ which means that R/J is Cohen-Macaulay if and only if R/I is Cohen-Macaulay. Assume now that R/I is Cohen-Macaulay, then $\operatorname{type}(R/I) = \beta_d(R/I) = \operatorname{rank}(U_d)$, where $d = \operatorname{pd}(R/I) = \operatorname{codim}(I)$.

Since $U'_{d+1} = U_d(-\mathbf{f})$ and $\operatorname{codim}(J) = \operatorname{codim}(I)+1$ it easily follows that we have $\operatorname{type}(R/J) = \beta_{d+1}(R/J) = \operatorname{rank}(U'_{d+1}) = \operatorname{type}(R/I)$. Since a Gorenstein ring is a Cohen-Macaulay ring of type 1, the last assertion follows. □

2.2. **Projective Extensions.**

Contrary to the case of affine extensions, it is not true in general that a minimal generating set of a projective extension of $V_{\overline{S}}$ contains a minimal generating set of $I_{\overline{S}}$ as illustrated by the following example.

Example 2.11. Recall from Example 2.3 that if $S = \mathbb{N}\{1, 4, 5\}$, then the projective monomial curve $V_{\overline{S}}$ in $\mathbb{P}^3$ is defined by $\overline{S} = \mathbb{N}\{(5, 0), (4, 1), (1, 4), (0, 5)\}$. Consider the projective extension $V_{\overline{S}_{1,10}}$ defined by the semigroup

$$\overline{S}_{1,10} = \mathbb{N}\{(10, 0), (9, 1), (6, 4), (5, 5), (0, 10)\}.$$

It is easy to see (use e.g. Macaulay [4]) that the set $\{F_1, F_2, F_3, F_4, F_5\}$ constitutes a reduced Gröbner basis (and a minimal generating set) for the ideal $I_{\overline{S}}$ with respect to the reverse lexicographic order with $x_1 > x_2 > x_3 > x_0$, where

$$\begin{aligned} F_1 &= x_1^4 - x_0^3 x_2 \\ F_2 &= x_2^4 - x_1 x_3^3 \\ F_3 &= x_1^2 x_3^2 - x_0 x_2^3 \\ F_4 &= x_1^3 x_3 - x_0^2 x_2^2 \\ F_5 &= x_1 x_2 - x_0 x_3. \end{aligned}$$

A computation shows that the set $\{F_1, F_4, F_5, F, F_6, F_7\}$ is a reduced Gröbner basis for $I_{\overline{S}_{1,10}}$ with respect to the reverse lexicographic order with $x_1 > x_2 > x_3 > x_4 > x_0$, where

$$\begin{aligned} F &= x_3^2 - x_0 x_4 \\ F_6 &= x_2^3 - x_1^2 x_4 \\ F_7 &= x_1^3 x_4 - x_0 x_2^2 x_3. \end{aligned}$$

We observe now that $F_7 = x_2^2 F_5 - x_1 F_6$ and that the set $\{F_1, F_4, F_5, F, F_6\}$ is a minimal generating set of $I_{\overline{S}_{1,10}}$. The fact that no minimal generating set of

$I_{\overline{S}}$ extends to a minimal generating set of $I_{\overline{S}_{1,10}}$ follows from the observation that $\mu(I_{\overline{S}}) = \mu(I_{\overline{S}_{1,10}})(= 5)$, where $\mu(\cdot)$ denotes the minimal number of generators.

Notice that the previous example reveals why minimal generating sets need not extend when $\ell < \delta(\mathbf{m})$. Next, we show that this can be avoided as long as $\ell \geq \delta(\mathbf{m})$. Now, we compute Gröbner basis for $I_{\overline{S}_{\ell,\mathbf{m}}}$ using the Proposition 2.4 and the fact that if $\mathcal{G}$ is a Gröbner basis for the ideal of an affine variety with respect to a term order refining the order by degree, then the homogenization of $\mathcal{G}$ is a Gröbner basis for the ideal of its projective closure.

Proposition 2.12. *If $\mathcal{G}$ is a reduced Gröbner basis for $I_{\overline{S}}$ with respect to a term order $\succ$ making x_0 the smallest variable and $\ell \geq \delta(\mathbf{m})$, then $\mathcal{G} \cup \{F\}$ is a reduced Gröbner basis for $I_{\overline{S}_{\ell,\mathbf{m}}}$ with respect to a term order refining $\succ$ and making x_{n+1} the biggest variable and thus $I_{\overline{S}_{\ell,\mathbf{m}}} = I_{\overline{S}} + \langle F \rangle$, where $F = x_{n+1}^{\ell} - x_0^{\ell-\delta(\mathbf{m})} x_1^{s_1} \cdots x_n^{s_n}$.*

Proof. Let $\mathcal{G} = \{F_1, \dots, F_k\}$. If we dehomogenize the polynomials in $\mathcal{G}$ by substituting $x_0 = 1$, we get a reduced Gröbner basis $\{G_1, \dots, G_k\}$ for I_S with respect to $\succ$ which refines the order by degree. From Proposition 2.4, we know that $I_{S_{\ell,\mathbf{m}}} = I_S + \langle G \rangle = \langle G_1, \dots, G_k, G \rangle$, where $G = F(1, x_1, \dots, x_n)$. Since $\mathrm{LM}(G_i) \in K[x_1, \dots, x_n]$ and $\mathrm{LM}(G) = x_{n+1}^{\ell}$, it follows that $gcd(\mathrm{LM}(G_i), \mathrm{LM}(G)) = 1$, for all i. This implies that the set $\{G_1, \dots, G_k, G\}$ is a Gröbner basis for $I_{S_{\ell,\mathbf{m}}}$ with respect to a term order refining the order by degree and $\succ$. Hence, their homogenizations constitute the required Gröbner basis for $I_{\overline{S}_{\ell,\mathbf{m}}}$ as claimed.

Now, if $\mathrm{LM}(F_i)$ does not divide $\mathrm{NLM}(F) := x_0^{\ell-\delta(\mathbf{m})} x_1^{s_1} \cdots x_n^{s_n}$, it follows that $\mathcal{G} \cup \{F\}$ is reduced as $\mathcal{G}$ is also. Otherwise, i.e., $\mathrm{NLM}(F) = \mathrm{LM}(F_i) x_0^{\ell-\delta(\mathbf{m})} M$, for some monomial M in $K[x_1, \dots, x_n]$, we replace $\mathrm{NLM}(F)$ by $T_i x_0^{\ell-\delta(\mathbf{m})} M$, since $\deg_S(\mathrm{LM}(F_i)) = \deg_S(T_i)$, which means that the new binomial $F = x_{n+1}^{\ell} - T_i x_0^{\ell-\delta(\mathbf{m})} M \in I_{\overline{S}_{\ell,\mathbf{m}}}$. Since $\mathcal{G}$ is reduced and F_i are irreducible binomials, no $LM(F_j)$ divides $T_i x_0^{\ell-\delta(\mathbf{m})} M$. Therefore, the set $\mathcal{G} \cup \{F\}$ is reduced as desired. Thus, we obtain $I_{\overline{S}_{\ell,\mathbf{m}}} = I_{\overline{S}} + \langle F \rangle$. □

Corollary 2.13. *If $V_{\overline{S}} \subset \mathbb{P}^n$ is a set theoretic complete intersection, arithmetically Cohen-Macaulay (Gorenstein), so are its extensions $V_{\overline{S}_{\ell,\mathbf{m}}} \subset \mathbb{P}^{n+1}$ provided that $\ell \geq \delta(\mathbf{m})$.*

Proof. Since $I_{\overline{S}_{\ell,\mathbf{m}}} = I_{\overline{S}} + \langle F \rangle$ while $\ell \geq \delta(\mathbf{m})$, we have $V_{\overline{S}_{\ell,\mathbf{m}}} = Z(I_{\overline{S}}, F)$. Since $V_{\overline{S}}$ is a set theoretic complete intersection, it follows that $V_{\overline{S}} = Z(F_1, \dots, F_c)$, where $c = \mathrm{codim}(I_{\overline{S}})$. Since $\mathrm{codim}(I_{\overline{S}_{\ell,\mathbf{m}}}) = c+1$ and $V_{\overline{S}_{\ell,\mathbf{m}}} = Z(F_1, \dots, F_c, F)$, it readily follows that $V_{\overline{S}_{\ell,\mathbf{m}}}$ is a set theoretic complete intersection.

The second claim follows directly from Proposition 2.12 and Corollary 2.10. □

3. Local Properties of Extensions

In this section, we study Cohen-Macaulayness of tangent cones of extensions of a toric variety having a Cohen-Macaulay tangent cone, see [1, 9, 12] for the literature about Cohen-Macaulayness of tangent cones. We also show that if the local ring of a toric variety is of homogeneous type or has a non-decreasing Hilbert function, then its extensions share the same property. As a main result, we demonstrate that in the framework of extensions it is very easy to create infinitely many new families of arbitrary dimensional and embedding codimensional local rings having non-decreasing Hilbert functions supporting Rossi's conjecture. This is important, as the conjecture is known only for local rings with small (co)dimension:

- Cohen-Macaulay rings of dimension 1 and embedding codimension 2, [6],

- Some Gorenstein rings of dimension 1 and embedding codimension 3, [2],
- Complete intersection rings of embedding codimension 2, [10],
- Some local rings of dimension 1, [3, 15],

where embedding codimension of a local ring is defined to be the difference between its embedding dimension and dimension. For instance, if $\mathbb{A}^n$ is the smallest affine space containing V_S, then embedding dimension of the local ring of V_S is n. Its dimension coincides with the dimension of V_S and its embedding codimension is nothing but the codimension of V_S, i.e. $n - \dim V_S$.

Before going further, we need to recall some terminology and fundamental results which will be used subsequently. If $V_S \subset \mathbb{A}^n$ is a toric variety, its associated graded ring is isomorphic to $K[x_1, \dots, x_n]/{I_S}^*$, where ${I_S}^*$ is the ideal of the tangent cone of V_S at the origin, that is the ideal generated by the polynomials f^* with $f \in I_S$ and f^* being the homogeneous summand of f of the smallest degree. Thus, the tangent cone is Cohen-Macaulay if this quotient ring is also. Similarly, we can study the Hilbert function of the local ring associated to V_S by means of this quotient ring, since the Hilbert function of the local ring is by definition the Hilbert function of the associated graded ring. Finally, we can find a minimal generating set for ${I_S}^*$ by computing a minimal standard basis of I_S with respect to a local order. For further inquiries and notations to be used, we refer to [7].

Assume now that $V_{S_{\ell,\mathbf{m}}} \subset A^{n+1}$ is an extension of V_S, for suitable ℓ and $\mathbf{m}$. Then, by Proposition 2.4, we know that $I_{S_{\ell,\mathbf{m}}} = I_S + \langle F \rangle$, where $F = x_{n+1}^{\ell} - x_1^{s_1} \cdots x_n^{s_n}$.

Proposition 3.1. *If $\mathcal{G}$ is a minimal standard basis of I_S with respect to a negative degree reverse lexicographic ordering $\succ$ and $\ell \leq \delta(\mathbf{m})$, then $\mathcal{G} \cup \{F\}$ is a minimal standard basis of $I_{S_{\ell,\mathbf{m}}}$ with respect to a negative degree reverse lexicographic ordering refining $\succ$ and making x_{n+1} the biggest variable.*

Proof. Let $\mathcal{G}' = \mathcal{G} \cup \{F\}$. Since $NF(\text{spoly}(f,g)|\mathcal{G}) = 0$, for all $f, g \in \mathcal{G}$, we have $NF(\text{spoly}(f,g)|\mathcal{G}') = 0$. Since $\text{LM}(f) \in K[x_1, \dots, x_n]$ and $\text{LM}(F) = x_{n+1}^{\ell}$, it follows at once that $gcd(\text{LM}(f), \text{LM}(F)) = 1$, for every $f \in \mathcal{G}$. Therefore we get $NF(\text{spoly}(f,F)|\mathcal{G}') = 0$, for any $f \in \mathcal{G}$. This reveals that $\mathcal{G}'$ is a standard basis with respect to the afore mentioned local ordering and it is minimal because of the minimality of $\mathcal{G}$. □

Theorem 3.2. *If $V_S \subset A^n$ has a Cohen-Macaulay tangent cone at 0, then so have its extensions $V_{S_{\ell,\mathbf{m}}} \subset A^{n+1}$, provided that $\ell \leq \delta(\mathbf{m})$.*

Proof. An immediate consequence of the previous result is that ${I_{S_{\ell,\mathbf{m}}}}^* = {I_S}^* + \langle F^* \rangle$, where F^* is x_{n+1}^{ℓ} whenever $\ell < \delta(\mathbf{m})$ and is F if $\ell = \delta(\mathbf{m})$. In any case F^* is a nonzerodivisor on $K[x_1, \dots, x_{n+1}]/{I_S}^*$. Since the graded ring $K[x_1, \dots, x_n]/{I_S}^*$ is Cohen-Macaulay by the assumption, clearly $K[x_1, \dots, x_{n+1}]/{I_S}^*$ is also Cohen-Macaulay and we can apply Corollary 2.10 to conclude that $K[x_1, \dots, x_{n+1}]/{I_{S_{\ell,\mathbf{m}}}}^*$ is Cohen-Macaulay as required. In particular, the tangent cone of a toric variety and tangent cones of its extensions have the same Cohen-Macaulay type. □

Remark 3.3. Theorem 3.2 generalizes the results appeared in [1, Proposition 4.1] and [2, Theorem 3.6] from toric curves to toric varieties of any dimension. Moreover, Hilbert functions of the local rings of these extensions are nondecreasing in this case supporting Rossi's conjecture.

According to [8], a local ring is of homogeneous type if its Betti numbers coincide with the Betti numbers of its associated graded ring, considered as a module over itself. It is interesting to obtain local rings of homogeneous type, since in this case, for example, the local ring and its associated ring will have the same depth and their Cohen-Macaulayness will be equivalent since they always have the same dimension.

It will also be easier to get information about the depth of the symmetric algebra in this case, see [8, 11].

Proposition 3.4. *If the local ring of $V_S \subset A^n$ is of homogeneous type, then its extensions will also have local rings of homogeneous type if and only if $\ell \leq \delta(\mathbf{m})$.*

Proof. Let $K[[S]]$ denote the local ring of V_S, i.e. the localization of the semigroup ring $K[S] = R/I_S$ at the origin, where $R = K[x_1, \ldots, x_n]$. The Betti numbers of $K[[S]]$ and $K[S]$ is the same, since localization is flat. For the convenience of notation let us use $GR[S]$ for the associated graded ring corresponding to V_S and $\beta_i(GR[S])$ for the Betti numbers of the minimal free resolution of $GR[S] = R/{I_S}^*$ over R.

Assume now that $K[[S]]$ is of homogeneous type, i.e. $\beta_i(K[[S]]) = \beta_i(GR[S])$, for all i. For any extension $V_{S_{\ell,\mathbf{m}}} \subset A^{n+1}$ of V_S, we have from Proposition 2.4 that $I_{S_{\ell,\mathbf{m}}} = I_S + \langle F \rangle$, where $F = x_{n+1}^{\ell} - x_1^{s_1} \cdots x_n^{s_n}$. Therefore, using Theorem 2.7, we can compute the Betti numbers as follows

- $\beta_1(K[[S_{\ell,\mathbf{m}}]]) = \beta_1(K[[S]]) + 1$
- $\beta_i(K[[S_{\ell,\mathbf{m}}]]) = \beta_i(K[[S]]) + \beta_{i-1}(K[[S]])$, $2 \leq i \leq d = pd(K[[S]])$
- $\beta_{d+1}(K[[S_{\ell,\mathbf{m}}]]) = \beta_d(K[[S]])$.

If furthermore $\ell \leq \delta(\mathbf{m})$, Proposition 3.1 yields ${I_{S_{\ell,\mathbf{m}}}}^* = {I_S}^* + \langle F^* \rangle$. Hence, we can use Theorem 2.7 again to compute Betti numbers of $GR[S_{\ell,\mathbf{m}}]$:

- $\beta_1(GR[S_{\ell,\mathbf{m}}]) = \beta_1(GR[S]) + 1$
- $\beta_i(GR[S_{\ell,\mathbf{m}}]) = \beta_i(GR[S]) + \beta_{i-1}(GR[S])$, $2 \leq i \leq d = pd(K[[S]])$
- $\beta_{d+1}(GR[S_{\ell,\mathbf{m}}]) = \beta_d(GR[S])$.

It is obvious now that $\beta_i(GR[S_{\ell,\mathbf{m}}]) = \beta_i(K[[S_{\ell,\mathbf{m}}]])$ for any i and that local rings of extensions are of homogeneous type.

The converse is rather trivial, since homogeneity of local rings of extensions force that $\beta_1(GR[S_{\ell,\mathbf{m}}]) = \beta_1(K[[S_{\ell,\mathbf{m}}]])$, i.e. ${I_{S_{\ell,\mathbf{m}}}}^* = {I_S}^* + \langle F^* \rangle$ which is possible only if $\ell \leq \delta(\mathbf{m})$. ☐

Finally, inspired by [3, Theorem 3.1], we consider extensions of a toric variety whose local ring has a non-decreasing Hilbert function and whose tangent cone is not necessarily Cohen-Macaulay.

Theorem 3.5. *If $V_S \subset A^n$ has a local ring with non-decreasing Hilbert function, then so have its extensions $V_{S_{\ell,\mathbf{m}}} \subset A^{n+1}$, provided that $\ell \leq \delta(\mathbf{m})$.*

Proof. Let $R = K[x_1, \ldots, x_n]$. If I is a graded ideal of R, then it is a standard fact that the Hilbert function of R/I is just the Hilbert function of $R/\mathrm{LM}(I)$, where $\mathrm{LM}(I)$ is a monomial ideal consisting of the leading monomials of polynomials in I. Now, Proposition 3.1 reveals that ${I_{S_{\ell,\mathbf{m}}}}^* = {I_S}^* + \langle F^* \rangle$, where $F = x_{n+1}^{\ell} - x_1^{s_1} \cdots x_n^{s_n}$ and that $\mathrm{LM}({I_{S_{\ell,\mathbf{m}}}}^*) = \mathrm{LM}({I_S}^*) + \langle \mathrm{LM}(F^*) \rangle$. Since $\mathrm{LM}({I_S}^*) \subset R$ and $\mathrm{LM}(F^*) = x_{n+1}^{\ell}$ with respect to the local order mentioned in Proposition 3.1, it follows from the proof of [5, Proposition 2.4] that $R' = R_1 \otimes_K R_2$, where

$$R' = R[x_{n+1}]/\mathrm{LM}({I_{S_{\ell,\mathbf{m}}}}^*),\ R_1 = R/\mathrm{LM}({I_S}^*) \text{ and } R_2 = K[x_{n+1}]/\langle x_{n+1}^{\ell} \rangle.$$

Hilbert series of R_1 can be given as $\sum_{k \geq 0} a_k t^k$, where $a_k \leq a_{k+1}$ for any $k \geq 0$, since from the assumption the local ring associated to V_S has non-decreasing Hilbert function. It is clear that the Hilbert series of R_2 is $h_2(t) = 1 + t + \cdots + t^{\ell-1}$. Since the Hilbert series of R' is the product of those of R_1 and R_2, we observe that the

Hilbert series of R' is given by

$$\begin{aligned}\sum_{k\geq 0} b_k t^k &= (1+t+\cdots+t^{\ell-1})\sum_{k\geq 0} a_k t^k\\ &= \sum_{k\geq 0} a_k t^k + \sum_{k\geq 0} a_k t^{k+1} + \cdots + \sum_{k\geq 0} a_k t^{k+\ell-1}\\ &= \sum_{k\geq 0} a_k t^k + \sum_{k\geq 1} a_{k-1} t^k + \cdots + \sum_{k\geq \ell-1} a_{k-\ell+1} t^k.\end{aligned}$$

Therefore, the Hilbert series $\sum_{k\geq 0} b_k t^k$ of R' is given by

$$a_0 + (a_0+a_1)t + \cdots + (a_0+\cdots+a_{\ell-2})t^{\ell-2} + \sum_{k\geq \ell-1}(a_k + a_{k-1} + \cdots + a_{k-\ell+1})t^k.$$

It is now clear that $b_k \leq b_{k+1}$, for any $0 \leq k \leq \ell-2$, from the first part of the last equality above, since $a_k \leq a_{k+1}$. For all the other values of k, i.e. $k \geq \ell-1$, we have $b_k - b_{k+1} = a_{k-\ell+1} - a_{k+1} \leq 0$ which accomplishes the proof. □

Example 3.6. In the following, we will say that the extension is *nice* if $\ell \leq \delta(\mathbf{m})$.

(1) The local ring of the affine cone of a projective toric variety is always of homogeneous type, for instance, $S = \{(3,0),(2,1),(1,2),(0,3)\}$ defines a projective toric curve in $\mathbb{P}^3$ and its affine cone is the toric surface $V_S \subset \mathbb{A}^4$ with the homogeneous toric ideal $I_S = \langle x_2^2 - x_1x_3, x_3^2 - x_2x_4, x_2x_3 - x_1x_4\rangle$. Thus by Proposition 3.4, its affine nice extensions will have homogeneous type local rings which are not necessarily homogeneous. Take for example, $\ell = 1$ and $\mathbf{m} = (0, 3s)$ for any $s > 1$. Then, although $I_{S_{\ell,\mathbf{m}}} = I_S + \langle x_4^s - x_5\rangle$ is not homogeneous, its local ring is of homogeneous type.

(2) Similarly, one can produce Cohen-Macaulay tangent cones using arithmetically Cohen-Macaulay projective toric varieties, since the toric ideal I_S of their affine cones are homogeneous and thus $I_S = {I_S}^*$. Therefore, all of their affine nice extensions will have Cohen-Macaulay tangent cones and local rings with non-decreasing Hilbert functions, by Theorem 3.2. The toric variety $V_S \subset \mathbb{A}^4$ considered in the previous item (1) and its nice extensions illustrate this as well.

(3) Take $S = \{(6,0),(0,2),(7,0),(6,4),(15,0)\}$. Then it is easy to see that $I_S = \langle x_1x_2^2 - x_4, x_3^3 - x_1x_5, x_1^5 - x_5^2\rangle$. Since $V_S \subset \mathbb{A}^5$ is a toric surface of codimension 3, I_S is a complete intersection and thus the local ring of V_S is Gorenstein. But, the tangent cone at the origin, is determined by ${I_S}^* = \langle x_5^2, x_4, x_3^3x_5, x_3^6, x_1x_5\rangle$ and thus is not Cohen-Macaulay. Nevertheless, its Hilbert function H_S is non-decreasing:
$H_S(0) = 1, H_S(1) = 4, H_S(2) = 8, H_S(3) = 13, H_S(r) = 6r - 6$, for $r \geq 4$.
Consider now all nice extensions of V_S; defined by the following semigroups $S_{\ell,\mathbf{m}} = \{(6\ell,0),(0,2\ell),(7\ell,0),(6\ell,4\ell),(15\ell,0),\mathbf{m}\}$. Therefore, Theorem 3.5 produces infinitely many new toric surfaces with local rings of dimension 2 and embedding codimension 4 whose Hilbert functions are non-decreasing even though their tangent cones are not Cohen-Macaulay. Indeed, one may produce this sort of examples in any embedding codimension by taking a sequence of nice extensions of the same example, since in each step the embedding codimension increases by one.

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