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BERNSTEIN PROCESSES, EUCLIDEAN QUANTUM MECHANICS AND INTEREST RATE MODELS

Paul Lescot

Laboratoire de Mathématiques Raphaël Salem
UMR 6085 CNRS-Université de Rouen
Bureau M2.24
Avenue de l'Université, B.P.12
Technopôle du Madrillet
F76801 Saint-Etienne-du-Rouvray
Tél. (+ 33) (0)2 32 95 52 24
Fax (+ 33) (0)2 32 95 52 86
Paul.Lescot@univ-rouen.fr
http://www.univ-rouen.fr/LMRS/Persopage/Lescot

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Abstract

We give an exposition, following joint works with J.-C. Zambrini, of the link between Euclidean Quantum Mechanics, Bernstein processes and isovectors for the heat equation. A new application to Mathematical Finance is then discussed.

1. Euclidean Quantum Mechanics

Schrödinger's equation for a (possibly time–dependent) potential $V(t,q)$:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V\psi \equiv H\psi$$

on $L^2(\mathbf{R}^d, dq)$ can be written, in space dimension $d = 1$, and for $m = 1$:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2}\frac{\partial^2\psi}{\partial q^2} + V\psi \ .$$

We shall henceforth treat $\theta = \sqrt{\hbar}$ as a new parameter.

In Zambrini's Euclidean Quantum Mechanics (see e.g.[1]), this equation splits into :

$$\theta^2\frac{\partial\psi}{\partial t} = -\frac{\theta^4}{2}\frac{\partial^2\psi}{\partial q^2} + V\psi \qquad (\mathcal{C}_1^{(V)})$$

and

$$-\theta^2\frac{\partial\psi}{\partial t} = -\frac{\theta^4}{2}\frac{\partial^2\psi}{\partial q^2} + V\psi \ , \qquad (\mathcal{C}_2^{(V)})$$

the probability density being given, not by $\psi\bar{\psi}$ as in the usual Quantum Mechanics, but by $\eta\eta_*$, η and η_* denoting respectively an everywhere strictly positive solution of $(\mathcal{C}_1^{(V)})$ and an everywhere strictly positive solution of $(\mathcal{C}_2^{(V)})$. To these data is associated a *Bernstein process* z, satisfying the Stochastic Differential Equation :

$$dz(t) = \theta dw(t) + \tilde{B}(t, z(t))dt \qquad ((\mathcal{B}))$$

relatively to the canonical increasing filtration of the Brownian w, and the Stochastic Differential Equation

$$d_*z(t) = \theta d_*w_*(t) + \tilde{B}_*(t, z(t))dt \qquad ((\mathcal{B}_*))$$

relatively to the canonical decreasing filtration of another Brownian w_*, where

$$\tilde{B} \equiv_{def} \theta^2\frac{\frac{\partial\eta}{\partial q}}{\eta}$$

and

$$\tilde{B}_* \equiv_{def} -\theta^2\frac{\frac{\partial\eta_*}{\partial q}}{\eta_*} \ .$$

Setting $S = -\theta^2\ln(\psi)$, equation $(\mathcal{C}_1^{(V)})$ becomes the Hamilton–Jacobi–Bellman equation :

$$\frac{\partial S}{\partial t} = -\frac{\theta^2}{2}\frac{\partial^2 S}{\partial q^2} + \frac{1}{2}(\frac{\partial S}{\partial q})^2 - V \ . \qquad (\mathcal{C}_3^{(V)})$$

Modulo the addition of the derivatives $E = -\frac{\partial S}{\partial t}$ and $B = -\frac{\partial S}{\partial q}$ as auxiliary unknown functions, $(\mathcal{C}_3^{(V)})$ is equivalent to the vanishing of the following differential forms :

$$\omega = dS + Edt + Bdq \ ,$$

$$\Omega = d\omega = dEdt + dBdq \ ,$$

and

$$\beta = (E + \frac{B^2}{2} - V)dqdt + \frac{\theta^2}{2}dBdt$$

on a 2–dimensional submanifold of $\mathbf{M} = \mathbf{R}^5$ $((t, q, S, E, B)$ being now considered as *independent* variables). Let then $L = \frac{1}{2}B^2 + V$ denote the *formal Lagrangian* ,

$$\omega_{PC} = Edt + Bdq = \omega - dS$$

the *Poincaré–Cartan* form, and I the ideal of $\mathcal{A} = \wedge T^*(\mathbf{M})$ generated by ω, $d\omega$ and β. By an *isovector* we shall mean a vector field N on $\mathbf{M}$ such that $\mathcal{L}_N(I) \subseteq I$; because of the linearity of $(\mathcal{C}_1^{(V)})$ the Lie algebra $\mathcal{G}_V$ of these isovectors contains an infinite–dimensional abelian ideal $\mathcal{J}_V$, that possesses a canonical supplement $\mathcal{H}_V$.

In the free case ($V = 0$) this canonical supplement has dimension 6 and admits a natural basis, each element of which corresponds to a symmetry of the underlying physical system.

Let $\Phi_N = -N(S)$ be the *phase* associated to N, and

$$D \equiv_{def} \frac{\partial}{\partial t} + B\frac{\partial}{\partial q} + \frac{\hbar}{2}\frac{\partial^2}{\partial q^2}$$

the *formal Ito differential* along the Bernstein process z. The following purely algebraic results are analogs of well–known theorems of Classical Analytical Mechanics:

Theorem 1.1. *For each $N \in \mathcal{H}_V$ one has :*

(1) $\mathcal{L}_N(\omega_{PC}) = d\Phi_N$;
(2) $\mathcal{L}_N(\Omega) = 0$;
(3) $\mathcal{L}_N(L) + L\frac{dN^t}{dt} = D\Phi_N$.

For a detailed proof see [7], and for complete calculations in the free case ($V = 0$) see [6].

2. Rosencrans' Theorem

Let N be an *isovector* (for $V = 0$), let ψ be a solution of $\mathcal{C}_1^{(0)}$, and let

$$S = -\theta^2 \ln(\psi);$$

then $e^{\alpha N}$ maps (t, q, S, E, B) to $(t_\alpha, q_\alpha, S_\alpha, E_\alpha, B_\alpha)$; setting

$$e^{-\frac{S_\alpha}{\theta^2}} = \psi_\alpha(t_\alpha, q_\alpha) \ ,$$

it follows that ψ_α is also a solution of $(\mathcal{C}_1^{(0)})$. We shall denote

$$e^{\alpha \hat{N}} : \psi \mapsto \psi_\alpha$$

the associated one–parameter group ; it is easily seen that, for

$$N = N^t \frac{\partial}{\partial t} + N^q \frac{\partial}{\partial q} - \Phi_N \frac{\partial}{\partial S} + ...$$

then

$$\hat{N} = -N^t \frac{\partial}{\partial t} - N^q \frac{\partial}{\partial q} + \frac{1}{\theta^2} \Phi_N \cdot \ ,$$

and it follows that $N \mapsto -\hat{N}$ is a homomorphism of Lie algebras.

Let η_u denote the solution de $(\mathcal{C}_1^{(0)})$ with initial condition u :

$$\frac{\partial \eta_u}{\partial t} = -\frac{\theta^2}{2} \frac{\partial^2 \eta_u}{\partial q^2} \ ,$$

and

$$\eta_u(0, q) = u(q) \ .$$

Let us set :

$$\rho_N(\alpha, t, q) = (e^{\alpha \hat{N}} \eta_u)(t, q)$$

and

$$\psi^N(\alpha, q) \equiv_{def} \rho_N(\alpha, 0, q) \ .$$

Then

Theorem 2.1. *([6], pp.321–322)*
ψ^N satisfies :

$$\frac{\partial \psi^N}{\partial \alpha} = -N^t(0, q)\left(-\frac{\theta^2}{2} \frac{\partial^2 \psi^N}{\partial \alpha^2}\right) - N^q(0, q) \frac{\partial \psi^N}{\partial q} + \frac{1}{\theta^2} \Phi_N(0, q) \psi^N$$

and

$$\psi^N(0, q) = u(q) \ .$$

Whence :

Corollary 2.2. *Let N be chosen so that $N^t(0,q) = -1$, $N^q(0,q) = -\frac{1}{\theta^2}(aq+b)$ and $\Phi_N(0,q) = cq^2 + dq + f$, where a, b, c, d, f denote real constants, then*

$$\eta_u^V(t,q) \equiv_{def} \psi^N(t,q)$$

satisfies the "backwards heat equation with drift term $D(q) = aq + b$ and quadratic potential $V(q) = cq^2 + dq + f + \frac{1}{2\hbar^2}(D(q))^2 - \frac{a}{2}$", corresponding to a vector potential

$$A = \frac{aq+b}{\theta^2} \ :$$

$$\theta^2 \frac{\partial \eta_u^V}{\partial t} = -\frac{\theta^4}{2}\frac{\partial^2 \eta_u^V}{\partial q^2} + (aq+b)\frac{\partial \eta_u^V}{\partial q} + (cq^2 + dq + f)\eta_u^V \qquad (\mathcal{C}_4^{(V)})$$

and

$$\eta_u^V(0,q) = u(q) \ .$$

(In the case $D(q) = 0$, the potential is given by $V(q) = cq^2 + dq + f$ and $\eta_u^{(V)}$ satisfies $\mathcal{C}_1^{(V)}$; for the general case, cf. [1], *pp.71–72).*

3. The case of a linear potential

Here $V(q) = \lambda q$; it appears that :

$$\eta_u^V(t,q) = e^{-\dfrac{\lambda^2}{6\theta^2}t^3} e^{\dfrac{\lambda t q}{\theta^2}} \eta_u(t, q - \lambda\frac{t^2}{2}) \ .$$

Then η_u^V satisfies $(\mathcal{C}_1^{(V)})$; the drift term can be written :

$$\begin{aligned}
\tilde{B_V}(t,q) &= \theta^2 \frac{\partial}{\partial q}(\ln(\eta_u^V)(t,q)) \\
&= \theta^2 \frac{\partial}{\partial q}(-\frac{\lambda^2}{6\theta^2}t^3 + \frac{\lambda t q}{\theta^2} + \ln(\eta_u)(t, q - \lambda\frac{t^2}{2})) \\
&= \lambda t + \theta^2 \frac{\partial}{\partial q}(\ln(\eta_u))(t, q - \lambda\frac{t^2}{2}) \\
&= \lambda t + \tilde{B}(t, q - \lambda\frac{t^2}{2}) \ .
\end{aligned}$$

Therefore, we have :

$$dz_V(t) = \theta dw(t) + \lambda t dt + \tilde{B}(t, z_V(t) - \lambda\frac{t^2}{2})dt \ .$$

Let us set $y(t) \equiv_{def} z_V(t) - \lambda\dfrac{t^2}{2}$; then

$$\begin{aligned}
dy(t) &= dz_V(t) - \lambda t dt \\
&= \theta dw(t) + \tilde{B}(t, y(t))dt \ ,
\end{aligned}$$

i.e. $y(t)$ is a Bernstein process $z(t)$ associated to solution η_u of the free equation $(\mathcal{C}_1^{(0)})$, and

$$z_V(t) = z(t) + \lambda\frac{t^2}{2} \ .$$

In other terms, the "perturbation" by a constant force λ produces a deterministic translation by $\lambda\dfrac{t^2}{2}$, which is logical on physical grounds.

Details are given in [7].

4. The case of a quadratic potential

For $V(t,q) = \dfrac{\omega^2 q^2}{2}$, one finds :

$$\eta_u^V(t,q) = \cosh(\omega t)^{-\frac{1}{2}} e^{\dfrac{\omega q^2}{2\theta^2}\tanh(\omega t)} \eta_u\left(\frac{\tanh(\omega t)}{\omega}, \frac{q}{\cosh(\omega t)}\right) .$$

Whence

$$\tilde{B_V}(t,q) = \omega q \tanh(\omega t) + \frac{1}{\cosh(\omega t)} \tilde{B}\left(\frac{\tanh(\omega t)}{\omega}, \frac{q}{\cosh(\omega t)}\right) .$$

Details are exposed in [9], §5, and a more general formula is proved in [5].

5. An example with $D \neq 0$

Here, we take $a = \theta^2\beta$, and $b = c = d = f = 0$. With the notations of [1], pp. 71–72 (but, of course, replacing $\mathbf{R}^3$ with $\mathbf{R}$), $A = \beta q$, and

$$V(q) = \frac{\beta^2 q^2}{2} - \frac{\beta\theta^2}{2} \ .$$

Then η_u^V satisfies

$$\frac{\partial \eta_u^V}{\partial t} = -\frac{\theta^2}{2}\frac{\partial^2 \eta_u^V}{\partial q^2} + \beta q \frac{\partial \eta_u^V}{\partial q} \ .$$

It is easy to see that :

$$\eta_u^V(t,q) = \eta_u\left(\frac{1}{2\beta}(e^{2\beta t}-1), e^{\beta t} q\right) \ .$$

The drift term (cf.[1], p.72) is given by :

$$\begin{aligned}\tilde{B_V}(t,q) &= \theta^2 \frac{\partial}{\partial q}(\ln(\eta_u^V)(t,q)) - A(t,q) \\ &= e^{\beta t}\tilde{B}\left(\frac{1}{2\beta}(e^{2\beta t}-1), e^{\beta t} q\right) - \beta q \ .\end{aligned}$$

In particular, for $\eta = 1$, one finds $\eta_u = 1$, $\tilde{B_V}(t,q) = -\beta q$, $z(t) = \theta w(t)$ and

$$dz_V(t) = \theta dw(t) - \beta z_V(t) dt \ ,$$

i.e. $z_V(t)$ is an Ornstein–Uhlenbeck process, as expected.

6. One-factor affine interest rate models

Such a model is characterized by the instantaneous rate $r(t)$, satisfying

$$dr(t) = \sqrt{\alpha r(t) + \beta}\ dw(t) + (\phi - \lambda r(t))\ dt$$

(cf. [3]).

Let us set $\tilde{\phi} = \phi + \frac{\lambda\beta}{\alpha}$; then :

Theorem 6.1. *Let*

$$z(t) = \sqrt{\alpha r(t) + \beta}\ ;$$

then $z(t)$ is a Bernstein process for

$$\theta = \frac{\alpha}{2}$$

and the potential

$$V(t,q) = \frac{A}{q^2} + Bq^2$$

where :

$$A = \frac{\alpha^2}{8}(\tilde{\phi} - \frac{\alpha}{4})(\tilde{\phi} - \frac{3\alpha}{4})$$

and

$$B = \frac{\lambda^2}{8}\ .$$

Corollary 6.2. *The isovector algebra $\mathcal{H}_V$ associated with V has dimension 6 if and only if $A = 0$; in the opposite case, it has dimension 4.*

But the condition $A = 0$ is equivalent to $\tilde{\phi} \in \{\frac{\alpha}{4}, \frac{3\alpha}{4}\}$, and these values of $\tilde{\phi}$ appear as special in Hénon's PhD thesis ([2]). I am now able to explain that coincidence([4],[5]).

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References

1. A.B.Cruzeiro and J.–C.Zambrini, *Malliavin calculus and Euclidean Quantum Mechanics.I.Functional Calculus*, J.Funct.Anal. **96(1)** (1991), 62–95.
2. S.Hénon, *Un modèle de taux avec volatilité stochastique*, (Thèse) (2005).
3. B. Leblanc , O. Scaillet, *Path dependent options on yields in the affine term structure model*, Finance and Stochastics **2(4)** (1998), 349–367.
4. P. Lescot, *(en préparation)*.
5. P. Lescot, P. Patie, *(en préparation)*.
6. P.Lescot, J.-C. Zambrini, *Isovectors for the Hamilton–Jacobi–Bellman Equation, Formal Stochastic Differentials and First Integrals in Euclidean Quantum Mechanics*, Proceedings of the Ascona conference (2002), Birkhaüser (Progress in Probability, vol 58), 2004, pp. 187–202.
7. P.Lescot, J.-C. Zambrini, *Probabilistic deformation of contact geometry, diffusion processes and their quadratures*, Seminar on Stochastic Analysis, Random Fields and applications V, Birkhaüser(Progress in Probability, vol. 59), 2008, pp. 203-226.
8. S.I.Rosencrans, *Perturbation Algebra of an Elliptic Operator*, Journal of Mathematical Analysis and Applications **56** (1976), 317–329.
9. J.-C. Zambrini, *From the geometry of parabolic PDE to the geometry of SDE*, Mathematical Analysis of Random Phenomena, World Sci., Hackensack, NJ, 2007, pp. 213–230.

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