Variance Optimal Hedging for continuous time processes with independent increments and applications

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Abstract

For a large class of vanilla contingent claims, we establish an explicit Föllmer-Schweizer decomposition when the underlying is a process with independent increments (PII) and an exponential of a PII process. This allows to provide an efficient algorithm for solving the mean variance hedging problem. Applications to models derived from the electricity market are performed.

Key words and phrases: Variance-optimal hedging, Föllmer-Schweizer decomposition, Lévy process, Cumulative generating function, Characteristic function, Normal Inverse Gaussian process, Electricity markets, Process with independent increments.

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1 Introduction

There are basically two main approaches to define the *mark to market* of a contingent claim: one relying on the *no-arbitrage assumption* and the other related to a *hedging portfolio*, those two approaches converging in the specific case of complete markets. A simple introduction to the different hedging and pricing models in incomplete markets can be found in chapter 10 of [16].

The fundamental theorem of Asset Pricing [18] implies that a pricing rule without arbitrage that moreover satisfies some usual conditions (linearity non anticipativity ...) can always be written as an expectation under a martingale measure. In general, the resulting price is not linked with a hedging strategy except in some specific cases such as complete markets. More precisely, it is proved [18] that the market completeness is equivalent to uniqueness of the equivalent martingale measure. Hence, when the market is not complete, there exist several equivalent martingale measures (possibly an infinity) and one has to specify a criterion to select one specific pricing measure: to recover some given option prices (by calibration) [27]; to simplify calculus and obtain a simple process under the pricing measure; to maintain the structure of the real world dynamics; to minimize a *distance* to the objective probability (entropy [26]) ... In this framework, the difficulty is to understand in a practical way the impact of the choice of the martingale measure on the resulting prices.

If the resulting price is in general not connected to a hedging strategy, yet it is possible to consider the hedging question in a second step, optimizing the hedging strategy for the given price. In this framework, one approach consists in deriving the hedging strategy minimizing the *global quadratic hedging error* under the pricing measure where the martingale property of the underlying highly simplifies calculations. This approach, is developed in [16], in the case of exponential-Lévy models: the optimal quadratic hedge is then expressed as a solution of an integro-differential equation involving the Lévy measure. Unfortunately, minimizing the quadratic hedging error under the pricing measure has no clear interpretation since the resulting hedging strategy can lead to huge quadratic error under the objective measure.

Alternatively, one can define option prices as a by product of the hedging strategy. In the case of complete markets, any option can be replicated perfectly by a self-financed hedging portfolio continuously rebalanced, then the option *hedging value* can be defined as the cost of the hedging strategy. When the market is not complete, it is not possible, in general, to hedge perfectly an option. One has to specify a risk criteria, and consider the hedging strategy that minimizes the distance (in terms of the given criteria) between the pay-off of the option and the terminal value of the hedging portfolio. Then, the price of the option is related to the cost of this imperfect hedging strategy to which is added in practice another prime related to the residual risk induced by incompleteness.

Several criteria can be adopted. The aim of super-hedging is to hedge all cases. This approach yields in general prices that are too expensive to be realistic [21]. Quantile hedging modifies this approach allowing for a limited probability of loss [23]. Indifference Utility pricing introduced by [29] defines the price of an option to sell (resp. to buy) as the minimum initial value s.t. the hedging portfolio with the option sold (resp. bought) is equivalent (in term of utility) to the initial portfolio. Quadratic hedging is developed in [48], [50]. The quadratic distance between the hedging portfolio and the pay-off is minimized. Then, contrarily to the case of utility maximization, losses and gains are treated in a symmetric manner, which yields a *fair price* for both the buyer and the seller of the option.

In this paper, we follow this last approach and our developments can be used in both the *no-arbitrage value* and the *hedging value* framework: either to derive the hedging strategy minimizing the *global quadratic hedging error* under the objective measure, for a given pricing rule; or to derive both the price and the

hedging strategy minimizing the *global quadratic hedging error* under the objective measure.
We spend now some words related to the global quadratic hedging approach which is also called *mean-variance hedging* or *global risk minimization*. Given a square integrable r.v. H, we say that the pair (V_0, φ) is optimal if $(c, v) = (V_0, \varphi)$ minimizes the functional $\mathbb{E}\left(H - c - \int_0^T v dS\right)^2$. The price V_0 represents the price of the contingent claim H and φ is the optimal strategy.
Technically speaking, the global risk minimization problem, is based on the so-called *Föllmer-Schweizer* decomposition (or FS decomposition) of a square integrable random variable (representing the contingent claim) with respect to an $(\mathcal{F}_t)$-semimartingale $S = M + A$ modeling the asset price. M is an $(\mathcal{F}_t)$-local martingale and A is a bounded variation process with $A_0 = 0$. Mathematically, the FS decomposition, constitutes the generalisation of the martingale representation theorem (Kunita-Watanabe representation) when S is a Brownian motion or a martingale. Given square integrable random variable H, the problem consists in expressing H as $H_0 + \int_0^T \xi dS + L_T$ where ξ is predictable and L_T is the terminal value of an orthogonal martingale L to M, i.e. the martingale part of S. The seminal paper is [24] where the problem is treated in the case that S is continuous. In the general case S is said to have the **structure condition** (SC) condition if there is a predictable process α such that $A_t = \int_0^t \alpha_s d\langle M\rangle_s$ and $\int_0^T \alpha_s^2 d\langle M\rangle_s < \infty$ a.s. In the sequel most of contributions were produced in the multidimensional case. Here for simplicity we will formulate all the results in the one-dimensional case.
An interesting connection with the theory of backward stochastic differential equations (BSDEs) in the sense of [39], was proposed in [48]. [39] considered BSDEs driven by Brownian motion; in [48] the Brownian motion is in fact replaced by M. The first author who considered a BSDE driven by a martingale was [11]. Suppose that $V_t = \int_0^t \alpha_s d\langle M\rangle_s$. The BSDE problem consists in finding a triple (V, ξ, L) where

$$V_t = H - \int_t^T \xi_s dM_s - \int_t^T \xi_s \alpha_s d\langle M\rangle_s - (L_T - L_t),$$

and L is an $(\mathcal{F}_t)$-local martingale orthogonal to M.
In fact, this decomposition provides the solution to the so called local risk minimization problem, see [24]. In this case, V_t represents the *price* of the contingent claim at time t and the price V_0 constitutes in fact the expectation under the so called *variance optimal measure* (VOM), as it will be explained at Section 2.5, with references to [51], [3] and [2].
In the framework of FS decomposition, a process which plays a significant role is the so-called **mean variance tradeoff** (MVT) process K. This notion is inspired by the theory in discrete time started by [46]; in the continuous time case K is defined as $K_t = \int_0^t \alpha_s^2 d\langle M\rangle_s,\ t \in [0, T]$. [48] shows the existence of the mean-variance hedging problem if the MVT process is deterministic. In fact, a slight more general condition was the (ESC) condition and the EMVT process but we will not discuss here further details. We remark that in the continuous case, treated by [24], no need of any condition on K is required. When the MVT process is deterministic, [48] is able to solve the global quadratic variation problem and provides an efficient relation, see Theorem 5.2 with the FS decomposition. He also shows that, for the obtention of the mentioned relation, previous condition is not far from being optimal. The next important step was done in [38] where under the only condition that K is uniformly bounded, the FS decomposition of any square integrable random variable admits existence and uniqueness and the global minimization problem admits a solution.
More recently has appeared an incredible amount of papers in the framework of global (resp. local) risk minimization, so that it is impossible to list all of them and it is beyond our scope. Two significant papers containing a good list of references are [51], [7] and [12]. The present paper puts emphasis on processes with independent increments (PII) and exponential of those processes. It provides explicit FS decompositions

when the process S is of that type when the contingent claims are provided by some Fourier transform (resp. Laplace-Fourier transform) of a finite measure. Some results of [31] concerning exponential of Lévy processes are generalized trying to investigate some peculiar properties behind and to consider the case of PII with possibly non stationary increments. The motivation came from hedging problems in the electricity market. Because of non-storability of electricity, the hedging instrument is in that case, a forward contract with value $S_t^0 = e^{-r(T_d-t)}(F_t^{T_d} - F_0^{T_d})$ where $F_t^{T_d}$ is the forward price given at time $t \leq T_d$ for delivery of 1MWh at time T_d. Hence, the dynamic of the underlying S^0 is directly related to the dynamic of forward prices. Now, forward prices show a volatility term structure that requires the use of models with non stationary increments and motivates the generalization of the pricing and hedging approach developed in [31] for Lévy processes to the case of PII with possibly non stationary increments.

The paper is organized as follows. After this introduction and some generalities about semimartingales, we introduce the notion of FS decomposition and describe local and global risk minimization. Then, we examine at Chapter 3 (resp. 4) the explicit FS decomposition for PII processes (resp. exponential of PII processes). Chapter 5 is devoted to the solution to the global minimization problem and Chapter 6 to the case of a model intervening in the electricity market. Chapter 7 is devoted to simulations. This paper will be followed by a companion paper, i. e. [28] which concentrates on the discrete time case leaving more space to numerical implementations.

2 Generalities on semimartingales and Föllmer-Schweizer decomposition

In the whole paper, $T > 0$, will be a fixed terminal time and we will denote by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ a filtered probability space, fulfilling the usual conditions.

2.1 Generating functions

Let $X = (X_t)_{t\in[0,T]}$ be a real valued stochastic process.

Definition 2.1. *The* **characteristic function** *of (the law of) X_t is the continuous mapping*

$$\varphi_{X_t} : \mathbb{R} \to \mathbb{C} \quad \textit{with} \quad \varphi_{X_t}(u) = \mathbb{E}[e^{iuX_t}] \ .$$

In the sequel, when there will be no ambiguity on the underlying process X, we will use the shortened notation φ_t for φ_{X_t}.

Definition 2.2. *The* **cumulant generating function** *of (the law of) X_t is the mapping $z \mapsto \mathrm{Log}(\mathbb{E}[e^{zX_t}])$ where $\mathrm{Log}(w) = \log(|w|) + i\mathrm{Arg(w)}$ where $\mathrm{Arg(w)}$ is the Argument of w, chosen in $]-\pi,\pi]$; Log is the principal value logarithm. In particular we have*

$$\kappa_{X_t} : D \to \mathbb{C} \quad \textit{with} \quad e^{\kappa_{X_t}(z)} = \mathbb{E}[e^{zX_t}] \ ,$$

where $D := \{z \in \mathbb{C} \mid \mathbb{E}[e^{Re(z)X_t}] < \infty, \ \forall t \in [0,T]\}$.

In the sequel, when there will be no ambiguity on the underlying process X, we will use the shortened notation κ_t for κ_{X_t}.

We observe that D includes the imaginary axis.

Remark 2.3. *1. For all $z \in D$, $\kappa_t(\bar{z}) = \overline{\kappa_t(z)}$, where $\bar{z}$ denotes the conjugate complex of $z \in \mathbb{C}$. Indeed, for any $z \in D$,*

$$\exp(\kappa_t(\bar{z})) = \mathbb{E}[\exp(\bar{z}X_t)] = \mathbb{E}[\overline{\exp(zX_t)}] = \overline{\mathbb{E}[\exp(zX_t)]} = \overline{\exp(\kappa_t(z))} = \exp(\overline{\kappa_t(z)}) \ .$$

2. For all $z \in D \cap \mathbb{R}$, $\kappa_t(z) \in \mathbb{R}$.

2.2 Semimartingales

An $(\mathcal{F}_t)$-semimartingale $X = (X_t)_{t\in[0,T]}$ is a process of the form $X = M + A$, where M is an $(\mathcal{F}_t)$-local martingale and A is a bounded variation adapted process vanishing at zero. $||A||_T$ will denote the total variation of A on $[0,T]$. Given two $(\mathcal{F}_t)$- local martingales M and N, $\langle M,N\rangle$ will denote the angle bracket of M and N, i.e. the unique bounded variation predictable process vanishing at zero such that $MN - \langle M,N\rangle$ is an $(\mathcal{F}_t)$-local martingale. If X and Y are $(\mathcal{F}_t)$-semimartingales, $[X,Y]$ denotes the square bracket of X and Y, i.e. the quadratic covariation of X and Y. In the sequel, if there is no confusion about the underlying filtration $(\mathcal{F}_t)$, we will simply speak about semimartingales, local martingales, martingales. All the local martingales admit a càdlàg version. By default, when we speak about local martingales we always refer to their càdlàg version.
More details about previous notions are given in chapter I.1. of [35].

Remark 2.4. *1. All along this paper we will consider $\mathbb{C}$-valued martingales (resp. local martingales, semimartingales). Given two $\mathbb{C}$-valued local martingales M^1, M^2 then $\overline{M^1}, \overline{M^2}$ are still local martingales. Moreover $\langle \overline{M^1}, \overline{M^2}\rangle = \overline{\langle M^1, M^2\rangle}$.*

2. If M is a $\mathbb{C}$-valued martingale then $\langle M, \overline{M}\rangle$ is a real valued increasing process.

Theorem 2.5. *$(X_t)_{t\in[0,T]}$ is a real semimartingale iff the characteristic function, $t \mapsto \varphi_t(u)$, has bounded variation over all finite intervals, for all $u \in \mathbb{R}$.*

Remark 2.6. *According to Theorem I.4.18 of [35], any local martingale M admits a unique (up to indistinguishability) decomposition,*

$$M = M_0 + M^c + M^d \ ,$$

where $M_0^c = M_0^d = 0$, M^c is a continuous local martingale and M^d is a purely discontinuous local martingale in the sense that $\langle N, M^d\rangle = 0$ for all continuous local martingales N. M^c is called the ***continuous part*** *of M and M^d the* ***purely discontinuous part****.*

Definition 2.7. *An $(\mathcal{F}_t)$-**special semimartingale** is an $(\mathcal{F}_t)$-semimartingale X with the decomposition $X = M + A$, where M is a local martingale and A is a bounded variation predictable process starting at zero.*

Remark 2.8. *The decomposition of a special semimartingale of the form $X = M + A$ is unique, see [35] definition 4.22.*

For any special semimartingale X we define

$$||X||^2_{\delta^2} = \mathbb{E}\left[[M,M]_T\right] + \mathbb{E}\left(||A||^2_T\right) \ .$$

The set δ^2 is the set of $(\mathcal{F}_t)$-special semimartingale X for which $||X||^2_{\delta^2}$ is finite.

A **truncation function** defined on $\mathbb{R}$ is a bounded function $h : \mathbb{R} \to \mathbb{R}$ with compact support such that $h(x) = x$ in a neighbourhood of 0.

An important notion, in the theory of semimartingales, is the notion of characteristics, defined in definition II.2.6 of [35]. Let $X = M + A$ be a real-valued semimartingale. A **characteristic** is a triplet, (b, c, ν), depending on a fixed truncation function, where

1. b is a predictable process with bounded variation;
2. $c = \langle M^c, M^c \rangle$, M^c being the continuous part of M according to Remark 2.6;
3. ν is a predictable random measure on $\mathbb{R}^+ \times \mathbb{R}$, namely the compensator of the random measure μ^X associated to the jumps of X.

2.3 Föllmer-Schweizer Structure Condition

Let $X = (X_t)_{t \in [0,T]}$ be a real-valued special semimartingale with canonical decomposition,

$$X = M + A \ .$$

For the clarity of the reader, we formulate in dimension one, the concepts appearing in the literature, see e.g. [48] in the multidimensional case.

Definition 2.9. *For a given local martingale M, the space $L^2(M)$ consists of all predictable $\mathbb{R}$-valued processes $v = (v_t)_{t \in [0,T]}$ such that*

$$\mathbb{E}\left[\int_0^T |v_s|^2 d\left\langle M \right\rangle_s\right] < \infty \ .$$

For a given predictable bounded variation process A, the space $L^2(A)$ consists of all predictable $\mathbb{R}$-valued processes $v = (v_t)_{t \in [0,T]}$ such that

$$\mathbb{E}\left[\left(\int_0^T |v_s| d||A||_s\right)^2\right] < \infty \ .$$

Finally, we set

$$\Theta := L^2(M) \cap L^2(A) \ .$$

For any $v \in \Theta$, the stochastic integral process

$$G_t(v) := \int_0^t v_s dX_s, \quad \text{for all } t \in [0,T] \ ,$$

is therefore well-defined and is a semimartingale in δ^2 with canonical decomposition

$$G_t(v) = \int_0^t v_s dM_s + \int_0^t v_s dA_s \ , \quad \text{for all } t \in [0,T] \ .$$

We can view this stochastic integral process as the gain process associated with strategy v on the underlying process X.

Definition 2.10. *The **minimization problem** we aim to study is the following.*
*Given $H \in \mathcal{L}^2$, an admissible strategy pair (V_0, φ) will be called **optimal** if $(c, v) = (V_0, \varphi)$ minimizes the expected squared hedging error*

$$\mathbb{E}[(H - c - G_T(v))^2] \ , \tag{2.1}$$

over all admisible strategy pairs $(c, v) \in \mathbb{R} \times \Theta$. V_0 will represent the **price** *of the contingent claim H at time zero.*

Definition 2.11. *Let $X = (X_t)_{t\in[0,T]}$ be a real-valued special semimartingale. X is said to satisfy the* ***structure condition (SC)*** *if there is a predictable $\mathbb{R}$-valued process $\alpha = (\alpha_t)_{t\in[0,T]}$ such that the following properties are verified.*

1. $A_t = \int_0^t \alpha_s d\langle M\rangle_s$, *for all $t \in [0,T]$, so that $dA \ll d\langle M\rangle$.*
2. $\int_0^T \alpha_s^2 d\langle M\rangle_s < \infty$, $P-a.s.$

Definition 2.12. *From now on, we will denote by $K = (K_t)_{t\in[0,T]}$ the càdlàg process*

$$K_t = \int_0^t \alpha_s^2 d\langle M\rangle_s \ , \quad \textit{for all } t \in [0,T] \ .$$

This process will be called the ***mean-variance tradeoff*** *(MVT) process.*

Remark 2.13. *In [48], the process $(K_t)_{t\in[0,T]}$ is denoted by $(\widehat{K}_t)_{t\in[0,T]}$.*

We provide here a technical proposition which allows to make the class Θ of integration of X explicit.

Proposition 2.14. *If X satisfies (SC) such that $\mathbb{E}[K_T] < \infty$, then $\Theta = L^2(M)$.*

Proof. Assume that $\mathbb{E}[K_T] < \infty$, we will prove that $L^2(M) \subseteq L^2(A)$. Let us consider a process $v \in L^2(M)$, then

$$\begin{aligned}
\mathbb{E}\left(\int_0^T |v_s|d||A||_s\right)^2 &= \mathbb{E}\left(\int_0^T |v_s||\alpha_s|d\langle M\rangle_s\right)^2 \leq \left[\mathbb{E}\left(\int_0^T |v_s|^2 d\langle M\rangle_s\right)\mathbb{E}\left(\int_0^T |\alpha|^2 d\langle M\rangle_s\right)\right]^{\frac{1}{2}} , \\
&= \left(\mathbb{E}\left(\int_0^T |v_s|^2 d\langle M\rangle_s\right)\mathbb{E}(K_T)\right)^{\frac{1}{2}} < \infty \ .
\end{aligned}$$

□

Schweizer in [48] also introduced the extended structure condition (ESC) on X and he provided the Föllmer-Schweizer decomposition in this more extended framework. We recall that notion (in dimension 1). Given a real càdlàg stochastic process X, the quantity ΔX_t will represent the jump $X_t - X_{t-}$.

Definition 2.15. *Let $X = (X_t)_{t\in[0,T]}$ be a real-valued special semimartingale. X is said to satisfy the* ***extended structure condition (ESC)*** *if there is a predictable $\mathbb{R}$-valued process $\alpha = (\alpha_t)_{t\in[0,T]}$ with the following properties.*

1. $A_t = \int_0^t \alpha_s d\langle M\rangle_s$, *for all $t \in [0,T]$, so that $dA \ll \langle dM\rangle$.*
2. *The quantity*
$$\int_0^T \frac{\alpha_s^2}{1 + \alpha_s^2 \Delta\langle M\rangle_s} d\langle M\rangle_s$$
is finite.

If condition (ESC) is fulfilled, then the process

$$\widetilde{K}_t := \int_0^t \frac{\alpha_s^2}{1 + \alpha_s^2 \Delta\langle M\rangle_s} \ , \quad \textit{for all } t \in [0,T] \ ,$$

is well-defined. It is called ***extended mean-variance tradeoff (EMVT)*** *process.*

Remark 2.16. *1. (SC) implies (ESC).*

2. If $\langle M \rangle$ is continuous then (ESC) and (SC) are equivalent and $K = \tilde{K}$.

3. $\tilde{K}_t = \int_0^t \frac{|\alpha_s|^2}{1+\Delta K_s} d\langle M \rangle_s = \int_0^t \frac{1}{1+\Delta K_s} dK_s$, for all $t \in [0,T]$.

4. $K_t = \int_0^t \frac{1}{1-\Delta \tilde{K}_s} d\tilde{K}_s$, for all $t \in [0,T]$.

5. If K is deterministic then $\tilde{K}$ is deterministic.

The structure condition (SC) appears quite naturally in applications to financial mathematics. In fact, it is mildly related to the no arbitrage condition. In fact (SC) is a natural extension of the existence of an equivalent martingale measure from the case where X is continuous. Next proposition will show that every adapted continuous process X admitting an equivalent martingale measure satisfies (SC). In our applications (ESC) will be equivalent to (SC) since in Section 3.2 and Section 4.2, $\langle M \rangle$ will always be continuous.

Proposition 2.17. *Let X be a $(P, \mathcal{F}_t)$ continuous semimartingale. Suppose the existence of a locally equivalent probability $Q \sim P$ under which X is an $(Q, \mathcal{F}_t)$-local martingale, then (SC) is verified.*

Proof. Let $(D_t)_{t\in[0,T]}$ be the strictly positive continuous Q-local martingale such that $dP = D_T dQ$. By Theorem VIII.1.7 of [42], $M = X - \langle X, L \rangle$ is a continuous P-local martingale, where L is the continuous Q-local martingale associated to the density process i.e.

$$D_t = \exp\{L_t - \frac{1}{2}\langle L \rangle_t\} , \quad \text{for all } t \in [0,T] .$$

According to Lemma IV.4.2 in [42], there is a progressively measurable process R such that for all $t \in [0,T]$,

$$L_t = \int_0^t R_s dX_s + O_t \quad \text{and} \quad \int_0^T R_s^2 d\langle X \rangle_s < \infty , \quad Q-\text{a.s.} ,$$

where O is a Q-local martingale such that $\langle X, O \rangle = 0$. Hence,

$$\langle X, L \rangle_t = \int_0^t R_s d\langle X \rangle_s \quad \text{and} \quad X_t = M_t + \int_0^t R_s d[X]_s , \quad \text{for all } t \in [0,T].$$

We end the proof by setting $\alpha_t = \frac{d\langle X, L \rangle_t}{d\langle X \rangle_t} = R_t$. □

2.4 Föllmer-Schweizer Decomposition and variance optimal hedging

Throughout this section, as in Section 2.3, X is supposed to be an $(\mathcal{F}_t)$-special semimartingale fulfilling the (SC) condition.

We recall here the definition stated in Chapter IV.3 p. 179 of [40].

Definition 2.18. *Two $(\mathcal{F}_t)$-martingales M, N are said to be **strongly orthogonal** if MN is a uniformly integrable martingale.*

Remark 2.19. *If M, N are strongly orthogonal, then they are (weakly) orthogonal in the sence that $\mathbb{E}[M_T N_T] = 0$.*

Lemma 2.20. *Let M, N be two square integrable martingales. Then M and N are strongly orthogonal if and only if $\langle M, N \rangle = 0$.*

Proof. Let $\mathcal{S}(M)$ be the stable subspace generated by M. $\mathcal{S}(M)$ includes the space of martingales of the form

$$M_t^f := \int_0^t f(s) dM_s \ , \quad \text{for all } t \in [0,T] \ ,$$

where $f \in L^2(dM)$ is deterministic. According to Lemma IV.3.2 of [40], it is enough to show that, for any $f \in L^2(dM)$, $g \in L^2(dN)$, M^f and N^g are weakly orthogonal in the sense that $\mathbb{E}[M_T^f N_T^g] = 0$. This is clear since previous expectation equals

$$\mathbb{E}[\langle M^f, N^g \rangle_T] = \mathbb{E}\left(\int_0^T f g d\langle M, N \rangle \right) = 0$$

if $\langle M, N \rangle = 0$. This shows the converse implication.
The direct implication follows from the fact that MN is a martingale, the definition of the angle bracket and uniqueness of special semimartingale decomposition. □

Definition 2.21. *We say that a random variable $H \in \mathcal{L}^2(\Omega, \mathcal{F}, P)$ admits a* ***Föllmer-Schweizer (FS) decomposition****, if it can be written as*

$$H = H_0 + \int_0^T \xi_s^H dX_s + L_T^H \ , \quad P-a.s. \ , \tag{2.2}$$

where $H_0 \in \mathbb{R}$ is a constant, $\xi^H \in \Theta$ and $L^H = (L_t^H)_{t \in [0,T]}$ is a square integrable martingale, with $\mathbb{E}[L_0^H] = 0$ and strongly orthogonal to M.

We formulate for this section one basic assumption.

Assumption 1. *We assume that X satisfies (SC) and that the MVT process K is uniformly bounded in t and ω.*

The first result below gives the existence and the uniqueness of the Föllmer-Schweizer decomposition for a random variable $H \in \mathcal{L}^2(\Omega, \mathcal{F}, P)$. The second affirms that subspaces $G_T(\Theta)$ and $\{\mathcal{L}^2(\mathcal{F}_0) + G_T(\Theta)\}$ are closed subspaces of $\mathcal{L}^2$. The last one provides existence and uniqueness of the solution of the minimization problem (2.1). We recall Theorem 3.4 of [38].

Theorem 2.22. *Under Assumption 1, every random variable $H \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$ admits a FS decomposition. Moreover, this decomposition is unique in the following sense:*
If

$$H = H_0 + \int_0^T \xi_s^H dX_s + L_T^H = H_0^{'} + \int_0^T \xi_s^{'H} dX_s + L_T^{'H},$$

where (H_0, ξ^H, L^H) and $(H_0^{'}, \xi^{'H}, L^{'H})$ satisfy the conditions of the FS decomposition, then

$$\left\{ \begin{array}{lcll} H_0 & = & H_0^{'} \ , & P-a.s. \ , \\ \xi^H & = & \xi^{'H} & \text{in } L^2(M) \ , \\ L_T^H & = & L_T^{'H}, & P-a.s. \ . \end{array} \right.$$

We recall Theorem 4.1 of [38].

Theorem 2.23. *Under Assumption 1, the subspaces $G_T(\Theta)$ and $\{\mathcal{L}^2(\mathcal{F}_0) + G_T(\Theta)\}$ are closed subspaces of $\mathcal{L}^2$.*

So we can project any random variable $H \in \mathcal{L}^2$ on $G_T(\Theta)$. By Theorem 2.22, we have the uniqueness of the solution of the minimization problem (2.1). This is given by Theorem 4.6 of [38], which is stated below.

Theorem 2.24. *We suppose Assumption 1.*

1. *For every $H \in \mathcal{L}^2(\Omega, \mathcal{F}, P)$ and every $c \in \mathcal{L}^2(\mathcal{F}_0)$, there exists a unique strategy $\varphi^{(c)} \in \Theta$ such that*

$$\mathbb{E}[(H - c - G_T(\varphi^{(c)}))^2] = \min_{v \in \Theta} \mathbb{E}[(H - c - G_T(v))^2] \ . \tag{2.3}$$

2. *For every $H \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$ there exists a unique $(c^{(H)}, \varphi^{(H)}) \in \mathcal{L}^2(\mathcal{F}_0) \times \Theta$ such that*

$$\mathbb{E}[(H - c^{(H)} - G_T(\varphi^{(H)}))^2] = \min_{(c,v) \in \mathcal{L}^2(\mathcal{F}_0) \times \Theta} \mathbb{E}[(H - c - G_T(v))^2] \ .$$

Next theorem gives the explicit form of the optimal strategy $\varphi^{(c)}$, which is valid even in the case where X satisfies the extended structure condition (ESC). For the purpose of the present work, this will not be useful, see considerations following Remark 2.16 2.

From Föllmer-Schweizer decomposition follows the solution to the global minimization problem (2.1).

Theorem 2.25. *Suppose that X satisifies (SC) and that the MVT process K of X is deterministic. If $H \in \mathcal{L}^2$ admits a FS decomposition of type (2.2), then the minimization problem (2.3) has a solution $\varphi^{(c)} \in \Theta$ for any $c \in \mathbb{R}$, such that*

$$\varphi_t^{(c)} = \xi_t^H + \frac{\alpha_t}{1 + \Delta K_t}(H_{t-} - c - G_{t-}(\varphi^{(c)})) \ , \quad \textit{for all } t \in [0, T] \tag{2.4}$$

where the process $(H_t)_{t \in [0,T]}$ is defined by

$$H_t := H_0 + \int_0^t \xi_s^H dX_s + L_t^H \ , \tag{2.5}$$

and the process α is the process appearing in Definition 2.11 of (SC).

Proof. Theorem 3 of [48] states the result under the (ESC) condition. We recall that (SC) implies (ESC), see Remark 2.16 and the result follows. □

To obtain the solution to the minimization problem (2.1), we use Corollary 10 of [48] that we recall.

Corollary 2.26. *Under the assumption of Theorem 2.25, the solution of the minimization problem (2.1) is given by the pair $(H_0, \varphi^{(H_0)})$.*

In the sequel of this paper we will only refer to the structure condition (SC) and to the MVT process K. The definition below can be found in section II.8 p. 85 of [40].

Definition 2.27. *The* **Doléans-Dade exponential** *of a semimartingale X is defined to be the unique càdlàg adapted solution Y to the stochastic differential equation,*

$$dY_t = Y_{t-} dX_t \ , \quad \textit{for all } t \in [0, T] \quad \textit{with } Y_0 = 1 \ .$$

This process is denoted by $\mathcal{E}(X)$.

This solution is a semimartingale given by

$$\mathcal{E}(X)_t = \exp\left(X_t - X_0 - [X]_t/2\right) \prod_{s \leq t} (1 + \Delta X_s) \exp(-\Delta X_s) \ .$$

Theorem below is stated in [48].

Theorem 2.28. *Under the assumptions of Theorem 2.25, for any $c \in \mathbb{R}$, we have*

$$\min_{v\in\Theta} \mathbb{E}[(H - c - G_T(v))^2] = \mathcal{E}(-\tilde{K}_T)\left((H_0 - c)^2 + \mathbb{E}[(L_0^H)^2] + \int_0^T \frac{1}{\mathcal{E}(-\tilde{K}_s)} d\left(\mathbb{E}[\langle L^H\rangle_s]\right)\right) . \qquad (2.6)$$

Proof. See the proof of Corollary 9 of [48] with Remark 2.16. □

Corollary 2.29. *If $\langle M, M\rangle$ is continuous*

$$\begin{aligned}\min_{v\in\Theta} \mathbb{E}[(H - c - G_T(v))^2] &= \exp(-K_T)\left((H_0 - c)^2 + \mathbb{E}[(L_0^H)^2]\right) \\ &+\mathbb{E}\left[\int_0^T \exp\{-(K_T - K_s)\} d\left\langle L^H\right\rangle_s\right] . \end{aligned} \qquad (2.7)$$

Proof. Remark 2.16 implies that $K = \tilde{K}$. Since K is continuous and with bounded variation, its Doléans-Dade exponential coincides with the classical exponential. The result follows from Theorem 2.28.

□

In the sequel, we will find an explicit expression of the FS decomposition for a large class of square integrable random variables, when the underlying process is a process with independent increments, or is an exponential of process with independent increments. For this, the first step will consist in verifying (SC) and the boundedness condition on the MVT process, see Assumption 1.

2.5 Link with the equivalent signed martingale measure

2.5.1 The Variance optimal martingale (VOM) measure

Definition 2.30. 1. *A signed measure, Q, on $(\Omega, \mathcal{F}_T)$, is called a* **signed Θ-martingale measure***, if*

(a) $Q(\Omega) = 1$;

(b) $Q \ll P$ *with* $\frac{dQ}{dP} \in \mathcal{L}^2(P)$;

(c) $\mathbb{E}[\frac{dQ}{dP} G_T(v)] = 0$ *for all* $v \in \Theta$.

We denote by $\mathbb{P}_s(\Theta)$, the set of all such signed Θ-martingale measures. Moreover, we define

$$\mathbb{P}_e(\Theta) := \{Q \in \mathbb{P}_s(\Theta) \mid Q \sim P \quad \textit{and } Q \textit{ is a probability measure}\} ,$$

and introduce the closed convex set,

$$\mathcal{D}_d := \{D \in \mathcal{L}^2(P) \mid D = \frac{dQ}{dP} \quad for\ some \quad Q \in \mathbb{P}_s(\Theta)\} .$$

2. *A signed martingale measure $\widetilde{P} \in \mathbb{P}_s(\Theta)$ is called variance-optimal martingale (VOM) measure if $\widetilde{D} = argmin_{D\in\mathcal{D}_d} Var[D^2] = argmin_{D\in\mathcal{D}_d}\left(\mathbb{E}[D^2] - 1\right)$, where $\widetilde{D} = \frac{d\widetilde{P}}{dP}$.*

The space $G_T(\Theta) := \{G_T(v) \mid v \in \Theta\}$ is a linear subspace of $\mathcal{L}^2(P)$. Then, we denote by $G_T(\Theta)^\perp$ its orthogonal complement, that is,

$$G_T(\Theta)^\perp := \{D \in \mathcal{L}^2(P) \mid \mathbb{E}[DG_T(v)] = 0 \quad \text{for any } v \in \Theta\} .$$

Furthermore, $G_T(\Theta)^{\perp\perp}$ denotes the orthogonal complement of $G_T(\Theta)^\perp$, which is the $\mathcal{L}^2(P)$-closure of $G_T(\Theta)$. A simple example when $\mathbb{P}_e(\Theta)$ is non empty is given by the following proposition, that anticipates some material treated in the next section.

Proposition 2.31. *Let X be a process with independent increments such that*

- *X_t has the same law as $-X_t$, for any $t \in [0,T]$;*
- *$\frac{1}{2}$ belongs to the domain D of the cumulative generating function $(t,z) \mapsto \kappa_t(z)$.*

Then, there is a probability $Q \sim P$ such that $S_t = \exp(X_t)$ is a martingale.

Proof. For all $t \in [0,T]$, we set $D_t = \exp\{-\frac{X_t}{2} - \kappa_t(-\frac{1}{2})\}$. Notice that D is a martingale so that the measure Q on $(\Omega, \mathcal{F}_T)$ defined by $dQ = D_T dP$ is an (equivalent) probability to P. On the other hand, the symmetry of the law of X_t implies for all $t \in [0,T]$,

$$S_t D_t = \exp\{\frac{X_t}{2} - \kappa_t(-\frac{1}{2})\} = \exp\{\frac{X_t}{2} - \kappa_t(\frac{1}{2})\} \ .$$

So SD is also a martingale. According to [35], chapter III, Proposition 3.8 a), S is a Q-martingale and so S is a Q-martingale. □

Example 2.32. *Let Y be a process with independent increments. We consider two copies Y^1 of Y and Y^2 of $-Y$. We set $X = Y^1 + Y^2$. Then X has the same law of $-X$.*

For simplicity, we suppose from now that Assumption 1 is verified, even if one could consider a more general framework, see [3] Therorem 1.28. This ensures that the linear space $G_T(\Theta)$ is closed in $\mathcal{L}^2(\Omega)$, therefore $G_T(\Theta) = \overline{G_T(\Theta)} = G_T(\Theta)^{\perp\perp}$. Moreover, Proposition 2.14 ensures that $\Theta = L^2(M)$. We recall an almost known fact cited in [3]. For completeness, we give a proof.

Proposition 2.33. *$\mathbb{P}_s(\Theta) \neq \emptyset$ is equivalent to $1 \notin G_T(\Theta)$.*

Proof. Let us prove the two implications.

- Let $Q \in \mathbb{P}_s(\Theta)$. If $1 \in G_T(\Theta)$, then $Q(\Omega) = \mathbb{E}^Q(1) = 0$ which leads to a contradiction since Q is a probability. Hence $1 \notin G_T(\Theta)$.
- Suppose that $1 \notin G_T(\Theta)$. We denote by f the orthogonal projection of 1 on $G_T(\Theta)$. Since $\mathbb{E}[f(1-f)] = 0$, then $\mathbb{E}[1-f] = \mathbb{E}[(1-f)^2]$. Recall that $1 \neq f \in G_T(\Theta)$, hence we have $\mathbb{E}[f] \neq 1$. Therefore, we can define the signed measure $\widetilde{P}$ by setting

$$\widetilde{P}(A) = \int_A \widetilde{D} dP \ , \quad \text{with} \quad \widetilde{D} = \frac{1-f}{1-\mathbb{E}[f]} \ . \tag{2.8}$$

  We check now that $\widetilde{P} \in \mathbb{P}_s(\Theta)$.

  - Trivially $\widetilde{P}(\Omega) = \mathbb{E}(\widetilde{D}) = 1$;
  - $\widetilde{P} \ll P$, by construction.
  - Let $v \in \Theta$, $\mathbb{E}[\widetilde{D} G_T(v)] = \frac{1}{1-\mathbb{E}[f]} \left(\mathbb{E}[(1-f)G_T(v)]\right) = 0$, since $1 - f \in G_T(\Theta)^{\perp}$.

  Hence, $\widetilde{P} \in \mathbb{P}_s(\Theta)$ which concludes the proof of the Proposition.

□

Remark 2.34. *If 1 is orthogonal to $G_T(\Theta)$, then $f = 0$ and $P \in \mathbb{P}_s(\Theta)$ so $\mathbb{P}_s(\Theta) \neq \emptyset$.*

In fact, $\widetilde{P}$ constructed in the proof of Proposition 2.33 coincides with the VOM measure.

Proposition 2.35. *Let $\widetilde{P}$ be the signed measure defined in (2.8). Then,*

$$\widetilde{D} = \arg\min_{D \in \mathcal{D}_d} \mathbb{E}[D^2] = \arg\min_{D \in \mathcal{D}_d} Var[D] \ .$$

Proof. Let $D \in \mathcal{D}_d$ and Q such that $dQ = DdP$. We have to show that $\mathbb{E}[D^2] \geq \mathbb{E}[\widetilde{D}^2]$. We write

$$\mathbb{E}[D^2] = \mathbb{E}[(D-\widetilde{D})^2] + \mathbb{E}[\widetilde{D}^2] + \frac{2}{1-\mathbb{E}[f]}\mathbb{E}[(D-\widetilde{D})(1-f)] \ .$$

Moreover, since $f \in G_T(\Theta)$ yields

$$\begin{aligned} \mathbb{E}[(D-\widetilde{D})(1-f)] &= \mathbb{E}[D] - \mathbb{E}[\widetilde{D}] - \mathbb{E}[Df] + \mathbb{E}[\widetilde{D}f] \ , \\ &= Q(\Omega) - \widetilde{Q}(\Omega) \ . \\ &= 0 \ . \end{aligned}$$

□

Remark 2.36.
1. *Arai [2] gives sufficient conditions under which the VOM measure is a probability, see Theorem 3.4 in [2].*
2. *Taking in account Proposition 2.33, the property $1 \notin G_T(\Theta)$ may be viewed as non-arbitrage condition. In fact, in [18], the existence of a martingale measure which is a probability is equivalent to a no free lunch condition.*

Next proposition can be easily deduced for a more general formulation, see [51].

Proposition 2.37. *We assume Assumption 1. Let $H \in \mathcal{L}^2(\Omega)$ and consider the solution (c^H, φ^H) of the minimization problem (2.1). Then, the price c^H equals the expectation under the VOM measure $\widetilde{P}$ of H.*

Proof. We have

$$H = c^H + G_T(\varphi^H) + R \ ,$$

where R is orthogonal to $G_T(\Theta)$ and $\mathbb{E}[R] = 0$. Since $\widetilde{P} \in \mathbb{P}_s(\Theta)$, taking the expectation with respect to $\widetilde{P}$, denoted by $\widetilde{\mathbb{E}}$ we obtain

$$\widetilde{\mathbb{E}}[H] = c^H + \widetilde{\mathbb{E}}[R] \ .$$

From the proof of Proposition 2.33, we have

$$\widetilde{\mathbb{E}}[R] = \frac{\mathbb{E}[(1-f)R]}{1-\mathbb{E}[f]} = \frac{1}{1-\mathbb{E}[f]}\left(\mathbb{E}[R] - \mathbb{E}[fR]\right) \ .$$

Since $f \in G_T(\Theta)$ and R is orthogonal to $G_T(\Theta)$, we get $\widetilde{\mathbb{E}}[R] = 0$. □

3 Processes with independent increments (PII)

This section deals with the case of Processes with Independent Increments. The preliminary part recalls some useful properties of such processes. Then, we obtain a sufficient condition on the characteristic function for the existence of the FS decomposition. Moreover, an explicit FS decomposition is derived.

Beyond its own theoretical interest, this work is motivated by its possible application to hedging and pricing energy derivatives and specifically electricity derivatives. Indeed, one way of modeling electricity forward prices is to use arithmetic models such as the Bachelier model which was developed for standard financial assets. The reason for using arithmetic models, is that the usual hedging intrument available on electricity markets are swap contracts which give a fixed price for the delivery of electricity over a contracted time period. Hence, electricty swaps can be viewed as a strip of forwards for each hour of the delivery period. In this framework, arithmetic models have the significant advantage to yield closed pricing formula for swaps which is not the case of geometric models.

However, in whole generality, an arithmetic model allows negative prices which could be underisable. Nevertheless, in the electricity market, negative prices may occur because it can be more expensive for a producer to switch off some generators than to pay someone to consume the resulting excess of production. Still, in [6], is introduced a class of arithmetic models where the positivity of spot prices is ensured, using a specific choice of increasing Lévy process. The parameters estimation of this kind of model is studied in [37].

3.1 Preliminaries

Definition 3.1. *$X = (X_t)_{t\in[0,T]}$ is a (real)* ***process with independent increments (PII)*** *iff*

1. *X is adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and has càdlàg paths.*
2. *$X_0 = 0$.*
3. *$X_t - X_s$ is independent of $\mathcal{F}_s$ for $0 \le s < t \le T$.*

   Moreover we will also suppose

4. *X is continuous in probability, i.e. X has no fixed time of discontinuties.*

We recall Theorem II.4.15 of [35].

Theorem 3.2. *Let $(X_t)_{t\in[0,T]}$ be a real-valued special semimartingale, with $X_0 = 0$. Then, X is a process with independent increments, iff there is a version (b, c, ν) of its characteristics that is deterministic.*

Remark 3.3. *In particular, ν is a (deterministic non-negative) measure on the Borel σ-field of $[0, T] \times \mathbb{R}$.*

From now on, given two reals a, b, we denote by $a \vee b$ (resp. $a \wedge b$) the maximum (resp. minimum) between a and b.

Proposition 3.4. *Suppose X is a semimartingale with independent increments with characteristics (b, c, ν), then there exists an increasing function $t \mapsto a_t$ such that*

$$db_t \ll da_t\ , \quad dc_t \ll da_t \quad \text{and} \quad \nu(dt, dx) = \widetilde{F}_t(dx)da_t\ , \tag{3.1}$$

where $\widetilde{F}_t(dx)$ is a non-negative kernel from $\big([0, T], \mathcal{B}([0, T])\big)$ into $(\mathbb{R}, \mathcal{B})$ verifying

$$\int_{\mathbb{R}} (|x|^2 \wedge 1)\tilde{F}_t(dx) \le 1\,, \quad \forall t \in [0, T]. \tag{3.2}$$

and

$$a_t = ||b||_t + c_t + \int_{\mathbb{R}} (|x|^2 \wedge 1)\nu([0, t], dx)\ , \tag{3.3}$$

Proof. The existence of (a_t) as a process fulfilling (3.3) and $\tilde{F}$ fulfilling (3.2) is provided by the statement and the proof of Proposition II. 2.9 of [35]. (3.3) and Theorem 3.2 guarantee that (a_t) is deterministic. $\square$

Remark 3.5. *In particular, (b_t), (c_t) and $t \mapsto \int_{[0,t]\times B}(|x|^2 \wedge 1)\nu(ds, dx)$ has bounded variation for any $B \in \mathcal{B}$.*

The proposition below provides the so called **Lévy-Khinchine Decomposition**.

Proposition 3.6. *Assume that $(X_t)_{t\in[0,T]}$ is a process with independent increments. Then*

$$\varphi_t(u) = e^{\Psi_t(u)} \ , \quad \textit{for all } u \in \mathbb{R} \ , \tag{3.4}$$

Ψ_t, is given by the Lévy-Khinchine decomposition of the process X,

$$\Psi_t(u) = iub_t - \frac{u^2}{2}c_t + \int_{\mathbb{R}} (e^{iux} - 1 - iuh(x))F_t(dx) \ , \quad \textit{for all } u \in \mathbb{R} \ , \tag{3.5}$$

where $B \mapsto F_t(B)$ is the positive measure $\nu([0,t]\times B)$ which integrates $1 \wedge |x|^2$ for any $t \in [0,T]$.

We introduce here a simplifying hypothesis for this section.

Assumption 2. *For any $t > 0$, X_t is never deterministic.*

Remark 3.7. *We suppose Assumption 2.*

1. *Up to a $2\pi i$ addition of $\kappa_t(e)$, we can write $\Psi_t(u) = \kappa_t(iu)$, $\forall u \in \mathbb{R}$. From now on we will always make use of this modification.*

2. *$\varphi_t(u)$ is never a negative number. Otherwise, there would be $u \in \mathbb{R}^*, t > 0$ such that $E(cos(uX_t)) = -1$. Since $cos(uX_t) + 1 \geq 0$ a.s. then $cos(uX_t) = -1$ a.s. and this is not possible since X_t is non-deterministic.*

3. *Previous point implies that all the differentiability properties of $u \mapsto \varphi_t(u)$ are equivalent to those of $u \mapsto \Psi_t(u)$.*

4. *If $\mathbb{E}[|X_t|^2] < \infty$, then for all $u \in \mathbb{R}$, $\Psi_t^{'}(u)$ and $\Psi_t^{''}(u)$ exist.*

We come back to the cumulant generating function κ and its domain D.

Remark 3.8. *In the case where the underlying process is a PII, then*

$$D := \{z \in \mathbb{C} \mid \mathbb{E}[e^{Re(z)X_t}] < \infty, \ \forall t \in [0,T]\} = \{z \in \mathbb{C} \mid \mathbb{E}[e^{Re(z)X_T}] < \infty\} \ .$$

In fact, for given $t \in [0,T], \gamma \in \mathbb{R}$ we have

$$\mathbb{E}(e^{\gamma X_T}) = \mathbb{E}(e^{\gamma X_t})\mathbb{E}(e^{\gamma(X_T - X_t)}) < \infty.$$

Since each factor is positive, and if the left-hand side is finite, then $\mathbb{E}(e^{\gamma X_t})$ is also finite.

We need now a result which extends the Lévy-Khinchine decomposition to the cumulant generating function. Similarly to Theorem 25.17 of [45] we have.

Proposition 3.9. *Let $D_0 = \left\{c \in \mathbb{R} \mid \int_{[0,T]\times\{|x|>1\}} e^{cx}\nu(dt, dx) < \infty\right\}$. Then,*

1. *D_0 is convex and contains the origin.*

2. $D_0 = D \cap \mathbb{R}$.

3. If $z \in \mathbb{C}$ *such that* $Re(z) \in D_0$*, i.e.* $z \in D$*, then*

$$\kappa_t(z) = zb_t + \frac{z^2}{2}c_t + \int_{[0,t]\times\mathbb{R}} (e^{zx} - 1 - zh(x))\nu(ds, dx) \ . \tag{3.6}$$

Proof. 1. is a consequence of Hölder inequality similarly as i) in Theorem 25.17 of [45] .

2. The characteristic function of the law of X_t is given by (3.5). According to Theorem II.8.1 (iii) of Sato [45], there is an infinitely divisible distribution with characteristics $(b_t, c_t, F_t(dx))$, fulfilling $F_t(\{0\}) = 0$ and $\int(1 \wedge x^2)F_t(dx) < \infty$ and $c_t \geq 0$. By uniqueness of the characteristic function, that law is precisely the law of X_t. By Corollary II.11.6, in [45], there is a Lévy process $(L^t_s, 0 \leq s \leq 1)$ such that L^t_1 and X_t are identically distributed. We define

$$C^t_0 = \{c \in \mathbb{R} \mid \int_{\{|x|>1\}} e^{cx}F_t(dx) < \infty\} \quad \text{and} \quad C^t = \{z \in \mathbb{C} \mid \mathbb{E}\left[\exp(Re(zL^t_1)\right] < \infty\} \ .$$

Remark 3.8 says that $C^T = D$, moreover clearly $C^T_0 = D_0$. Theorem V.25.17 of [45] implies $D_0 = D \cap \mathbb{R}$, i.e. point 2. is established.

3. Let $t \in [0, T]$ be fixed; let $w \in D$. We apply point (iii) of Theorem V.25.17 of [45] to the Lévy process L^t.

□

Proposition 3.10. *Let X be a semimartingale with independent increments. For all $z \in D$, $t \mapsto \kappa_t(z)$ has bounded variation and*

$$\kappa_{dt}(z) \ll da_t \ . \tag{3.7}$$

Proof. Using (3.6), it remains to prove that

$$t \mapsto \int_{[0,T]\times\mathbb{R}} (e^{zx} - 1 - zh(x))\nu(ds, dx)$$

is absolutely continuous with respect to (da_t). We can conclude

$$\kappa_t(z) = \int_0^t \frac{db_s}{da_s} da_s + \frac{z^2}{2}\int_0^t \frac{dc_s}{da_s} da_s + \int_0^t da_s \int_{\mathbb{R}} (e^{zx} - 1 - zh(x))\, \widetilde{F}_s(dx) \ ,$$

if we show that

$$\int_0^T da_s \int_{\mathbb{R}} |e^{zx} - 1 - zh(x)| \widetilde{F}_s(dx) < \infty \ . \tag{3.8}$$

Without restriction of generality we can suppose $h(x) = x1_{|x|\leq 1}$. (3.8) can be bounded by the sum $I_1 + I_2 + I_3$ where

$$I_1 = \int_0^T da_s \int_{|x|>1} |e^{zx}| \widetilde{F}_s(dx) \ , \quad I_2 = \int_0^T da_s \int_{|x|>1} \widetilde{F}_s(dx) \ , \quad \text{and} \quad I_3 = \int_0^T da_s \int_{|x|\leq 1} |e^{zx} - 1 - zx| \widetilde{F}_s(dx) \ .$$

Using Proposition 3.4, we have

$$\begin{aligned} I_1 &= \int_0^T da_s \int_{|x|>1} |e^{zx}| \widetilde{F}_s(dx) \\ &= \int_0^T da_s \int_{|x|>1} |e^{Re(z)x}| \widetilde{F}_s(dx) \\ &= \int_{[0,T]\times|x|>1} |e^{Re(z)x}| \nu(ds, dx); \end{aligned}$$

this quantity is finite because $Re(z) \in D_0$ taking into account Proposition 3.9. Concerning I_2 we have

$$\begin{aligned} I_2 &= \int_0^T da_s \int_{|x|>1} \widetilde{F}_s(dx) \\ &= \int_0^T da_s \int_{|x|>1} (1 \wedge |x^2|) \widetilde{F}_s(dx) \\ &\leq a_T \end{aligned}$$

because of (3.2). As far as I_3 is concerned, we have

$$\begin{aligned} I_3 &\leq e^{Re(z)} \frac{z^2}{2} \int_{[0,T]\times|x|\leq 1} da_s (x^2 \wedge 1) \widetilde{F}_s(dx) \\ &= e^{Re(z)} \frac{z^2}{2} a_T \end{aligned}$$

again because of (3.2). This concludes the proof the Proposition.

□

The converse of the first part of previous corollary also holds. For this purpose we formulate first a simple remark.

Remark 3.11. *For every $z \in D$, $(\exp(zX_t - \kappa_t(z)))$ is a martingale. In fact, for all $0 \leq s \leq t \leq T$, we have*

$$\mathbb{E}[\exp(z(X_t - X_s))] = \exp(\kappa_t(z) - \kappa_s(z)) \ . \tag{3.9}$$

Proposition 3.12. *Let X be a PII. Let $z \in D \cap \mathbb{R}^\star$. $(X_t)_{t\in[0,T]}$ is a semimartingale iff $t \mapsto \kappa_t(z)$ has bounded variation.*

Proof. It remains to prove the converse implication.
If $t \mapsto \kappa_t(z)$ has bounded variation then $t \mapsto e^{\kappa_t(z)})$ has the same property. Remark 3.11 says that $e^{zX_t} = M_t e^{\kappa_t(z)}$ where (M_t) is a martingale. Finally, (e^{zX_t}) is a semimartingale and taking the logarithm (zX_t) has the same property. □

Remark 3.13. *Let $z \in D$. If (X_t) is a semimartingale with independent increments then (e^{zX_t}) is necessarily a special semimartingale since it is the product of a martingale and a bounded variation continuous deterministic function, by use of integration by parts.*

Lemma 3.14. *Suppose that (X_t) is a semimartingale with independent increments. Then for every $z \in Int(D)$, $t \mapsto \kappa_t(z)$ is continuous.*

Remark 3.15. *The conclusion remains true for any process which is continuous in probability, whenever $t \mapsto \kappa_t(z)$ is (locally) bounded.*

Proof of Lemma 3.14. Since $z \in Int(D)$, there is $\gamma > 1$ such that $\gamma z \in D$; so

$$\mathbb{E}[\exp(z\gamma X_t)] = \exp(\kappa_t(\gamma z)) \leq \exp(\sup_{t\leq T}(\kappa_t(\gamma z))) \ ,$$

because $t \mapsto \kappa_t(\gamma z)$ is bounded, being of bounded variation. This implies that $(\exp(zX_t))_{t\in[0,T]}$ is uniformly integrable. Since (X_t) is continuous in probability, then $(\exp(zX_t))$ is continuous in $\mathcal{L}^1$. The result easily follows. □

Proposition 3.16. *The function $(t, z) \mapsto \kappa_t(z)$ is continuous. In particular, $(t, z) \mapsto \kappa_t(z)$, $t \in [0, T]$, z belonging to a compact real subset, is bounded.*

Proof.

- Proposition 3.9 implies that $z \mapsto \kappa_t(z)$ is continuous uniformly with respect to $t \in [0, T]$.
- By Lemma 3.14, for $z \in \mathrm{IntD}$, $t \mapsto \kappa_t(z)$ is continuous.
- To conclude it is enough to show that $t \mapsto \kappa_t(z)$ is continuous for every $z \in D$. Since $\bar{D} = \overline{\mathrm{IntD}}$, there is a sequence (z_n) in the interior of D converging to z. Since a uniform limit of continuous functions on $[0, T]$ converges to a continuous function, the result follows.

□

3.2 Structure condition for PII (which are semimartingales)

Let $X = (X_t)_{t \in [0,T]}$ be a real-valued semimartingale with independent increments and $X_0 = 0$. We assume that $\mathbb{E}[|X_t|^2] < \infty$. We denote by $\varphi_t(u) = \mathbb{E}[\exp(iuX_t)]$ the characteristic function of X_t and by $u \mapsto \Psi_t(u)$ its log-characteristic function introduced in Proposition 3.6. We recall that $\varphi_t(u) = \exp(\Psi_t(u))$.
X has the property of independent increments; therefore

$$\exp(iuX_t)/\mathbb{E}[\exp(iuX_t)] = \exp(iuX_t)/\exp(\Psi_t(u)) \ , \tag{3.10}$$

is a martingale.

Remark 3.17. *Notice that the two first order moments of X are related to the log-characterisctic function of X, as follows*

$$\mathbb{E}[X_t] = -i\Psi_t^{'}(0) \ , \quad \mathbb{E}[X_t - X_s] = -i(\Psi_t^{'}(0) - \Psi_s^{'}(0)), \tag{3.11}$$

$$Var(X_t) = -\Psi_t^{''}(0) \ , \quad Var(X_t - X_s) = -[\Psi_t^{''}(0) - \Psi_s^{''}(0)] \ . \tag{3.12}$$

Proposition 3.18. *Let $X = (X_t)_{t \in [0,T]}$ be a real-valued semimartingale with independent increments.*

1. *X is a special semimartingale with decomposition $X = M + A$ with the following properties:*

$$\langle M \rangle_t = -\Psi_t^{''}(0) \quad \text{and} \quad A_t = -i\Psi_t^{'}(0) \ . \tag{3.13}$$

   In particular $t \mapsto -\Psi_t^{''}(0)$ is increasing and therefore of bounded variation.

2. *X satisfies condition (SC) of Definition 2.11 if and only if*

$$\Psi_t^{'}(0) \ll \Psi_t^{''}(0) \quad \text{and} \quad \int_0^T \left| \frac{d_t \Psi_s^{'}}{d_t \Psi_s^{''}}(0) \right|^2 |d\Psi_s^{''}(0)| < \infty \ . \tag{3.14}$$

   In that case

$$A_t = \int_0^t \alpha_s d\langle M \rangle_s \quad \text{with} \quad \alpha_t = i \frac{d_t \Psi_t^{'}(0)}{d_t \Psi_t^{''}(0)} \quad \text{for all } t \in [0, T] \ . \tag{3.15}$$

3. *Under condition (3.14), FS decomposition exists (and it is unique) for every square integrable random variable.*

In the sequel, we will provide an explicit decomposition for a class of contingent claims, under condition (3.14).

Proof. 1. Let us first determine A and M in terms of the log-characteristic function of X. Using (3.11) of Remark 3.17, we get

$$\begin{aligned}
\mathbb{E}[X_t|\mathcal{F}_s] &= \mathbb{E}[X_t - X_s + X_s \mid \mathcal{F}_s] \ , = \mathbb{E}[X_t - X_s] + X_s \ , \\
&= -i\Psi_t^{'}(0) + i\Psi_s^{'}(0) + X_s \ , \quad \text{then} \ , \\
\mathbb{E}[X_t + i\Psi_t^{'}(0)|\mathcal{F}_s] &= X_s + i\Psi_s^{'}(0) \ .
\end{aligned}$$

Hence, $(X_t + i\Psi_t^{'}(0))$ is a martingale and the canonical decomposition of X follows

$$X_t = \underbrace{X_t + i\Psi_t^{'}(0)}_{M_t} \underbrace{-i\Psi_t^{'}(0)}_{A_t} \ ,$$

where M is a local martingale and A is a locally bounded variation process thanks to the semimartingale property of X. Let us now determine $\langle M \rangle$, in terms of the log-characteristic function of X.

$$\begin{aligned}
M_t^2 &= [X_t + i\Psi_t^{'}(0)]^2 \ , \\
\mathbb{E}[M_t^2|\mathcal{F}_s] &= \mathbb{E}[(X_t + i\Psi_t^{'}(0))^2|\mathcal{F}_s] \ , \\
&= \mathbb{E}[(X_s + i\Psi_s^{'}(0) + X_t - X_s + i\Psi_t^{'}(0) - i\Psi_s^{'}(0))^2|\mathcal{F}_s] \ , \\
&= \mathbb{E}[(M_s + X_t - X_s + i(\Psi_t^{'}(0) - \Psi_s^{'}(0)))^2|\mathcal{F}_s] \ ,
\end{aligned}$$

Using (3.11) and (3.12) of Remark 3.17, yields

$$\begin{aligned}
M_t^2 &= \mathbb{E}[(M_s - \mathbb{E}[X_t - X_s] + X_t - X_s)^2] \ , \\
&= M_s^2 + Var(X_t - X_s) = M_s^2 - \Psi_t^{''}(0) + \Psi_s^{''}(0) \ .
\end{aligned}$$

Hence, $(M_t^2 + \Psi_t^{''}(0))$ is a $(\mathcal{F}_t)$-martingale, and point 1. is established.

$$A_t = \int_0^t \alpha_s d\left\langle M \right\rangle_s \quad \text{with} \quad \alpha_t = i\frac{d_t\Psi_t^{'}(0)}{d_t\Psi_t^{''}(0)} \quad \text{for all } t \in [0,T] \ .$$

2. is a consequence of point 1. and of Definition 2.11.

3. follows from Theorem 2.22. In fact $K_t = -\int_0^T \left(\frac{d_t\Psi_s^{'}}{d_t\Psi_s^{''}}(0)\right)^2 d\Psi_s^{''}(0)$ is deterministic and so Assumption 1 is fulfilled.

□

3.3 Examples

3.3.1 A continuous process example

Let $\psi : [0,T] \to \mathbb{R}$ be a continuous strictly increasing function, $\gamma : [0,T] \to \mathbb{R}$ be a bounded variation function such that $d\gamma \ll d\psi$. We set $X_t = W_{\psi(t)} + \gamma(t)$, where W is the standard Brownian motion on $\mathbb{R}$. Clearly, $X_t = M_t + \gamma(t)$, where $M_t = W_{\psi(t)}$, defines a continuous martingale, such that $\langle M \rangle_t = [M]_t = \psi(t)$. Since $X_t \sim \mathcal{N}(\gamma(t), \psi(t))$ for all $u \in \mathbb{R}$ and $t \in [0,T]$, we have

$$\Psi_t(u) = i\gamma(t)u - \frac{u^2\psi(t)}{2} \ ,$$

which yields

$$\Psi_t^{'}(0) = i\gamma(t) \quad \text{and} \quad \Psi_t^{''}(0) = -\psi(t) \ ,$$

Therefore, if $\dfrac{d\gamma}{d\psi} \in \mathcal{L}^2(d\psi)$, then X satisfies condition (SC) of Definition 2.11 with

$$A_t = \int_0^t \alpha_s d\langle M\rangle_s \quad \text{and} \quad \alpha_t = \left.\frac{d\gamma}{d\psi}\right|_t \quad \text{for all } t \in [0,T] \ .$$

3.3.2 Processes with independent and stationary increments (Lévy processes)

Definition 3.19. $X = (X_t)_{t\in[0,T]}$ *is called* **Lévy process** *or process with stationary and independent increments if the following properties hold.*

1. *X is adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and has càdlàg trajectories.*
2. *$X_0 = 0$.*
3. *The distribution of $X_t - X_s$ depends only on $t-s$ for $0 \le s \le t \le T$.*
4. *$X_t - X_s$ is independent of $\mathcal{F}_s$ for $0 \le s \le t \le T$.*
5. *X is continuous in probability.*

For details on Lévy processes, we refer the reader to [40], [45] and [35].
Let $X = (X_t)_{t\in[0,T]}$ be a real-valued Lévy process, with $X_0 = 0$. We assume that $\mathbb{E}[|X_t|^2] < \infty$ and we do not consider the trivial case where L_1 is deterministic.

Remark 3.20. 1. *Since $X = (X_t)_{t\in[0,T]}$ is a Lévy process then $\Psi_t(u) = t\Psi_1(u)$. In the sequel, we will use the shortened notation $\Psi := \Psi_1$.*

2. *Ψ is a function of class C^2 and $\Psi^{''}(0) = Var(X_1)$ which is strictly positive if X has no stationary increments.*

3.4 Cumulative and characteristic functionals in some particular cases

We recall some cumulant and log-caracteristic functions of some typical Lévy processes.

Remark 3.21. 1. Poisson Case: *If X is a Poisson process with intensity λ, we have that $\kappa^\Lambda(z) = \lambda(e^z - 1)$. Moreover, in this case the set $D = \mathbb{C}$.*
Concerning the log-characteristic function we have

$$\Psi(u) = \lambda(e^{iu} - 1) \ , \quad \Psi^{'}(0) = i\lambda \quad \text{and} \quad \Psi^{''}(0) = -\lambda, u \in \mathbb{R}.$$

2. NIG Case: *This process was introduced by Barndorff-Nielsen in [4]. Then X is a Lévy process with $X_1 \sim NIG(\alpha, \beta, \delta, \mu)$, with $\alpha > |\beta| > 0$, $\delta > 0$ and $\mu \in \mathbb{R}$. We have $\kappa^\Lambda(z) = \mu z + \delta(\gamma_0 - \gamma_z)$ and $\gamma_z = \sqrt{\alpha^2 - (\beta+z)^2}$, $D =]-\alpha-\beta, \alpha-\beta[+i\mathbb{R}$.*
Therefore

$$\Psi(u) = \mu iu + \delta(\gamma_0 - \gamma_{iu}) \ , \quad \text{where} \quad \gamma_{iu} = \sqrt{\alpha^2 - (\beta+iu)^2} \ .$$

By derivation, one gets

$$\Psi^{'}(0) = i\mu + \delta\frac{i\beta}{\gamma_0} \quad and \quad \Psi^{''}(0) = -\delta(\frac{1}{\gamma_0} + \frac{\beta^2}{\gamma_0^3}),$$

Which yields $\alpha = i\frac{\Psi^{'}(0)}{\Psi''(0)} = \frac{\gamma_0^2(\gamma_0\mu + \delta\beta)}{\delta(\gamma_0^2 + \beta)}$.

3. <u>*Variance Gamma case:*</u> *Let* $\alpha, \beta > 0, \delta \neq 0$. *If* X *is a Variance Gamma process with* $X_1 \sim VG(\alpha, \beta, \delta, \mu)$ *with* $\kappa^{\Lambda}(z) = \mu z + \delta Log\left(\frac{\alpha}{\alpha - \beta z - \frac{z^2}{2}}\right)$, *where Log is again the principal value complex logarithm defined in Section 2. The expression of* $\kappa^{\Lambda}(z)$ *can be found in [31, 36] or also [16], table IV.4.5 in the particular case* $\mu = 0$. *In particular an easy calculation shows that we need* $z \in \mathbb{C}$ *such that* $Re(z) \in]-\beta - \sqrt{\beta^2 + 2\alpha}, -\beta + \sqrt{\beta^2 + 2\alpha}[$ *so that* $\kappa^{\Lambda}(z)$ *is well defined so that*

$$D =]-\beta - \sqrt{\beta^2 + 2\alpha}, -\beta + \sqrt{\beta^2 + 2\alpha}[+ i\mathbb{R}.$$

Finally we obtain

$$\Psi(u) = \mu i u + \delta Log\left(\frac{\alpha}{\alpha - \beta i u + \frac{u^2}{2}}\right).$$

After derivation it follows

$$\Psi^{'}(0) = i(\mu - \delta\beta), \quad \Psi^{''}(0) = \frac{\delta}{\alpha}(\alpha^2 - \beta^2).$$

3.5 Structure condition in the Lévy case

By application of Proposition 3.18 and Remark 3.20, we get the following result.

Corollary 3.22. *Let* $X = M + A$ *be the canonical decomposition of* X, *then for all* $t \in [0, T]$,

$$\langle M \rangle_t = -t\Psi^{''}(0) \quad and \quad A_t = -it\Psi^{'}(0) \ . \tag{3.16}$$

Moreover X *satisfies condition (SC) of Definition 2.11 with*

$$A_t = \int_0^t \alpha d\langle M \rangle_s \quad with \quad \alpha = i\frac{\Psi^{'}(0)}{\Psi''(0)} \quad for\ all\ t \in [0, T] \ . \tag{3.17}$$

Hence, FS decomposition exists for every square integrable random variable.

Remark 3.23. *We have the following in previous three examples of subsubsection 3.4*

1. **Poisson case:** $\alpha = 1$.
2. **NIG process:** $\alpha = \frac{\gamma_0^2(\gamma_0\mu + \delta\beta)}{\delta(\gamma_0^2 + \beta)}$.
3. **VG process:** $\alpha = \frac{\mu - \delta\beta}{\alpha^2 - \beta^2}\frac{\alpha}{\delta}$.

3.5.1 Wiener integrals of Lévy processes

We take $X_t = \int_0^t \gamma_s d\Lambda_s$, where Λ is a square integrable Lévy process as in Section 3.3.2. Then, $\int_0^T \gamma_s d\Lambda_s$ is well-defined for at least $\gamma \in \mathcal{L}^\infty([0,T])$. It is then possible to calculate the characteristic function and the cumulative function of $\int_0^\cdot \gamma_s d\Lambda_s$. Let $(t,z) \mapsto t\Psi_\Lambda(z)$, (resp. $(t,z) \mapsto t\kappa^\Lambda(z)$) denoting the log-characteristic function (resp. the cumulant generating function) of Λ.

Lemma 3.24. *Let $\gamma : [0,T] \to \mathbb{R}$ be a Borel bounded function.*

1. *The log-characteristic function of X_t is such that for all $u \in \mathbb{R}$,*

$$\Psi_{X_t}(u) = \int_0^t \Psi_\Lambda(u\gamma_s)ds \ , \quad \textit{where} \quad \mathbb{E}[\exp(iuX_t)] = \exp\left(\Psi_{X_t}(u)\right) \ ;$$

2. *Let D_Λ be the domain related to κ^Λ in the sense of Definition 2.2. The cumulant generating function of X_t is such that for all $z \in \{z \,|\, Rez\gamma_t \in D_\Lambda \text{ for all } t \in [0,T]\}$,*

$$\kappa_{X_t}(z) = \int_0^t \kappa^\Lambda(z\gamma_s)ds.$$

Proof. We only prove 1. since 2. follows similarly. Suppose first γ to be continuous, then $\int_0^T \gamma_s d\Lambda_s$ is the limit in probability of $\sum_{j=0}^{p-1} \gamma_{t_j}(\Lambda_{t_{j+1}} - \Lambda_{t_j})$ where $0 = t_0 < t_1 < ... < t_p = T$ is a subdivision of $[0,T]$ whose mesh converges to zero. Using the independence of the increments, we have

$$\begin{aligned}
\mathbb{E}\left[\exp\{i\sum_{j=0}^{p-1}\gamma_{t_j}(\Lambda_{t_{j+1}} - \Lambda_{t_j})\}\right] &= \prod_{j=0}^{p-1}\mathbb{E}\left[\exp\{i\gamma_{t_j}(\Lambda_{t_{j+1}} - \Lambda_{t_j})\}\right] \ , \\
&= \prod_{j=0}^{p-1}\exp\{\Psi_\Lambda(\gamma_{t_j})(t_{j+1} - t_j)\} \ , \\
&= \exp\{\sum_{j=0}^{p-1}(t_{j+1} - t_j)\Psi_\Lambda(\gamma_{t_j})\} \ .
\end{aligned}$$

This converges to $\exp\left(\int_0^T \Psi_\Lambda(\gamma_s)ds\right)$, when the mesh of the subdivision goes to zero.
Suppose now that γ is only bounded and consider, using convolution, a sequence γ_n of continuous functions, such that $\gamma_n \to \gamma$ a.e. and $\sup_{t\in[0,T]}|\gamma_n(t)| \le \sup_{t\in[0,T]}|\gamma(t)|$. We have proved that

$$\mathbb{E}\left[\exp\left(i\int_0^T \gamma_n(s)d\Lambda_s\right)\right] = \exp\left(\int_0^T \Psi_\Lambda(\gamma_n(s))ds\right) \tag{3.18}$$

Now, Ψ_Λ is continuous therefore bounded, so Lebesgue dominated convergence and continuity of stochastic integral imply statement 1. □

Remark 3.25. 1. *Previous proof, which is left to the reader, also applies for statement 2. This statement in a slight different form is proved in [9]*

2. *We prefer to formulate a direct proof. In particular statement 1. holds with the same proof even if Λ has no moment condition and γ is a continuous function with bounded variation. Stochastic integrals are then defined using integration by parts.*

We suppose now that Λ is a Lévy process such that Λ_1 is not deterministic. In particular $Var(\Lambda_1) \neq 0$ and so $\Psi_\Lambda'' \neq 0$.
In this case

$$\Psi_t^{'}(u) = \int_0^t \Psi_\Lambda^{'}(u\gamma_s)\gamma_s ds \quad \text{and} \quad \Psi_t^{''}(u) = \int_0^t \Psi_\Lambda^{''}(u\gamma_s)\gamma_s^2 ds \ .$$

So

$$\Psi_t^{'}(0) = \Psi_\Lambda^{'}(0)\int_0^t \gamma_s ds \quad \text{and} \quad \Psi_t^{''}(0) = \Psi_\Lambda^{''}(0)\int_0^t \gamma_s^2 ds \ .$$

Condition (SC) is verified since $d\Psi_t^{'}(0) \ll d\Psi_t^{''}(0)$ with

$$\alpha_t = i\frac{d\Psi_t^{'}(0)}{d\Psi_t^{''}(0)} = \frac{\Psi_\Lambda^{'}(0)}{\Psi_\Lambda^{''}(0)}\frac{i}{\gamma_t}1_{\{\gamma_t\neq 0\}} \quad \text{and} \quad \int_0^T \alpha_s^2\,|\Psi_s^{''}(0)|\gamma_s^2 ds = T\frac{|\Psi_\Lambda^{'}(0)|^2}{|\Psi_\Lambda^{''}(0)|} < \infty \ .$$

3.6 Explicit Föllmer-Schweizer decomposition in the PII case

3.6.1 Preliminaries

Let $X = (X_t)_{t\in[0,T]}$ be a semimartingale with independent increments with log-characteristic function $(t,u) \mapsto \Psi_t(u)$. We assume that $(X_t)_{t\in[0,T]}$ is square integrable and satisfies Assumption 2.

Remark 3.26. *1. $u \mapsto \Psi_t(u)$ is of class C^2, for any $t \in [0,T]$ because X_t is square integrable.*

2. $t \mapsto \Psi_t^{''}(0)$ and $t \mapsto \Psi_t^{'}(0)$ have bounded variation because of Proposition 3.18. Therefore, they are bounded.

3. $t \mapsto \Psi_t^{'}(u)$ is continuous for every $u \in \mathbb{R}$. In fact, first $t \mapsto X_t$ is continuous in probability. Since $M_t = X_t - \Psi_t'(0)$ is a square integrable martingale and $t \mapsto \Psi_t^{'}(0)$ is bounded, then the family $(E(X_t^2))$ is bounded and so (X_t) is uniformly integrable. So $t \mapsto \varphi_t'(u)$ is continuous and the result follows by Assumption 2

4. $t \mapsto \Psi_t^{''}(0)$ is continuous. In fact, again it is enough to prove $t \mapsto \varphi_t''(0)$ is continuous. This follows if we prove that (M_t) is continuous in $\mathcal{L}^2$. This is true because M is continuous in probability and for any $N > 0$, $t \in [0,T]$, Chebyshev implies that

$$P\{|M_t^2| > N\} \leq \frac{\mathrm{Var}(X_t)}{N} \leq \frac{\mathrm{Var}(X_T)}{N},$$

and so the family (M_t^2) is again uniformly integrable.

We suppose the following.

Assumption 3. *1. $t \mapsto \Psi_t^{'}(u)$ is absolutely continuous with respect to $d\Psi_t^{''}(0)$.*

2. For every $u \in \mathbb{R}$, we suppose that the following quantity

$$K(u) := \int_0^T \left|\frac{d\Psi_t^{'}(u)}{d\Psi_t^{''}(0)}\right|^2 d(-\Psi_s^{''}(0)) \tag{3.19}$$

is finite.

Remark 3.27. *If $u = 0$, the previous quantity* (3.19) *is finite because of the (SC) condition.*

We consider a contingent claim which is given as a Fourier transform of X_T,

$$H = f(X_T) \quad \text{with} \quad f(x) = \int_{\mathbb{R}} e^{iux} \mu(du) \ , \quad \text{for all } x \in \mathbb{R} \ , \tag{3.20}$$

for some finite signed measure μ.

Assumption 4.

$$\int_{\mathbb{R}} K(u) d|\mu(u)| < \infty.$$

Remark 3.28. *We observe that the function*

$$(u,t) \mapsto \exp(\Psi_T(u) - \Psi_t(u))$$

is uniformly bounded because the characteristic function is bounded.

We will first evaluate an explicit Kunita-Watanabe decomposition of H w.r.t. the martingale part M of X. Later, we will finally obain the decomposition with respect to X.

3.6.2 Explicit elementary Kunita-Watanabe decomposition

By Propostion 3.18, X admits the following semimartingale decomposition, $X_t = A_t + M_t$, where

$$A_t = -i\Psi_t^{'}(0) \quad \text{and} \quad \langle M \rangle_t = -\Psi_t^{''}(0) \ . \tag{3.21}$$

Proposition 3.29. *Let $H = f(X_T)$ where f is of the form (3.20). We suppose that the PII X satisfies Assumptions 2, 3 and 4. Then, H admits the decomposition*

$$\left\{ \begin{array}{lcl} V_t & = & V_0 + \int_0^t Z_s dM_s + O_t \\ V_T & = & H \ , \end{array} \right. \tag{3.22}$$

with the following properties.

1. *For all $t \in [0,T]$,*

$$Z_t = i \int_{\mathbb{R}} e^{iuX_{t-}} \frac{d\left(\Psi_t^{'}(u) - \Psi_t^{'}(0)\right)}{d\Psi_t^{''}(0)} \exp\left\{\Psi_T(u) - \Psi_t(u)\right\} \mu(du) \ ; \tag{3.23}$$

2. *O is a square integrable $(\mathcal{F}_t)$-martingale such that $\langle O, M \rangle = 0$;*

3. *$H = V_T$ where $(V_t)_{t\in[0,T]}$ is the $(\mathcal{F}_t)$-martingale defined by*

$$V_t = \mathbb{E}[H|\mathcal{F}_t] = \int_{\mathbb{R}} e^{iuX_t} \exp\left\{\Psi_T(u) - \Psi_t(u)\right\} \mu(du) \ . \tag{3.24}$$

Remark 3.30. *In particular,*

1. $V_0 = \mathbb{E}[H]$;
2. $\mathbb{E}\left[\int_0^T Z_s^2 d\langle M \rangle_s\right] < \infty$.

Proof. We start with the case $\mu = \delta_u(dx)$ for some $u \in \mathbb{R}$ so that $f(x) = e^{iux}$. We consider the $(\mathcal{F}_t)$-martingale $V_t = \mathbb{E}[f(X_T)|\mathcal{F}_t] = \mathbb{E}[e^{iuX_T}|\mathcal{F}_t]$.

1. Clearly $V_0 = \mathbb{E}[e^{iuX_T}]$.
2. We calculate explicitely V_t, which gives

$$\begin{aligned} V_t &= \mathbb{E}[e^{iuX_T}|\mathcal{F}_t] \ = e^{iuX_t}\mathbb{E}[e^{iu(X_T-X_t)}] \ = \exp(iuX_t - \Psi_t(u))\exp(\Psi_T(u)) \\ &= \widetilde{V}_t \exp(\Psi_T(u)) \ , \end{aligned}$$

where $\widetilde{V}_t = \exp(iuX_t - \Psi_t(u))$ defines an $(\mathcal{F}_t)$-martingale.
3. We evaluate $\langle V, M\rangle$.

Lemma 3.31. $\langle V, M\rangle_t = -i\int_0^t V_s(\Psi'_{ds}(u) - \Psi'_{ds}(0))$.

Proof. We evaluate $\mathbb{E}[\widetilde{V}_t M_t|\mathcal{F}_s]$. Since $\widetilde{V}$ and M are $(\mathcal{F}_t)$-martingales and using the property of independent increments we get

$$\begin{aligned} \mathbb{E}[\widetilde{V}_t M_t|\mathcal{F}_s] &= \mathbb{E}[\widetilde{V}_t M_s|\mathcal{F}_s] + \mathbb{E}[\widetilde{V}_t(M_t - M_s)|\mathcal{F}_s] \ , \\ &= M_s\widetilde{V_s} + \widetilde{V_s}\mathbb{E}[\exp\{iu(X_t - X_s) - (\Psi_t(u) - \Psi_s(u))\}(M_t - M_s)] \ , \\ &= M_s\widetilde{V_s} + \widetilde{V_s}e^{-(\Psi_t(u)-\Psi_s(u))}\mathbb{E}[e^{iu(X_t-X_s)}(M_t - M_s)] \ . \end{aligned}$$

Previous expectation gives

$$\begin{aligned} \mathbb{E}[e^{iu(X_t-X_s)}(M_t - M_s)] &= \mathbb{E}[e^{iu(X_t-X_s)}(X_t - X_s)] + \mathbb{E}[e^{iu(X_t-X_s)}i(\Psi'_t(0) - \Psi'_s(0))] \ , \\ &= -i\frac{\partial}{\partial u}\mathbb{E}[e^{iu(X_t-X_s)}] + i(\Psi'_t(0) - \Psi'_s(0))\mathbb{E}[e^{iu(X_t-X_s)}] \ , \\ &= -ie^{\Psi_t(u)-\Psi_s(u)}(\Psi'_t(u) - \Psi'_s(u)) + i(\Psi'_t(0) - \Psi'_s(0))e^{\Psi_t(u)-\Psi_s(u)} \ . \end{aligned}$$

Consequently,

$$\begin{aligned} \mathbb{E}[\widetilde{V}_t M_t|\mathcal{F}_s] &= M_s\widetilde{V_s} - i\widetilde{V_s}(\Psi'_t(u) - \Psi'_s(u)) + i\widetilde{V_s}(\Psi'_t(0) - \Psi'_s(0)) \\ &= M_s\widetilde{V_s} - i\widetilde{V_s}\left(\Psi'_t(u) - \Psi'_t(0) - (\Psi'_s(u) - \Psi'_s(0))\right) \ . \end{aligned}$$

This implies that $\left(\widetilde{V}_t M_t + i\widetilde{V}_t(\Psi'_t(u) - \Psi'_t(0))\right)_t$ is an $(\mathcal{F}_t)$-martingale. Then by integration by parts,

$$\widetilde{V}_t(\Psi'_t(u) - \Psi'_t(0)) = \int_0^t \widetilde{V_s}(\Psi'_{ds}(u) - \Psi'_{ds}(0)) + \int_0^t (\Psi'_s(u) - \Psi'_s(0))d\widetilde{V_s} \ .$$

The second integral term of the right-hand side being a martingale, it follows that

$$\left\langle \widetilde{V}, M\right\rangle_t = -i\int_0^t \widetilde{V_s}(\Psi'_{ds}(u) - \Psi'_{ds}(0)) \ .$$

and so

$$\langle V, M\rangle_t = -i\int_0^t V_s(\Psi'_{ds}(u) - \Psi'_{ds}(0)) \ . \tag{3.25}$$

□

4. We continue the proof of the Proposition 3.29. For given (Z_t) we have

$$\left\langle \int_0^t ZdM, M \right\rangle_t = \int_0^t Z_{s-} d\left\langle M \right\rangle_s = -\int_0^t Z_s \Psi''_{ds}(0) \ .$$

5. We want to identify

$$-\int_0^t Z_s \Psi''_{ds}(0) = -i \int_0^t V_s(\Psi'_{ds}(u) - \Psi'_{ds}(0)) \ .$$

   This naturally leads to

$$Z_s = i\frac{d(\Psi'_s(u) - \Psi'_s(0))}{d\Psi''_s(0)} V_{s-} \ . \tag{3.26}$$

6. Finally, we obtain the general case, for general finite signed measure μ, similarly to the proof of Theorem 4.23 (in the sequel) in the case of exponential of PII processes. The use of Fubini's theorem is essential.

□

Example 3.32. *We take $X = M = W$ the classical Wiener process. We have $\Psi_s(u) = -\frac{u^2 s}{2}$ so that $\Psi'_s(u) = -us$ and $\Psi''_s(u) = -s$. So $Z_s = iuV_s$. We recall that*

$$V_s = \mathbb{E}[\exp(iuW_T)|\mathcal{F}_s] = \exp(iuW_s)\exp\left(-u^2\frac{T-s}{2}\right) \ .$$

In particular, $V_0 = \exp(-\frac{u^2T}{2})$ and so

$$\exp(iuW_T) = i\int_0^T u\exp(iuW_s)\exp\left(-u^2\frac{T-s}{2}\right)dW_s + \exp(-\frac{u^2T}{2}).$$

In fact that expression is classical and it can be derived from Clark-Ocone formula.

3.6.3 Explicit Föllmer-Schweizer decomposition

We introduce a quantity which will be useful in the sequel. For $t \in [0,T], u \in \mathbb{R}$ we set

$$\eta(u,t) = \int_0^t \frac{d(\Psi'_s(u) - \Psi_s'(0))}{d(\Psi''_s(0))} \Psi'_{ds}(0) \ . \tag{3.27}$$

Remark 3.33. *1. η is defined unambiguously since $d\left(\Psi'_t(u) - \Psi'_t(0)\right)$ is absolutely continuous with respect to $d\Psi''_t(0)$.*

2. η is well-defined, because for any $u \in \mathbb{R}$,

$$\eta(u,t) = \int_0^t \frac{d(\Psi'_s(u) - \Psi_s'(0))}{d(\Psi''_s(0))}\frac{d(\Psi_s'(0))}{d(\Psi''_s(0))} d\Psi''_{ds}(0)$$

is bounded by Cauchy-Schwarz, taking into account Assumption 3 point 2.

We are now able to evaluate the FS decomposition of $H = f(X_T)$ where f is given by (4.27).
We introduce now a supplementary hypothesis.

Assumption 5. *The quantity*

$$\sup_{u\in \mathrm{supp}\mu, t\in[0,T]} (\mathrm{Re}(\eta(\mathrm{u},\mathrm{t})) < \infty \ .$$

Theorem 3.34. *Under the assumptions of Proposition 3.29 and Assumption 5, the FS decomposition of H is the following*

$$H_t = H_0 + \int_0^t \xi_s dX_s + L_t \quad \textit{with} \quad H_T = H \tag{3.28}$$

and

$$\begin{aligned} H_t &= \int_{\mathbb{R}} H(u)_t \mu(du) \ , \\ \xi_t &= \int_{\mathbb{R}} \xi(u)_t \mu(du) \ , \end{aligned} \tag{3.29}$$

where

$$\begin{aligned} \xi(u)_t &= i\frac{d(\Psi_t^{'}(u) - \Psi_t^{'}(0))}{d\Psi_t^{''}(0)} H(u)_{t-} \ , \\ H(u)_t &= \exp\left\{\eta(u,T) - \eta(u,t) + \Psi_T(u) - \Psi_t(u)\right\} e^{iuX_t} \ . \end{aligned} \tag{3.30}$$

Proof. Using Fubini's theorem, we reduce the problem to show that

$$H(u)_t = H(u)_0 + \int_0^t \xi(u)_s dX_s + L(u)_t \quad \text{with} \quad H(u)_T = \exp(iuX_T) \ ,$$

for fixed $u \in \mathbb{R}$ where $L(u)$ is a square integrable martingale and $\langle L(u), M\rangle = 0$, where M is the martingale part of the special semimartingale X. Notice that by equation (3.30),

$$H(u)_t = H(u)_0 + e^{\int_t^T \eta(u,ds)} V(u)_t \quad \text{with} \quad V(u)_t = \exp(iuX_t + \Psi_T(u) - \Psi_t(u)) \ .$$

Integrating by parts, gives

$$H(u)_t = H(u)_0 - \int_0^t e^{\int_r^T \eta(u,ds)} V(u)_r \eta(u,dr) + \int_0^t e^{\int_r^T \eta(u,ds)} dV(u)_r \ .$$

We denote again by $Z(u)$ the expression provided by (3.26). We recall that

$$dV(u)_r = Z(u)_r dM_r + dO(u)_r = Z(u)_r(dX_r - dA_r) + dO(u)_r \ ,$$

where A is given by (3.21) and O is a square integrable martingale strongly orthogonal to M (i.e. $\langle M, O\rangle_\cdot = 0$).

$$H(u)_t = H(u)_0 + L(u)_t + \int_0^t e^{\int_r^T \eta(u,ds)} Z(u)_r dX_r - \int_0^t e^{\int_r^T \eta(u,ds)} Z(u)_r(-i\Psi_{dr}^{'}(0)) - \int_0^t e^{\int_r^T \eta(u,ds)} V(u)_r \eta(u,dr) \ ,$$

where

$$L(u)_t = \int_0^t e^{\int_r^T \eta(u,ds)} dO(u)_r \ ,$$

is a martingale strongly orthogonal to M. To conclude, we need to choose η so that

$$\int_0^t Z(u)_r e^{\int_r^T \eta(u,ds)}(-i\Psi_{dr}^{'}(0)) = \int_0^t e^{\int_r^T \eta(u,ds)} V(u)_r \eta(u,dr)) \ .$$

This requires

$$\eta(u,dr) = \frac{d(\Psi_r^{'}(u) - \Psi_r{'}(0))}{d(\Psi_r^{''}(0))}\Psi_{dr}^{'}(0) \ .$$

So we define η as in (3.27). $\square$

3.6.4 The Lévy case

Let X be a square integrable Lévy process, with characteristic function $\exp(\Psi(u)t)$. In particular, Ψ is of class $C^2(\mathbb{R})$. We have

$$\frac{d\Psi_t^{'}(u)}{d\Psi_t^{''}(0)} = \frac{\Psi^{'}(u)}{\Psi^{''}(0)} \quad \text{and} \quad \eta(u,t) = t\frac{\Psi^{'}(u) - \Psi^{'}(0)}{\Psi^{''}(0)}\Psi^{'}(0) \ .$$

We remark that Assumptions 2 is verified. Concerning Assumption 3, point 1. is trivial; point 2. is verified because $K(u) = \left|\frac{\Psi^{'}(u)}{\Psi^{''}(u)}\right|^2 (-T\Psi^{''}(u))$. On the other hand Assumption 5 is verified if

$$Re\left(\frac{\Psi^{'}(u)\Psi^{'}(0)}{\Psi^{''}(0)}\right) < \infty \ . \tag{3.31}$$

Since $\Psi^{'}(0) = i\mathbb{E}[X_1]$ and $\Psi^{''}(0) < 0$, (3.31) is fulfilled if

$$\mathbb{E}[X_1]Im(\Psi^{'}(u)) > -\infty \ . \tag{3.32}$$

Assumption 4 is verified if

$$\int_{\mathbb{R}} \left|\Psi^{'}(u)\right|^2 d|\mu(u)| < \infty \ . \tag{3.33}$$

Example 3.35. *We start with the toy model $X_t = \sigma W_t + mt$, $\sigma, m \in \mathbb{R}$. We have $\Psi(u) = -\frac{u^2}{2}\sigma^2 + imu$ so $\Psi^{'}(u) = -u\sigma^2 + im$ and $Im(\Psi^{'}(u)) = m$. Condition (3.32) is always verified and Condition (3.33) is verified if*

$$\int_{\mathbb{R}} u^2 d\mu(u) < \infty \ . \tag{3.34}$$

(3.34) is verified for instance in the example of the beginning of subsection 3.7 since $\int_{-\infty}^{c} u^2 e^u du < \infty$ for $c > 0$.

Remark 3.36. *In the examples introduced in Remark 3.21, we can show that $u \mapsto \left|\Psi^{'}(u)\right|$ is bounded and so (3.32) and (3.33) are always verified for the following reasons.*

1. *<u>Poisson case</u>*
   We have $\Psi^{'}(u) = i\lambda e^{iu}$.

2. *<u>NIG case</u>*
   We have $\Psi'(u) = i\mu + i\delta\,(\beta + iu)\left(\alpha^2 - (\beta + iu)^2\right)^{-\frac{1}{2}}$. Now

   $$|\Psi'(u)| \leq 2\left(|\mu|^2 + 2\delta\sqrt{\frac{\beta^2 + u^2}{(\alpha^2 - \beta^2 + u^2)^2 + 4u^2\beta^2}}\right) \ .$$

   Since $|\alpha| > |\beta|$, $u \mapsto |\Psi'(u)|$ is bounded.

3. *Variance Gamma case*
We have $\Psi^{'}(u) = i\mu - \frac{u - i\beta}{\alpha - iu\beta + \frac{u^2}{2}}$ *Clearly* $|\Psi'(u)|$ *is again bounded.*

In conclusion, we can apply Theorem 3.34 and we obtain

$$\begin{aligned} V(u)_t &= \exp(iuX_t + (T-t)\Psi(u)) \ , \\ H(u)_t &= \exp\left((T-t)\Psi(u) + \eta(u,T) - \eta(u,t)\right) e^{iuX_t} \ , \\ \xi(u)_t &= H_t(u) i \frac{\Psi^{'}(u) - \Psi^{'}(0)}{\Psi''(0)} \ . \end{aligned}$$

3.7 Representation of some contingent claims by Fourier transforms

In general, it is not possible to find a Fourier representation, of the form (3.20), for a given payoff function which is not necessarily bounded or integrable. Hence, it can be more convenient to use the bilateral Laplace transform that allows an extended domain of definition including non integrable functions. We refer to [17], [41] and more recently [20] for such characterizations of payoff functions. This will be done in the next section. However, to illustrate the results of this section restricted to payoff functions represented as classical Fourier transforms, we give here two simple examples of such representation extracted from [20]:

1. A variant of the digital option is the so-called *asset-or-nothing digital*, where the option holder receives one unit of the *asset*, instead of *currency*, depending on wether the underlying reaches some barrier or not. Hence, the payoff of the asset-or-nothing digital put with barrier is
$$f(x) = e^x \mathbf{1}_{e^x < B} \quad \text{and} \quad \hat{f}(u) = \int_{\mathbb{R}} e^{iux} f(x)\, dx = \frac{B^{1+iu}}{1+iu} \ .$$
2. The payoff of a *self quanto put option* with strike K is
$$f(x) = e^x (K - e^x)_+ \quad \text{and} \quad \hat{f}(u) = \int_{\mathbb{R}} e^{iux} f(x)\, dx = \frac{K^{2+iu}}{(1+iu)(2+iu)} \ .$$

In both cases the measure μ is finite.

4 Föllmer Schweizer decomposition for exponential of PII processes

In this section, we consisder the case of exponential of PII corresponding to geometric models (such as the Black-Scholes model) much more used in finance than arithmetic models (such as the Bachelier model). The aim of this section is to generalize the results of [31] to the case of PII with possibly non stationary increments. Here again, this generalization is motivated by applications to energy derivatives where forward prices show a volatility term structure that requires the use of models with non stationary increments.

4.1 A reference variance measure

We come back to the main optimization problem which was formulated in Section 2. We assume that the process S is the discounted price of the non-dividend paying stock which is supposed to be of the form,

$$S_t = s_0 \exp(X_t) \ , \quad \text{for all } t \in [0,T] \ ,$$

where s_0 is a strictly positive constant and X is a semimartingale process with independent increments (PII), in the sense of Definition 3.1, but not necessarily with stationary increments.

Remark 4.1. *Let $\gamma \in \mathbb{R}^*$,*

1. $\mathbb{E}[\exp(\gamma(X_t - X_s))] > 0$, *since* $X_t - X_s > -\infty$ *a.s.*

2. $\exp(\gamma(X_t - X_s))$ *has a strictly positive variance if* $(X_t - X_s)$ *is non-deterministic.*

We introduce a new function that will be useful in the sequel.

Definition 4.2. *For any $t \in [0, T]$, let ρ_t denote the complex valued function such that for all $z, y \in D$*

$$\rho_t(z, y) = \kappa_t(z + y) - \kappa_t(z) - \kappa_t(y) \ . \tag{4.1}$$

For all $z \in D$, then $\bar{z} \in D$ and $\rho_t(z, \bar{z})$ is well defined. To shorten notations ρ_t will also denote the real valued function defined on D such that,

$$\rho_t(z) = \rho_t(z, \bar{z}) = \kappa_t(2Re(z)) - 2Re(\kappa_t(z)) \ . \tag{4.2}$$

Notice that the last equality results from Remark 2.3.

An important technical lemma follows below.

Lemma 4.3. *Let $z \in D$, with $z \neq 0$, then, $t \mapsto \rho_t(z)$ is strictly increasing if and only if X has no deterministic increments.*

Proof. It is enough to show that X has no deterministic increment if and only if for any $0 \leq s < t \leq T$, the following quantity is positive,

$$\rho_t(z) - \rho_s(z) = \left[\kappa_t\big(2Re(z)\big) - \kappa_s\big(2Re(z)\big)\right] - 2Re\big(\kappa_t(z) - \kappa_s(z)\big) \ . \tag{4.3}$$

By Remark 3.11, for all $z \in D$, we have

$$\exp[\kappa_t(z) - \kappa_s(z)] = \mathbb{E}[\exp(z\Delta_s^t)] \ , \quad \text{where } \Delta_s^t = X_t - X_s \ .$$

Applying this property and Remark 2.3 1., to the exponential of the first term on the right-hand side of (4.3) yields

$$\begin{aligned} \exp\left[\kappa_t\big(2Re(z)\big) - \kappa_s\big(2Re(z)\big)\right] &= \mathbb{E}[\exp(2Re(z)\Delta_s^t)] = \mathbb{E}[\exp((z + \bar{z})\Delta_s^t)] \\ &= \mathbb{E}[\big|\exp(z\Delta_s^t)\big|^2] \ . \end{aligned}$$

Similarly, for the exponential of the second term on the right-hand side difference of (4.3), one gets

$$\exp\left[2Re\big(\kappa_t(z) - \kappa_s(z)\big)\right] = \exp\left[\big(\kappa_t(z) - \kappa_s(z)\big) + \overline{\big(\kappa_t(z) - \kappa_s(z)\big)}\right] = \big|\mathbb{E}[\exp(z\Delta_s^t)]\big|^2 \ .$$

Hence taking the exponential of $\rho_t(z) - \rho_s(z)$ yields

$$\begin{aligned} \exp[\rho_t(z) - \rho_s(z)] - 1 &= \frac{\mathbb{E}[|\exp(z\Delta_s^t)|^2]}{|\mathbb{E}[\exp(z\Delta_s^t)]|^2} - 1 \ , \\ &= \frac{\mathbb{E}[|\Gamma_s^t(z)|^2]}{|\mathbb{E}[\Gamma_s^t(z)]|^2} - 1 \ , \quad \text{where } \Gamma_s^t(z) = \exp(z\Delta_s^t) \ , \\ &= \frac{Var\left[Re\big(\Gamma_s^t(z)\big)\right] + Var\left[Im\big(\Gamma_s^t(z)\big)\right]}{|\mathbb{E}[\Gamma_s^t(z)]|^2} \ . \end{aligned} \tag{4.4}$$

- If X has a deterministic increment $\Delta_s^t = X_t - X_s$, then $\Gamma_s^t(z)$ is again deterministic and (4.4) vanishes and hence $t \rightarrow \rho_t(z)$ is not strictly increasing.

- If X has never deterministic increments, then the nominator is never zero, otherwise $Re\big(\Gamma_s^t(z)\big)$, $Im\big(\Gamma_s^t(z)\big)$ and therefore $\Gamma_s^t(z)$ would be deterministic.

$\square$

From now on, we will always suppose the following assumption.

Assumption 6. *1. (X_t) has no deterministic increments.*

2. $2 \in D$.

Remark 4.4. *1. In particular for $\gamma \in D$, $\gamma \neq 0$, the function $t \mapsto \rho_t(\gamma)$ is strictly increasing.*

2. If $z = 1$, (4.4) equals $\dfrac{Var\big(\exp(\Delta_s^t)\big)}{\big(\mathbb{E}[\exp(\Delta_s^t)]\big)^2}$, which is a mean-variance quantity.

We continue with a simple observation.

Lemma 4.5. *Let I be a compact real interval included in D.*

$$\sup_{\gamma \in I} \sup_{t \leq T} \mathbb{E}[S_t^\gamma] < \infty \ .$$

Proof. Let $t \in [0, T]$ and $\gamma \in I$, we have

$$\mathbb{E}[S_t^\gamma] = s_0^\gamma \exp\{\kappa_t(\gamma)\} \leq \max(1, s_0^{\sup I}) \exp\big(\sup_{t \leq T, \gamma \in I} |\kappa_t(\gamma)|\big) \ .$$

since κ is continuous. $\square$

We state now a result that will help us to show that $\kappa_{dt}(z)$ is absolutely continuous with respect to $\rho_{dt}(1) = \kappa_{dt}(2) - 2\kappa_{dt}(1)$.

Lemma 4.6. *We consider two positive finite non-atomic Borel measures on $E \subset \mathbb{R}^n$, μ and ν. We suppose the following:*

1. $\mu \ll \nu$;

2. $\mu(I) \neq 0$ for every open ball of E.

Then $\dfrac{d\mu}{d\nu} := h \neq 0$ ν a.e. In particular μ and ν are equivalent.

Proof. We consider the Borel set

$$B = \{x \in E | h(x) = 0\} \ .$$

We want to prove that $\nu(B) = 0$. So we suppose that there exists a constant $c > 0$ such that $\nu(B) = c > 0$ and another constant ϵ such that $0 < \epsilon < c$. Since ν is a Radon measure, there are compact subsets K_ϵ and $K_{\frac{\epsilon}{2}}$ of E such that

$$K_\epsilon \subset K_{\frac{\epsilon}{2}} \subset B \quad \text{and} \quad \nu(B - K_\epsilon) < \epsilon \ , \quad \nu(B - K_{\frac{\epsilon}{2}}) < \frac{\epsilon}{2} \ .$$

Setting $\epsilon = \frac{c}{2}$, we have

$$\nu(K_\epsilon) > \frac{c}{2} \quad \text{and} \quad \nu(K_{\frac{\epsilon}{2}}) > \frac{3c}{4} \ .$$

By Urysohn lemma, there is a continuous function $\varphi : E \to \mathbb{R}$ such that, $0 \le \varphi \le 1$ with

$$\varphi = 1 \text{ on } K_\epsilon \quad \text{and} \quad \varphi = 0 \text{ on } K^c_{\frac{\epsilon}{2}} \ .$$

Now

$$\int_E \varphi(x)\nu(dx) \ge \nu(K_\epsilon) > \frac{c}{2} > 0 \ .$$

By continuity of φ there is an open set $O \subset E$ with $\varphi(x) > 0$ for $x \in O$. Clearly $O \subset K_{\frac{\epsilon}{2}} \subset B$; since O is relatively compact, it is a countable union of balls, and so B contains a ball I. The fact that $h = 0$ on I implies $\mu(I) = 0$ and this contradicts Hypothesis 2. of the statement. Hence the result follows.

□

Remark 4.7. *From now on, in this section, $d\rho_t = \rho_{dt}$ will denote the measure*

$$d\rho_t = \rho_{dt}(1) = d(\kappa_t(2) - 2\kappa_t(1)) \ . \tag{4.5}$$

According to Remark 4.4 1., it is a positive measure which is strictly positive on each interval. This measure will play a fundamental role.

Remark 4.8. *1. If $E = [0, T]$, then point 2. of Lemma 4.6 becomes $\mu(I) \neq 0$ for every open interval $I \subset [0, T]$.*

2. The result holds for every normal metric locally connected space E, provided ν are Radon measures.

Proposition 4.9. *Under Assumption 6*

$$d(\kappa_t(z)) \ll d\rho_t \ , \quad \textit{for all } z \in D \ . \tag{4.6}$$

Proof. We apply Lemma 4.6, with $d\mu = d\rho_t$ and $d\nu = da_t$. Indeed, Corollary 3.10 implies Condition 1. of Lemma 4.6 and Lemma 4.3 implies Condition 2. of Lemma 4.6. Therefore, da_t is equivalent to $d\rho_t$. □

Remark 4.10. *Notice that this result also holds with $d\rho_t(y)$ instead of $d\rho_t = d\rho_t(1)$, for any $y \in D$ such that $Re(y) \neq 0$.*

4.2 On some semimartingale decompositions and covariations

Proposition 4.11. *Let $y, z \in D$ such that $y + z, 2Re(y), Re(y) + 1, 2Re(z)$ and $Re(z) + 1 \in D$. Then S^z is a special semimartingale whose canonical decomposition $S^z_t = M(z)_t + A(z)_t$ satisfies*

$$A(z)_t = \int_0^t S^z_{u-}\kappa_{du}(z) \ , \quad \langle M(y), M(z)\rangle_t = \int_0^t S^{y+z}_{u-}\rho_{du}(z, y) \ , \quad M(z)_0 = s^z_0, \tag{4.7}$$

where $d\rho_u(z)$ is defined by equation (4.2). In particular we have the following:

1. $\langle M(z), M\rangle_t = \int_0^t S^{z+1}_{u-}\rho_{du}(z, 1)$
2. $\langle M(z), M(\bar{z})\rangle_t = \int_0^t S^{2Re(z)}_{u-}\rho_{du}(z) \ .$

Proof. For simplicity, we will only treat the case $y = 1$ in (4.7), i.e. statement 1. The general case will follow similarly. By Remark 3.11, $N(z)_t := e^{-\kappa_t(z)} S_t^z$ is a martingale. Integration by parts yields

$$S_t^z = e^{\kappa_t(z)} N(z)_t = M(z)_t + A(z)_t \quad \text{with} \quad M_0(z) = s_0^z, \quad A(z)_t = \int_0^t S_{u-}^z \kappa_{du}(z) \quad \text{and}$$

$$\begin{aligned}
[M(z), M]_t &= [S^z, S]_t \ , \\
&= S_t^z S_t^1 - S_0^z S_0^1 - \int_0^t S_{s-}^z dS_s^1 \int_0^t S_{s-}^1 dS_s^z \ , \\
&= S_t^z S_t^1 - S_0^z S_0^1 - \int_0^t S_{s-}^z dM_s - \int_0^t S_{s-}^z dA_s - \int_0^t S_{s-}^1 dM(z)_s - \int_0^t S_{s-}^1 dA(z)_s \ , \\
&= S_t^{z+1} - S_0^{z+1} - \int_0^t S_{s-}^z dM_s - \int_0^t S_{s-}^1 dM(z)_s - \int_0^t S_{s-}^{z+1} \kappa_{ds}(1) - \int_0^t S_{s-}^{z+1} \kappa_{ds}(z) \ , \\
&= M(z+1)_t - \int_0^t S_{s-}^z dM_s - \int_0^t S_{s-} dM(z)_s + \int_0^t (\kappa_{ds}(z+1) - \kappa_{ds}(z) - \kappa_{ds}(1)) S_{s-}^{z+1} \ .
\end{aligned}$$

Note that the first three terms on the right-hand side are local martingales. Since $\langle M(z), M \rangle_t$ is the predictable part of finite variation of the special semimartingale $M(z)M$, equation (1) follows.

□

Remark 4.12. *Lemma 4.5 implies that* $\mathbb{E}[| \langle M(y), M(z) \rangle |] < \infty$ *and so* $M(z)$ *is a square integrable martingale for any* $z \in D$ *such that* $2Re(z), Re(z) + 1 \in D$.

4.3 On the Structure Condition

If we apply Proposition 4.11 with $y = z = 1$, we obtain $S = M + A$ where M is a martingale and

$$A_t = \int_0^t S_{u-} \kappa_{du}(1) \ , \tag{4.8}$$

and

$$\langle M, M \rangle_t = \int_0^t S_{u-}^2 (\kappa_{du}(2) - 2\kappa_{du}(1)) = \int_0^t S_{u-}^2 \rho_{du} \ . \tag{4.9}$$

At this point, the aim is to exhibit a predictable $\mathbb{R}$-valued process α such that

1. $A_t = \int_0^t \alpha_s d \langle M \rangle_s$;
2. $K_T = \int_0^T \alpha_s^2 d \langle M \rangle_s$ is bounded.

In that case, according Theorem 2.22, there will exist a unique FS decomposition for any $H \in \mathcal{L}^2$ and so the minimization problem (2.1) will have a unique solution, by Theorem 2.25.

Proposition 4.13. *Under Assumption 6, we have*

$$A_t = \int_0^t \alpha_s d \langle M \rangle_s \ , \tag{4.10}$$

where α is given by

$$\alpha_u := \frac{\lambda_u}{S_{u-}} \quad \textit{with} \quad \lambda_u := \frac{d\kappa_u(1)}{d\rho_u} \ , \quad \textit{for all } u \in [0,T] \ . \tag{4.11}$$

Moreover the MVT process is given by

$$K_t = \int_0^t \left(\frac{d(\kappa_u(1))}{d\rho_u} \right)^2 d\rho_u \ . \tag{4.12}$$

Corollary 4.14. *Under Assumption 6, the structure condition (SC) is verified if and only if*

$$K_T = \int_0^T \left(\frac{d(\kappa_u(1))}{d\rho_u} \right)^2 d\rho_u < \infty \ .$$

In particular, (K_t) is deterministic therefore bounded.

Proof of Proposition 4.13. By Proposition 4.9, $d\kappa_t(1)$ is absolutely continuous with respect to $d\rho_t$. Setting α_u as in (4.11), relation (4.12) follows from Proposition 4.11, expressing $K_t = \int_0^t \alpha_u^2 d\left\langle M \right\rangle_u$. □

Lemma 4.15. *The space Θ is constituted by all predictable processes v such that*

$$\mathbb{E}[\int_0^T v_t^2 S_{t-}^2 d\rho_t] < \infty \ .$$

Proof. According to Proposition 2.14, the fact that K is bounded and S satisfies (SC), then $v \in \Theta$ holds if and only if v is predictable and $\mathbb{E}[\int_0^T v_t^2 d\left\langle M,M \right\rangle_t] < \infty$. Since

$$\left\langle M,M \right\rangle_t = \int_0^t S_{s-}^2 d\rho_s \ ,$$

the assertion follows. □

4.4 Explicit Föllmer-Schweizer decomposition

We denote by $\mathcal{D}$ the set of $z \in D$ such that

$$\int_0^T \left| \frac{d\kappa_u(z)}{d\rho_u} \right|^2 d\rho_u < \infty. \tag{4.13}$$

From now on, we formulate another assumption which will be in force for the whole section.

Assumption 7. $1 \in \mathcal{D}$.

Remark 4.16. *1. Because of Proposition 4.9, $\dfrac{d\kappa_t(z)}{d\rho_t}$ exists for every $z \in D$.*

2. Assumption 7 implies that K is uniformly bounded.

The proposition below will constitute an important step for determining the FS decomposition of the contingent claim $H = f(S_T)$ for a significant class of functions f, see Section 4.5.

Proposition 4.17. *Let $z \in \mathcal{D}$ with $z+1 \in \mathcal{D}$ and $2Re(z) \in D$.*

1. $S_T^z \in \mathcal{L}^2(\Omega, \mathcal{F}_T)$.

2. *We suppose Assumptions 6 and 7 and we define*

$$\gamma(z,t) := \frac{d(\rho_t(z,1))}{d\rho_t}, \; t \in [0,T]. \tag{4.14}$$

$\int_0^T |\gamma(z,t)|^2 \rho_{dt} < \infty$ *and*

$$\begin{aligned} \eta(z,t) &:= \kappa_t(z) - \int_0^t \gamma(z,s)\kappa_{ds}(1) \\ &= \kappa_t(z) - \int_0^t \gamma(z,s)\frac{d\kappa_s(1)}{d\rho_s}\rho_{ds} \end{aligned} \tag{4.15}$$

is well-defined and $\eta(z,\cdot)$ is absolutely continuous with respect to ρ_{ds} and therefore bounded.

3. *Under the same assumptions $H(z) = S_T^z$ admits a FS decomposition $H(z) = H(z)_0 + \int_0^T \xi(z)_t dS_t + L(z)_T$ where*

$$H(z)_t := e^{\int_t^T \eta(z,ds)} S_t^z \;, \tag{4.16}$$

$$\xi(z)_t := \gamma(z,t) e^{\int_t^T \eta(z,ds)} S_{t-}^{z-1} \;, \tag{4.17}$$

$$L(z)_t := H(z)_t - H(z)_0 - \int_0^t \xi(z)_u dS_u \;. \tag{4.18}$$

Proof. 1. is a consequence of Lemma 4.5.

2. $\gamma(z,\cdot)$ is square integrable because Assumption 7 and $z, z+1 \in \mathcal{D}$. Moreover η is well-defined since

$$\left(\int_0^T |\gamma(z,s)| \left|\frac{d\kappa_s(1)}{d\rho_s}\right| \rho_{ds}\right)^2 \leq \int_0^T |\gamma(z,s)|^2 \rho_{ds} \int_0^T |\frac{d\kappa_s(1)}{d\rho_s}|^2 \rho_{ds}. \tag{4.19}$$

3. In order to prove that (4.16),(4.17) and (4.18) constitute the FS decomposition of $H(z)$, taking into account Remark 2.19 we need to show that

   (a) $H(z)_0$ is $\mathcal{F}_0$-measurable,

   (b) $\langle L(z), M\rangle = 0$,

   (c) $\xi(z) \in \Theta$,

   (d) $L(z)$ is a square integrable martingale.

   Point (a) is obvious. Partial integration and point 1 of Proposition 4.11 yield

$$\begin{aligned} H(z)_t &= H(z)_0 + \int_0^t e^{\int_u^T \eta(z,ds)} dS_u^z + \int_0^t S_u^z d(e^{\int_u^T \eta(z,ds)}) \\ &= H(z)_0 + \int_0^t e^{\int_u^T \eta(z,ds)} dM(z)_u + \int_0^t e^{\int_u^T \eta(z,ds)} dA(z)_u + \int_0^t S_u^z d(e^{\int_u^T \eta(z,ds)}) \\ &= H(z)_0 + \int_0^t e^{\int_u^T \eta(z,ds)} dM(z)_u + \int_0^t e^{\int_u^T \eta(z,ds)} dA(z)_u - \int_0^t e^{\int_u^T \eta(z,ds)} S_u^z \eta(z,du) \\ &= H(z)_0 + \int_0^t e^{\int_u^T \eta(z,ds)} dM(z)_u - \int_0^t e^{\int_u^T \eta(z,ds)} S_u^z \eta(z,du) + \int_0^t e^{\int_u^T \eta(z,ds)} S_{u-}^z \kappa_{du}(z) \;. \end{aligned}$$

On the other hand

$$
\begin{aligned}
\int_0^t \xi(z)_u dS_u &= \int_0^t \xi(z)_u dM_u + \int_0^t \xi(z)_u dA_u \ , \\
&= \int_0^t \xi(z)_u dM_u + \int_0^t \xi(z)_u S_{u-}\kappa_{du}(1) \ , \\
&= \int_0^t \xi(z)_u dM_u + \int_0^t \gamma(z,u) e^{\int_u^T \eta(z,ds)} S^z_{u-}\kappa_{du}(1) \ .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
L(z)_t &= H(z)_t - H(z)_0 - \int_0^t \xi(z)_u dS_u \ , \\
&= \int_0^t e^{\int_u^T \eta(z,ds)} dM(z)_u - \int_0^t e^{\int_u^T \eta(z,ds)} S^z_u \eta(z,du) + \int_0^t e^{\int_u^T \eta(z,ds)} S^z_{u-}\kappa_{du}(z) \\
&- \int_0^t \xi(z)_u dM_u - \int_0^t \gamma(z,u) e^{\int_u^T \eta(z,ds)} S^z_{u-}\kappa_{du}(1) \ , \\
&= \int_0^t e^{\int_u^T \eta(z,ds)} dM(z)_u - \int_0^t \xi(z)_u dM_u \\
&+ \int_0^t e^{\int_u^T \eta(z,ds)} S^z_{u-}[\kappa_{du}(z) - \eta(z,du) - \gamma(z,u)\kappa_{du}(1)].
\end{aligned}
$$

Then, by definition of η in (4.15), $\eta(z,du) = \kappa_{du}(z) - \gamma(z,u)\kappa_{du}(1)$, hence,

$$
L(z)_t = \int_0^t e^{\int_u^T \eta(z,ds)} dM(z)_u - \int_0^t \xi(z)_u dM_u \ , \tag{4.20}
$$

which implies that $L(z)$ is a local martingale.
From point 1 of Proposition 4.11, it follows that

$$
\begin{aligned}
\langle L(z), M\rangle_t &= \int_0^t e^{\int_u^T \eta(z,ds)} d\langle M(z), M\rangle_u - \int_0^t \xi(z)_u d\langle M, M\rangle_u \ , \\
&= \int_0^t e^{\int_u^T \eta(z,ds)} S^{z+1}_{u-}\rho_{du}(z,1) - \int_0^t \xi(z)_u S^2_{u-}\rho_{du} \ , \\
&= \int_0^t e^{\int_u^T \eta(z,ds)} S^{z+1}_{u-}\rho_{du}(z,1) - \int_0^t \gamma(z,u) e^{\int_u^T \eta(z,ds)} S^{z+1}_{u-}\rho_{du} \ .
\end{aligned}
$$

Consequently,

$$
\langle L(z), M\rangle_t = \int_0^t e^{\int_u^T \eta(z,ds)} S^{z+1}_{u-}[\rho_{du}(z,1) - \gamma(z,u)\rho_{du}] \ .
$$

Then by definition of γ in (4.14), $\rho_{dt}(z,1) = \gamma(z,t)\rho_{dt}$, which yields,

$$
\langle L(z), M\rangle_t = 0 \ . \tag{4.21}
$$

Consequently, point (b) follows. To continue the proof of this proposition we need the lemma below.

Lemma 4.18. *For all $z \in \mathbb{C}$ as in Proposition 4.17, $d\rho_t$ a.e. we have*

1. $\overline{\gamma(z,t)} = \gamma(\bar{z},t)$;
2. $\overline{\eta(z,t)} = \eta(\bar{z},t)$.

Proof. Using Remark 2.3 1) we observe $\bar{z}, \bar{z}+1 \in \mathcal{D}$.

1. By definition of γ in (4.14), $\gamma(z,t)\rho_{dt} = \rho_{dt}(z,1)$. Then, taking the complex conjugate of the integral from 0 to t and using Remark 2.3.1 yields,

$$\begin{aligned}
\overline{\int_0^t \gamma(z,s)\rho_{ds}} &= \int_0^t \overline{\gamma(z,s)}\rho_{ds} \ , \\
&= \overline{\rho_t(z,1)} = \overline{\kappa_t(z+1) - \kappa_t(z) - \kappa_t(1)} \ , \\
&= = \kappa_t(\bar{z}+1) - \kappa_t(\bar{z}) - \kappa_t(1) = \rho_t(\bar{z},1) \ , \\
&= \int_0^t \gamma(\bar{z},s)\rho_{ds} \ .
\end{aligned}$$

2. By definition of η in (4.15), $\eta(z,t) = \kappa_t(z) - \int_0^t \gamma(z,u)\kappa_{du}(1)$, so taking the complex conjugate,

$$\begin{aligned}
\overline{\eta(z,t)} &= \kappa_t(\bar{z}) - \int_0^t \overline{\gamma(z,s)}\kappa_{ds}(1) \ , \\
&= \kappa_t(\bar{z}) - \int_0^t \gamma(\bar{z},s)\kappa_{ds}(1) \ , \\
&= \eta(\bar{z},t) \ .
\end{aligned}$$

□

We continue with the proof of point 3. of Proposition 4.17. It remains to prove that $L(z)$ is a square-integrable martingale for all $z \in D$ and that $Re(\xi(z))$ and $Im(\xi(z))$ are in Θ. (4.20) says that

$$L(z)_t = \int_0^t e^{\int_s^T \eta(z,du)} dM_s(z) - \int_0^t \xi(z)_s dM_s \ .$$

By Proposition 4.11 and Lemma 4.18, it follows

$$\overline{L(z)_t} = L(\bar{z})_t \ , \tag{4.22}$$

hence,

$$\begin{aligned}
\left\langle L(z), \overline{L(z)} \right\rangle_t &= \langle L(z), L(\bar{z}) \rangle_t \ , \\
&= \left\langle L(z), \int_0^\cdot e^{\int_s^T \eta(\bar{z},du)} dM_s(\bar{z}) \right\rangle_t \ , \\
&= \int_0^t e^{\int_s^T \eta(z,du)} e^{\int_s^T \eta(\bar{z},du)} d\langle M(z), M(\bar{z}) \rangle_s \\
&\quad - \int_0^t \xi(z)_s e^{\int_s^T \eta(\bar{z},du)} d\langle M, M(\bar{z}) \rangle_s \ .
\end{aligned} \tag{4.23}$$

By Proposition 4.11 we have

$$\begin{aligned}\left\langle L(z),\overline{L(z)}\right\rangle_t &= \int_0^t e^{\int_s^T \eta(z,du)} e^{\int_s^T \eta(\bar z,du)} S_{s-}^{2Re(z)} \rho_{ds}(z) \\ &\quad - \int_0^t \xi(z)_s e^{\int_s^T \eta(\bar z,du)} S_{s-}^{1+\bar z} \rho_{ds}(\bar z,1)\ .\end{aligned}$$

Using Lemma 4.18 and expressions (4.14), and (4.17) of $\gamma(z,s)$ and $\xi(z)_s$, we have

$$\begin{aligned}\left\langle L(z),\overline{L(z)}\right\rangle_t &= \int_0^t e^{\int_s^T 2Re(\eta(z,du))} S_{s-}^{2Re(z)} \rho_{ds}(z) - \int_0^t \xi(z)_s e^{\int_s^T \eta(\bar z,du)} S_{s-}^{1+\bar z} \gamma(\bar z,s)\rho_{ds}(1)\ , \\ &= \int_0^t e^{\int_s^T 2Re(\eta(z,du))} S_{s-}^{2Re(z)} \rho_{ds}(z) - \int_0^t \gamma(z,s) e^{\int_s^T 2Re(\eta(z,du))} S_{s-}^{2Re(z)} \gamma(\bar z,s)\rho_{ds}\ , \\ &= \int_0^t e^{\int_s^T 2Re(\eta(z,du))} S_{s-}^{2Re(z)} \rho_{ds}(z) - \int_0^t e^{\int_s^T 2Re(\eta(z,du))} S_{s-}^{2Re(z)} |\gamma(z,s)|^2\rho_{ds}\ .\end{aligned}$$

Consequently

$$\left\langle L(z),\overline{L(z)}\right\rangle_t = \int_0^t e^{\int_s^T 2Re(\eta(z,du)})S_{s-}^{2Re(z)}[\rho_{ds}(z) - |\gamma(z,s)|^2\rho_{ds}]\ . \tag{4.24}$$

Then, point 2. implies

$$\begin{aligned}\int_0^T |\xi(z)_s|^2 S_{s-}^2 \rho_{ds} &= \int_0^T \xi(z)_s \xi(\bar z)_s S_{s-}^2\rho_{ds} \\ &= \int_0^T \gamma(z,s) e^{\int_s^T \eta(z,du)} S_{s-}^{z-1} \gamma(\bar z,s) e^{\int_s^T \eta(\bar z,du)} S_{s-}^{\bar z-1} S_{s-}^2 \rho_{ds}\ , \\ &= \int_0^T |\gamma(z,s)|^2 e^{\int_t^T 2Re(\eta(z,du))} S_{s-}^{2Re(z)} \rho_{ds}\ .\end{aligned} \tag{4.25}$$

Taking the expectation in (4.25), using again point 2., (4.14), (4.15) and Lemma 4.5, we obtain

$$\mathbb{E}\left[\left\langle L(z),\overline{L(z)}\right\rangle_T\right] < \infty\ . \tag{4.26}$$

Therefore, L is a square-integrable martingale. Similarly, (4.25) yields that $Re(\xi(z)) \in \Theta$ and $Im(\xi(z)) \in \Theta$. This concludes the proof of Proposition 4.17. □

4.5 FS decomposition of special contingent claims

Now, we will proceed to the FS decomposition of more general contingent claims. We consider now options of the type

$$H = f(S_T) \quad \text{with} \quad f(s) = \int_{\mathbb{C}} s^z \Pi(dz)\ , \tag{4.27}$$

where Π is a (finite) complex measure in the sense of Rudin [44], Section 6.1. An integral representation of some basic European calls can be found later.
We need now the new following assumption.

Assumption 8. *Let* $I_0 = \mathrm{supp}\Pi \cap \mathbb{R}$. *We denote* $I := [\inf I_0 \wedge 2\inf I_0, 2\sup I_0 \vee \sup I_0 + 1]$.

1. $\forall z \in \mathrm{supp}\Pi, \quad z, z+1 \in \mathcal{D}$.

2. $I \subset D$ *and* $\sup_{x\in I\cup\{1\}} \left\| \frac{d(\kappa_t(x))}{d\rho_t} \right\|_\infty < \infty$.

Remark 4.19. *1. Point 2. of Assumption 8 implies* $\sup_{z\in I+i\mathbb{R}} \|\kappa_{dt}(Re(z))\|_T < \infty$.

2. *Under Assumption 8, $H = f(S_T)$ is square integrable. In particular it admits an FS decomposition.*

3. *Because of (4.6) in Proposition 4.9, the Radon-Nykodim derivative at Point 2. of Assumption 8, always exists.*

We need now to obtain upper bounds on z for the quantity (4.26). We will first need the following lemma.

Lemma 4.20. *There are positive constants c_1, c_2, c_3 such that $d\rho_s$ a.e.*

1.
$$\sup_{z\in \mathrm{supp}\Pi} \frac{dRe(\eta(z,s))}{d\rho_s} \le c_1.$$

2. *For any $z \in \mathrm{supp}\Pi$*
$$|\gamma(z,s)|^2 \le \frac{d\rho_s(z)}{d\rho_s} \le c_2 - c_3 \frac{dRe(\eta(z,s))}{d\rho_s}$$

3.
$$-\sup_{z\in \mathrm{supp}\Pi} \int_0^T 2Re(\eta(z,dt)) \exp(\int_t^T Re(\eta(z,ds))) < \infty.$$

Remark 4.21. *According to Proposition 4.17, $t \mapsto Re(\eta(z,t))$ is absolutely continuous with respect to $d\rho_t$.*

Proof (**of Lemma** 4.20).
The proof is inspired by Lemma 3.9 of [31]. According to Point 2. of Assumption 8 we denote

$$c_{11} := \sup_{x\in I} \left\| \frac{d(\kappa_t(x))}{d\rho_t} \right\|_\infty . \tag{4.28}$$

For $z \in \mathrm{supp}\Pi, t \in [0,T]$, we have

$$\eta(z,t) = \kappa_t(z) - \int_0^t \gamma(z,s) d\kappa_s(1) \quad \text{and} \quad \eta(\bar{z},t) = \kappa_t(\bar{z}) - \int_0^t \gamma(\bar{z},s) d\kappa_s(1).$$

Then, we get $Re(\eta(z,t)) = Re(\kappa_t(z)) - \int_0^t Re(\gamma(z,s)) d\kappa_s(1)$. We obtain

$$\begin{aligned} \int_t^T Re(\eta(z,ds)) &\le Re\left(\kappa_T(z) - \kappa_t(z)\right) + \left| \int_t^T \gamma(z,s) d\kappa_s(1) \right| \\ &= \int_t^T \frac{Re(d\kappa_s(z))}{d\rho_s} d\rho_s + \left| \int_t^T \gamma(z,s) d\kappa_s(1) \right| . \end{aligned} \tag{4.29}$$

Since $\left\langle L(z), \overline{L(z)} \right\rangle_t$ is increasing, and taking into account (4.24), the measure, $\left(d\rho_s(z) - |\gamma(z,s)|^2 d\rho_s\right)$, is non-negative. It follows that

$$\frac{d\rho_s(z)}{d\rho_s} - |\gamma(z,s)|^2 \ge 0 , \quad d\rho_s \ a.e. \tag{4.30}$$

Remark 4.22. *By Lemma* (4.30)*, in particular the density* $\frac{d\rho_s(z)}{d\rho_s}$ *is non-negative* $d\rho_s$ *a.e.*

Consequently,

$$2\frac{dRe(\kappa_s(z))}{d\rho_s} \leq \frac{d\kappa_s(2Re(z))}{d\rho_s} \ , \quad d\rho_s \ a.e. \tag{4.31}$$

In order to prove 1. it is enough to verify that, for some $c_0 > 0$,

$$\frac{dRe(\eta(z,s))}{d\rho_s} \leq c_0 + \frac{1}{2}\frac{dRe(\kappa_s(z))}{d\rho_s} \quad d\rho_s \ a.e. \tag{4.32}$$

In fact, (4.31) and Assumption 8 point 2. and (4.28), imply that

$$\frac{dRe(\eta(z,s))}{d\rho_s} \leq c_0 + \frac{1}{2}c_{11} =: c_1. \tag{4.33}$$

To prove (4.32) it is enough to show that

$$Re(\eta(z,T) - \eta(z,t)) \leq c_0(\rho_T - \rho_t) + \frac{1}{2}Re(\kappa_T(z) - \kappa_t(z)), \quad \forall t \in [0,T]. \tag{4.34}$$

Again Assumption 8 point 2. implies that

$$\left| \int_t^T \gamma(z,s) d\kappa_s(1) \right| \leq c_{12} \int_t^T |\gamma(z,s)| d\rho_s \ , \tag{4.35}$$

where $c_{12} = \| \frac{d\kappa_s(1)}{d\rho_s} \|_\infty$. Using (4.30), and Assumption 8 it follows

$$\begin{aligned} |\gamma(z,s)|^2 \quad &\leq \quad \frac{d\rho_s(z)}{d\rho_s} = \frac{d\kappa(2Re(z))}{d\rho_s} - \frac{2dRe(\kappa_s(z))}{d\rho_s} \ , \\ &\leq \quad c_{11} - \frac{2dRe(\kappa_s(z))}{d\rho_s} \ . \end{aligned} \tag{4.36}$$

This implies that

$$c_{12}^2 \, |\gamma(z,s)|^2 \leq \left(c_{13}^2 + \frac{1}{4}\left(\frac{dRe(\kappa_s(z))}{d\rho_s} \right)^2 \right) \ ,$$

where $c_{13} > 0$ is chosen such that $c_{13}^2 \geq 4c_{12}^4 + c_{12}^2 c_{11}$. Consequently

$$\left| \int_t^T \gamma(z,s) d\kappa_s(1) \right| \leq \int_t^T d\rho_s \left(c_{13} + \frac{1}{2}\left| \frac{dRe(\kappa_s(z))}{d\rho_s} \right| \right) \ .$$

Coming back to (4.29), we obtain

$$\begin{aligned} Re(\eta(z,T) - \eta(z,t)) \quad &\leq \quad \int_t^T \left(\frac{Re(d\kappa_s(z))}{d\rho_s} + c_{13} + \frac{1}{2}\left| \frac{Re(d\kappa_s(z))}{d\rho_s} \right| \right) d\rho_s \\ &\leq \quad \int_t^T \left(\frac{1}{2}\frac{Re(d\kappa_s(z))}{d\rho_s} + \left(\frac{Re(d\kappa_s(z))}{d\rho_s} \right)^+ + c_{13} \right) d\rho_s \end{aligned}$$

(4.31) and Assumption 8 allow to establish

$$Re(\eta(z,T) - \eta(z,t)) \leq \int_t^T d\rho_s \left(c_0 + \frac{1}{2}\frac{dRe(\kappa_s(z))}{d\rho_s} \right) \ , \tag{4.37}$$

where $c_0 = \frac{c_{11}}{2} + c_{13}$. This concludes the proof of point 1.
In order to prove point 2. we first observe that (4.32) implies

$$-\frac{dRe(\kappa_s(z))}{d\rho_s} \le 2\left(c_0 - \frac{dRe(\eta(z,s))}{d\rho_s}\right) \tag{4.38}$$

$d\rho_s$ a.e. (4.36) implies

$$|\gamma(z,s)|^2 \le c_{21} - 4\frac{dRe(\eta(z,s))}{d\rho_s} \;, \tag{4.39}$$

where $c_{21} = c_{11} + 4c_0$. Point 2. is now established with $c_2 = c_{21}$ and $c_3 = 4$.
We continue with the proof of point 3. We decompose

$$Re(\eta(z,t)) = A^+(z,t) - A^-(z,t) \;,$$

where

$$A^+(z,t) = \int_0^t \left(\frac{dRe(\eta(z,s))}{d\rho_s}\right)_+ d\rho_s \;, \quad \text{and} \quad A^-(z,t) = \int_0^t \left(\frac{dRe(\eta(z,s))}{d\rho_s}\right)_- d\rho_s \;.$$

$A^+(z,.)$ and $A^-(z,.)$ are increasing non negative functions. Moreover point 1. implies

$$A^+(z,t) \le c_1\rho_t \;.$$

At this points for $z \in I + i\mathbb{R}$

$$\begin{aligned}
-\int_0^T Re(\eta(z,dt))e^{\int_t^T 2Re(\eta(z,ds))} &= \int_0^T \left(A^-(z,dt) - A^+(z,dt)\right) e^{2\int_t^T Re(\eta(z,ds))} \\
&\le \int_0^T A^-(z,dt)e^{2\left(A^+(z,T)-A^+(z,t)\right)}e^{-2\left(A^-(z,T)-A^-(z,t)\right)} \\
&\le e^{2c_1\rho_T}\int_0^T e^{-2\left(A^-(z,T)-A^-(z,t)\right)}A^-(z,dt) \\
&= \frac{e^{2c_1\rho_T}}{2}\left\{1 - e^{-2A^-(z,T)}\right\} \\
&\le \frac{e^{2c_1\rho_T}}{2} \;,
\end{aligned}$$

which concludes the proof of point 3 of Lemma 4.20.

□

Let $\gamma = \sup_{z\in I}(2Re(z))$, by Lemma 4.5, it follows

$$\sup_{z\in I, s\le T} \mathbb{E}\left[S_s^{2Re(z)}\right] < \infty \;. \tag{4.40}$$

Theorem 4.23. *Let Π be a finite complex-valued Borel measure on $\mathbb{C}$.*
Suppose Assumptions 6, 7, 8. Any complex-valued contingent claim $H = f(S_T)$, where f is of the form (4.27), and $H \in \mathcal{L}^2$, admits a unique FS decomposition $H = H_0 + \int_0^T \xi_t dS_t + L_T$ with the following properties.

1. *$H \in \mathcal{L}^2$ and*
   - $H_t = \int H(z)_t \Pi(dz)$,
   - $\xi_t = \int \xi(z)_t \Pi(dz)$,

- $L_t = \int L(z)_t \Pi(dz)$,

where for $z \in \mathrm{supp}(\Pi)$, $H(z), \xi(z)$ and $L(z)$ are the same as those introduced in Proposition 4.17 and we convene that they vanish if $z \notin \mathrm{supp}(\Pi)$.

2. *Previous decomposition is real-valued if f is real-valued.*

Remark 4.24. *Taking $\Pi = \delta_{z_0}(dz)$, $z_0 \in \mathbb{C}$, Assumption 8 is equivalent to the assumptions of Proposition 4.17.*

Proof. a) $f(S_T) \in \mathcal{L}^2$ since by Jensen,

$$E\left|\int_{\mathbb{C}} \Pi(dz) S_T^z\right|^2 \leq \int_{\mathbb{C}} |\Pi|(dz) E|S_T^{2Rez}||\Pi|(\mathbb{C}) \leq \sup_{x\in I_0} E(S_T^{2x})|\Pi|(\mathbb{C})^2,$$

where $|\Pi|$ denotes the total variation of the finite measure Π. Previous quantity is bounded because of Lemma 4.17.

We go on with the FS decomposition. We would like to prove first that H and L are well defined square-integrable processes and $E(\int_0^T |\xi_s|^2 d\langle M\rangle_s) < \infty$.

We denote $K = supp(\Pi)$. By Jensen's inequality, we have

$$\mathbb{E}\left|\int_{\mathbb{C}} L(z)_t \Pi(dz)\right|^2] \leq \mathbb{E}\left(\int_{\mathbb{C}} |\Pi|(dz)|L_t(z)|_t^2\right) |\Pi(\mathbb{C})| = \int_{\mathbb{C}} |\Pi|(dz)\mathbb{E}[|L_t(z)|_t^2]|\Pi|$$

Similar calculations allow to show that

$$\mathbb{E}[\xi_t^2] \leq |\Pi|(\mathbb{C}) \int_{\mathbb{C}} |\Pi|dz)\mathbb{E}[|\xi_t(z)|^2] \quad \text{and} \quad \mathbb{E}[L_t^2] \leq |\Pi(\mathbb{C})| \int_{\mathbb{C}} |\Pi|(dz)\mathbb{E}[|L_t(z)|^2] \ .$$

We will show now that

- (A1): $\sup_{t\leq T, z\in I+i\mathbb{R}} \mathbb{E}[|H_t(z)|^2] < \infty$;
- (A2): $\int_{\mathbb{C}} |\Pi|(dz)\mathbb{E}[|L_t(z)|_t^2] < \infty$;
- (A3):
$$E\left(\int_0^T d\rho_t S_t^2 \int_{\mathbb{C}} |\xi_t(z)|^2 |\Pi|(dz)\right) < \infty \, .$$

(A1): Since $H(z)_t = e^{\int_t^T \eta(z,ds)} S_t^z$, we have

$$|H(z)_t|^2 = H(z)_t \overline{H(z)_t} = e^{\int_t^T 2Re(\eta(z,ds))} S_t^{2Re(z)} \ ,$$

so

$$\mathbb{E}[|H(z)_t|^2] = e^{\int_t^T 2Re(\eta(z,ds))}\mathbb{E}[S_t^{2Re(z)}] \leq e^{\int_t^T 2Re(\eta(z,ds))} \sup_{t\leq T} \mathbb{E}[S_t^{\gamma}] \ ,$$

with $\gamma = \sup_{z\in I} 2Re(z)$. Inequality (4.40) and Lemma 4.20 imply (A1). Therefore (H_t) is a well-defined square-integrable process.

(A2): $\mathbb{E}[|L_t(z)|^2] \leq \mathbb{E}[|L_T(z)|^2] = \mathbb{E}[\left\langle L(z), \overline{L(z)}\right\rangle_T]$, where the first inequality is due to the fact that $|L_t(z)|^2$ is a submartingale.

$$\mathbb{E}\left[\left\langle L(z), \overline{L(z)}\right\rangle_T\right] = \mathbb{E}\left[\int_0^T e^{\int_s^T 2Re(\eta(z,du)} S_{s-}^{2Re(z)} \left[d\rho_s(z) - |\gamma(z,s)|^2 d\rho_s\right]\right] \ .$$

By Fubini, Lemma 4.5 and (4.24), we have

$$\begin{aligned}\mathbb{E}\left[\left\langle L(z),\overline{L(z)}\right\rangle_T\right] &= \int_0^T e^{\int_s^T 2Re(\eta(z,du)}\mathbb{E}[S_{s-}^{2Re(z)}]\left[\frac{d\rho_s(z)}{d\rho_s}-|\gamma(z,s)|^2\right]d\rho_s\\ &\leq \int_0^T e^{\int_s^T 2Re(\eta(z,du)}\left[\frac{d\rho_s(z)}{d\rho_s}-|\gamma(z,s)|^2\right]\mathbb{E}[S_{s-}^{2Re(z)}]d\rho_s\\ &\leq c_4\int_0^T e^{\int_s^T 2Re(\eta(z,du)}\left[\frac{d\rho_s(z)}{d\rho_s}\right]d\rho_s\ ,\end{aligned}$$

where $c_4=\sup_{s\leq T}\mathbb{E}[S_s^{2Re(z)}]$.

According to Lemma 4.20 point 2, previous expression is bounded by $c_4 I(z)$, where

$$\begin{aligned}I(z) &:= \int_0^T d\rho_t \exp\left(\int_t^T 2Re(\eta(z,ds))\left[c_2-c_3\frac{dRe(\eta(z,t))}{d\rho_t}\right]\right)\\ &= c_2 I_1(z)+c_3 I_2(z)\ ,\end{aligned}\tag{4.41}$$

where

$$I_1(z)=\int_0^T d\rho_t\exp\left(\int_t^T 2Re(\eta(z,ds))\right)\quad\text{and}\quad I_2(z)=\int_0^T \exp\left(\int_t^T 2Re(\eta(z,ds))\right)dRe(\eta(z,ds))\ .$$

Using Lemma 4.20, we obtain

$$\sup_{z\in I+i\mathbb{R}}|I_1(z)|\leq \rho_T\exp\left(2c_1\rho_T\right)\quad\text{and}\quad \sup_{z\in I+i\mathbb{R}}|I_2(z)|<\infty\ ,\tag{4.42}$$

and so

$$\sup_{z\in I+i\mathbb{R}}\mathbb{E}\left[\left\langle L(z),\overline{L(z)}\right\rangle_T\right]<\infty\ .\tag{4.43}$$

This concludes (A2).

We verify now the validity of (A3). This requires to control

$$\mathbb{E}\left[\int_0^T \rho_{dt}S_t^2\left(\int_{\mathbb{C}}|\Pi|(dz)|\xi(z)_t|^2\right)\right]\leq \mathbb{E}\left[\int_0^T \rho_{dt}S_t^2\left(\int_{\mathbb{C}}|\Pi|(dz)\left|\gamma(z,t)\exp\left(\int_t^T Re(\eta(z,ds))\right)S_t^{z-1}\right|^2\right)\right]\ .$$

Using Jensen inequality, this is smaller or equal than

$$|\Pi(\mathbb{C})|\int_{\mathbb{C}}|\Pi|(dz)\int_0^T\rho_{dt}\mathbb{E}\left[S_t^{2Re(z)}\right]|\gamma(z,t)|^2\exp\left(2\int_t^T Re(\eta(z,ds))\right)\ .$$

Lemma 4.20 gives the upper bound

$$|\Pi|(\mathbb{C})\sup_{t\leq T,\gamma\in I}\mathbb{E}\left[S_t^{2Re(z)}\right]\int_{\mathbb{C}}|\Pi|(dz)I(z)\ ,$$

where $I(z)$ was defined in (4.42). Since Π is finite and because of (4.43), (A3) is now established.

We show now that (L_t) is an $(\mathcal{F}_t)$-martingale. Let $0\leq s\leq t\leq T$, $\mathcal{B}\in\mathcal{F}_s$. By Proposition 4.17, since $(L(z)_t)$ is a martingale, we obtain

$$\mathbb{E}[(L_t-L_s)1_{\mathcal{B}}]=\mathbb{E}[\int_{\mathbb{C}}(L(z)_t-L(z)_s)\Pi(dz)1_{\mathcal{B}}]\ .$$

By Fubini's theorem we conclude that

$$\mathbb{E}[(L_t - L_s)1_{\mathcal{B}}] = \int_{\mathbb{C}} \mathbb{E}[(L(z)_t - L(z)_s)1_{\mathcal{B}}]\Pi(dz) \ ,$$

and $\mathbb{E}[(L(z)_t - L(z)_s)1_{\mathcal{B}}] = 0$. So

$$\mathbb{E}[(L_t - L_s)1_{\mathcal{B}}] = 0 \ .$$

Hence, L is a square-integrable martingale.
Similarly, it can be shown that $\mathbb{E}[(M_tL_t - M_sL_s)1_{\mathcal{B}}] = 0$ and so ML is a square-integrable martingale as well. Hence L is orthogonal to M. By Fubini's theorem for stochastic integrals, cf. [40], Theorem IV.46, we have

$$\int \int_0^t \xi(z)_s dS_s \Pi(dz) = \int_0^t \int \xi(z)_s \Pi(dz) dS_s = \int_0^t \xi_s dS_s \ .$$

Consequently, (H_0, ξ, L) provide a (possibly complexe) FS decomposition of H.

b) It remains to prove that the decomposition is real-valued. Let (H_0, ξ, L) and $(\overline{H_0}, \overline{\xi}, \overline{L})$ be two FS decomposition of H. Consequently, since H and (S_t) are real-valued, we have

$$0 = H - \overline{H} = (H_0 - \overline{H}_0) + \int_0^T (\xi_s - \overline{\xi}_s) dS_s + (L_T - \overline{L}_T) \ ,$$

which implies that $0 = Im(H_0) + \int_0^T Im(\xi_s)dS_s + Im(L_T)$. By Theorem 2.22, the uniqueness of the real-valued Föllmer-Schweizer decomposition yields that the processes (H_t),(ξ_t) and (L_t) are real-valued.

□

4.6 Representation of some typical contingent claims

We used some integral representations of payoffs of the form (4.27). We refer to [17], [41] and more recently [20], for some characterizations of classes of functions which admit this kind of representation. In order to apply the results of this paper, we need explicit formulae for the complex measure Π in some example of contingent claims.

4.6.1 Call

The first example is the European Call option $H = (S_T - K)_+$. We have two representations of the form (4.27) which result from the following lemma.

Lemma 4.25. *Let $K > 0$, the European Call option $H = (S_T - K)_+$ has two representations of the form (4.27):*

1. *For arbitrary $R > 1$, $s > 0$, we have*

$$(s - K)_+ = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} s^z \frac{K^{1-z}}{z(z-1)} dz \ . \tag{4.44}$$

2. *For arbitrary $0 < R < 1$, $s > 0$, we have*

$$(s - K)_+ - s = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} s^z \frac{K^{1-z}}{z(z-1)} dz \ . \tag{4.45}$$

4.6.2 Put

Lemma 4.26. *Let $K > 0$, the European Put option $H = (K - S_T)_+$ gives for an arbitrary $R < 0$, $s > 0$*

$$(K - s)_+ = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} s^z \frac{K^{1-z}}{z(z-1)} dz \ . \tag{4.46}$$

5 The solution to the minimization problem

5.1 Mean-Variance Hedging

FS decomposition will help to provide the solution to the global minimization problem. Next theorem deals with the case where the underlying process is a PII.

Theorem 5.1. *Let $X = (X_t)_{t\in[0,T]}$ be a process with independent increments with log-characteristic function Ψ_t. Let $H = f(X_T)$ where f is of the form (3.20). We suppose that the PII, X, satisfies Assumptions 2, 3, 4 and 5. Then, the variance-optimal capital V_0 and the variance-optimal hedging strategy φ, solution of the minimization problem (2.1), are given by*

$$V_0 = H_0 \ , \tag{5.1}$$

and the implicit expression

$$\varphi_t = \xi_t + \frac{\lambda_t}{S_{t-}}(H_{t-} - V_0 - \int_0^t \varphi_s dS_s) \ , \tag{5.2}$$

where the processes (H_t),(ξ_t) and (λ_t) are defined by

$$H_t = \int_{\mathbb{R}} H(u)_t \mu(du) \ , \quad \xi_t = \int_{\mathbb{R}} i\frac{d(\Psi'_t(u) - \Psi'_t(0))}{d\Psi''_t(0)} H(u)_t \mu(du) \quad and \quad \lambda_t = i\frac{d\Psi'_t(0)}{d\Psi''_t(0)} \ , \tag{5.3}$$

and

$$H(u)_t = e^{\eta(u,T)-\eta(u,t)+\Psi_T(u)-\Psi_t(u)} \, e^{iuX_{t-}} \quad with \quad \eta(u,t) = i\int_0^t \frac{d\Psi'_t(0)}{d\Psi''_t(0)} d\left(\Psi'_s(u) - \Psi'_s(0)\right) \ . \tag{5.4}$$

The optimal initial capital is unique. The optimal hedging strategy $\varphi_t(\omega)$ is unique up to some $(P(d\omega) \otimes dt)$-null set.

Proof. Since K is deterministic, the optimality follows from Theorem 3.34, Theorem 2.25 and Corollary 2.26. Uniqueness follows from Theorem 2.24. □

Next theorem deals with the case where the payoff to hedge is given as a bilateral Laplace transform of the exponential of a PII. It is an extension of Theorem 3.3 of [31] to PII with no stationary increments.

Theorem 5.2. *Let $X = (X_t)_{t\in[0,T]}$ be a process with independent increments with cumulant generating function κ. Let $H = f(e^{X_T})$ where f is of the form (4.27). We assume the validity of Assumptions 6, 7, 8. The variance-optimal capital V_0 and the variance-optimal hedging strategy φ, solution of the minimization problem (2.1), are given by*

$$V_0 = H_0 \tag{5.5}$$

and the implicit expression

$$\varphi_t = \xi_t + \frac{\lambda_t}{S_{t-}}(H_{t-} - V_0 - \int_0^t \varphi_s dS_s) \ , \tag{5.6}$$

where the processes (H_t), (ξ_t) *and* (λ_t) *are defined by*

$$\gamma(z,t) := \frac{d\rho_t(z,1)}{d\rho_t} \quad \textit{with} \quad \rho_t(z,y) = \kappa_t(z+y) - \kappa_t(z) - \kappa_t(y) \ , \tag{5.7}$$

$$\eta(z,dt) := \kappa_{dt}(z) - \gamma(z,t)\kappa_{dt}(1) \ , \tag{5.8}$$

$$\lambda_t := \frac{d(\kappa_t(1))}{d\rho_t} \ , \tag{5.9}$$

$$H_t := \int_{\mathbb{C}} e^{\int_t^T \eta(z,ds)} S_t^z \Pi(dz) \ , \tag{5.10}$$

$$\xi_t := \int_{\mathbb{C}} \gamma(z,t) e^{\int_t^T \eta(z,ds)} S_{t-}^{z-1} \Pi(dz) \ . \tag{5.11}$$

The optimal initial capital is unique. The optimal hedging strategy $\varphi_t(\omega)$ *is unique up to some* $(P(d\omega)\otimes dt)$*-null set.*

Remark 5.3. *The mean variance tradeoff process can be expressed as follows, see (4.12):*

$$K_t = \int_0^t \frac{d\kappa_u(1)}{d\rho_u} \kappa_{du}(1) \ .$$

Proof of Theorem 5.2. Since K is deterministic, the optimality follows from Theorem 4.23, Theorem 2.25 and Corollary 2.26. Uniqueness follows from Theorem 2.24.

□

5.2 The quadratic error

Again, ρ_{dt} denotes the measure $\kappa_{dt}(2) - 2\kappa_{dt}(1)$. Let V, φ and H appearing in Theorem 5.2. The quantity $\mathbb{E}[(V_0 + G_T(\varphi) - H)^2]$ will be called **the variance of the hedging error**.

Theorem 5.4. *Under the assumptions of Theorem 5.2, the variance of the hedging error equals*

$$J_0 := \left(\int_{\mathbb{C}} \int_{\mathbb{C}} J_0(y,z) \Pi(dy) \Pi(dz) \right) \ ,$$

where

$$J_0(y,z) := \begin{cases} s_0^{y+z} \int_0^T \beta(y,z,dt) e^{\kappa_t(y+z)+\alpha(y,z,t)} dt & : \quad y,z \in \mathrm{supp}\Pi \\ 0 & : \quad \textit{otherwise.} \end{cases}$$

and

$$\begin{aligned} \alpha(y,z,t) \quad &:= \quad \eta(z,T) - \eta(z,t) - (\eta(y,T) - \eta(y,t)) - \int_t^T \left(\frac{d\kappa_s(1)}{d\rho_s} \right)^2 d\rho_s \ , \\ \beta(y,z,t) \quad &:= \quad \rho_t(y,z) - \int_0^t \gamma(z,s) \rho_{ds}(y,1) \ . \end{aligned}$$

Remark 5.5. *We have*

$$\alpha(y,z,t) = (\eta(z,T) - \eta(z,t)) - (\eta(y,T) - \eta(y,t)) - (K_T - K_t) \ ,$$

where K is the MVT process.

Proof. The quadratic error can be calculated using Corollary 2.29 and Corollary 2.26. It gives

$$\mathbb{E}\left[\int_0^T \exp\{-(K_T - K_s)\} \, d\langle L\rangle_s\right] \ , \tag{5.12}$$

where L is the remainder martingale in the FS decomposition of H. We proceed now to the evaluation of $\langle L\rangle$.

Using (4.22), (4.23), Remark 2.4 and the bilinearity of the covariation give

$$\begin{aligned} Re\left(\langle L(y), L(z)\rangle\right) &= \frac{1}{2}\left(\langle L(y), L(z)\rangle + \left\langle \overline{L(y)}, \overline{L(z)}\right\rangle\right) \\ &= \frac{1}{2}\left(\left\langle L(y) + L(\bar z), \overline{L(y) + L(\bar z)}\right\rangle - \left\langle L(y), \overline{L(y)}\right\rangle - \left\langle L(z), \overline{L(z)}\right\rangle\right) \ , \end{aligned}$$

and

$$\begin{aligned} \left\langle L(y) + L(\bar z), \overline{L(y) + L(\bar z)}\right\rangle &\le \left\langle L(y) + L(\bar z), \overline{L(y) + L(\bar z)}\right\rangle + \left\langle L(y) - L(\bar z), \overline{L(y) - L(\bar z)}\right\rangle \ , \\ &= 2\left\langle L(y), \overline{L(y)}\right\rangle + 2\left\langle L(z), \overline{L(z)}\right\rangle \ . \end{aligned}$$

(4.43) in the proof of Theorem 4.23, and considerations above allow to prove that

$$\sup_{y,z \in I + i\mathbb{R}} \left| Re\left(\langle L(y), L(z)\rangle\right)\right| < \infty.$$

Similarly we can bound $Im(\langle L(y), L(z)\rangle_t)$, writing

$$Im\left(\langle L(y), L(z)\rangle\right) = \frac{1}{2}\left(\left\langle L(y) - L(\bar z), \overline{L(y) - L(\bar z)}\right\rangle - \left\langle L(y), \overline{L(y)}\right\rangle - \left\langle L(z), \overline{L(z)}\right\rangle\right) \ ,$$

so that we obtain

$$Im\left(\langle L(y), L(z)\rangle\right) \le \left\langle L(y), \overline{L(y)}\right\rangle + \left\langle L(z), \overline{L(z)}\right\rangle$$

and

$$\sup_{y,z \in I + i\mathbb{R}} \left| Im\left(\langle L(y), L(z)\rangle\right)\right| < \infty \ .$$

Therefore

$$\int\int \langle L(y), L(z)\rangle_t \, \Pi(dy)\Pi(dz)$$

is a well-defined, continuous, predictable, with bounded variation complex-valued process.
We recall that $L_t = \int L(z)_t \Pi(dz)$ so

$$L_t^2 = \int\int L(y)_t L(z)_t \Pi(dy)\Pi(dz) \ .$$

An application of Fubini's theorem yields that

$$L_t^2 - \int\int \langle L(y), L(z)\rangle_t \, \Pi(dy)\Pi(dz) \ ,$$

is a martingale. This implies

$$\langle L, L \rangle_t = \int \int \langle L(y), L(z) \rangle_t \, \Pi(dy)\Pi(dz) \ ,$$

by definition of oblique bracket. It remains to evaluate $\langle L(y), L(z) \rangle$ for $y, z \in \mathrm{supp}(\Pi)$.
We know by Proposition 4.11 that for all $y, z, y+z \in D$,

$$\langle M(y), M(z) \rangle_t = \int_0^t S_{u-}^{y+z} \rho_{du}(y,z) \ .$$

Using the same terminology of Proposition 4.17, (4.21) says $\langle L(z), M \rangle_t = 0$ and (4.20) imply

$$\begin{aligned}
\langle L(y), L(z) \rangle_t &= \int_0^t e^{\int_s^T (\eta(z,du)+\eta(y,du))} d\langle M(y), M(z) \rangle_s - \int_0^t \xi(z)_s e^{\int_s^T \eta(y,du)} d\langle M, M(y) \rangle_s \ , \\
&= \int_0^t e^{\int_s^T (\eta(z,du)+\eta(y,du))} d\langle M(y), M(z) \rangle_s - \int_0^t \gamma(z,s) e^{\int_s^T (\eta(z,du)+\eta(y,du))} S_{s-}^{z-1} d\langle M, M(y) \rangle_s \ , \\
&= \int_0^t e^{\int_s^T (\eta(z,du)+\eta(y,du))} S_{s-}^{y+z} \rho_{ds}(y,z) - \int_0^t \gamma(z,t) e^{\int_s^T (\eta(z,du)+\eta(y,du))} S_{s-}^{z-1} S_{s-}^{y+1} \rho_{ds}(y,1) \ , \\
&= \int_0^t e^{\int_s^T (\eta(z,du)+\eta(y,du))} S_{s-}^{y+z} \left[\rho_{ds}(y,z) - \gamma(z,s)\rho_{ds}(y,1) \right] \ .
\end{aligned}$$

Hence,

$$\langle L(y), L(z) \rangle_t = \int_0^t e^{\int_s^T (\eta(z,du)+\eta(y,du))} S_{s-}^{y+z} \beta(y,z,ds) \ .$$

We come back to (5.12). Recalling Remark 5.3 we have

$$\begin{aligned}
\int_0^T e^{-(K_T-K_t)} d\langle L(y), L(z) \rangle_t &= \int_0^T e^{-(K_T-K_t)+\int_t^T (\eta(z,du)+\eta(y,du))} S_{t-}^{y+z} \beta(y,z,dt) \ , \\
&= \int_0^T e^{\alpha(y,z,t)} S_{t-}^{y+z} \beta(y,z,dt) \ .
\end{aligned}$$

Since $\mathbb{E}[S_{t-}^{y+z}] = s_0^{y+z} e^{\kappa_t(y+z)}$, an application of Fubini's theorem yields

$$\begin{aligned}
\mathbb{E}\left(\int_0^T e^{-(K_T-K_t)} d\langle L(y), L(z) \rangle_t \right) &= \mathbb{E}\left(\int_0^T e^{\alpha(y,z,t)} S_{t-}^{y+z} \beta(y,z,dt) \right) \ , \\
&= s_0^{y+z} \int_0^T e^{\alpha(y,z,t)+\kappa_t(y+z)} \beta(y,z,dt) \ .
\end{aligned} \tag{5.13}$$

which equals $J_0(y,z)$.

Another application of Fubini's theorem gives

$$\int_0^T e^{-(K_T-K_t)} d\langle L, L \rangle_t = \int_{\mathbb{C}} \int_{\mathbb{C}} \int_0^T e^{-(K_T-K_t)} d\langle L(y), L(z) \rangle_t \, \Pi(dy)\Pi(dz) \ ,$$

hence

$$\begin{aligned}
\mathbb{E}\left[\int_0^T e^{-(K_T-K_t)} d\langle L, L \rangle_t \right] &= \int_{\mathbb{C}} \int_{\mathbb{C}} \mathbb{E}\left[\int_0^T e^{-(K_T-K_t)} d\langle L(y), L(z) \rangle_t \right] \Pi(dy)\Pi(dz) \ , \\
&= \int_{\mathbb{C}} \int_{\mathbb{C}} J_0(y,z) \Pi(dy)\Pi(dz) \ .
\end{aligned}$$

Corollaries 2.29 and 2.26 imply that the left-hand side of the previous equation provides the variance of the hedging error. $\square$

5.3 The exponential Lévy case

In this section, we specify rapidly the results concerning FS decomposition and the minimization problem when (X_t) is a Lévy process (Λ_t). Using the fact that (Λ_t) is a process with independent stationary increments it is not difficult to show that

$$\kappa_t(z) = t\kappa^\Lambda(z) \ , \tag{5.14}$$

where $\kappa^\Lambda(z) = \kappa_1(z)$, $\kappa^\Lambda : D \to \mathbb{C}$. Since for every $z \in D$, $t \mapsto \kappa_t(z)$ has bounded variation then $X = \Lambda$ is a semimartingale and Proposition 3.16 implies that $(t, z) \mapsto \kappa_t(z)$ is continuous.

We make the following hypothesis.

Assumption 9. *1.* $2 \in D$;

2. $\kappa^\Lambda(2) - 2\kappa^\Lambda(1) \neq 0$.

Remark 5.6. *1.* $\rho_{dt} = \left(\kappa^\Lambda(2) - 2\kappa^\Lambda(1)\right) dt$;

2. $\dfrac{d\kappa_t}{d\rho_t}(z) = \dfrac{1}{\kappa^\Lambda(2) - 2\kappa^\Lambda(1)}\kappa^\Lambda(z)$ *for any* $t \in [0, T], z \in D$; *so* $D = \mathcal{D}$.

3. Assumptions 6, and 7 are verified.

Again we denote the process S as

$$S_t = s_0 \exp(X_t) = s_0 \exp(\Lambda_t) \ .$$

It remains to verify Assumption 8 which of course depends on the contingent claim.

Example 5.7. *1.* $H = (S_T - K)_+$. *We choose the second representation for the call. So, for* $0 < R < 1$,

$$I_0 = supp(\Pi) \cap \mathbb{R} = \{R, 1\} \, .$$

In this case Assumption 8.1 becomes $I = [R, R + 1] \subset D$. *This is always satisfied since* $D \supset [0, 2]$ *and it is convex. Assumption 8.2 is always verified because* I *is compact and* κ^Λ *is continuous.*

2. $H = (K - S_T)_+$. *We recall that* $R < 0$ *and so*

$$I_0 = supp(\Pi) \cap \mathbb{R} = \{R\}.$$

In this case, Assumption 8.1, gives again $I = [2R, 1] \subset D$. *Since* $[0, 2]$ *is always included in* D*, we need to suppose here that* $2R$ *(which is a negative value) belongs to* D*.*
This is not a restriction provided that D *contains some negative values since we have the degree of freedom for choosing* R*.*

In this subsection, we reobtain results obtain in [31]. From Proposition 4.13, we obtain the following.

Corollary 5.8. *Under Assumption 9, the process* (S_t) *can be written as*

$$S_t = M_t + A_t \ ,$$

where

$$A_t = \kappa(1)\int_0^t S_{u-}du \quad \text{and} \quad \langle M, M\rangle_t = (\kappa(2) - 2\kappa(1))\int_0^t S_{u-}^2 du \ .$$

The mean-variance tradeoff process equals

$$K_t = \int_0^t \alpha_u^2 d\langle M, M\rangle_u = \frac{\kappa(1)^2}{\kappa(2) - 2\kappa(1)}t \ . \tag{5.15}$$

From Theorem 4.23 and Theorem 5.2, we obtain the following result.

Theorem 5.9. *We suppose the validity of Assumption 9. We consider an option* H *of the type (4.27). The following properties hold true.*

1. *The FS decomposition is given by* $H_T = H_0 + \int_0^T \xi_t dS_t + L_T$ *where*
   - $H_t = \int H(z)_t \Pi(dz)$ *with* $H(z)_t = \exp(\eta^\Lambda(z)(T-t))S_t^z$ *and* $z \in I$, $t \in [0,T]$;
   - $\xi_t = \int \xi(z)_t \Pi(dz)$ *with* $\xi(z)_t = \gamma^\Lambda(z)\exp(\eta^\Lambda(z)(T-t))S_{t-}^{z-1}$ *and* $z \in I$, $t \in [0,T]$;
   - $L_t = H_t - H_0 - \int_0^t \xi_u dS_u$.

   Moreover, for $z \in \mathrm{supp}\Pi$,
   - $\gamma^\Lambda(z) = \dfrac{\kappa(z+1) - \kappa(z) - \kappa(1)}{\kappa(2) - 2\kappa(1)}$;
   - $\eta^\Lambda(z) = \kappa(z) - \kappa(1)\gamma^\Lambda(z)$.

   According to the notations of Lemma 4.17, we have

   $$\eta(z,t) = \eta^\Lambda(z)t, \quad \gamma(z,t) = \gamma^\Lambda(z).$$

2. *The solution of the minimization problem is given by a pair* (V_0, φ) *where*

   $$V_0 = H_0 \quad \text{and} \quad \varphi_t = \xi_t + \frac{\lambda}{S_{t-}}(H_{t-} - V_0 - G_{t-}(\varphi)) \quad \text{with} \quad \lambda = \frac{\kappa(1)}{\kappa(2) - 2\kappa(1)} \ .$$

Remark 5.10. *Lemma 2.14 implies that* Θ *is the linear space of predictable processes* v *such that* $\mathbb{E}\left(\int_0^T v_t^2 S_{t-}^2 dt\right) < \infty$.

Remark 5.11. *We come back to the examples introduced in Remark 3.21. In all the three cases, Assumption 9 is verified if* $2 \in D$. *This is happens in the following situations:*

1. *always in the Poisson case;*
2. *if* $\Lambda = X$ *is a NIG process and if* $2 < \alpha - \beta$;
3. *if* $\Lambda = X$ *is a VG process and if* $2 < -\beta + \sqrt{\beta^2 + 2\alpha}$.

Remark 5.12. *If X is a Poisson process with parameter $\lambda > 0$ then the quadratic error is zero. In fact, the quantities*

$$\begin{aligned}
\kappa^{\Lambda}(z) &= \lambda(\exp(z) - 1)) \\
\rho_t(y,z) &= \lambda t(\exp(y) - 1)(\exp(z) - 1) \\
\gamma(z,t) &= \frac{\exp(z) - 1}{e - 1}
\end{aligned}$$

imply that $\beta(y,z,t) = 0$ for every $y, z \in \mathbb{C}, t \in [0,T]$.

Therefore $J_0(y,z,t) \equiv 0$. In particular all the options of type (4.27) are perfectly hedgeable.

5.4 Exponential of a Wiener integral driven by a Lévy process

Let Λ be a Lévy process. The cumulant function of Λ_t equals $\kappa_t^{\Lambda}(z) = t\kappa_1^{\Lambda}(z)$ for $\kappa_1^{\Lambda} = \kappa^{\Lambda} : D_{\Lambda} \to \mathbb{C}$. We formulate the following hypothesis:

Assumption 10. *1. There is $r > 0$ such that $r \in D_{\Lambda}$.*

2. Λ has no deterministic increments.

Remark 5.13. *According to Lemma 4.3 for every $\gamma > 0$, such that $\gamma \in D$,*

$$\kappa^{\Lambda}(2\gamma) - 2\kappa^{\Lambda}(\gamma) > 0 \; . \tag{5.16}$$

We consider the PII process $X_t = \int_0^t l_s d\Lambda_s$ where $l : [0,T] \to [\varepsilon, r/2]$ is a (deterministic continuous) function and $\varepsilon, r > 0$ such that $2\varepsilon \le r$.

Remark 5.14. *1. Lemma 3.24 says that D contains $D_{\varepsilon,r} := \left\{x \in \mathbb{R} \,|\, \varepsilon x, \frac{rx}{2} \in D_{\Lambda}\right\} + i\mathbb{R}\,$, and $\kappa_t(z) = \int_0^t \kappa^{\Lambda}(zl_s)ds\,$.*

2. $\rho_t = \int_0^t \left(\kappa^{\Lambda}(2l_s) - 2\kappa^{\Lambda}(l_s)\right) ds\,$;

3. $2 \in D\,$; X is a PII semimartingale since $t \mapsto \kappa_t(2)$ has bounded variation, see Lemma 3.14.

4. $1 \in D_{\varepsilon,r}$ since $0, r \in D_{\Lambda}$.

Proposition 5.15. *Assumptions 6 and 7 are verified. Moreover $D_{\varepsilon,r} \subset \mathcal{D}$.*

Proof. 1. Using Lemma 4.3, Assumption 6 is verified if we show that $t \mapsto \rho_t(1) = \kappa_t(2) - 2\kappa_t(1)$ is strictly increasing. Now

$$\kappa_t(2) - 2\kappa_t(1) = \int_0^t \left(\kappa^{\Lambda}(2l_s) - 2\kappa^{\Lambda}(l_s)\right) ds \; .$$

Inequality (5.16) and Lemma 4.3 imply that $\forall s \in [0,T]$

$$\kappa^{\Lambda}(2l_s) - 2\kappa^{\Lambda}(l_s) > 0 \; .$$

In fact, Λ has no deterministic increments. This shows Assumption 6.

2. For $z \in D_{\varepsilon,r}$, by Remark 5.14 1. we have

$$\left|\frac{d\kappa_t(z)}{d\rho_t}\right| = \left|\frac{\kappa^{\Lambda}(zl_t)}{\kappa^{\Lambda}(2l_t) - 2\kappa^{\Lambda}(l_t)}\right| \le \frac{\sup_{x\in[\varepsilon,r]} |\kappa^{\Lambda}(xz)|}{\inf_{x\in[\varepsilon,r/2]} \left(\kappa^{\Lambda}(2x) - 2\kappa^{\Lambda}(x)\right)} \; .$$

Previuous supremum and infimum exist since $x \mapsto \kappa^\Lambda(zx)$ is continuous and it attains a maximum and a minimum on a compact interval. So, $D_{\varepsilon,r} \subset \mathcal{D}$ and Assumption 7 is verified because of Remark 5.15 4.

□

Remark 5.16. *1. Point 1. of Assumption 8 is also verified if we show that $I \subset D_{\varepsilon,r}$; in fact $D_{\varepsilon,r} \subset \mathcal{D}$ and $I_0 \cup (I_0 + 1) \subset I$.*

2. From previous proof it follows that

$$\frac{d\kappa_t(z)}{d\rho_t} = \frac{\kappa^\Lambda(zl_t)}{\kappa^\Lambda(2l_t) - 2\kappa^\Lambda(l_t)} \ .$$

3. Since I is compact and $t \mapsto \frac{d\kappa_t(z)}{d\rho_t}$ is continuous, point 2. of Assumption 8 would be verified again for all cases provided that $I \subset D_{\varepsilon,r}$.

It remains to verify Assumption 8 for the same class of options as in previous subsections. The only point to establish will be to show

$$I \subset \{x | \varepsilon x, \frac{rx}{2} \in D_\Lambda\}. \tag{5.17}$$

Example 5.17. *1. $H = (S_T - K)_+$. Similarly to the case where X is a Lévy process, we take the second representation of the European Call. In this case $I = [R, R+1]$ and (5.17) is verified.*

2. $H = (K - S_T)_+$. Again, here $R < 0$, $I = [2R, R+1]$.
We only have to require that D_Λ contains some negative values, which is the case for the three examples introduced at Section 3.4. Selecting R in a proper way, (5.17)*is fulfilled.*

We provide now the solution to the minimization problem under Assumption 10. . By Theorem 5.2, we have

$$\lambda(s) = \frac{\kappa^\Lambda(l_s)}{\kappa^\Lambda(2l_s) - 2\kappa^\Lambda(ls)} \ ,$$

$$\gamma(z,s) = \frac{\kappa^\Lambda((z+1)l_s) - \kappa^\Lambda(zl_s) - \kappa^\Lambda(l_s)}{\kappa^\Lambda(2l_s) - 2\kappa^\Lambda(l_s)} \ ,$$

$$\eta(z,s) = \kappa^\Lambda(zl_s) - \frac{\kappa^\Lambda(l_s)}{\kappa^\Lambda(2l_s) - 2\kappa^\Lambda(l_s)} \left(\kappa^\Lambda((z+1)l_s) - \kappa^\Lambda(zl_s) - \kappa^\Lambda(l_s)\right) \ ,$$

hence

$$\eta(z,s) = \kappa^\Lambda(zl_s) - \lambda(s) \left(\kappa^\Lambda((z+1)l_s) - \kappa^\Lambda(zl_s) - \kappa^\Lambda(l_s)\right) \ ,$$

We obtain finally the optimal hedging

$$\varphi_t = \xi_t + \frac{\lambda_t}{S_{t-}}(H_{t-} - V_0 - \int_0^t \varphi_s dS_s) \ ,$$

where the processes (H_t),(ξ_t) are defined by

$$H_t = \int_{\mathbb{C}} e^{\int_t^T \eta(z,ds)} S_t^z \Pi(dz) \ ,$$

$$\xi_t = \int_{\mathbb{C}} \gamma(z,t) e^{\int_t^T \eta(z,ds)} S_{t-}^{z-1} \Pi(dz) \ .$$

5.5 A toy example

Let (W_t) be a standard Brownian motion, we consider $X_t = W_{\psi(t)}$, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly increasing function, including the pathological case where $\psi_t^{'} = 0$ a.e. We have

$$\mathbb{E}[e^{zX_t}] = \mathbb{E}[e^{zW_{\psi(z)}}] = e^{\kappa_t(z)} = e^{\frac{z^2}{2}\psi(t)} ,$$

so that

$$\kappa_t(z) = \frac{z^2}{2}\psi(t) , \quad \kappa_t(2) - 2\kappa_t(1) = \psi(t) \quad \text{and} \quad \kappa_t(z+1) - \kappa_t(z) - \kappa_t(1) = z\psi(t) .$$

So

$$\langle M, M \rangle_t = \int_0^t S_{s-}^2 \psi(ds) \quad \text{and} \quad A_t = \int_0^t \frac{1}{2S_{s-}} d\langle M, M \rangle_s = \int_0^t \frac{1}{2} S_{s-}\psi(ds) ,$$

and the MVT process verifies

$$K_t = \int_0^t \frac{1}{4S_{s-}^2} d\langle M, M \rangle_s = \int_0^t \frac{1}{4}\psi(ds) = \frac{1}{4}\psi(t) .$$

All the conditions to apply Theorem 5.2 are satisfied so the function $\gamma(z,t)$ is equal to the Radon-Nykodim derivative of $\kappa_t(z+1) - \kappa_t(z) - \kappa_t(1)$ with respect to $\kappa_t(2) - 2\kappa_t(1)$, so

$$\gamma(z,t) = z , \quad \eta(z,t) = \frac{\psi(t)}{2}(z^2 - z) \quad \text{and} \quad \lambda(t) = \lambda = \frac{1}{2} .$$

Hence we can compute the variance-optimal hedging strategy φ and the variance-optimal initial capital V_0 in this case

$$\varphi_t = \xi_t + \frac{1}{2S_{t-}}(H_{t-} - V_0 - \int_0^t \varphi_s dS_s)$$

and

$$H_t = \int_{\mathbb{C}} e^{\int_t^T \eta(z,ds)} S_t^z \Pi(dz) = \int_{\mathbb{C}} \exp\left\{\frac{z^2 - z}{2}(\Psi(T) - \Psi(t))\right\} S_t^z \Pi(dz)$$

$$\xi_t = \int_{\mathbb{C}} \gamma(z,t) e^{\int_t^T \eta(z,ds)} S_{t-}^{z-1} \Pi(dz) = \int_{\mathbb{C}} z \exp\left\{\frac{z^2 - z}{2}(\Psi(T) - \Psi(t))\right\} S_{t-}^{z-1} \Pi(dz)$$

Remark 5.18. *Calculating $\beta(y,z,t)$ of the quadratic error section, we find $\beta \equiv 0$. Therefore here also the quadratic error is zero. This confirms the fact that the market is **complete**, at least for the considered class of options.*

6 Application to Electricity

6.1 Hedging electricity derivatives with forward contacts

Electricity markets are composed by the Spot market setting prices for each delivery hour of the next day and the forward or futures market setting prices for more distant delivery periods. For simplicity, we will assume that interest rates are deterministic and zero so that futures prices are equivalent to forward prices. Forward prices given by the market correspond to a fixed price of one MWh of electricity for delivery in a given future period, typically a month, a quarter or a year. Hence, the corresponding term contracts are in fact swaps (i.e. forward contracts with delivery over a period) but are improperly named forward. However, the strong assumption that there are tradable forward contracts for all future time points $T_d \geq 0$ is usual and will be assumed here.

Because of non-storability of electricity, no dynamic hedging strategy can be performed on the spot market. Hedging instruments for electricity derivatives are then futures or forward contracts. The value of a forward contract offering the fixed price $F_0^{T_d}$ at time 0 for delivery of 1MWh at time T_d is by definition of the forward price, $S_0^{0,T_d} = 0$. Indeed, there is no cost to enter at time 0 the forward contract with the current market forward price $F_0^{T_d}$. Then, the value of the same forward contract S^{0,T_d} at time $t \in [0, T_d]$ is deduced by an argument of Absence of (static) Arbitrage as $S_t^{0,T_d} = e^{-r(T-t)}(F_t^{T_d} - F_0^{T_d})$. Hence, the dynamic of the hedging instrument $(S_t^{0,T_d})_{0 \leq t \leq T_d}$ is directly related (for deterministic interest rates) to the dynamic of forward prices $(F_t^{T_d})_{0 \leq t \leq T_d}$. Consequently, in the sequel we will focus on the dynamic of forward prices.

6.2 Electricity price models for pricing and hedging application

Observing market data, one can notice two main stylised features of electricity spot and forward prices:

- Volatility term structure of forward prices: the volatility increases when the time to maturity decreases;
- Non-Gaussianity of log-returns: log-returns can be considered as Gaussian for long-term contracts but they are clearly leptokurtic for short-term contratcs with huge spikes on the Spot market.

Hence, a challenge is to be able to describe with a single model, both the spikes on the short term and the volatility term structure of the forward curve. One reasonable attempt to do so is to consider the exponential Lévy factor model, proposed by Benth and Benth [9], or [15]. The forward price given at time t for delivery at time $T_d \geq t$, denoted $F_t^{T_d}$ is then modeled by a p-factors model, such that

$$F_t^{T_d} = F_0^{T_d} \exp(m_t^{T_d} + \sum_{k=1}^{p} X_t^{k,T_d}) \ , \quad \text{for all } t \in [0, T_d] \ \text{,where} \tag{6.18}$$

- $(m_t^{T_d})_{0 \leq t \leq T_d}$ is a real deterministic trend;
- For any $k = 1, \cdots p$, $(X_t^{k,T_d})_{0 \leq t \leq T_d}$ is such that $X_t^{k,T_d} = \int_0^t \sigma_k e^{-\lambda_k (T_d - s)} d\Lambda_s^k$, where $\Lambda = (\Lambda^1, \cdots, \Lambda^p)$ is a Lévy process on $\mathbb{R}^d$, with $\mathbb{E}[\Lambda_1^k] = 0$ and $Var[\Lambda_1^k] = 1$;
- $\sigma_k > 0$, $\lambda_k \geq 0$, are called respectively the *volatilities* and the *mean-reverting rates*.

Hence, forward prices are given as exponentials of PII with *non-stationary increments*. Then, the spot model is derived by setting $S_{T_d} = F_{T_d}^{T_d}$ and reduces to the exponential of a sum of possibly non-Gaussian Ornstein-Uhlenbeck processes. In practice, we consider the case of a one or a two factors model ($p = 1$ or 2), where the first factor X^1 is a non-Gaussian PII and the second factor X^2 is a Brownian motion with $\sigma_1 \gg \sigma_2$. Notice that this kind of model was originally developed and studied in details for interest rates in [41], as an extension of the Heath-Jarrow-Morton model where the Brownian motion has been replaced by a general Lévy process. Recent contributions in the subject are [22, 43].

Of course, this modeling procedure (6.18), implies incompleteness of the market. Hence, if we aim at pricing and hedging a European call on a forward with maturity $T \leq T_d$, it won't be possible, in general, to hedge perfectly the payoff $(F_T^{T_d} - K)_+$ with a hedging portfolio of forward contracts. Then, a natural approach could consist in looking for the variance optimal price and hedging portfolio. In this framework, the results of Section 4 generalizing the results of Hubalek & al in [31] to the case of non stationary PII can be useful. Similarly, some arithmetic models proposed in [6] for electricity prices, consists of replacing the right-hand side of (6.18) by its logarithm. Hence, with this kind of models the results of Section 3.6 can also be useful.

6.3 The non Gaussian two factors model

To simplify let us forget the upperscript T_d denoting the delivery period (since we will consider a fixed delivery period). We suppose that the forward price F follows the two factors model

$$F_t = F_0 \exp(m_t + X_t^1 + X_t^2)\ , \quad \text{for all } t \in [0, T_d]\ , \text{where} \tag{6.19}$$

- m is a real deterministic trend starting at 0. It is supposed to be absolutely continuous with respect to Lebesgue;
- $X_t^1 = \int_0^t \sigma_s e^{-\lambda(T_d-u)} d\Lambda_u$, where Λ is a Lévy process on $\mathbb{R}$ with Λ following a Normal Inverse Gaussian (NIG) distribution or a Variance Gamma (VG) distribution. Moreover, we will assume that $\mathbb{E}[\Lambda_1] = 0$ and $Var[\Lambda_1] = 1$;
- $X^2 = \sigma_l W$ where W is a standard Brownian motion on $\mathbb{R}$;
- Λ and W are independent.
- σ_s and σ_l standing respectively for the short-term volatilty and long-term volatility.

6.4 Verification of the assumptions

The result below helps to extend Theorem 5.2 to the case where X is a finite sum of independent PII semimartingales, each one verifying Assumptions 6, 7 and 8 for a given payoff $H = f(s_0 e^{X_T})$.

Lemma 6.1. *Let X^1, X^2 be two independent PII semimartingales with cumulant generating functions κ^i and related domains $D^i, \mathcal{D}^i, i = 1, 2$ characterized in Remark 3.8 and (4.13). Let $f : \mathbb{C} \to \mathbb{C}$ of the form (4.27). For $X = X^1 + X^2$ with related domains $D, \mathcal{D}$ and cumulant generating function κ, we have the following.*

1. *$D = D^1 \cap D^2$.*
2. *$\mathcal{D}^1 \cap \mathcal{D}^2 \subset \mathcal{D}$.*
3. *If X^1, X^2 verify Assumptions 6, 7 and 8, then X has the same property.*

Proof. Since X^1, X^2 are independent and taking into account Remark 3.8 we obtain 1. and

$$\kappa_t(z) = \kappa_t^1(z) + \kappa^2(z),\ \forall z \in D.$$

We denote by $\rho^i, i = 1, 2$, the reference variance measures defined in Remark 4.7. Clearly $\rho = \rho^1 + \rho^2$ and $d\rho^i \ll d\rho$ with $\|\frac{d\rho^i}{d\rho}\|_\infty \le 1$.
If $z \in \mathcal{D}^1 \cap \mathcal{D}^2$, we can write

$$\begin{aligned}
\int_0^T \left|\frac{d\kappa_t(z)}{d\rho_t}\right|^2 d\rho_t &\le 2\int_0^T \left|\frac{d\kappa_t^1(z)}{d\rho_t^1}\frac{d\rho_t^1}{d\rho_t}\right|^2 d\rho_t + 2\int_0^T \left|\frac{d\kappa_t^2(z)}{d\rho_t^2}\frac{d\rho_t^2}{d\rho_t}\right|^2 d\rho_t \\
&= 2\int_0^T \left|\frac{d\kappa_t^1(z)}{d\rho_t^1}\right|^2 \frac{d\rho_t^1}{d\rho_t} d\rho_t^1 + 2\int_0^T \left|\frac{d\kappa_t^2(z)}{d\rho_t^2}\right|^2 \frac{d\rho_t^2}{d\rho_t} d\rho_t^2 \\
&\le 2\left(\int_0^T \left|\frac{d\kappa_t^1(z)}{d\rho_t^1}\right|^2 d\rho_t^1 + \int_0^T \left|\frac{d\kappa_t^2(z)}{d\rho_t^2}\right|^2 d\rho_t^2\right).
\end{aligned}$$

This concludes the proof of $\mathcal{D}^1 \cap \mathcal{D}^2 \subset \mathcal{D}$ and therefore of the of Point 2.
Finally Point 3. follows then by inspection. $\square$

With the two factors model, the forward price F is then given as the exponential of a PII, X, such that for all $t \in [0, T_d]$,

$$X_t = m_t + X^1_t + X^2_t = m_t + \sigma_s \int_0^t e^{-\lambda(T_d - u)} d\Lambda_u + \sigma_l W_t \ . \tag{6.20}$$

For this model, we formulate the following assumption.

Assumption 11. *1. $2\sigma_s \in D_\Lambda$.*

2. If $\sigma_l = 0$, we require Λ not to have deterministic increments.

3. We define $\varepsilon = \sigma_s e^{-\lambda T_d}$, $\quad r = 2\sigma_s$.
$f : \mathbb{C} \to \mathbb{C}$ is of the type (4.27) fulfilling (5.17).

Proposition 6.2. *1. The cumulant generating function of X defined by (6.20), $\kappa : [0, T_d] \times D \to \mathbb{C}$ is such that for all $z \in D_{\varepsilon,r} := \{x \in \mathbb{R} \,|\, x\sigma_s \in D_\Lambda\} + i\mathbb{R}$, then for all $t \in [0, T_d]$,*

$$\kappa_t(z) = z m_t + \frac{z^2 \sigma_l^2 t}{2} + \int_0^t \kappa^\Lambda(z\sigma_s e^{-\lambda(T_d - u)}) du \ . \tag{6.21}$$

In particular for fixed $z \in D_{\varepsilon,r}$, $t \mapsto \kappa_t(z)$ is absolutely continuous with respect to Lebesgue measure.

2. Assumptions 6, 7 and 8 are verified.

Proof. We set $\tilde{X}^2 = m + X^2$. We observe that

$$D^2 = \mathcal{D}^2 = \mathbb{C}, \quad \kappa^2_t(z) = \exp(z m_t + z^2 \sigma_l^2 \frac{t}{2}).$$

We recall that Λ and W are independent so that $\tilde{X}^2$ and X^1 are independent.
X^1 is a process of the type studied at Section 5.4. According to Proposition 5.15, Remark 5.16 and (5.17) it follows that Assumptions 6, 7 and 8 are verified for X^1.
Both statements 1. and 2. are now a consequence of Lemma 6.1. $\square$

Remark 6.3. *For examples of f fulfilling* (5.17)*, we refer to Example 5.17.*

The solution to the mean-variance problem is provided by Theorem 5.2.

Theorem 6.4. *The variance-optimal capital V_0 and the variance-optimal hedging strategy φ, solution of the minimization problem (2.1), are given by*

$$V_0 = H_0 \tag{6.22}$$

and the implicit expression

$$\varphi_t = \xi_t + \frac{\lambda_t}{S_{t-}}(H_{t-} - V_0 - \int_0^t \varphi_s dS_s), \tag{6.23}$$

where the processes (H_t),(ξ_t) and (λ_t) are defined as follows:

$$\begin{aligned}
\widetilde{z}_t : &= \sigma_s e^{-\lambda(T_d - t)}, \\
\gamma(z,t) : &= \frac{z\sigma_l^2 + \kappa^\Lambda((z+1)\widetilde{z}) - \kappa^\Lambda(z\widetilde{z}) - \kappa^\Lambda(\widetilde{z})}{\sigma_l^2 + \kappa^\Lambda(2\widetilde{z}) - 2\kappa^\Lambda(\widetilde{z})}, \\
\eta(z,t) : &= \left[z m_t + \frac{z^2\sigma_l^2}{2} + \kappa^\Lambda(z\widetilde{z}) - \gamma(z,t)\big(m_t + \frac{\sigma_l^2}{2} + \kappa^\Lambda(\widetilde{z})\big) \right] dt \ ,
\end{aligned}$$

$$
\begin{aligned}
\lambda_t &= \frac{m_t + \frac{\sigma_l^2}{2} + \kappa^\Lambda(\widetilde{z})}{\sigma_l^2 + \kappa^\Lambda(2\widetilde{z}) - 2\kappa^\Lambda(\widetilde{z})}, \\
H_t &= \int_{\mathbb{C}} e^{\int_t^T \eta(z,ds)} S_t^z \Pi(dz), \\
\xi_t &= \int_{\mathbb{C}} \gamma(z,t) e^{\int_t^T \eta(z,ds)} S_{t-}^{z-1} \Pi(dz) \ .
\end{aligned}
$$

The optimal initial capital is unique. The optimal hedging strategy $\varphi_t(\omega)$ is unique up to some $(P(d\omega) \otimes dt)$-null set.

Remark 6.5. *Previous formulae are practically exploitable numerically. The last condition to be checked is*

$$2\sigma_s \in D_\Lambda. \tag{6.24}$$

In our classical examples, this is always verified.

1. *Λ_1 is a Normal Inverse Gaussian random variable. If $\sigma_s \leq \frac{\alpha-\beta}{2}$ then (6.24) is verified.*
2. *Λ_1 is a Variance Gamma random variable then (6.24) is verified. if for instance $\sigma_s \leq \frac{-\beta+\sqrt{\beta^2+2\alpha}}{2}$.*

7 Simulations

7.1 Exponential Lévy

We consider the problem of pricing a European call, with payoff $(S_T - K)_+$, where the underlying process S is given as the exponential of a NIG Lévy process i.e. for all $t \in [0,T]$,

$$S_t = e^{X_t} \ , \quad \text{where } X \text{ is a Lévy process with } \ X_1 \sim NIG(\alpha, \beta, \delta, \mu) \ .$$

The time unit is the year and the interest rate is zero in all our simulations. The initial value of the underlying is $S_0 = 100$ Euros. The maturity of the option is $T = 0.25$ i.e. three months from now. Five different sets of parameters for the NIG distribution have been considered, going from the case of *almost Gaussian* returns corresponding to standard equities, to the case of *highly non Gaussian* returns. The standard set of parameters is estimated on the *Month-ahead base* forward prices of the French Power market in 2007:

$$\alpha = 38.46 \, , \ \beta = -3.85 \, , \ \delta = 6.40 \, , \ \mu = 0.64 \ . \tag{7.25}$$

Those parameters imply a zero mean, a standard deviation of 41%, a skewness (measuring the asymmetry) of -0.02 and an excess kurtosis (measuring the *fatness* of the tails) of 0.01. The other sets of parameters are obtained by multiplying parameter α by a coefficient C, (β, δ, μ) being such that the first three moments are unchanged. Note that when C grows to infinity the tails of the NIG distribution get closer to the tails of the Gaussian distribution. For instance, Table 1 shows how the excess kurtosis (which is zero for a Gaussian distribution) is modified with the five values of C chosen in our simulations.

We have compared on simulations the Variance Optimal strategy (VO) using the real NIG incomplete market model with the real values of parameters to the Black-Scholes strategy (BS) assuming Gaussian returns with the real values of mean and variance. Of course, the VO strategy is by definition theoritically optimal in continuous time, w.r.t. the quadratic norm. However, both strategies are implemented in discrete time, hence the performances observed in our simulations are spoiled w.r.t. the theoritical continuous rebalancing framework.

Coefficient	$C = 0.08$	$C = 0.14$	$C = 0.2$	$C = 1$	$C = 2$
α	3.08	5.38	7.69	38.46	76.92
Excess kurtosis	1.87	0.61	0.30	0.01	4.10^{-3}

Figure 1: Excess kurtosis of X_1 for different values of α, (β, δ, μ) insuring the same three first moments.

7.1.1 Strike impact on the pricing value and the hedging ratio

Figure 2 shows the Initial Capital (on the left graph) and the initial hedge ratio (on the right graph) produced by the VO and the BS strategies as functions of the strike, for three different sets of parameters $C = 0.08\,,\ C = 1\,,\ C = 2$. We consider $N = 12$ trading dates, which corresponds to operational practices on electricity markets, for an option expiring in three months. One can observe that BS results are very similar to VO results for $C \geq 1$ which corresponds to *almost Gaussian returns*. However, for small values of C, for $C = 0.08$, corresponding to highly non Gaussian returns, BS approach under-estimates *out-of-the-money* options and over-estimates *at-the-money* options. For instance, on Figure 3, one can observe that for $K = 99$ Euros the Black-Scholes Initial Capital (IC_{BS}) represents 122% of the variance optimal Initial Capital (IC_{VO}), while for $K = 150$ it represents only 57% of the variance optimal price. Moreover, the hedging strategy differs sensibly for $C = 0.08$, while it is quite similar to BS's ratio for $C \geq 1$.

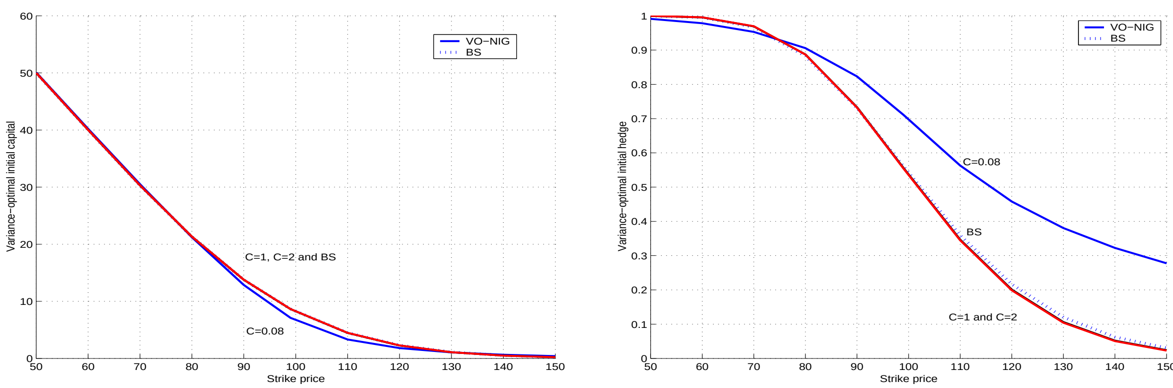

Figure 2: Initial Capital (on the left) and hedge ratio (on the right) w.r.t. the strike, for $C = 0.08\,,\ C = 1\,,\ C = 2$.

Strikes	$K = 60$	$K = 99$	$K = 150$
IC_{VO}	50.08	7.11	0.40
IC_{BS} (vs IC_{VO})	50.00 (99.56%)	8.65 (121.73%)	0.23 (57.30%)

Figure 3: Initial Capital of VO pricing (IC_{VO}) vs Initial Capital of BS pricing (IC_{BS}) for $C = 0.08$.

7.1.2 Hedging error and number of trading dates

Figure 4 considers the hedging error (the difference between the terminal value of the hedging portfolio and the payoff) as a function of the number of trading dates, for a strike $K = 99$ Euros (at the money) and for five different sets of parameters C described on Figure 1. The bias (on the left graph) and standard deviation (on the right graph) of the hedging error have been estimated by Monte Carlo method on 5000 runs. Note that we could have used the formula stated in Theorem 5.4 to compute the variance of the error, but this would have give us the limiting error which does not take into account the additional error due to the finite number of trading dates.

In terms of standard deviation, the VO strategy seems to outperform sensibly the BS strategy, for small values of C. For instance, one can observe on Figure 5, for $C = 0.08$ that the VO strategy allows to reduce 10% of the standard deviation of the error. As expected, one can observe that the VO error converges to the BS error when C increases. This is due to the convergence of NIG log-returns to Gaussian log-returns when C increases (recall that the simulated log-returns are almost symmetric). One can distinguish two sources of incompleteness, the *rebalancing error* due to the dicrete rebalancing strategy and the *intrinsic error* due to the model incompleteness. On Figure 4, the hedging error (both for BS and VO) decreases with the number of trading dates and seems to converge to a limiting error corresponding to the intrinsic error. For $C = 1$ and for a small number of trading dates $N \leq 5$, the rebalancing error represents the most part of the hedging error, then it seems to vanish over $N = 30$ trading dates, where the intrinsic error is predominant. For small values of $C \leq 0.2$, even for small numbers of trading dates, the intrinsic error seems to be predominant. For $C \leq 0.2$ and $N \geq 12$ trading dates, it seems useless to increase the number of trading dates. Moreover, one can observe that for a small number of trading dates $N \leq 12$ and for large values of $C \geq 1$, BS seems to outperform the VO strategy, in terms of standard deviation. This can be interpreted as a consequence of the central limit theorem. Indeed, when the time between two trading dates increases the corresponding increments of the Lévy process converge to a Gaussian variable. Hence, the model error comitted by the BS approach decreases when the number of trading dates decreases.

In term of bias, the over-estimation of at-the-money options (observed for $C = 0.08$, on Figures 2, 3) seems to induce a positive bias for the BS error (see Figure 4), whereas the Bias of the VO error is negligeable (as expected from the theory). However, one can observe on Figure 5, that the difference between VO and BS bias error is smaller than the difference between the Initial Capitals, therefore one can conclude that, in our simulations, the BS hedging strategy induces more losses in average than the VO strategy.

However, to be more relevant in our analysis, we have compared on Figure 7, the performances of the BS hedging portfolio with the VO hedging portfolio starting with the same Initial Capital as the BS hedging portfolio. One can observe on Figure 5 that this approach allows to reduce the standard deviation of the VO hedging error (increasing the bias and of course the global quadratic error w.r.t. the VO strategy with optimal Initial capital).

It is interesting to notice that, in terms of skewness and kurtosis, the VO strategy seems to outperform sensibly the BS strategy for small values of C. Figure 6 shows that for $C = 0.08$, the skewness of the BS hedging error is strongly negative (3 times greater than the VO error using the same Initial Capital) and the kurtosis is high (14 times greater than the VO error). Hence, in our simulations, BS strategy seems to imply more extreme losses than the VO strategy.

In conclusion, the VO approach provides initial capital and hedging strategies which are not significantly different from the BS approach except for log-returns with high excess kurtosis (with small values of parameter α in the NIG case). Similarly, we can observe (though the figures are not reported here) the same behaviour w.r.t. to the asymmetry of the distribution: the VO approach allows to outperform significantly the BS approach for strongly asymmetric log-returns (with high (absolute) values of parameter β in the NIG case). On the other hand, in more standard cases, the VO strategy seems to be comparable with the BS strategy in terms of quadratic error and to have the significant and unexpected advantage to limit extreme losses (skewness and kurtosis) compared to the BS strategy.

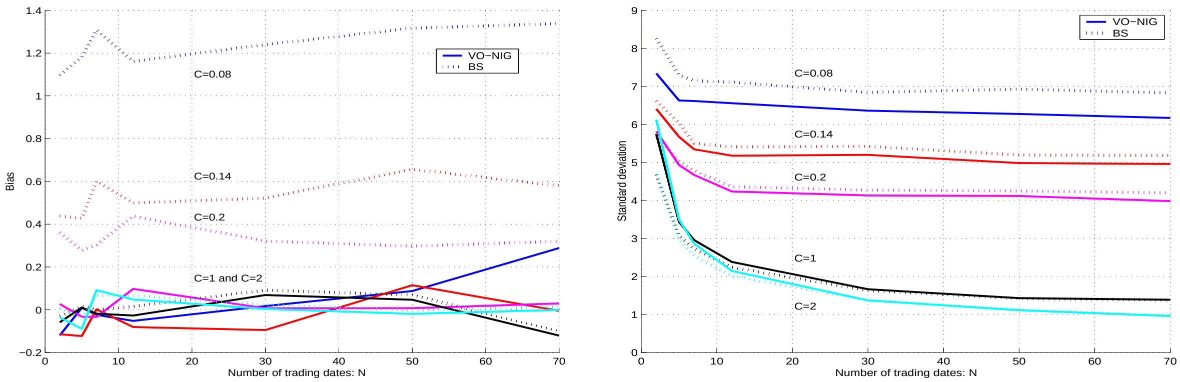

Figure 4: Hedging Error w.r.t. the number of trading dates for different values of C and for $K = 99$ Euros (Bias, on the left and standard deviation, on the right).

Coefficient	$C = 0.08$	$C = 0.14$	$C = 0.2$	$C = 1$	$C = 2$
$\mathrm{Std}_{VO}/\mathrm{Std}_{BS}$	91.19%	95.88%	97.63%	107.52%	109.39%
$\mathrm{Bias}_{BS} - \mathrm{Bias}_{VO}$	1.20	0.57	0.32	0.022	0.019
$\mathrm{IC}_{BS} - \mathrm{IC}_{VO}$	1.55	0.7	0.39	0.01	0

Figure 5: Variance optimal hedging error vs Black-Scholes hedging error for different values of C and for $K = 99$ Euros (averaged values for different numbers of trading dates).

Moments	Mean	Standard deviation	Skewness	Kurtosis
VO	−0.049	6.59	−3.50	31.51
BS	1.27	7.25	−7.65	152.09
VO with $\mathrm{IC}_{VO} = \mathrm{IC}_{BS}$	1.39	6.47	−2.37	10.70

Figure 6: Empirical moments of the hedging error for $C = 0.08$, $N = 12$ and $K = 99$ Euros (averaged values for different number of trading dates).

7.2 Exponential PII

We consider the problem of hedging and pricing a European call on an electricity forward, with a maturity $T = 0.25$ of three month. The maturity is equal to the delivery date of the forward contract $T = T_d$. As stated in Section 6, the natural hedging instrument is the corresponding forward contract with value $S_t^0 = e^{-r(T-t)}(F_t^T - F_0^T)$ for all $t \in [0, T]$, where $F^T = F$ is supposed to follow the NIG one factor model:

$$F_t = e^{X_t} \ , \quad \text{where } X_t = \int_0^t \sigma_s e^{-\lambda(T-u)} d\Lambda_u \quad \text{where } \Lambda \text{ is a NIG process with} \quad \Lambda_1 \sim NIG(\alpha, \beta, \delta, \mu) \ .$$

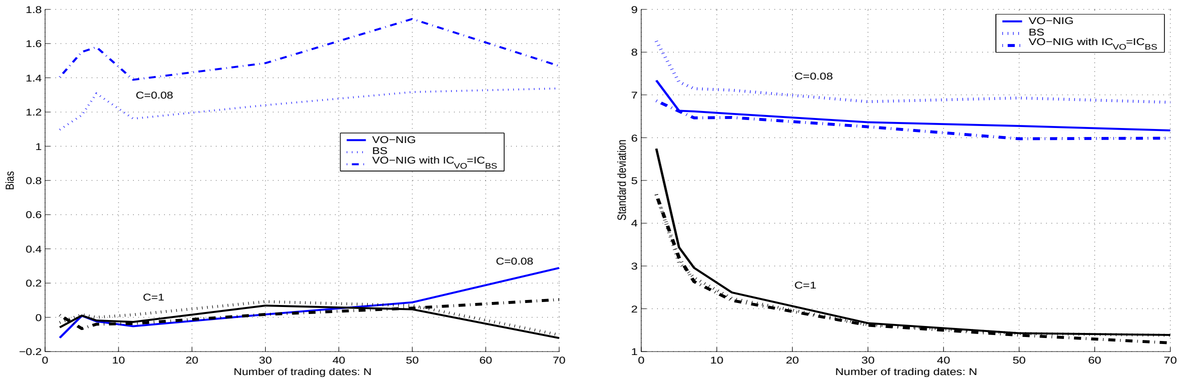

Figure 7: Hedging Error of BS strategy v.s. the VO strategy with the same initial capital as BS w.r.t. the number of trading dates for different values of C and for $K = 99$ Euros (Bias, on the left and standard deviation, on the right).

The standard set of parameters ($C = 1$) for the distribution of Λ_1 is estimated on the same data as in the previous section (*Month-ahead base* forward prices of the French Power market in 2007):

$$\alpha = 15.81\,,\ \beta = -1.581\,,\ \delta = 15.57\,,\ \mu = 1.56\ .$$

Those parameters correspond to a standard and centered NIG distribution with a skewness of -0.019. The estimated annual short-term volatility and mean-reverting rate are $\sigma_s = 57.47\%$ and $\lambda = 3$. The other sets of parameters considered in simulations are obtained by multiplying parameter α by a coefficient C, (β, δ, μ being such that the first three moments are unchanged). Table 1 shows how the excess kurtosis is modified with C.

Coefficient	$C = 0.08$	$C = 1$
α	1.26	15.81
Excess kurtosis	1.87	0.013

Figure 8: Excess kurtosis of Λ_1 for different values of α (β, δ, μ) insuring the same three first moments

Figure 9 shows the Bias and Standard deviation of the hedging error as a function of the number of trading dates estimated by Monte Calo method on 5000 runs. The results are comparable to those obtained in the case of the Lévy process, on Figure 9. However, one can notice that the BS strategy does no more outperform the VO strategy for small numbers of trading dates as observed in the Lévy case. This is due to the fact that X_t is no more a sum of i.i.d. variables.

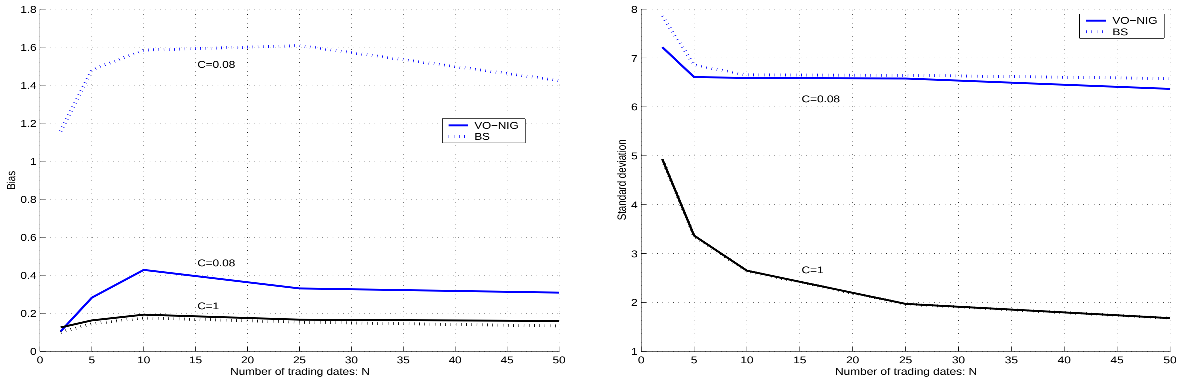

Figure 9: Hedging Error w.r.t. the number of trading dates for $C = 0.08$ and $C = 1$, for $K = 99$ Euros (Bias, on the left and standard deviation, on the right).

Moments	Mean	Standard deviation	Skewness	Kurtosis
VO	0.43	6.59	-2.89	16.24
BS	1.58	6.65	-3.79	25.53

Figure 10: Empirical moments of the hedging error for $C = 0.08$, $N = 10$ and $K = 99$ Euros.

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