Asymptotic behavior of prices of path dependent options

Yuji Hishida (Mizuho Securities Co., Ltd), Kenji Yasutomi (Ritsumeikan University)
e-mail: yuji.hishida@mizuho-sc.com yasutomi@se.ritsumei.ac.jp

November 30, 2009

Abstract

In this paper, we give a numerical method for pricing long maturity, path dependent options by using the Markov property for each underlying asset. This enables us to approximate a path dependent option by using some kinds of plain vanillas. We give some examples whose underlying assets behave as some popular Levy processes. Moreover, we give some payoffs and functions used to approximate them.

Key Words: path dependent option, Markov property, Levy process, Asian option, partial barrier option, asymptotic behavior

1 Introduction

In this paper, we give a numerical method for pricing some path dependent options by using the Markov property for each underlying asset process. Path dependent options are options whose payoff at maturity depend on the past history of the underlying asset as well as the price at maturity. Asian options, Lookback options and barrier options are thier well-known examples. Some of them as the above are difficult to calculate analytically and numerically, and there have been numerous studies on how to do this.

One numerical method of pricing path dependent options uses Monte Carlo simulation. It has an advantage in that with it, we can simulate the expectation of the payoff without detailed discussion of the payoff type, but only with the distribution of the underlying asset. N. Hilber, N. Reich, C. Schwab and C. Winter [11] or Cont, R and Tankov, P [7] give the Monte Carlo method when market models are extended to Levy processes. They mention that most of Levy processes can be simulated approximately. As seen from them, the Monte Carlo method has the applicability for many payoff types and many underlying asset models. However, the longer a maturity time is, the more the calculation time is consumed.

Several analytical methods of path dependent options have been also developed. An analytical formula for standard barrier options is given in papers by Merton [17] or Reiner and Rubinstein [18] when the underlying asset behaves as the geometric Brownian motion. An analytical formula for partial barrier options is also given by Armstrong [3] or Heynen and Kat [10] under the same conditions. When the underlying asset is more general such as a Levy process, it is not easy to find the exact value of them. There are some studies about this problem, for example, Kudryavstev and Levendorski [13] give the Fast Wiener-Hopf factorization method. Asian options are difficult to calculate analytically. Geman and Yor [9] give a semi-analytical formula by using the Laplace transformation. Linetsky [15] uses a spectral expansion approach, and Benhamou [4] uses a convolution method for pricing them. Albrecher and Predota [2] approximate the arithmetic option price based on the moments of the average.

Most approaches of these methods depend on the payoff functions. In contrast, a method using the Malliavin-Watanabe calculus given in Kumitomo and Takahashi [14] is valid for some path dependent options. Bermin [6] and [5] show that the Malliavin calculus approach can be applied for any square integrable payoff.

Our method is an analytical approximation approach. In [12], we give the asymptotic behavior of the prices of an Asian option about maturity time T as the underlying asset behaves as the geometric Brownian motion. When the maturity time $T \to \infty$, the expectation of the payoff for an Asian option can be approximated by that of the payoff for some linear combinations of plain vanillas; i.e., there exists a non-zero constant, D, such that when $T \to \infty$,

$$E[(\frac{1}{N}\sum_{i=1}^{N} S_{T-\tau_i} - K)^+] \simeq \frac{1}{N}\sum_{i=1}^{N} E[(S_{T-\tau_i} - K)^+] + D\alpha(T)$$

where the underlying asset is defined by $S_t = S_0 \exp(\sigma W_t + (r - \frac{1}{2}\sigma^2)t)$, K is a strike price, $\{\tau_i\}$ is fixed time, $W = \{W_t\}_{0\le t\le T}$ is a one-dimensional Brownian motion, and α is defined by

$$\alpha(T) = \frac{1}{\sqrt{T}} \exp\left(-\frac{1}{2\sigma^2}(r - \frac{1}{2}\sigma^2)T\right). \tag{1.1}$$

In this paper, we generalize the above idea to an almost universal situation; i.e., to some payoff functions and a class of the underlying asset process.

To summarize our result of this paper, let the underlying asset behave as positive Markov process S_t. We represent an expectation of a path dependent option with maturity time T as

$$A^*(T) = E[g((S_s)_{T-\tau\le s<T})]$$

where g is a function from path space which represents the payoff of the path dependent option. Roughly speaking, if S_t decays uniformly in $\mathbb{R}$; i.e., there exists $\alpha^* : [0,\infty) \to (0,\infty)$ such that $\frac{P(dS_T)}{\alpha^*(T)}$ “convergence” to non-trivial measure as $T \to \infty$, and also if g satisfys “good” conditions, there exists $\widetilde{A}^*(T)$ and a constant, C, such that the error term between A^* and $\widetilde{A}^*$ can be estimated when $T \to \infty$; i.e.,

$$\frac{A^*(T) - \widetilde{A}^*(T)}{\alpha^*(T)} \to C. \tag{1.2}$$

In practical sense, if $\widetilde{A}^*(T)$ is either easy to calculate or is quoted in the market, we easily obtain a numerical approximation of $A^*(T)$ from

$$A^*(T) \simeq \widetilde{A}^*(T) + C\alpha^*(T).$$

This means that we can approximate a path dependent option by using $\widetilde{A}^*$ and an error term.

Although this asymptotic approach is only valid for calculating the value of long maturity options, it is beneficial because

1. We can get the value of $A^*(T)$ instantaneously since α^* is written as elementary functions in most cases.

2. We can apply this method to many path dependent options.

3. We can apply this method to a large class of the underlying asset.

As an example of the first case, when S_t is geometric Brownian motion, α^* is given by (1.1) and written as elementary functions. As an example of the second case, we give the result in the case of an Asian option, the Lookback option, and the barrier option. As an example of the third case, we give the result in case of a geometric Levy process, particularly for Brownian motion (**BM**) and for the Normal Inverse Gaussian process (**NIG**). In our method, the longer a maturity time is, the better the computation accuracy is. That is because our method is based on the asymptotic behavior of price as T tends to ∞. This is a notable result since most methods, like Monte Carlo simulation, are effective when the variance of the underlying asset is small; i.e., the maturity time is short.

This paper is organized as follows. In Section 2, we present a theorem which gives the principle of our method and give the short proof. In Section 3, we give some exmaples of α^* when the underlying asset behaves as some popular Levy processes, and we see that these α^* are written by elementary functions in these cases. In Section 4, we give two theorems which help us how to find the approximation function $\widetilde{A}^*(T)$ when given $A^*(T)$. Also, we show that some path dependent options, such as an Asian option, a Lookback option and a barrier option, can be applied these theorems. In Section 5, theorems of Section 4 are proved, and in Section 6, we present some properties of (**BM**) and (**NIG**) used in the proof.

2 Generalized Principle

We set the notations again. Let an underlying asset process, S_s be a positive Markov process. For a fixed time $\tau > 0$, we denote the set of all cádlág functions with domain $[0,\tau]$ by $D([0,\tau])$. We also define an expectation of a path dependent option whose maturity time is T and whose monitoring period is $[T-\tau,T]$ by

$$A^*(T) := E[g((S_s)_{T-\tau\le s\le T})]$$

where g is a function from $D([0,\tau])$ into $\mathbb{R}$. For given $A^*(T)$, we also define $\widetilde{A}^*(T)$ by

$$\widetilde{A}^*(T) := E[\widetilde{g}((S_s)_{T-\tau\le s\le T})]$$

where $\widetilde{g}$ is also a function from $D([0,\tau])$ into $\mathbb{R}$. In this paper, we treat this $\widetilde{g}$ as the approximation function of the path dependent option. For the sake of simplicity, we use the notation $t := T-\tau$,

$$\begin{aligned} A(t) &:= A^*(T) = E[g((S_s)_{t\le s\le t+\tau})], \\ \widetilde{A}(t) &:= \widetilde{A}^*(T) = E[\widetilde{g}((S_s)_{t\le s\le t+\tau})], \end{aligned}$$

and we see the asymptotic behavior as $t\to\infty$ instead of the asymptotic behavior as $T\to\infty$. Under these conditions, we derive the following theorem.

Theorem 2.1. *We assume two assumptions, A1 and A2 about $(g,\widetilde{g},S)$;*

A1. Process S decays uniformly; that is, there exist measures ν and $\bar{\nu}$ and a funcion, $\alpha:[0,\infty)\to(0,\infty)$, such that

$$\begin{aligned} &\nu_t \gg \bar{\nu}, \quad \nu \gg \bar{\nu}, \\ &\frac{d\nu_t}{d\bar{\nu}} \to \frac{d\nu}{d\bar{\nu}} \quad (t\to\infty) \end{aligned}$$

where $\nu_t(M) := \frac{P(S_t\in M)}{\alpha(t)}$. We use the notaion $\gg$ to mean absolutely continuous, and $\frac{d\nu_t}{d\bar{\nu}}$ it the Radon Nycodim derivative.

A2.

$$\int |\epsilon(x)| \sup_t \frac{d\nu_t}{d\bar{\nu}}(x)\,\bar{\nu}(dx) < \infty$$

where

$$\epsilon(x) := E[g((S^x_s)_{0\le s\le\tau}) - \widetilde{g}((S^x_s)_{0\le s\le\tau})]$$

and S^x_s is a process starting at x; i.e., $S^x_s := \frac{x}{S_0}S_s$.

Then it follows that when $t\to\infty$,

$$\frac{A(t)-\widetilde{A}(t)}{\alpha(t)} \to \int \epsilon(x)\,\nu(dx). \tag{2.1}$$

We remark that, for $\alpha^*(T) := \alpha(t+\tau)$, since

$$\frac{A^*(T) - \widetilde{A}^*(T)}{\alpha^*(T)} = \frac{A(t) - \widetilde{A}(t)}{\alpha(t)},$$

(1.2) means (2.1).

Proof of the theorem. From the Markov property of S, we obtain

$$\begin{aligned} A(t) - \widetilde{A}(t) &= E[g((S_s)_{t\le s\le t+\tau}) - \widetilde{g}((S_s)_{t\le s\le t+\tau})] \\ &= E[E[g((S_s)_{t\le s\le t+\tau}) - \widetilde{g}((S_s)_{t\le s\le +\tau})|S_t]] \\ &= E[\epsilon(S_t)]. \end{aligned}$$

Since $\int |\epsilon(x)| \sup_t \frac{d\nu_t}{d\bar{\nu}}(x)\,\bar{\nu}(dx) < \infty$, the Lebesgue convergence theorem implies that

$$\begin{aligned} \frac{A(t) - \widetilde{A}(t)}{\alpha(t)} &= \int \epsilon(x) \frac{d\nu_t}{d\bar{\nu}}(x)\,\bar{\nu}(dx) \\ &\to \int \epsilon(x) \frac{d\nu}{d\bar{\nu}}(x)\,\bar{\nu}(dx) \\ &= \int \epsilon(x)\,\nu(dx). \end{aligned}$$

□

3 Asymptotic Order of each model

In this section, we consider A1 in Theorem 2.1. The existence of $\alpha(t)$ depends only on each process, S. It is clear that α is unique in the sense of order; i.e., $\alpha(t) \sim \alpha'(t)$ as $t \to \infty$ if α and α' satisfy A1 for the same process. We give some examples of S_t satisfying A1 represented as $S_t = e^{Z_t}$, where Z_t is a popular Levy process. In the following argument, we regard measure $\bar{\nu}$ as a Lebesgue measure.

For the sake of simplicity, we discuss Z instead of S. Let $\hat{\nu}_t$ be a modified distribution of Z_t; i.e., $\hat{\nu}_t(A) := \frac{P(Z_t \in A)}{\alpha(t)}$. Then, since $S_t = e^{Z_t}$, we have that

$$\frac{d\hat{\nu}_t}{d\bar{\nu}}(z) = \frac{d}{dz}\frac{P(Z_t \le z)}{\alpha(t)} = \frac{d}{dz}\frac{P(e^{Z_t} \le e^z)}{\alpha(t)} = \frac{de^z}{dz}\frac{d}{de^y}\frac{P(S_t \le e^z)}{\alpha(t)} = e^z \frac{d\nu_t}{d\bar{\nu}}(e^z).$$

Thus, by substituting $e^y = x$, the condition A1 is replaced by a condition about Z;

Lemma 3.1. *$\hat{\nu}_t \gg \bar{\nu}$, $\hat{\nu} \gg \bar{\nu}$, and*

$$\frac{d\hat{\nu}_t}{d\bar{\nu}}(z) \to \frac{d\hat{\nu}}{d\bar{\nu}}(z) \tag{3.1}$$

for any z imply A1 for $\frac{d\nu}{d\bar{\nu}}(x) := \frac{1}{x}\frac{d\hat{\nu}}{d\bar{\nu}}(\log x)$.

3.1 Brownian Model

Let us define $Z_t = z_0 + \sigma W_t + \mu t$ where z_0, $\mu \in \mathbb{R}$, $\sigma > 0$, and $\{W_t\}_{0\le t<\infty}$ is a one-dimensional Brownian motion. Then

$$\begin{aligned} \alpha(t) &= \frac{1}{\sqrt{t}} \exp(-\frac{\mu^2 t}{2\sigma^2}) \\ \text{and} \quad \frac{d\hat{\nu}}{d\bar{\nu}}(z) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp(\frac{\mu z}{\sigma^2}) \end{aligned}$$

satisfy (3.1).

3.2 Normal Inverse Gaussian Model

Let us define $Z_t = z_0 + W_{\mathrm{IG}(t)} + \theta \mathrm{IG}(t) + bt$ where z_0, $b \in \mathbb{R}$, and we denote the inverse Gaussian subordinator, IG, by $\mathrm{IG}(t) = \inf\{s > 0; B_s + \mu s > \delta t\}$, where $\mu \in \mathbb{R}$, $\delta > 0$, and B is another Brownian motion independent of W. This process is called normal inverse Gaussian, see [8]. Then

$$\begin{aligned} \alpha(t) &= t^{-\frac{1}{2}} \exp\Big(t\Big(\mu\delta - \theta b - \sqrt{(b^2+\delta^2)(\theta^2+\mu^2)}\Big)\Big) \\ \text{and} \quad \frac{d\hat{\nu}}{d\bar{\nu}}(z) &= \frac{\delta}{\sqrt{2\pi(b^2+\delta^2)}} \left(\frac{\theta^2+\mu^2}{b^2+\delta^2}\right)^{\frac{1}{4}} e^{\left(\theta + b\sqrt{\frac{\theta^2+\mu^2}{b^2+\delta^2}}\right)(z-z_0)} \end{aligned}$$

satisfy (3.1).

3.3 Variance Gamma Model

Let us define $Z_t = z_0 + \sigma W_{\gamma(t)} + \theta\gamma(t) + mt$, where $z_0 \ge 0$, θ, $m \in \mathbb{R}$, and we denote the gamma process by $\gamma(t)$ with variance rate λ, see [16]. Then

$$\begin{aligned} \alpha(t) &= t^{-\frac{1}{2}} \left(\frac{1+\eta}{2+\nu\theta^2/\sigma^2}\right)^{\frac{t}{\nu}} e^{(\frac{1-\eta}{\lambda} - \frac{\theta m}{\sigma^2})t} \\ \text{and} \quad \frac{d\hat{\nu}}{d\bar{\nu}}(z) &= \frac{1}{2}\sqrt{\frac{2+\lambda\theta^2/\sigma^2}{2\pi\eta(1+\eta)\sigma^2}} \exp\left((\frac{\theta}{\sigma^2} + \frac{\eta-1}{m\lambda})(z-z_0)\right) \end{aligned}$$

satisfy (3.1) where $\eta := \sqrt{1 + \frac{m^2\lambda^2}{\sigma^2}(\frac{2}{\lambda} + \frac{\theta^2}{\sigma^2})}$.

4 Path dependent Payoff and its Approximation function

In this section, we consider A2 in Theorem 2.1. To satisfy A2 for $(g, \widetilde{g}, Z)$, we need to select a "good" $\widetilde{g}$ for a given path dependent payoff, g. The following theorems give classes of $(g, \widetilde{g})$, which satisfy A2. The first class includes, as examples, Asian options and a Lookback option.

Theorem 4.1. *Let $h : D([0,\tau]) \to \mathbb{R}$ satisfy*

$$\inf_{0\le s\le\tau} w(s) \le h(w) \le \sup_{0\le s\le\tau} w(s), \tag{4.1}$$

$$h(aw) = ah(w), \tag{4.2}$$

for any $w \in D([0,\tau])$ and $a \in \mathbb{R}$, and let $(g, \widetilde{g})$ represent

$$g(w) = (h(w) - K)^+, \quad \widetilde{g}(w) = C(e^{w(0)} - K/C)^+,$$

where $C := E[h((S_t)_{0\le t\le\tau})]$. Then A2 holds in cases of (**BM**) *and* (**NIG**) *with the parameter condition,*

$$\frac{2b}{\delta + \sqrt{b^2+\delta^2}} < 1.$$

The second class includes a barrier option. It also includes the Asian options with another $\widetilde{g}$.

Theorem 4.2. *Let $(g, \widetilde{g})$ satisfy there exist M_U, M_L and M such that*

$$\inf_{0\le s\le\tau} w(s) > M_L \Rightarrow g(w) = \widetilde{g}(w), \tag{4.3}$$

$$\sup_{0\le s\le\tau} w(s) < M_U \Rightarrow g(w) = \widetilde{g}(w), \tag{4.4}$$

$$|g(w) - \widetilde{g}(w)| \le \sup_{0\le s\le\tau} w(s) + M. \tag{4.5}$$

Then A2 holds in cases of (**BM**) *and* (**NIG**) *with the parameter condition,*

$$\frac{2b}{\delta + \sqrt{b^2+\delta^2}} < 1.$$

We note that the classes given by Theorem 4.1 and Theorem 4.2 are just examples of class satisfying A2 and that there may exist many other classes. Especially, for given payoff g, the $\widetilde{g}$ is not unique.

4.1 Discrete Asian option

The payoff for a discrete Asian option whose strike price is K and whose maturity time is T is defined by

$$\left(\frac{1}{N}\sum_{i=1}^{N} S_{T-\tau_i} - K\right)^+$$

where $0 \le \tau_1 \le \cdots \le \tau_N = \tau$. In this case, let h be defined by

$$h(w) := \frac{1}{N}\sum_{i=1}^{N} w(\tau - \tau_i).$$

We can then express the option as

$$
\begin{aligned}
g((S_s)_{T-\tau\le s\le T}) &= (h((S_s)_{T-\tau\le s\le T}) - K)^+ \\
&= \left(\frac{1}{N}\sum_{i=1}^{N} S_{T-\tau_i} - K\right)^+ .
\end{aligned}
$$

Then it is easy to see that h satisfies the conditions of Theorem 4.1.

To apply Theorem 4.2, let $(g, \widetilde{g})$ be defined by

$$
\begin{aligned}
g(w) &:= \left(\frac{1}{N}\sum_{i=1}^{N} w(\tau-\tau_i) - K\right)^+ , \\
\widetilde{g}(w) &:= \frac{1}{N}\sum_{i=1}^{N} (w(\tau-\tau_i) - K_i)^+
\end{aligned}
$$

where $\frac{1}{N}\sum_{i=1}^{N} K_i = K$. Then $(g, \widetilde{g})$ satisfys the conditions of Theorem 4.2 with respect to $M_U := \min_i K_i$ and $M_L := \max_i K_i$.

4.2 Integral Asian Option

Similar to the payoff for a discrete Asian option, we define that for an Asian option as

$$
\left(\frac{1}{\tau}\int_{T-\tau}^{T} S_s\, ds - K\right)^+ .
$$

In this case, let h be defined by

$$
h(w) := \frac{1}{\tau}\int_0^{\tau} w(s) ds.
$$

Then h satisfies the conditions of Theorem 4.1.

To apply Theorem 4.2, let $(g, \widetilde{g})$ be defined by

$$
\begin{aligned}
g(w) &:= \left(\frac{1}{\tau}\int_0^{\tau} w(s)\, ds - K\right)^+ , \\
\widetilde{g}(w) &:= \frac{1}{\tau}\int_0^{\tau} (w(s) - K_s)^+\, ds
\end{aligned}
$$

where $\frac{1}{\tau}\int_0^{\tau} K_s\, ds = K$. Then $(g, \widetilde{g})$ satisfys the conditions of Theorem 4.2 with respect to $M_U := \inf_s K_s$ and $M_L := \sup_s K_s$.

4.3 Discrete Lookback Option

The payoff for a discrete Lookback option whose strike price is K and whose maturity time is T is defined by

$$
\left(\max_{i=1,\dots,N} S_{T-\tau_i} - K\right)^+
$$

where $0 \leq \tau_1 \leq \cdots \leq \tau_N = \tau$. In this case, let h defined by

$$h(w) := \max_{i=1,\ldots,N} w(\tau_i).$$

Then h satisfies the conditions of Theorem 4.1.

4.4 Partial Barrier Option

The payoff for a knock-in partial barrier option with the strike price K, the barrier level L, the maturity time T, and monitoring period $[T-\tau, T]$ is defined by

$$(S_T - K)^+ 1_{\{\inf_{T-\tau \leq t \leq T} S_t \geq L\}}$$

where $L > K$. Let $(g, \widetilde{g})$ be defined by

$$\begin{aligned} g(w) &:= (w(\tau) - K)^+ 1_{\{\inf w \geq L\}}, \\ \widetilde{g}(w) &:= (w(\tau) - L)^+. \end{aligned}$$

Then $(g, \widetilde{g})$ satisfies the conditions of Theorem 4.2 with respect to $M_U := K$ and $M_L := L$.

5 Proof of Theorem 4.1 and Theorem 4.2

To prove theorems, we use Lebesgue integrability. By substituting $e^z = x$, the condition A2 is replaced by a condition about Z;

Lemma 5.1.

$$\int_{-\infty}^{\infty} |\hat{\epsilon}(z)| \sup_t \frac{d\hat{\nu}_t}{d\bar{\nu}}(z)\, \bar{\nu}(dz) < \infty \tag{5.1}$$

implies A2 for $\hat{\epsilon}(z) := \epsilon(e^z)$.

To see (5.1), it is enough to demonstrate that the integrand vanishes fast when $|z| \to \infty$. At first, we start with descriptions of the bound of $\sup_t \frac{d\hat{\nu}_t}{d\bar{\nu}}(x)$. Note that this density function, $\sup_t \frac{d\nu_t}{d\bar{\nu}}(x)$, depends only on the process Z but not payoff and its approximation function, $(g, \widetilde{g})$;

$$\textbf{(BM)} \qquad \sup_t \frac{d\hat{\nu}_t}{d\bar{\nu}}(z) \leq C_{(\mathbf{BM})} \exp\left(\frac{\mu}{\sigma^2} z\right), \tag{5.2}$$

$$\textbf{(NIG)} \qquad \sup_t \frac{d\hat{\nu}_t}{d\bar{\nu}}(z) \leq C_{(\mathbf{NIG})} \exp\left(\left(\theta + b\frac{2\sqrt{\theta^2 + \mu^2}}{\delta + \sqrt{b^2 + \delta^2}}\right) z\right) \tag{5.3}$$

where $C_{(\mathbf{BM})}$ and $C_{(\mathbf{NIG})}$ are constants.

We next give a lemma about the order for

$$\hat{\epsilon}(z) = E[g((e^{Z_s^z})_{0\le s\le\tau}) - \widetilde{g}((e^{Z_s^z})_{0\le s\le\tau})]$$

to prove Theorem 4.1 where $Z_s^z := Z_s - z_0 + z$. The order is described by the tail order for each process, Z.

Lemma 5.2. *Let h and $(g,\widetilde{g})$ satisfy the same conditions as in Theorem 4.1. Let Z_t^0 be a Levy procces starting with 0 such that*

$$0 < \min\{\inf_{0<u\le\tau} P(Z_u^0 > 0), \inf_{0<u\le\tau} P(Z_u^0 < 0)\}.$$

Then it holds that as $z\to\infty$,

$$|\hat{\epsilon}(z)| \quad = \quad O\left(P(Z_\tau^0 < \log K - z)\right).$$

Also, for any $\varepsilon > 0$, it holds that as $z\to-\infty$,

$$|\hat{\epsilon}(z)| \quad = \quad O\left(P(Z_\tau^0 > \log K - z)^{1-\varepsilon}\right)$$

if $E[e^{\frac{1}{\varepsilon}\sup Z_s^0}] < \infty$.

We next give a description of the order when Z is either (**BM**) or (**NIG**);

(5.4) (**BM**) $\quad P(Z_t < \log K - z) = O\big(\frac{1}{z}e^{-\frac{(z+|\log K+\mu\tau|)^2}{2\sigma^2\tau}}\big) \qquad (z\to\infty)$

(5.5) (**BM**) $\quad P(Z_t > \log K - z) = O\big(\frac{1}{z}e^{-\frac{(z-|\log K+\mu\tau|)^2}{2\sigma^2\tau}}\big) \qquad (z\to-\infty)$

(5.6) (**NIG**) $\quad P(Z_t < \log K - z) = O(|z|^{-\frac{3}{2}}e^{-\theta z-\sqrt{\theta^2+\mu^2}|z|}) \qquad (z\to\infty)$

(5.7) (**NIG**) $\quad P(Z_t > \log K - z) = O(|z|^{-\frac{3}{2}}e^{-\theta z-\sqrt{\theta^2+\mu^2}|z|}) \qquad (z\to-\infty)$

We now prove Theorem 4.1.

Proof of Theorem 4.1. In the case of (**BM**), we have the following from the results of Lemma 5.2, (5.2), (5.4), and (5.5) that

$$|\epsilon(z)|\sup_t \frac{d\nu_t}{d\bar{\nu}}(z) \quad = \quad O\left(e^{-\frac{z^2\rho}{2\sigma^2\tau}}\right)$$

for any $\rho < 1$. This guarantees the integrability of the integrand in (5.1).

In the case of (**NIG**), we also have the following from the results of Lemma 5.2, (5.3), (5.6), and (5.7) that

$$|\epsilon(z)|\sup_t \frac{d\nu_t}{d\bar{\nu}}(z) \quad = \quad O(|z|^{-\frac{3}{2}\rho}e^{\theta(1-\rho)z+b\frac{2\sqrt{\theta^2+\mu^2}}{\delta+\sqrt{b^2+\delta^2}}z-\rho\sqrt{\theta^2+\mu^2}|z|})$$

for any $\rho < 1$. If

$$\frac{2b}{\delta+\sqrt{b^2+\delta^2}} < 1,$$

this guarantees the integrability of the integrand in (5.1). □

Proof of Theorem 4.2. Similarly to Theorem 4.1, the theorem is proved by using a following lemma instead of Lemma 5.2. □

Lemma 5.3. *Let $(g, \widetilde{g})$ satisfy the same conditions as in Theorem 4.2. Let Z_t^0 be a Levy procces starting with* 0 *such that*

$$0 < \min\{\inf_{0<u\leq\tau} P(Z_u^0 > 0), \inf_{0<u\leq\tau} P(Z_u^0 < 0)\}.$$

Then for $\epsilon > 0$, it holds that as $|z| \to \infty$,

$$|\hat{\epsilon}(z)| \quad = \quad O\left(\min\{P(Z_\tau^0 > \log M_U - z)^{1-\varepsilon}, P(Z_\tau^0 < \log M_L - z)^{1-\varepsilon}\}\right)$$

if $E[e^{\frac{1}{\varepsilon}(\sup Z_s^0 - \inf Z_s^0)}] < \infty$.

5.1 Proof of lemmas

Note that Lemma 5.2 and Lemma 5.3 do not require that Z be exact for either the (**BM**) or the (**NIG**) case, but that Z has a "reflection principle";

Lemma 5.1. *Let Z_t^0 be a Levy procces starting with* 0*. Then for any $a > 0$,*

$$\begin{aligned} P(Z_\tau^0 > a) &\geq DP(\sup_{0\leq s\leq\tau} Z_s^0 > a) \quad \textit{and} \\ P(Z_\tau^0 < -a) &\geq DP(\inf_{0\leq s\leq\tau} Z_s^0 < -a). \end{aligned}$$

where

$$D := \min\{\inf_{0<u\leq\tau} P(Z_u^0 > 0), \inf_{0<u\leq\tau} P(Z_u^0 < 0)\}.$$

Proof. Let τ_a be the hitting time of (a, ∞); i.e., $\tau_a := \inf\{s \mid Z_s^0 \in (a, \infty)\}$. Since the path of Z^0 is right continous, $Z_{\tau_a}^0 \geq a$ if $\tau_a < \infty$. Therefore, the strong Markov property implies

$$\begin{aligned} P(Z_\tau^0 > a) &= P(Z_\tau^0 > a, \tau_a \leq \tau) \\ &\geq P(Z_\tau^0 > Z_{\tau_a}^0, \tau_a \leq \tau) \\ &= P(Z_\tau^0 - Z_{\tau_a}^0 > 0, \tau_a \leq \tau) \\ &= \int_0^\tau P(Z_{\tau-s}^0 > 0) P(\tau_a = ds) \\ &\geq \inf_{0<u\leq\tau} P(Z_u^0 > 0) \int_0^\tau P(\tau_a = ds) \\ &= \inf_{0<u\leq\tau} P(Z_u^0 > 0) P(\tau_a \leq \tau) \\ &= \inf_{0<u\leq\tau} P(Z_u^0 > 0) P(\sup_{0\leq s\leq\tau} Z_s^0 > a). \end{aligned}$$

The proof of the other case is similar. □

We now prove Lemma 5.2 and Lemma 5.3:

Proof of Lemma 5.2. From the definition of $\hat{\epsilon}(z)$ and the assumption of $(g, \widetilde{g})$,

$$\begin{aligned}\hat{\epsilon}(z) &= E[g((e^{Z_s^z})_{0\le s\le\tau}) - \widetilde{g}((e^{Z_s^z})_{0\le s\le\tau})] \\ &= E[(h((e^{Z_s^z})_{0\le s\le\tau}) - K)^+ - C(e^{Z_0^z} - K/C)^+] \\ &= E[(h((e^{z+Z_s^0})_{0\le s\le\tau}) - K)^+] - C(e^z - K/C)^+.\end{aligned}$$

When $z \le \log \frac{K}{C}$, we have the following inequality from (4.1), the Hölder inequality and the reflection principle that

$$\begin{aligned}\hat{\epsilon}(z) &= E[(h((e^{z+Z_s^0})_{0\le s\le\tau}) - K)^+] \\ &\le E[(\sup_{0\le s\le\tau} e^{z+Z_s^0} - K)^+] \\ &\le E[(e^z \exp(\sup_{0\le s\le\tau} Z_s^0) - K)^p]^{1/p} P(\sup_{0\le s\le\tau} Z_s^0 > \log K - z)^{1/q} \\ &\le E[(K/C \exp(\sup_{0\le s\le\tau} Z_s^0) - K)^p]^{1/p} P(\sup_{0\le s\le\tau} Z_s^0 > \log K - z)^{1/q} \\ &\le D^{-1/q} E[(K/C \exp(\sup_{0\le s\le\tau} Z_s^0) - K)^p]^{1/p} P(Z_\tau^0 > \log K - z)^{1/q},\end{aligned}$$

where $1 < q < \infty$ with $\frac{1}{q} + \frac{1}{p} = 1$. This yields the result

$$\hat{\epsilon}(z) = O\left(P(Z_\tau^0 > \log K - z)^{1/q}\right)$$

as $z \to -\infty$.

On the other hand, when $z > \log \frac{K}{C}$, (4.2), (4.1) and the reflection principle imply that

$$\begin{aligned}\hat{\epsilon}(z) &= E[(h((e^{Z_s^z})_{0\le s\le\tau}) - K)^+] - (Ce^z - K) \\ &= E[(h((e^{z+Z_s^0})_{0\le s\le\tau}) - K)^+] - (E[h((e^{Z_s^0})_{0\le s\le\tau})]e^z - K) \\ &= E[(e^z h((e^{Z_s^0})_{0\le s\le\tau}) - K)^-] \\ &\le E[(e^z \inf_{0\le s\le\tau} e^{Z_s^0} - K)^-] \\ &\le KP(\inf_{0\le s\le\tau} Z_s^0 < \log K - z) \\ &\le \frac{K}{D} P(Z_\tau^0 < \log K - z).\end{aligned}$$

This yields the result $\hat{\epsilon}(z) = O(P(Z_\tau^0 < \log K - z))$ as $z \to \infty$. This completes the proof. □

Proof of Lemma 5.3. Note that

$$\sup e^{Z_s^z} = e^z \sup e^{Z_s^0} = e^z \inf e^{Z_s^0} \frac{\sup e^{Z_s^0}}{\inf e^{Z_s^0}} = \inf e^{Z_s^z} e^{\sup Z_s^0 - \inf Z_s^0}.$$

Then from the definition of $\hat{\epsilon}(z)$, (4.3), (4.4) and (4.5), we have

$$\begin{aligned}
\hat{\epsilon}(z) &= E[g((e^{Z_s^z})_{0\le s\le\tau}) - \widetilde{g}((e^{Z_s^z})_{0\le s\le\tau})] \\
&= E[\Big(g((e^{Z_s^z})_{0\le s\le\tau}) - \widetilde{g}((e^{Z_s^z})_{0\le s\le\tau})\Big) 1_{\{\sup e^{Z_s^z}>M_U, M_L>\inf e^{Z_s^z}\}}] \\
&\le E[\Big(\sup e^{Z_s^z} + M\Big) 1_{\{\sup e^{Z_s^z}>M_U, M_L>\inf e^{Z_s^z}\}}] \\
&\le E[\Big(M_L e^{\sup Z_s^0 - \inf Z_s^0} + M\Big) 1_{\{\sup e^{Z_s^z}>M_U, M_L>\inf e^{Z_s^z}\}}].
\end{aligned}$$

Therefore, the Hölder inequality and the reflection principle imply

$$\begin{aligned}
\hat{\epsilon}(z) &\le (M_L E[e^{p(\sup Z_s^0 - \inf Z_s^0)}]^{\frac{1}{p}} + M) P(\sup e^{Z_s^z} > M_U, M_L > \inf e^{Z_s^z})^{\frac{1}{q}} \\
&= (M_L E[e^{p(\sup Z_s^0 - \inf Z_s^0)}]^{\frac{1}{p}} + M) P(\sup Z_s^0 > \log M_U - z, \log M_L - z > \inf Z_s^0)^{\frac{1}{q}} \\
&\le (M_L E[e^{p(\sup Z_s^0 - \inf Z_s^0)}]^{\frac{1}{p}} + M) D^{-\frac{1}{q}} \min\{P(Z_\tau^0 > \log M_U - z), P(\log M_L - z > Z_\tau^0)\}^{\frac{1}{q}}.
\end{aligned}$$

□

6 Appendix

In this section, we present some properties of (**BM**) and (**NIG**); α and $\hat{\nu}$ of each process, (5.2), (5.3), (5.4), (5.5), (5.6) and (5.7).

6.1 Brownian Motion

Recall that $Z_t^z = z + \sigma W_t + \mu t$ and $Z_t = Z_t^{z_0}$. Then the density function of Z_t^z with respect to the Lebesgue measure is

$$y \mapsto \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(y-\mu t - z)^2}{2\sigma^2 t}}.$$

By setting $\alpha : [0,\infty) \to [0,\infty)$ as $\alpha(t) = t^{-\frac{1}{2}} e^{-\frac{\mu^2 t}{2\sigma^2}}$, we have

$$\frac{d\hat{\nu}_t}{d\bar{\nu}}(z) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-z_0)^2}{2\sigma^2 t} + \frac{\mu(z-z_0)}{\sigma^2}}.$$

From the fact that $0 < e^{-\frac{(z-z_0)^2}{2\sigma^2 t}} \le 1$ for all t, we have

$$\sup_t \frac{d\hat{\nu}_t}{d\bar{\nu}}(z) \le \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{\mu(z-z_0)}{\sigma^2}}.$$

This inequality shows (5.2).

For $y > 0$, we have

$$P(|Z_\tau^0| > y) = \Phi(\frac{y-\mu\tau}{\sigma\sqrt{\tau}}) + \Phi(\frac{y+\mu\tau}{\sigma\sqrt{\tau}}),$$

where

$$\Phi(\xi) = \int_{\xi}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx.$$

It holds from L'Hopital's theorem that $1 - \Phi(\xi) = \Phi(\xi) \sim \frac{1}{\sqrt{2\pi}\xi} e^{-\frac{\xi^2}{2}}$. This shows (5.4) and (5.5).

6.2 Normal Inverse Gaussian

Recall that $Z_t^z = z + W_{\mathrm{IG}(t)} + \theta \mathrm{IG}(t) + bt$ where $\mu \in \mathbb{R}$, and we denote the inverse Gaussian subordinator $\mathrm{IG}(t) = \inf\{s > 0; B_s + \mu s > \delta t\}$ where $\delta > 0$. Then the density function of Z_t^z with respect to the Lebesgue measure is

$$y \mapsto \frac{1}{\pi} \sqrt{\frac{\theta^2 + \mu^2}{\left(\frac{y-z-bt}{\delta t}\right)^2 + 1}} e^{\mu \delta t + \theta(y-z-bt)} K_1 \left(\delta t \sqrt{(\theta^2 + \mu^2)(1 + \left(\frac{y - z - bt}{\delta t}\right)^2)} \right)$$

where K_1 is the Bessel function of the third kind (See [8]).

We use a estimation of the Bessel function from page378 of [1];

$$K_1(y) \quad \sim \quad \sqrt{\frac{\pi}{2y}} e^{-y}. \tag{6.1}$$

We set $\alpha(t)$ as

$$\alpha(t) = \frac{1}{\sqrt{t}} e^{t\left(\mu\delta - \theta b - \sqrt{(b^2+\delta^2)(\theta^2+\mu^2)}\right)}.$$

Then it holds that

$$\begin{aligned} \frac{d\hat{\nu}_t}{d\bar{\nu}}(x) \quad &\to \quad \frac{\delta}{\sqrt{2\pi}} \left(\frac{\theta^2 + \mu^2}{(b^2 + \delta^2)^3} \right)^{\frac{1}{4}} e^{\left(\theta + b\sqrt{\frac{\theta^2+\mu^2}{b^2+\delta^2}}\right)(x - z_0)}, \\ \sup_t \frac{d\hat{\nu}_t}{d\bar{\nu}}(x) \quad &\leq \quad \frac{1}{\sqrt{2\pi}} \left(\frac{\theta^2 + \mu^2}{\delta^2} \right)^{\frac{1}{4}} e^{\left(\theta + b\frac{2\sqrt{\theta^2+\mu^2}}{\delta + \sqrt{b^2+\delta^2}}\right)(x - z_0)}. \end{aligned}$$

This shows (5.2).

Note that $\delta t \sqrt{(\theta^2 + \mu^2)(1 + \left(\frac{z - z_0 - bt}{\delta t}\right)^2)} \to \infty$ when $|z| \to \infty$. By using (6.1) again, we obtain that the density function of Z_τ^0 decays in the order as

$$f_{Z_\tau^0}(z) \sim |z|^{-\frac{3}{2}} e^{\theta z - \sqrt{\theta^2+\mu^2}|z|}$$

when $|z| \to \infty$. By using L'Hopital's theorem, we have (5.6) and (5.7).

References

[1] Abramowith, M. and Stegun, I. A. : Handbook of mathematical functions.

[2] Albrecher, H. and Predota, M. (2004): Op Asian option pricing for NIG Lévy processes. Journal of Computational and Applied Mathematics, 172, 153-168

[3] Armstrong, G. F. (2001): Valuation formulae for window barrier options. Applied Mathematics Finance, 8, 197-208

[4] Benhamou, E. (2002): Fast Fourier transform for discrete Asian options. Journal of Computational Finance, 6,

[5] Bermin, H. P. (2000): Hedging lookback and partial lookback options using Malliavin calculus. Applied Mathematical Finance, 7, 75-100

[6] Bermin, H. P. (2002): A General Approach to Hedging Options:Applications to Barrier and Partial Barrier Options. Mathematical Finance, 12, 199-218

[7] Cont, R. and Tankov, P. (2004): Financial Modeling with Jump Processes (Champan and Hall/CRC Financial Mathematics Series) :Crc Pr I Llc

[8] David, A. (2004): Levy processes and stochastic calculus. Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge

[9] Geman, H. and Yor, M. (1993): Bessel Process, Asian Options, and Perpetuities. Mathematical Finance, 3, 349-375

[10] Heynen, R. C. and Kat, H. M. (1994): Partial Barrier Options. The Journal of Financial Engineering, 3, 253-274

[11] Hilber, H., Reich, N., Schwab, C. and Winter, C. (2009): Numerical methods for Lévy processes. Finance and Stochastic, 13, 471-500

[12] Hishida, Y. and Yasutomi K. (2005): On the asymptotic behavior of the prices of Asian Option. Asia-Pacific Financial Markets, 12, 289-306

[13] Kudryavstev, O. and Levendorskii, S. (2009): Fast and accurate pricing of barrier options under Lévy processes. Finance and Stochastic, 13, 531-562

[14] Kunitomo, N. Takahashi, A. (2003): On Validity of the Asymptotic Expansion Approach in Contigent Claim Analysis. Annals of Applied Probability, 13, 914-952

[15] Linetsky, V. (2004): Spectral Expansions for Asian (Average Price) Options. Operations Research, 52. 856-867

[16] Madan, D., Carr, P. and Chang, E. (1998): The Variance Gamma Process and Option Pricing. Review of Finance, 2, 79-105

[17] Merton, R. C. (1973): Theory of rational option pricing. Bell Journal of Economics, 4, 141-183

[18] Reiner, E. and Rubinstein, M. (1991): Breaking Down the Barriers. Risk, 4, 28-35