Spaces of Type BLO on Non-homogeneous Metric Measure Spaces

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Abstract. Let $(\mathcal{X}, d, \mu)$ be a metric measure space and satisfy the so-called upper doubling condition and the geometrically doubling condition. In this paper, the authors introduce the space RBLO(μ) and prove that it is a subset of the known space RBMO(μ) in this context. Moreover, the authors establish several useful characterizations for the space RBLO(μ). As an application, the authors obtain the boundedness of the maximal Calderón-Zygmund operators from $L^\infty(\mu)$ to RBLO(μ).

1 Introduction

Spaces of homogeneous type were introduced by Coifman and Weiss [3] as a general framework in which many results from real and harmonic analysis on Euclidean spaces have their natural extensions; see, for example, [4, 6, 5]. Recall that a metric space $(\mathcal{X}, d)$ equipped with a Borel measure μ is called a *space of homogeneous type* if $(\mathcal{X}, d, \mu)$ satisfies the following *measure doubling condition* that there exists a positive constant C_μ such that for all balls $B \subset \mathcal{X}$,

$$0 < \mu(2B) \leq C_\mu \mu(B), \tag{1.1}$$

where and in what follows, a ball $B \equiv B(c_B, r_B) \equiv \{x \in \mathcal{X} : d(x, c_B) < r_B\}$, and for any ball B and $\rho \in (1, \infty)$, $\rho B \equiv B(c_B, \rho r_B)$. We point out that in [3] (see also [4]), the metric d appeared in the definition of spaces of homogeneous type was assumed only to be a *quasi-metric*. However, in this paper, for simplicity, we *always assume* that d is a metric.

Meanwhile, many classical results concerning the theory of Calderón-Zygmund operators and function spaces have been proved still valid for non-doubling measures. In particular, let μ be a non-negative Radon measure on $\mathbb{R}^n$ which only satisfies the *polynomial growth condition* that there exist positive constants C and $\kappa \in (0, n]$ such that for

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all $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, $\mu(\{y \in \mathbb{R}^n : |x-y| < r\}) \le Cr^\kappa$. Such a measure does not need to satisfy the doubling condition (1.1). The $L^q(\mu)$-boundedness with $q \in (1, \infty)$ of Calderón-Zygmund operators modeled on the Cauchy integral operator with respect to such a measure, as well as the endpoint spaces of $L^q(\mu)$ scale and the related mapping properties of operators, have been successfully developed in this context. Some highlights of this theory, are the introduction of the Hardy space H^1 and its dual space, the regularized BMO space, by Tolsa [17], the proof of Tb theorem by Nazarov, Treil and Volberg [15], and the solution of the Painlevé problem by Tolsa [18].

However, as pointed out by Hytönen in [8], notwithstanding these impressive achievements, the Calderón -Zygmund theory with non-doubling measures is not in all respects a generalization of the corresponding theory of spaces of homogeneous type. The measures satisfying the polynomial growth condition are different from, not general than, the doubling measures.

To include the spaces of homogeneous type and Euclidean spaces with a non-negative Radon measure satisfying a polynomial condition, Hytönen [8] introduced a new class of metric measure spaces which satisfy the so-called upper doubling condition and the geometrically doubling condition (see, respectively, Definitions 1.1 and 1.2 below), and a notion of spaces of regularized BMO. Later, Hytönen and Martikainen [10] further established a version of Tb theorem in this setting.

Let $(\mathcal{X}, d, \mu)$ be a metric space satisfying the upper doubling condition and geometrically doubling condition. The main purpose of this paper is to introduce the space $\mathrm{RBLO}(\mu)$ and prove that it is a subset of the known space $\mathrm{RBMO}(\mu)$ in this context. Moreover, we establish several useful characterizations, including the one in terms of the natural maximal operator, for the space $\mathrm{RBLO}(\mu)$. As an application, we prove that if the Calderón-Zygmund operator is bounded on $L^2(\mu)$, then the corresponding maximal operator is bounded from $L^\infty(\mu)$ to $\mathrm{RBLO}(\mu)$.

Recently, an atomic Hardy space $H^1(\mu)$ in this setting was introduced in [11] and it was proved in [11] that $(H^1(\mu))^* = \mathrm{RBMO}(\mu)$. As an application, the boundedness of Calderón-Zygmund operators from $H^1(\mu)$ to $L^1(\mu)$ was obtained in [11].

We now recall the upper doubling space in [8].

Definition 1.1. A metric measure space $(\mathcal{X}, d, \mu)$ is called *upper doubling* if μ is a Borel measure on $\mathcal{X}$ and there exists a dominating function $\lambda : \mathcal{X} \times (0, \infty) \to (0, \infty)$ and a positive constant C_λ such that for each $x \in \mathcal{X}$, $r \to \lambda(x, r)$ is non-decreasing, and for all $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mu(B(x, r)) \le \lambda(x, r) \le C_\lambda \lambda(x, r/2). \tag{1.2}$$

In what follows, we write $\nu \equiv \log_2 C_\lambda$ which can be thought of as a dimension of the measure in some sense.

Remark 1.1. (i) Obviously, a space of homogeneous type is a special case of the upper doubling spaces, where one can take the dominating function $\lambda(x, r) \equiv \mu(B(x, r))$. Moreover, let μ be a non-negative Radon measure on $\mathbb{R}^n$ which only satisfies the polynomial growth condition. By taking $\lambda(x, r) \equiv Cr^\kappa$, we see that $(\mathbb{R}^n, |\cdot|, \mu)$ is also an upper doubling measure space.

(ii) It was proved in [11] that there exists a dominating function $\widetilde{\lambda}$ related to λ satisfying the property that there exists a positive constant C such that for all $x, y \in \mathcal{X}$ with $d(x, y) \le r$,

$$\widetilde{\lambda}(x, r) \le C\widetilde{\lambda}(y, r). \tag{1.3}$$

Based on this, in this paper, we *always assume* that the dominating function λ also satisfies (1.3).

Throughout the whole paper, we *also assume* that the underlying metric space $(\mathcal{X}, d)$ satisfies the following geometrically doubling condition.

Definition 1.2. A metric space $(\mathcal{X}, d)$ is called *geometrically doubling* if there exists some $N_0 \in \mathbb{N} \equiv \{1, 2, \cdots\}$ such that for any ball $B(x, r) \subset \mathcal{X}$, there exists a finite ball covering $\{B(x_i, r/2)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most N_0.

Remark 1.2. Let $(\mathcal{X}, d)$ be a metric space. In [8, Lemma 2.3], Hytönen showed that the following statements are mutually equivalent:

(i) $(\mathcal{X}, d)$ is geometrically doubling.

(ii) For any $\epsilon \in (0, 1)$ and any ball $B(x, r) \subset \mathcal{X}$, there exists a finite ball covering $\{B(x_i, \epsilon r)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most $N_0\epsilon^{-n}$, where and in what follows, N_0 is as in Definition 1.2 and $n \equiv \log_2 N_0$.

(iii) For every $\epsilon \in (0, 1)$, any ball $B(x, r) \subset \mathcal{X}$ can contain at most $N_0\epsilon^{-n}$ centers $\{x_i\}_i$ of disjoint balls with radius ϵr.

(iv) There exists $M \in \mathbb{N}$ such that any ball $B(x, r) \subset \mathcal{X}$ can contain at most M centers $\{x_i\}_i$ of disjoint balls $\{B(x_i, r/4)\}_{i=1}^M$.

It is well known that spaces of homogeneous type are geometrically doubling spaces; see [3, p. 67]. Conversely, if $(\mathcal{X}, d)$ is a complete geometrically doubling metric spaces, then there exists a Borel measure μ on $\mathcal{X}$ such that $(\mathcal{X}, d, \mu)$ is a space of homogeneous type; see [14] and [20].

A metric measure space $(\mathcal{X}, d, \mu)$ is called a *non-homogeneous metric measure space* in this paper, if μ is upper doubling and $(\mathcal{X}, d)$ is geometrically doubling. The motivation to develop a harmonic analysis on non-homogeneous metric measure spaces can be found in [8] and also in [19, 4, 3].

The paper is organized as follows. Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous metric measure space. In Section 2, we introduce the space $\mathrm{RBLO}(\mu)$ and obtain some useful properties of this space. In Section 3, a characterization of $\mathrm{RBLO}(\mu)$ in terms of the natural maximal operator is established. In Section 4, we obtain the boundedness of the maximal Calderón-Zygmund operators from $L^\infty(\mu)$ to $\mathrm{RBLO}(\mu)$.

Finally, we make some convention on symbols. Throughout the paper, we denote by C, $\widetilde{C}$, c and $\widetilde{c}$ *positive constants* which are independent of the main parameters, but they may vary from line to line. *Constant with subscript*, such as C_1, does not change in different occurrences. If $f \le Cg$, we then write $f \lesssim g$ or $g \gtrsim f$; and if $f \lesssim g \lesssim f$, we then write $f \sim g$. Also, for any subset $E \subset \mathcal{X}$, χ_E denotes the *characteristic function* of E.

2 The spaces $\mathrm{RBLO}(\mu)$

In this section, we introduce the space $\mathrm{RBLO}(\mu)$ and establish its several equivalent characterizations.

We begin with the coefficients $\delta(B, S)$ for all balls $B \subset S$ which were introduced by Hytönen in [8] as analogues of Tolsa's numbers $K_{Q,R}$ from [17]; see also [11].

Definition 2.1. For all balls $B \subset S$, let

$$\delta(B, S) \equiv \int_{(2S)\setminus B} \frac{d\mu(x)}{\lambda(c_B, d(x, c_B))}.$$

The following useful properties of δ were proved in [11].

Lemma 2.1. *(i) For all balls $B \subset R \subset S$, $\delta(B, R) \le \delta(B, S)$.*

(ii) For any $\rho \in [1, \infty)$, there exists a positive constant C, depending on ρ, such that for all balls $B \subset S$ with $r_S \le \rho r_B$, $\delta(B, S) \le C$.

(iii) For any $\alpha \in (1, \infty)$, there exists a positive constant $\widetilde{C}$, depending on α, such that for all balls B, $\delta(B, \widetilde{B}^{\alpha}) \le \widetilde{C}$.

(iv) There exists a positive constant c such that for all balls $B \subset R \subset S$, $\delta(B, S) \le \delta(B, R) + c\delta(R, S)$. In particular, if B and R are concentric, then $c = 1$.

(v) There exists a positive constant $\widetilde{c}$ such that for all balls $B \subset R \subset S$, $\delta(R, S) \le \widetilde{c}[1 + \delta(B, S)]$; moreover, if B and R are concentric, then $\delta(R, S) \le \delta(B, S)$.

Though the measure condition (1.1) is not assumed uniformly for all balls in the non-homogeneous metric measure space $(\mathcal{X}, d, \mu)$, it was shown in [8] that there are still many small and large balls that have the following (α, β)-doubling property.

Definition 2.2. Let $\alpha, \beta \in (1, \infty)$. A ball $B(x, r) \subset \mathcal{X}$ is called (α, β)-*doubling* if $\mu(\alpha B) \le \beta\mu(B)$.

To be precise, it was proved in [8] that if a metric measure space $(\mathcal{X}, d, \mu)$ is upper doubling and $\beta > C_\lambda^{\log_2 \alpha} = \alpha^{\nu}$, then for every ball $B(x, r) \subset \mathcal{X}$, there exists some $j \in \mathbb{Z}_+ \equiv \mathbb{N} \cup \{0\}$ such that $\alpha^j B$ is (α, β)-doubling. Moreover, let $(\mathcal{X}, d)$ be geometrically doubling, $\beta > \alpha^n$ with $n \equiv \log N_0$ and μ a Borel measure on $\mathcal{X}$ which is finite on bounded sets. Hytönen [8] also showed that for μ-almost every $x \in \mathcal{X}$, there exist arbitrarily small (α, β)-doubling balls centered at x. Furthermore, the radius of these balls may be chosen to be of the form $\alpha^{-j} r$ for $j \in \mathbb{N}$ and any preassigned number $r \in (0, \infty)$. Throughout this paper, for any $\alpha \in (1, \infty)$ and ball B, $\widetilde{B}^{\alpha}$ denotes the *smallest* (α, β_α)-*doubling ball of the form* $\alpha^j B$ *with* $j \in \mathbb{Z}_+$, where

$$\beta_\alpha \equiv \max\{\alpha^n, \alpha^\nu\} + 30^n + 30^\nu = \alpha^{\max\{n, \nu\}} + 30^n + 30^\nu. \tag{2.1}$$

Inspired by the work of [12, 7, 8], we introduce the space $\mathrm{RBLO}(\mu)$ as follows. In what follows, $L^1_{\mathrm{loc}}(\mu)$ denotes the *space of all μ-locally integrable functions*.

Definition 2.3. Let η, $\rho \in (1, \infty)$, and β_ρ be as in (2.1). A real-valued function $f \in L^1_{\rm loc}(\mu)$ is said to be in the *space* $\mathrm{RBLO}(\mu)$ if there exists a non-negative constant C such that for all balls B,

$$\frac{1}{\mu(\eta B)} \int_B \left[f(y) - \operatorname*{essinf}_{\widetilde{B}^\rho} f \right] d\mu(y) \le C, \tag{2.2}$$

and that for all $(\rho,\, \beta_\rho)$-doubling balls $B \subset S$,

$$\operatorname*{essinf}_B f - \operatorname*{essinf}_S f \le C[1 + \delta(B,\, S)]. \tag{2.3}$$

Moreover, the $\mathrm{RBLO}(\mu)$ *norm* of f is defined to be the minimal constant C as above and denoted by $\|f\|_{\mathrm{RBLO}(\mu)}$.

Remark 2.1. (i) It is obvious that $L^\infty(\mu) \subset \mathrm{RBLO}(\mu)$. Moreover, if $f \in \mathrm{RBLO}(\mu)$, then $f + C$ with any fixed $C \in \mathbb{R}$ also belongs to $\mathrm{RBLO}(\mu)$ and $\|f + C\|_{\mathrm{RBLO}(\mu)} = \|f\|_{\mathrm{RBLO}(\mu)}$. Based on this, in this paper, we identify f with its *equivalent class* $\{f + C :\ C \in \mathbb{R}\}$, namely, we regard $\mathrm{RBLO}(\mu)$ as the *quotient space* $\mathrm{RBLO}(\mu)/\mathbb{R}$.

(ii) The classical space $\mathrm{BLO}(\mathbb{R}^n)$ is defined by Coifman and Rochberg [2]. Let μ be a non-negative Radon measure on $\mathbb{R}^n$ which only satisfies the polynomial growth condition. In the setting of $(\mathbb{R}^n,\, |\cdot|,\, \mu)$, the space $\mathrm{RBLO}(\mu)$ was first introduced by Jiang [12] and improved by [7]. Moreover, in this setting, the space $\mathrm{RBLO}(\mu)$ defined as in Definition 2.3 is just the one introduced in [7].

(iii) The definition of $\mathrm{RBLO}(\mu)$ is independent of the choice of the constants η, $\rho \in (1, \infty)$; see Propositions 2.1 and 2.2 below.

Let $\eta \in (1, \infty)$. Suppose that for any given $f \in L^1_{\rm loc}(\mu)$, there exist a non-negative constant $\widetilde{C}$ and a real number f_B for any ball B such that for all balls B,

$$\frac{1}{\mu(\eta B)} \int_B [f(y) - f_B]\, d\mu(y) \le \widetilde{C}, \tag{2.4}$$

that for all balls $B \subset S$,

$$|f_B - f_S| \le \widetilde{C}\, [1 + \delta(B,\, S)], \tag{2.5}$$

and that for all balls B,

$$f_B \le \operatorname*{essinf}_B f. \tag{2.6}$$

We then define the *norm* $\|f\|_{**,\,\eta} \equiv \inf\{\widetilde{C}\}$, where the infimum is taken over all the non-negative constants $\widetilde{C}$ as above.

Proposition 2.1. *The norm* $\|\cdot\|_{**,\,\eta}$ *is independent of the choice of the constant* $\eta \in (1, \infty)$.

Proof. Let $\rho > \eta > 1$ be some fixed constants. Obviously, $\|f\|_{**,\,\rho} \le \|f\|_{**,\,\eta}$. So we only have to show that $\|f\|_{**,\,\eta} \lesssim \|f\|_{**,\,\rho}$.

For the norm $\|f\|_{**,\,\rho}$, there exists a fixed collection $\{f_B\}_B$ of real numbers satisfying (2.4) through (2.6) with the constant $\widetilde{C}$ replace by $\|f\|_{**,\,\rho}$. Fix $\epsilon \in (0, (\eta-1)/\rho)$ and consider a fixed ball $B_0 \equiv B(x_0, r)$. Then, by Remark 1.2(ii), there exists a family $\{B_i \equiv B(x_i, \epsilon r) : x_i \in B_0\}_{i \in I}$ of balls, which cover B_0, where $\sharp I \le N_0 \epsilon^{-n}$. Here and in what follows, for any set I, we use $\sharp I$ to denote the *cardinality* of I. Moreover, $\rho B_i = B(x_i, \epsilon\rho r) \subset B(x_0, \eta r) = \eta B_0$, since $r + \epsilon\rho r < \eta r$. By this, (2.5) and (ii) and (iv) of Lemma 2.1, we have that

$$\begin{aligned}|f_{B_i} - f_{B_0}| &\le |f_{B_i} - f_{\eta B_0}| + |f_{\eta B_0} - f_{B_0}| \le \|f\|_{**,\,\rho}[2 + \delta(B_i, \eta B_0) + \delta(B_0, \eta B_0)]\\ &\lesssim \|f\|_{**,\,\rho}[1 + \delta(B_i, \rho B_i) + \delta(\rho B_i, \eta B_0)] \lesssim \|f\|_{**,\,\rho}.\end{aligned}$$

Thus, by this estimate and $\rho B_i \subset \eta B_0$ again, we obtain

$$\begin{aligned}\int_{B_0} |f(y) - f_{B_0}|\, d\mu(y) &\le \sum_{i \in I} \int_{B_i} |f(y) - f_{B_0}|\, d\mu(y)\\ &\le \sum_{i \in I} \left\{ \int_{B_i} |f(y) - f_{B_i}|\, d\mu(y) + \mu(B_i)|f_{B_i} - f_{B_0}| \right\}\\ &\lesssim \sum_{i \in I} \|f\|_{**,\,\rho}\mu(\rho B_i) \lesssim \|f\|_{**,\,\rho}\mu(\eta B_0),\end{aligned}$$

which, together with (2.6) and the fact that (2.5) holds with the constant $\widetilde{C}$ replaced by $\|f\|_{**,\,\rho}$, yields that $\|f\|_{**,\,\eta} \lesssim \|f\|_{**,\,\rho}$. This finishes the proof of Proposition 2.1. □

Based on Proposition 2.1, from now on, we write $\|\cdot\|_{**}$ instead of $\|\cdot\|_{**,\,\eta}$.

Proposition 2.2. *Let $\eta, \rho \in (1, \infty)$, and β_ρ be as in* (2.1)*. Then the norms $\|\cdot\|_{**}$ and $\|\cdot\|_{\mathrm{RBLO}(\mu)}$ are equivalent.*

Proof. Suppose that $f \in L^1_{\mathrm{loc}}(\mu)$. We first show that

$$\|f\|_{**} \lesssim \|f\|_{\mathrm{RBLO}(\mu)}. \tag{2.7}$$

For any ball B, let $f_B \equiv \operatorname{essinf}_{\widetilde{B}^\rho} f$. Then (2.4) and (2.6) hold with $\widetilde{C} \equiv \|f\|_{\mathrm{RBLO}(\mu)}$. For any two balls $B \subset S$, to show (2.5), we consider two cases.

Case (i) $r_{\widetilde{S}^\rho} \ge r_{\widetilde{B}^\rho}$. In this case, $\widetilde{B}^\rho \subset 2\widetilde{S}^\rho$. Let $S_0 \equiv \widetilde{2\widetilde{S}^\rho}^\rho$. It follows from Lemma 2.1 that $\delta(\widetilde{S}^\rho, S_0) \lesssim 1$ and $\delta(\widetilde{B}^\rho, S_0) \lesssim 1 + \delta(B, S)$, which together with (2.3) shows that

$$\begin{aligned}|f_B - f_S| = \left| \operatorname*{essinf}_{\widetilde{B}^\rho} f - \operatorname*{essinf}_{\widetilde{S}^\rho} f \right| &\le \left| \operatorname*{essinf}_{\widetilde{B}^\rho} f - \operatorname*{essinf}_{S_0} f \right| + \left| \operatorname*{essinf}_{S_0} f - \operatorname*{essinf}_{\widetilde{S}^\rho} f \right|\\ &\le [2 + \delta(\widetilde{B}^\rho, S_0) + \delta(\widetilde{S}^\rho, S_0)]\|f\|_{\mathrm{RBLO}(\mu)}\\ &\lesssim [1 + \delta(B, S)]\|f\|_{\mathrm{RBLO}(\mu)}.\end{aligned}$$

Case (ii) $r_{\widetilde{S}^\rho} < r_{\widetilde{B}^\rho}$. In this case, $\widetilde{S}^\rho \subset 2\widetilde{B}^\rho$. Notice that $r_{\widetilde{S}^\rho} \geq r_B$. Thus, there exists a unique $m \in \mathbb{N}$ such that $r_{\rho^{m-1}B} \leq r_{\widetilde{S}^\rho} < r_{\rho^m B}$ and $r_{\rho^m B} \leq r_{\widetilde{B}^\rho}$, since $r_{\widetilde{S}^\rho} < r_{\widetilde{B}^\rho}$. Therefore, $\widetilde{S}^\rho \subset 2\rho^m B \subset 2\widetilde{B}^\rho$. Set $B_0 \equiv \widetilde{2\widetilde{B}^\rho}^{\rho}$. Then another application of Lemma 2.1 implies that $\delta(\widetilde{B}^\rho, B_0) \lesssim 1$ and

$$\delta(\widetilde{S}^\rho, B_0) \lesssim \delta(\widetilde{S}^\rho, 2\rho^m B) + \delta(2\rho^m B, B_0) \lesssim 1.$$

An argument similar to Case (i) also establishes (2.5) in this case. Thus, (2.5) always holds.

Now let us show the converse of (2.7). For $f \in L^1_{\mathrm{loc}}(\mu)$, assume that there exists a sequence $\{f_B\}_B$ of real numbers satisfying (2.4) through (2.6) with the non-negative constant $\widetilde{C}$ replaced by $\|f\|_{**}$. For any ball B, by (2.5), (2.6) and Lemma 2.1,

$$f_B - \operatorname*{essinf}_{\widetilde{B}^\rho} f = f_B - f_{\widetilde{B}^\rho} + f_{\widetilde{B}^\rho} - \operatorname*{essinf}_{\widetilde{B}^\rho} f \leq [1 + \delta(B, \widetilde{B}^\rho)]\|f\|_{**} \lesssim \|f\|_{**}.$$

This together with (2.4) yields that for any ball B,

$$\begin{aligned}
&\frac{1}{\mu(\eta B)} \int_B \left[f(y) - \operatorname*{essinf}_{\widetilde{B}^\rho} f\right] d\mu(y) \\
&\quad = \frac{1}{\mu(\eta B)} \int_B [f(y) - f_B]\, d\mu(y) + \frac{\mu(B)}{\mu(\eta B)} \left[f_B - \operatorname*{essinf}_{\widetilde{B}^\rho} f\right] \lesssim \|f\|_{**}.
\end{aligned}$$

On the other hand, for any (ρ, β_ρ)-doubling ball B, since (2.4) holds with ρ by Proposition 2.1, we then have

$$\frac{1}{\mu(B)} \int_B [f(y) - f_B]\, d\mu(y) \leq \frac{\mu(\rho B)}{\mu(B)} \|f\|_{**} \lesssim \|f\|_{**}.$$

Then from (2.5) and (2.6), it follows that for any two (ρ, β_ρ)-doubling balls $B \subset S$,

$$\begin{aligned}
\operatorname*{essinf}_{B} f - \operatorname*{essinf}_{S} f &\leq \operatorname*{essinf}_{B} f - f_B + f_B - f_S \\
&\leq \frac{1}{\mu(B)} \int_B [f(y) - f_B]\, d\mu(y) + [1 + \delta(B, S)]\|f\|_{**} \\
&\lesssim [1 + \delta(B, S)]\|f\|_{**}.
\end{aligned}$$

This establishes the converse of (2.7), and hence finishes the proof of Proposition 2.2. □

Remark 2.2. In [8], the space $\mathrm{RBMO}(\mu)$ was defined in the following way, namely, let $\eta \in (1, \infty)$, a function $f \in L^1(\mu)$ is said to be in the *space* $\mathrm{RBMO}(\mu)$ if there exists a non-negative constant C and a complex number f_B for any ball B such that for all balls B,

$$\frac{1}{\mu(\eta B)} \int_B |f(y) - f_B|\, d\mu(y) \leq C$$

and that for all balls $B \subset S$,

$$|f_B - f_S| \leq C[1 + \delta(B, S)].$$

Moreover, the RBMO(μ) *norm* of f is defined to be the minimal constant C as above and denoted by $\|f\|_{\mathrm{RBMO}(\mu)}$. From [8, Lemma 4.6], Propositions 2.1 and 2.2, it is easy to follow that RBLO(μ) $\subset$ RBMO(μ).

Proposition 2.3. *Let η, $\rho \in (1, \infty)$, and β_ρ be as in (2.1). For $f \in L^1_{\mathrm{loc}}(\mu)$, the following statements are equivalent:*

(i) $f \in \mathrm{RBLO}(\mu)$.

(ii) There exists a non-negative constant C_1 satisfying (2.3) and that for all (ρ, β_ρ)-doubling balls B,

$$\frac{1}{\mu(B)} \int_B \left[f(y) - \operatorname*{essinf}_B f \right] d\mu(y) \le C_1. \tag{2.8}$$

(iii) There exists a non-negative constant C_2 satisfying (2.8) and that for all (ρ, β_ρ)-doubling balls $B \subset S$,

$$m_B(f) - m_S(f) \le C_2[1 + \delta(B, S)], \tag{2.9}$$

where and in what follow, $m_B(f)$ denotes the mean of f over B, namely, $m_B(f) \equiv \frac{1}{\mu(B)} \int_B f(y)\, d\mu(y)$.

Moreover, the minimal constants C_1 and C_2 as above are equivalent to $\|f\|_{\mathrm{RBLO}(\mu)}$.

To prove Proposition 2.3, we need the following lemma, which is a simple corollary of [6, Theorem 1.2] and [8, Lemma 2.5]; see also [11, Lemma 2.2].

Lemma 2.2. *Let $(\mathcal{X}, d)$ be a geometrically doubling metric space. Then every family $\mathcal{F}$ of balls of uniformly bounded diameter contains an at most countable disjointed subfamily $\mathcal{G}$ such that $\cup_{B\in\mathcal{F}} B \subset \cup_{B\in\mathcal{G}} 5B$.*

Proof of Proposition 2.3. By Propositions 2.1 and 2.2, it suffices to show Proposition 2.3 with $\eta \equiv 6/5$ and $\rho = 6$. It is easy to see that (i) implies (ii) automatically.

We now prove that (ii) implies (iii). From (2.3) together with (2.8), it follows that for any two $(6, \beta_6)$-doubling balls $B \subset S$,

$$m_B(f) - m_S(f) \le m_B(f) - \operatorname*{essinf}_B f + \operatorname*{essinf}_B f - \operatorname*{essinf}_S f \lesssim C_1[1 + \delta(B, S)],$$

which implies (iii).

Finally, assuming that (iii) holds, we show $f \in \mathrm{RBLO}(\mu)$ by Definition 2.3. If B is a $(6, \beta_6)$-doubling ball, then by (2.8), (2.2) holds. Let B be any ball which is not $(6, \beta_6)$-doubling. For μ-almost every $x \in B$, let B_x be the *biggest* $(30, \beta_6)$*-doubling ball with center x and radius $30^{-k} r_B$ for some $k \in \mathbb{N}$*. Recall that such ball exists by [8, Lemma 3.3]. Moreover, B_x and $5B_x$ are also $(6, \beta_6)$-doubling balls. Since B is not $(6, \beta_6)$-doubling, then $\widetilde{B}^6$ has the radius at least $6r_B$. From this, it follows that $B_x \subset (6/5)B \subset \widetilde{B}^6$. Let

A_x be the *smallest* $(30, \beta_6)$*-doubling ball of the form* $30^k B_x$ *for some* $k \in \mathbb{N}$, which exists by [8, Lemma 3.2]. Then $r_{A_x} \geq r_B$. To verify (2.2), we first claim that

$$\operatorname*{essinf}_{B_x} f - \operatorname*{essinf}_{\widetilde{B}^6} f \lesssim C_2. \tag{2.10}$$

To show (2.10), we consider the following two cases.

Case (i) $r_{\widetilde{B}^6} \leq r_{A_x}$. In this case, $\widetilde{B}^6 \subset 2A_x$. Notice that B_x is also $(6, \beta_6)$-doubling. From (iv), (ii) and (iii) of Lemma 2.1, we deduce that $\delta(B_x, \widetilde{2A_x}^6) \lesssim 1$. This combined with (2.9) and (2.8) yields that

$$\begin{aligned}
\operatorname*{essinf}_{B_x} f - \operatorname*{essinf}_{\widetilde{B}^6} f &\leq m_{B_x}(f) - m_{\widetilde{2A_x}^6}(f) + m_{\widetilde{2A_x}^6}(f) - \operatorname*{essinf}_{\widetilde{2A_x}^6} f \\
&\lesssim C_2 \left[1 + \delta(B_x, \widetilde{2A_x}^6)\right] \lesssim C_2.
\end{aligned}$$

Case (ii) $r_{\widetilde{B}^6} > r_{A_x}$. In this case, since $r_{A_x} \geq r_B$, then $B \subset 2A_x \subset 3\widetilde{B}^6$. This together with (2.9), (2.8), the fact that B_x is also $(6, \beta_6)$-doubling and Lemma 2.1, we have that

$$\begin{aligned}
\operatorname*{essinf}_{B_x} f - \operatorname*{essinf}_{\widetilde{B}^6} f &\leq m_{B_x}(f) - m_{\widetilde{3\widetilde{B}^6}^6}(f) + m_{\widetilde{3\widetilde{B}^6}^6}(f) - \operatorname*{essinf}_{\widetilde{3\widetilde{B}^6}^6} f \\
&\lesssim C_2 \left[1 + \delta(B_x, \widetilde{3\widetilde{B}^6}^6)\right] \lesssim C_2 \left[1 + \delta(B_x, 2A_x) + \delta(2A_x, \widetilde{3\widetilde{B}^6}^6)\right] \\
&\lesssim C_2 \left[1 + \delta(B, \widetilde{3\widetilde{B}^6}^6)\right] \lesssim C_2.
\end{aligned}$$

Thus, (2.10) holds. That is, the claim is true.

Now, by Lemma 2.2, there exists a countable disjoint subfamily $\{B_i\}_i$ of $\{B_x\}_x$ such that for μ-almost every $x \in B$, $x \in \cup_i 5B_i$. Moreover, since for any i, B_i and $5B_i$ are $(6, \beta_6)$-doubling, by (2.8) and (2.10), we have

$$\begin{aligned}
&\int_B \left[f(y) - \operatorname*{essinf}_{\widetilde{B}^6} f\right] d\mu(y) \\
&\quad \leq \sum_i \int_{5B_i} \left|f(y) - \operatorname*{essinf}_{\widetilde{B}^6} f\right| d\mu(y) \\
&\quad \leq \sum_i \int_{5B_i} \left[f(y) - \operatorname*{essinf}_{5B_i} f\right] d\mu(y) + \sum_i \left[\operatorname*{essinf}_{5B_i} f - \operatorname*{essinf}_{\widetilde{B}^6} f\right] \mu(5B_i) \\
&\quad \lesssim C_2 \sum_i \mu(5B_i) + \sum_i \left[\operatorname*{essinf}_{B_i} f - \operatorname*{essinf}_{\widetilde{B}^6} f\right] \mu(5B_i) \\
&\quad \lesssim C_2 \sum_i \mu(5B_i) \lesssim C_2 \sum_i \mu(B_i) \lesssim C_2 \mu\left(\frac{6}{5}B\right).
\end{aligned} \tag{2.11}$$

On the other hand, from (2.8) and (2.9), it follows that for any two $(6, \beta_6)$-doubling balls $B \subset S$,

$$\operatorname*{essinf}_{B} f - \operatorname*{essinf}_{S} f \leq m_B(f) - m_S(f) + m_S(f) - \operatorname*{essinf}_{S} f \lesssim C_2[1 + \delta(B, S)].$$

This together with (2.11) shows that $f \in \mathrm{RBLO}(\mu)$ and $\|f\|_{\mathrm{RBLO}(\mu)} \lesssim C_2$, which implies (i), and hence completes the proof of Proposition 2.3. □

3 A characterization of $\mathrm{RBLO}(\mu)$ in terms of the natural maximal operator

In this section, we give a characterization of $\mathrm{RBLO}(\mu)$ in terms of the *natural maximal operator*. This characterization in $\mathbb{R}^n$ equipped with the n-dimensional Lebesgue measure was obtained by Bennett [1]. In $\mathbb{R}^n$ equipped with a non-doubling measure with polynomial growth, this characterization was first established by Jiang [12] and was improved in [7].

We begin with the notion of the natural maximal operator, which is a variant of the maximal operator introduced by Hytönen in [8]. In the non-doubling context, the natural maximal operator was introduce by Jiang in [12]. For any $f \in L^1_{\mathrm{loc}}(\mu)$ and $x \in \mathcal{X}$, define

$$\mathcal{M}(f)(x) \equiv \sup_{\substack{B \ni x \\ B\,(6,\,\beta_6)-\text{doubling}}} \frac{1}{\mu(B)} \int_B f(y)\,d\mu(y).$$

Obviously, $\mathcal{M}(f)(x) \lesssim \widetilde{M} f(x)$, where the maximal operator $\widetilde{M}$ is defined by setting, for all $x \in \mathcal{X}$,

$$\widetilde{M}(f)(x) \equiv \sup_{B \ni x} \frac{1}{\mu(6B)} \int_B |f(y)|\,d\mu(y).$$

By [8, Proposition 3.5], we know that $\widetilde{M}$ is of weak type (1, 1) and bounded on $L^p(\mu)$ with $p \in (1, \infty]$. As a consequence, $\mathcal{M}$ is also of weak type (1, 1) and bounded on $L^p(\mu)$ with $p \in (1, \infty]$.

Lemma 3.1. *$f \in \mathrm{RBLO}(\mu)$ if and only if $\mathcal{M}(f) - f \in L^\infty(\mu)$ and f satisfies* (2.9)*. Furthermore,*

$$\|\mathcal{M}(f) - f\|_{L^\infty(\mu)} \sim \|f\|_{\mathrm{RBLO}(\mu)}. \tag{3.1}$$

Proof. By [8, Corollary 3.6], we know that for any $f \in L^1_{\mathrm{loc}}(\mu)$ and μ-almost every $x \in \mathcal{X}$,

$$f(x) = \lim_{\substack{B \downarrow x \\ B\,(6,\,\beta_6)-\text{doubling}}} \frac{1}{\mu(B)} \int_B f(y)\,d\mu(y),$$

where the limit is along the decreasing family of all $(6, \beta_6)$-doubling balls containing x, ordered by set inclusion. Using this fact and following the proof of [12, Lemma 1], we can show Lemma 3.1. We omit the details, which completes the proof of Lemma 3.1. □

Theorem 3.1. *Let $f \in \mathrm{RBMO}(\mu)$. Then $\mathcal{M}(f)$ is either infinite everywhere or finite almost everywhere, and in the later case, there exists a positive constant C, independent of f, such that*

$$\|\mathcal{M}(f)\|_{\mathrm{RBLO}(\mu)} \le C\|f\|_{\mathrm{RBMO}(\mu)}.$$

From Lemma 3.1 and Theorem 3.1, we immediately deduce the following result. We omit the details.

Theorem 3.2. *A locally integrable function f belongs to* RBLO(μ) *if and only if there exist $h \in L^\infty(\mu)$ and $g \in \mathrm{RBMO}(\mu)$ with $\mathcal{M}(g)$ finite μ-almost everywhere such that*

$$f = \mathcal{M}(g) + h. \tag{3.2}$$

Furthermore, $\|f\|_{\mathrm{RBLO}(\mu)} \sim \inf(\|g\|_{\mathrm{RBMO}(\mu)} + \|h\|_{L^\infty(\mu)})$, where the infimum is taken over all representations of f as in (3.2).

To prove Theorem 3.1, we need the following characterization of RBMO(μ).

Lemma 3.2. *Let $\eta, \rho \in (1, \infty)$, and β_ρ be as in* (2.1). *For $f \in L^1_{\mathrm{loc}}(\mu)$, the following statements are equivalent:*

(i) $f \in \mathrm{RBMO}(\mu)$.

(ii) There exists a non-negative constant C_3 such that for all (ρ, β_ρ)-doubling balls B,

$$\frac{1}{\mu(B)} \int_B |f(y) - m_B(f)|\, d\mu(y) \le C_3, \tag{3.3}$$

and that for all (ρ, β_ρ)-doubling balls $B \subset S$,

$$|m_B(f) - m_S(f)| \le C_3[1 + \delta(B, S)]. \tag{3.4}$$

(iii) There exists a non-negative constant C_4 satisfying (3.4) *and that for all balls B,*

$$\frac{1}{\mu(\eta B)} \int_B \left|f(y) - m_{\widetilde{B}^\rho}(f)\right| d\mu(y) \le C_4, \tag{3.5}$$

(iv) Let $p \in [1, \infty)$. There exists a non-negative constant C_5 satisfying (3.4) *and that for all balls B,*

$$\left\{\frac{1}{\mu(\eta B)} \int_B \left|f(y) - m_{\widetilde{B}^\rho}(f)\right|^p d\mu(y)\right\}^{1/p} \le C_5, \tag{3.6}$$

Moreover, the minimal constants C_3, C_4 and C_5 as above are equivalent to $\|f\|_{\mathrm{RBMO}(\mu)}$.

Proof. The equivalent of (i) and (ii) is a special case of [11, Proposition 2.2]. Obviously, (iii) implies (ii). By an argument similar to that used in the proof of [11, Proposition 2.2], we have that (ii) implies (iii). Hence, (i), (ii) and (iii) are equivalent.

We now prove the equivalent of (iii) and (iv). By the Hölder inequality, it is easy to see that (iv) implies (iii). Conversely, it follows from [8, Corollary 6.3] that for any ball B,

$$\left\{\frac{1}{\mu(\eta B)} \int_B |f(y) - f_B|^p\, d\mu(y)\right\}^{1/p} \lesssim \|f\|_{\mathrm{RBMO}(\mu)}.$$

On the other hand, from the equivalence of (i) and (iii), we deduce that the number f_B in the definition of RBMO(μ) can be chosen to be $m_{\widetilde{B}^\rho}$. Therefore,

$$\left\{\frac{1}{\mu(\eta B)}\int_B \left|f(y)-m_{\widetilde{B}^\rho}(f)\right|^p\,d\mu(y)\right\}^{1/p} \lesssim \|f\|_{\mathrm{RBMO}(\mu)} \sim \min\{C_4\},$$

which shows that (iii) implies (iv) and hence completes the proof of Lemma 3.2. $\square$

Proof of Theorem 3.1. Suppose that $f\in \mathrm{RBMO}(\mu)$ and there exists a point $x_0\in\mathcal{X}$ such that $\mathcal{M}(f)(x_0)<\infty$. First, we claim that there exists a positive constant C independent of f such that for all $(6,\,\beta_6)$-doubling balls $B\ni x_0$,

$$\frac{1}{\mu(B)}\int_B \mathcal{M}(f)(y)\,d\mu(y)\le C\|f\|_{\mathrm{RBMO}(\mu)}+\inf_{x\in B}\mathcal{M}(f)(x). \tag{3.7}$$

To prove this, we decompose f as

$$f=[f-m_B(f)]\,\chi_{3B}+[m_B(f)\chi_{3B}+f\chi_{\mathcal{X}\setminus(3B)}]\equiv f_1+f_2.$$

We choose $\eta\equiv 6/5$ and $\rho\equiv 6$ in Lemma 3.2. Since $\mathcal{M}$ is bounded on $L^2(\mu)$, by the Hölder inequality, (3.6), (3.4), and (ii) and (iii) of Lemma 2.1, we have

$$\begin{aligned}
&\int_B \mathcal{M}(f_1)(y)\,d\mu(y) \qquad (3.8)\\
&\quad\le [\mu(B)]^{1/2}\left\{\int_{\mathcal{X}}|\mathcal{M}(f_1)(y)|^2\,d\mu(y)\right\}^{1/2}\lesssim [\mu(B)]^{1/2}\left\{\int_{\mathcal{X}}|f_1(y)|^2\,d\mu(y)\right\}^{1/2}\\
&\quad\lesssim [\mu(B)]^{1/2}\left\{\int_{3B}|f(y)-m_{\widetilde{3B}^6}(f)|^2\,d\mu(y)+\int_{3B}|m_B(f)-m_{\widetilde{3B}^6}(f)|^2\,d\mu(y)\right\}^{1/2}\\
&\quad\lesssim [\mu(B)]^{1/2}\left\{\left[\mu\left(\frac{18}{5}B\right)\right]^{1/2}+[\mu(3B)]^{1/2}\left[1+\delta(B,\,\widetilde{3B}^6)\right]\right\}\|f\|_{\mathrm{RBMO}(\mu)}\\
&\quad\lesssim \mu(6B)\|f\|_{\mathrm{RBMO}(\mu)}\lesssim \mu(B)\|f\|_{\mathrm{RBMO}(\mu)}.
\end{aligned}$$

Next, we show that

$$\frac{1}{\mu(B)}\int_B \mathcal{M}(f_2)(y)\,d\mu(y)\lesssim \|f\|_{\mathrm{RBMO}(\mu)}+\inf_{x\in B}\mathcal{M}(f)(x). \tag{3.9}$$

It suffices to show that for any $y\in B$,

$$\mathcal{M}(f_2)(y)\lesssim \|f\|_{\mathrm{RBMO}(\mu)}+\inf_{x\in B}\mathcal{M}(f)(x).$$

To this end, it is enough to show that for any $(6,\,\beta_6)$-doubling ball $S\ni y$ and $y\in B$,

$$\frac{1}{\mu(S)}\int_S f_2(z)\,d\mu(z)\lesssim \|f\|_{\mathrm{RBMO}(\mu)}+\inf_{x\in B}\mathcal{M}(f)(x). \tag{3.10}$$

If $S \subset 3B$, we immediately have that

$$\frac{1}{\mu(S)} \int_S f_2(z)\, d\mu(z) = m_B(f) \le \inf_{x\in B} \mathcal{M}(f)(x).$$

If $S \cap [\mathcal{X} \setminus (3B)] \neq \emptyset$. Then $r_S > r_B$ and $3B \subset (5S)$. Write

$$f_2 = \left[m_B(f) - m_{\widetilde{5S}^6}(f)\right] \chi_{3B} + \left[f - m_{\widetilde{5S}^6}(f)\right] \chi_{\mathcal{X}\setminus(3B)} + m_{\widetilde{5S}^6}(f).$$

Obviously, $m_{\widetilde{5S}^6}(f) \le \inf_{x\in B} \mathcal{M}(f)(x)$. From (3.5), it follows that

$$\begin{aligned}
&\int_S \left\{\left[m_B(f) - m_{\widetilde{5S}^6}(f)\right] \chi_{3B}(z) + \left[f(z) - m_{\widetilde{5S}^6}(f)\right] \chi_{\mathcal{X}\setminus(3B)}(z)\right\} d\mu(z)\\
&\quad \le \mu(3B) \left|m_B(f) - m_{\widetilde{5S}^6}(f)\right| + \int_{5S} \left|f(z) - m_{\widetilde{5S}^6}(f)\right| \chi_{\mathcal{X}\setminus(3B)}(z)\, d\mu(z)\\
&\quad \le \frac{\mu(6B)}{\mu(B)} \int_B \left|f(z) - m_{\widetilde{5S}^6}(f)\right| d\mu(z) + \int_{5S\setminus(3B)} \left|f(z) - m_{\widetilde{5S}^6}(f)\right| d\mu(z)\\
&\quad \lesssim \int_{5S} \left|f(z) - m_{\widetilde{5S}^6}(f)\right| d\mu(z) \lesssim \mu(6S)\|f\|_{\mathrm{RBMO}(\mu)} \lesssim \mu(S)\|f\|_{\mathrm{RBMO}(\mu)},
\end{aligned}$$

which implies (3.10). Hence, (3.9) holds. Combining the estimates for (3.8) and (3.9) yields (3.7).

From (3.7), it follows that for $f \in \mathrm{RBMO}(\mu)$, if $\mathcal{M}(f)(x_0) < \infty$ for some point $x_0 \in \mathcal{X}$, then $\mathcal{M}(f)$ is μ-finite almost everywhere and in this case,

$$\frac{1}{\mu(B)} \int_B \left[\mathcal{M}(f)(y) - \operatorname*{essinf}_{x\in B} \mathcal{M}(f)(x)\right] d\mu(y) \lesssim \|f\|_{\mathrm{RBMO}(\mu)}, \tag{3.11}$$

provided that B is a $(6,\, \beta_6)$-doubling ball. To prove $\mathcal{M}(f) \in \mathrm{RBLO}(\mu)$, by Proposition 2.3, we still need to prove that for any $(6,\, \beta_6)$-doubling balls $B \subset S$,

$$m_B[\mathcal{M}(f)] - m_S[\mathcal{M}(f)] \lesssim [1 + \delta(B,\, S)]\|f\|_{\mathrm{RBMO}(\mu)}. \tag{3.12}$$

To prove (3.12), for any point $x \in B$, we set

$$\mathcal{M}_1(f)(x) \equiv \sup_{\substack{P\ni x,\, P\ (6,\, \beta_6)-\text{doubling}\\ r_P \le 4r_S}} \frac{1}{\mu(P)} \int_P f(y)\, d\mu(y),$$

$$\mathcal{M}_2(f)(x) \equiv \sup_{\substack{P\ni x,\, P\ (6,\, \beta_6)-\text{doubling}\\ r_P > 4r_S}} \frac{1}{\mu(P)} \int_P f(y)\, d\mu(y),$$

$\mathcal{U}_{1,\, B} \equiv \{x \in B : \mathcal{M}_1(f)(x) \ge \mathcal{M}_2(f)(x)\}$ and $\mathcal{U}_{2,\, B} \equiv B \setminus \mathcal{U}_{1,\, B}$. Then for any $x \in B$, $\mathcal{M}(f)(x) = \max[\mathcal{M}_1(f)(x),\, \mathcal{M}_2(f)(x)]$. By writing

$$f = [f - m_S(f)]\chi_{3B} + [f - m_S(f)]\chi_{\mathcal{X}\setminus(3B)} + m_S(f)$$

and using the fact that $m_S(f) \le m_S[\mathcal{M}(f)]$, we see that

$$\begin{aligned} m_B[\mathcal{M}(f)] - m_S[\mathcal{M}(f)] &\le \frac{1}{\mu(B)} \int_{\mathcal{U}_{1,\,B}} \mathcal{M}_1([f - m_S(f)]\chi_{3B})(x)\,d\mu(x) \\ &\quad + \frac{1}{\mu(B)} \int_{\mathcal{U}_{1,\,B}} \mathcal{M}_1([f - m_S(f)]\chi_{\mathcal{X}\setminus(3B)})(x)\,d\mu(x) \\ &\quad + \frac{1}{\mu(B)} \int_{\mathcal{U}_{2,\,B}} \{\mathcal{M}_2(f)(x) - m_S[\mathcal{M}(f)]\}\,d\mu(x) \\ &\equiv \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3. \end{aligned}$$

Notice that $\mathcal{M}$ is bounded on $L^2(\mu)$. From this, the Hölder inequality, Lemma 3.2, and (ii) and (iii) of Lemma 2.1, it follows that

$$\begin{aligned} \mathrm{I}_1 &\le \left\{\frac{1}{\mu(B)} \int_B |\mathcal{M}_1([f - m_S(f)]\chi_{3B})(x)|^2\,d\mu(x)\right\}^{1/2} \\ &\lesssim \left\{\frac{1}{\mu(B)} \int_{3B} |f(x) - m_S(f)|^2\,d\mu(x)\right\}^{1/2} \\ &\lesssim \left\{\frac{1}{\mu(B)} \int_{3B} \left|f(x) - m_{\widetilde{3B}^6}(f)\right|^2\,d\mu(x)\right\}^{1/2} + \left|m_{\widetilde{3B}^6}(f) - m_B(f)\right| \\ &\quad + |m_B(f) - m_S(f)| \\ &\lesssim [1 + \delta(B,\,S)]\|f\|_{\mathrm{RBMO}(\mu)}, \end{aligned}$$

To estimate I_2, we first claim that for any point $x \in B$ and any $(6,\,\beta_6)$-doubling ball $P \ni x$ with $r_P \le 4r_S$,

$$\mathrm{J} \equiv \frac{1}{\mu(P)} \int_P |f(y) - m_S(f)|\chi_{\mathcal{X}\setminus(3B)}(y)\,d\mu(y) \lesssim [1 + \delta(B,\,S)]\|f\|_{\mathrm{RBMO}(\mu)}. \tag{3.13}$$

If $P \subset 3B$, then $\mathrm{J} = 0$ and (3.13) holds automatically. Assume that $P \not\subset 3B$. We then have that $r_P > r_B$, which together with the fact that $r_P \le 4r_S$ implies that $B \subset 3P \subset 17S$. Thus, (3.3) and (3.4), together with (ii), (iii) and (iv) of Lemma 2.1, yield that

$$\begin{aligned} \mathrm{J} &\le \frac{1}{\mu(P)} \int_P |f(y) - m_P(f)|\,d\mu(y) + \left|m_P(f) - m_{\widetilde{3P}^6}(f)\right| \\ &\quad + \left|m_{\widetilde{3P}^6}(f) - m_B(f)\right| + |m_B - m_S(f)| \lesssim [1 + \delta(B,\,S)]\|f\|_{\mathrm{RBMO}(\mu)}, \end{aligned}$$

which further implies that for all $x \in B$,

$$\mathcal{M}_1([f - m_S(f)]\chi_{\mathcal{X}\setminus(3B)})(x) \lesssim [1 + \delta(B,\,S)]\|f\|_{\mathrm{RBMO}(\mu)}.$$

From this, we deduce that $\mathrm{I}_2 \lesssim [1 + \delta(B,\,S)]\|f\|_{\mathrm{RBMO}(\mu)}$.

Now we estimate I_3. Notice that for any $x \in B$, any $(6,\,\beta_6)$-doubling ball P containing x with $r_P > 4r_S$ and $B \subset S$, $S \subset 3P$. Then from (3.4) and the fact $m_{\widetilde{3P}^6}(f) \le m_S[\mathcal{M}(f)]$, it follows that

$$m_P(f) - m_S[\mathcal{M}(f)] \le \left|m_P(f) - m_{\widetilde{3P}^6}(f)\right| + m_{\widetilde{3P}^6}(f) - m_S[\mathcal{M}(f)] \lesssim \|f\|_{\mathrm{RBMO}(\mu)}.$$

Taking the supremum over all $(6, \beta_6)$-doubling balls P containing x with $r_P > 4r_S$, we have that for all $x \in B$,

$$\mathcal{M}_2(f)(x) - m_S[\mathcal{M}(f)] \lesssim \|f\|_{\mathrm{RBMO}(\mu)}.$$

This implies that $\mathrm{I}_3 \lesssim \|f\|_{\mathrm{RBMO}(\mu)}$.

Combining the estimates for I_1 through I_3 leads to (3.12), which together with (3.11) implies that $\mathcal{M}$ is bounded from $\mathrm{RBMO}(\mu)$ to $\mathrm{RBLO}(\mu)$ and hence completes the proof of Theorem 3.1. $\square$

4 Boundedness of the maximal Calderón-Zygmund operators

This section is devoted to the boundedness of the maximal operators associated with the Calderón-Zygmund operators introduced in [10].

Let $\triangle \equiv \{(x, x) : x \in \mathcal{X}\}$ and $L^\infty_b(\mathcal{X})$ denote the space of all functions in $L^\infty(\mathcal{X})$ with bounded support. A standard kernel is a mapping $K : (\mathcal{X} \times \mathcal{X})\backslash\triangle \to \mathbb{C}$ for which, there exist some positive constants σ and C such that for all $x, y \in \mathcal{X}$ with $x \neq y$,

$$|K(x, y)| \le C\frac{1}{\lambda(x, d(x, y))}, \tag{4.1}$$

and that for all $x, \widetilde{x}, y \in \mathcal{X}$ with $d(x, \widetilde{x}) \le \frac{d(x, y)}{2}$,

$$|K(x, y) - K(\widetilde{x}, y)| + |K(y, x) - K(y, \widetilde{x})| \le C\frac{[d(x, \widetilde{x})]^\sigma}{[d(x, y)]^\sigma \lambda(x, d(x, y))}. \tag{4.2}$$

A linear operator T is called a *Calderón-Zygmund operator* with kernel K satisfying (4.1) and (4.2) if for all $f \in L^\infty_b(\mathcal{X})$ and $x \notin \mathrm{supp}\,(f)$,

$$Tf(x) \equiv \int_{\mathcal{X}} K(x, y)f(y)\,d\mu(y). \tag{4.3}$$

Now, we define the corresponding maximal Calderón-Zygmund operator associated with the kernel K. For any $\epsilon \in (0, \infty)$, define the *truncated operator* T_ϵ by setting, for all $x \in \mathcal{X}$,

$$T_\epsilon f(x) \equiv \int_{d(x, y)>\epsilon} K(x, y)f(y)\,d\mu(y). \tag{4.4}$$

The *maximal Calderón-Zygmund operator* T_* is defined by setting, for all $x \in \mathcal{X}$,

$$T_* f(x) \equiv \sup_{\epsilon>0} |T_\epsilon f(x)|. \tag{4.5}$$

Remark 4.1. Let $\mathcal{X} \equiv \mathbb{R}^n$. It is well known that if μ is the n-dimensional Lebesgue measure and T is bounded on $L^2(\mathbb{R}^n)$, then T_* is bounded from $L^\infty(\mu)$ to $\mathrm{BMO}(\mathbb{R}^n)$ (see [16]), and furthermore, is bounded from $L^\infty(\mu)$ to $\mathrm{BLO}(\mathbb{R}^n)$ (see [13]). When μ is a non-doubling measure with polynomial growth, Tolsa [17] proved that if T is bounded on $L^2(\mu)$, then T is bounded from $L^\infty(\mu)$ to $\mathrm{RBMO}(\mu)$, and moreover, the boundedness of T_* from $L^\infty(\mu)$ to $\mathrm{RBLO}(\mu)$ was obtained by Jiang [12].

It was proved in [9] that if the Calderón-Zygmund operator T is bounded on $L^2(\mu)$, then the maximal operator T_* is of weak type (1, 1) and is bounded on $L^p(\mu)$ for any $p \in (1, \infty)$. On the boundedness of T_* when $p = \infty$, we have the following conclusion.

Theorem 4.1. *Let T be the Calderón-Zygmund operator as in* (4.3) *with kernel K satisfying* (4.1) *and* (4.2). *If T is bounded on $L^2(\mu)$, then the maximal operator T_* as in* (4.5) *is bounded from $L^\infty(\mu)$ to* $\mathrm{RBLO}(\mu)$.

Proof. First we claim that there exists a positive constant C such that for all $f \in L^\infty(\mu) \cap L^{p_0}(\mu)$, $p_0 \in [1, \infty)$, and $(6, \beta_6)$-doubling balls B,

$$\frac{1}{\mu(B)} \int_B T_* f(x)\, d\mu(x) \le C\|f\|_{L^\infty(\mu)} + \inf_{y\in B} T_* f(y). \tag{4.6}$$

To prove this, we decompose f as

$$f = f\chi_{5B} + f\chi_{\mathcal{X}\setminus(5B)} \equiv f_1 + f_2.$$

By the Hölder inequality and the $L^2(\mu)$-boundedness of T_*, we have

$$\begin{aligned}\frac{1}{\mu(B)} \int_B T_* f_1(x)\, d\mu(x) &\le \frac{1}{[\mu(B)]^{1/2}} \left\{\int_{\mathcal{X}} [T_*(f\chi_{5B})(x)]^2\, d\mu(x)\right\}^{1/2} \\ &\lesssim \frac{1}{[\mu(B)]^{1/2}} \left\{\int_{\mathcal{X}} |f\chi_{5B}(x)|^2\, d\mu(x)\right\}^{1/2} \\ &\lesssim \frac{[\mu(5B)]^{1/2}}{[\mu(B)]^{1/2}} \|f\|_{L^\infty(\mu)} \lesssim \|f\|_{L^\infty(\mu)}.\end{aligned} \tag{4.7}$$

From (1.3) and (1.2), we deduce that for any ball B, $y \notin 5B$ and $x \in B$,

$$\lambda(c_B, d(y, c_B)) \sim \lambda(y, d(y, c_B)) \sim \lambda(y, d(y, x)) \sim \lambda(x, d(y, x)). \tag{4.8}$$

Notice that

$$\{y \in \mathcal{X} : d(x, y) > 6r_B \text{ for some } x \in B\} \subset [\mathcal{X} \setminus (5B)].$$

It then follows from (4.1), (4.8) and Lemma 2.1(ii) that for all $y \in B$,

$$\begin{aligned}T_* f_2(y) &\le \max\left\{\sup_{\epsilon\ge 6r_B} |T_\epsilon f_2(y)|,\ \sup_{0<\epsilon<6r_B} |T_\epsilon f_2(y)|\right\} \\ &\le \max\left\{T_* f(y),\ \sup_{0<\epsilon<6r_B} \left|\int_{d(y,z)>6r_B} K(y, z) f_2(z)\, d\mu(z)\right.\right.\end{aligned} \tag{4.9}$$

$$
\begin{aligned}
&\quad + \int_{\epsilon<d(y,z)\le 6r_B} K(y,\,z) f_2(z)\,d\mu(z)\Bigg|\Bigg\}\\
&\le T_*f(y) + C\|f\|_{L^\infty(\mu)} \sup_{0<\epsilon<6r_B} \int_{(7B)\setminus(5B)} \frac{1}{\lambda(y,\,d(y,\,z))}\,d\mu(z)\\
&\le T_*f(y) + C\|f\|_{L^\infty(\mu)} \int_{(8B)\setminus B} \frac{1}{\lambda(c_B,\,d(z,\,c_B))}\,d\mu(z)\\
&= T_*f(y) + C\|f\|_{L^\infty(\mu)}\delta(B,\,4B) \le T_*f(y) + C\|f\|_{L^\infty(\mu)},
\end{aligned}
$$

where C is a positive constant independent of f and y. Thus, the proof of the estimate (4.6) is reduced to proving that for all $x,\,y\in B$,

$$
|T_*f_2(x) - T_*f_2(y)| \lesssim \|f\|_{L^\infty(\mu)}. \tag{4.10}
$$

To this end, for any $\epsilon\in(0,\,\infty)$, write

$$
\begin{aligned}
|T_\epsilon f_2(x) - T_\epsilon f_2(y)| &= \left|\int_{d(x,\,z)>\epsilon} K(x,\,z) f_2(z)\,d\mu(z) - \int_{d(y,\,z)>\epsilon} K(y,\,z) f_2(z)\,d\mu(z)\right|\\
&\le \int_{\substack{d(x,\,z)>\epsilon\\ d(y,\,z)>\epsilon}} |K(x,\,z)-K(y,\,z)||f_2(z)|\,d\mu(z)\\
&\quad + \int_{\substack{d(x,\,z)>\epsilon\\ d(y,\,z)\le\epsilon}} |K(x,\,z)f_2(z)|\,d\mu(z)\\
&\quad + \int_{\substack{d(y,\,z)>\epsilon\\ d(x,\,z)\le\epsilon}} |K(y,\,z)f_2(z)|\,d\mu(z) \equiv \mathrm{J}_1+\mathrm{J}_2+\mathrm{J}_3.
\end{aligned}
$$

By (4.2), (4.8) and (1.2), we have that for all $x,\,y\in B$,

$$
\begin{aligned}
\mathrm{J}_1 &\le \int_{\mathcal{X}\setminus(5B)} |K(x,\,z)-K(y,\,z)||f(z)|\,d\mu(z)\\
&\lesssim \|f\|_{L^\infty(\mu)} \int_{\mathcal{X}\setminus(5B)} \frac{[d(x,\,y)]^\sigma}{[d(x,\,z)]^\sigma\lambda(x,\,d(x,\,z))}\,d\mu(z)\\
&\lesssim \|f\|_{L^\infty(\mu)} \int_{\mathcal{X}\setminus(5B)} \left[\frac{r_B}{d(z,\,c_B)}\right]^\sigma \frac{1}{\lambda(c_B,\,d(z,\,c_B))}\,d\mu(z) \lesssim \|f\|_{L^\infty(\mu)}.
\end{aligned}
$$

Now we estimate J_2. Notice that if $z\notin 5B$ and $x\in B$, then $d(x,\,z)>4r_B$. Therefore, for any $\epsilon\in(0,\,4r_B]$ and $x,\,y\in B$, $\{z\notin 5B:\ d(x,\,z)>\epsilon \text{ and } d(y,\,z)\le\epsilon\}=\emptyset$. So, we only need to consider the case that $\epsilon\in(4r_B,\,\infty)$. In this case, there exists a unique $m\in\mathbb{N}$ such that $2^{m-1}r_B<\epsilon\le 2^m r_B$, which leads to that

$$
\{z\notin 5B:\ d(x,\,z)>\epsilon \text{ and } d(y,\,z)\le\epsilon\}\subset[2^{m+1}B\setminus(\max(2,\,2^{m-1}-1)B)].
$$

This, together with (4.1) and (4.8), and (ii) of Lemma 2.1 shows that

$$
\mathrm{J}_2 \lesssim \|f\|_{L^\infty(\mu)} \int_{2^{m+1}B\setminus(\max(2,\,2^{m-1}-1)B)} \frac{1}{\lambda(c_B,\,d(z,\,c_B))}\,d\mu(z)
$$

$$\lesssim \|f\|_{L^\infty(\mu)}\delta(\max(2,\, 2^{m-1}-1)B,\, 2^m B) \lesssim \|f\|_{L^\infty(\mu)}.$$

An argument similar to the estimate of J_2 also yields that $J_3 \lesssim \|f\|_{L^\infty(\mu)}$. Combining the estimates for J_1 through J_3 implies (4.10) and hence (4.6) holds.

Thus, by (4.6), we know that if $f \in L^\infty(\mu) \cap L^{p_0}(\mu)$ with $p_0 \in [1, \infty)$, then $T_* f$ is μ-finite almost everywhere and in this case, by (4.6) again, we have that

$$\frac{1}{\mu(B)}\int_B \left[T_* f(x) - \operatorname*{essinf}_{y\in B} T_* f(y)\right] d\mu(x) \lesssim \|f\|_{L^\infty(\mu)},$$

provided that B is a $(6, \beta_6)$-doubling ball. To prove $T_* f \in \mathrm{RBLO}(\mu)$, by Proposition 2.3, we still need to prove that $T_* f$ satisfies (2.9). Let $B \subset S$ be any two $(6, \beta_6)$-doubling balls. For any $\epsilon \in (0, \infty)$, $x \in B$ and $y \in S$, we set

$$\begin{aligned} T_\epsilon f(x) &= T_\epsilon(f\chi_{5B})(x) + T_\epsilon(f\chi_{(5S)\setminus(5B)})(x) \\ &\quad + \left[T_\epsilon(f\chi_{\mathcal{X}\setminus(5S)})(x) - T_\epsilon(f\chi_{\mathcal{X}\setminus(5S)})(y)\right] + T_\epsilon(f\chi_{\mathcal{X}\setminus(5S)})(y). \end{aligned}$$

By an estimate similar to that of (4.9), we have that for all $y \in S$,

$$T_*(f\chi_{\mathcal{X}\setminus(5S)})(y) \le T_* f(y) + C\|f\|_{L^\infty(\mu)},$$

where C is a positive constant independent of f and y. On the other hand, by the estimate same as that of (4.10), we have that for all $x, y \in S$,

$$\left|T_\epsilon(f\chi_{\mathcal{X}\setminus(5S)})(x) - T_\epsilon(f\chi_{\mathcal{X}\setminus(5S)})(y)\right| \lesssim \|f\|_{L^\infty(\mu)}.$$

For all $x \in B$, if $z \notin 5B$, then $d(x, z) \ge 4r_B$, which together with (4.4), (4.1) and (4.8) shows that

$$\begin{aligned} T_\epsilon(f\chi_{(5S)\setminus(5B)})(x) &= \int_{d(x,\,z)>\epsilon} K(x, z) f\chi_{(5S)\setminus(5B)}(z)\, d\mu(z) \\ &\lesssim \|f\|_{L^\infty(\mu)} \int_{(5S)\setminus(5B)} |K(x, z)|\, d\mu(z) \\ &\lesssim \|f\|_{L^\infty(\mu)} \int_{(5S)\setminus(5B)} \frac{1}{\lambda(x, d(x, z))}\, d\mu(z) \\ &\lesssim \|f\|_{L^\infty(\mu)} \int_{(5S)\setminus B} \frac{1}{\lambda(c_B, d(z, c_B))}\, d\mu(z) \lesssim [1 + \delta(B, S)]\|f\|_{L^\infty(\mu)}. \end{aligned}$$

Thus,

$$T_* f(x) \lesssim T_*(f\chi_{5B})(x) + [1 + \delta(B, S)]\|f\|_{L^\infty(\mu)} + T_* f(y).$$

Taking mean value over B for x, and over S for y, then yields

$$m_B(T_* f) - m_S(T_* f) \lesssim [1 + \delta(B, S)]\|f\|_{L^\infty(\mu)},$$

where we used (4.7). This finishes the proof of Theorem 4.1 in the case of $f \in L^\infty(\mu) \cap L^{p_0}(\mu)$ with $p_0 \in [1, \infty)$.

If $f \in L^\infty(\mu)$ and $f \notin L^p(\mu)$ for all $p \in [1, \infty)$, then the integral

$$\int_{d(x,\,y)>\epsilon} K(x,\,y)f(y)\,d\mu(y)$$

may not be convergent. The operator T_ϵ can be extended to the whole space $L^\infty(\mu)$ by following the standard arguments (see, for example, [17, p. 105]): Fix any point $x_0 \in \mathcal{X}$. For any given ball $B(x_0, r)$ centered at $x_0 \in \mathcal{X}$ with the radius $r > 3\epsilon$, we write $f = f_1 + f_2$, with $f_1 \equiv f\chi_{B(x_0,3r)}$. For $x \in B(x_0, r)$, we then define

$$T_\epsilon f(x) = T_\epsilon f_1(x) + \int_{d(x,\,y)>\epsilon} [K(x,\,y) - K(x_0,\,y)]f_2(y)\,d\mu(y).$$

Now both integrals in this equation are convergent. Using this definition, Remark 2.1(i) and then repeating the argument as above then completes the proof of Theorem 4.1. □

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