Maps close to identity and universal maps in the Newhouse domain

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Abstract

Given an n-dimensional C^r-diffeomorphism g, its renormalized iteration is an iteration of g, restricted to a certain n-dimensional ball and taken in some C^r-coordinates in which the ball acquires radius 1. We show that for any $r \geq 1$ the renormalized iterations of C^r-close to identity maps of an n-dimensional unit ball B^n ($n \geq 2$) form a residual set among all orientation-preserving C^r-diffeomorphisms $B^n \to R^n$. In other words, any generic n-dimensional dynamical phenomenon can be obtained by iterations of C^r-close to identity maps, with the same dimension of the phase space. As an application, we show that any C^r-generic two-dimensional map which belongs to the Newhouse domain (i.e., it has a wild hyperbolic set, so it is not uniformly-hyperbolic, nor uniformly partially-hyperbolic) and which neither contracts, nor expands areas, is C^r-universal in the sense that its iterations, after an appropriate coordinate transformation, C^r-approximate every orientation-preserving two-dimensional diffeomorphism arbitrarily well. In particular, every such universal map has an infinite set of coexisting hyperbolic attractors and repellers.

1 Ruelle-Takens problem and universal maps

A long standing open problem in the theory of dynamical systems is to describe which kind of dynamical phenomena can be expected in close to identity maps. It started with a much celebrated paper [1] where it was shown that any n-dimensional dynamics can be implemented by a C^n-small perturbation of the identity map of an n-dimensional torus. The paper seized a lot of attention by physicists, because it proposed a new view on the onset of hydrodynamical turbulence; at the same time it caused a lot of criticism. One of the reasons for the critique was that the C^n-small perturbations constructed in [1] were not small in C^{n+1}, which is quite unphysical. The controversy was resolved in [2] where it was shown that for any r, given any C^r-diffeomorphism F of a closed n-dimensional ball, one can find a C^r-close to identity map g of the $(n+1)$-dimensional closed unit ball B^{n+1} such that the diffeomorphism F coincides with some iteration of the map g restricted to some n-dimensional invariant manifold. Thus, the restriction on smoothness of perturbations was removed by sacrificing one dimension of the phase space; anyway, other scenarios of the transitions to turbulence had already been known.

From the purely mathematical point of view, the question still remained unsolved: can an arbitrary n-dimensional dynamics be obtained by iterations of a C^r-close to identity map of B^n, i.e. in the same dimension of the phase space? The difficulty is that the straightforward construction proposed in [1] does not work for high r in principle. Indeed, given an orientation-preserving diffeomorphism $F : B^n \to R^n$, one can imbed it into a continuous family $\mathcal{F}_t$ of the diffeomorphisms such that $\mathcal{F}_1 = F$ and $\mathcal{F}_0 = id$. Then, given any N, the map F

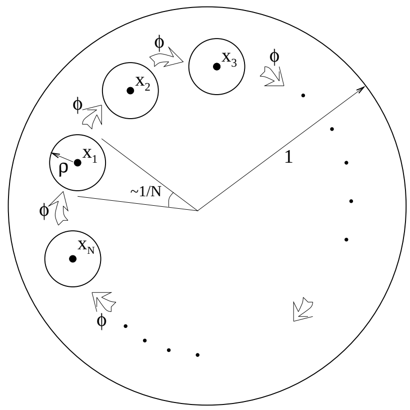

Figure 1: An illustration to Ruelle-Takens construction.

can be represented as a superposition of N maps

$$F = F_N \circ \ldots \circ F_1, \quad \text{where} \quad F_s = \mathcal{F}_{s/N} \circ \mathcal{F}^{-1}_{(s-1)/N}, \tag{1.1}$$

that are $O(1/N)$-close to identity. One can then choose N pairwise disjoint small balls $D_s \in B^n$ of radius $\rho \sim N^{-1/(n-1)}$ and define a map $\phi : B^n \to B^n$ such that $\phi(x)|_{x\in D_s} \equiv x_{s+1} + \rho F_s(\frac{x-x_s}{\rho})$ where x_s is the center of D_s; the positions of the centers are chosen in such a way (Fig.1) that the distances $\|x_{s+1} - x_s\|$ are uniformly close to zero for all s, hence the map ϕ is C^0-close to identity. By construction, $\phi^N|_{D_0}$ is linearly conjugate to F, i.e. the dynamics of $\phi^N|_{D_0}$ coincides with the dynamics of F. However, the derivatives of ϕ of order k behave as $N^{-1}\rho^{1-k} \sim N^{\frac{k-n}{n-1}}$, i.e. at $k \geq n$ they do not, in general, tend to zero as $N \to +\infty$. Thus, an arbitrary n-dimensional dynamics can be implemented by iterations of C^{n-1}-close to identity maps of B^n, but the construction gives no clue of whether the same can be said about the C^n-close to identity maps.

One could try to position the regions D_s differently, or make their radii vary, or change their shape. This, however, hardly can lead to an essential increase in the maximal order of the derivatives (of ϕ) which tend to zero as $N \to +\infty$. The reason lies in a well-known fact from the averaging theory that the $O(\delta)$-close to identity map

$$\bar{x} = x + \delta f(x)$$

approximates a time shift of a certain autonomous flow with the accuracy $O(\delta^m)$ for an arbitrarily large m (if $f \in C^\infty$). Hence, the number of iterations necessary in order to obtain a dynamics which is far from that of an autonomous flow, has to grow faster than $O(\delta^{-m})$, for every m. As we see, in order to obtain such kind of dynamics, one has to control a very large number of iterations of close to identity maps, hence decompositions much longer than that given by (1.1) have to be considered.

In this paper we propose such a decomposition (Theorem 4), using which we show that
an arbitrary C^r-generic orientation-preserving n-dimensional dynamics can be obtained by iterations of C^r-close to identity maps of B^n, $n \geq 2$.

To make the formulations precise, we borrow some definitions from [3]. Let g be a C^r-diffeomorphism of a closed n-dimensional ball D. Take an integer $m > 0$ and any C^r-diffeomorphism ψ of R^n such that $\psi(B^n) \subseteq D$. The map $g_{m,\psi} = \psi^{-1} \circ g^m \circ \psi|_{B^n}$ is a C^r-diffeomorphism that maps B^n into R^n. We will call the maps $g_{m,\psi}$ obtained by this procedure *renormalized iterations of g*.

Theorem 1 . *In the space of C^r-smooth orientation-preserving diffeomorphisms of B^n into R^n ($n \geq 2$) there is a residual set $\mathcal{S}_r$ such that for every map $F \in \mathcal{S}_r$, for every $\delta > 0$ and for every n-dimensional ball D there exists a map $g : R^n \to R^n$, equal to identity outside D, such that $||g - id||_{C^r} < \delta$ and F is a renormalized iteration of g.*

In other words, a generic C^r-diffeomorphism $B^n \to R^n$ is, up to a smooth coordinate transformation, an iteration of an arbitrarily close to identity map. This theorem is proven in Section 4. Actually, we prove there that
for any $\delta > 0$ and $\varepsilon > 0$, for every orientation-preserving C^r-diffeomorphism $F : B^n \to R^n$ there exists a δ-close to identity map g, equal to identity outside a given ball D, such that $||F - g_{m,\psi}||_{C^r} < \varepsilon$ for some renormalized iteration $g_{m,\psi} = \psi^{-1} \circ g^m \circ \psi|_{B^n}$ (it is enough to prove this for one particular ball D that may depend on F, but the choice of D should be fixed in advance, not depending on δ nor on the accuracy ε of the approximation; then for other balls the claim will remain true because there always exists an affine conjugacy that takes one ball to the other). Moreover, we construct the map g and the coordinate transformation ψ in such a way that the balls $g^i(\psi(B^n))$ ($i = 0, \ldots, m-1$) do not intersect each other — hence, by adding small, localized in $g^{m-1}(\psi(B^n))$, perturbations to g, we may make $g_{m,\psi}$ run an open subset in the space of C^r-diffeomorphisms $B^n \to R^n$. Thus, the set $\mathcal{S}_r(\delta)$ of all renormalized iterations of the maps $g : D \to D$ such that $||g - id||_{C^r} < \delta$ contains a subset which is open and dense in the space of C^r-smooth orientation-preserving diffeomorphisms of B^n into R^n, for every $\delta > 0$. Hence, the intersection $\mathcal{S}_r$ of these sets over all $\delta > 0$ is residual (and independent of the choice of D), which gives us the theorem.

The first step in our construction of the approximations of the given map F by renormalized iterations is a representation of F as a superposition of a pair of certain special maps and some volume-preserving diffeomorphisms (Lemma 1). Each of the special maps can be realized as a flow map through a kind of saddle-node bifurcation (see Fig.2), reminiscent of the so-called "Iljashenko lips" (see [4]). For the volume-preserving diffeomorphisms one may adjust the results obtained in [3] for symplectic diffeomorphisms and prove (Lemma 2) the existence of an arbitrarily good, in the C^r-norm on any compact, polynomial approximation by a superposition of volume-preserving Hénon-like maps. It is known that Hénon-like maps often appear as rescaled first-return maps near a homoclinic tangency (cf. [5, 6, 7]). In this paper we find a kind of homoclinic tangency which does incorporate all the Hénon-like maps that appear in our volume-preserving polynomial approximations.

Thus, we show that the map F can be approximated arbitrarily well by a superposition of maps related to certain homoclinic bifurcations. The last step is to build a close to identity map which displays these bifurcations simultaneously. This is achieved by an arbitrarily small perturbation of the time-δ map of a certain C^∞ flow (the time δ map of a flow is, obviously, $O(\delta)$-close to identity).

Note that the approximation (that we construct in Lemma 2) of any volume-preserving diffeomorphism of a unit ball into R^n by a polynomial volume-preserving diffeomorphism is not straightforward, because the Jacobian of the approximating diffeomorphism should be equal to 1 everywhere, and this constrain is quite strong for polynomial maps. Had the approximation result been true for all volume-preserving maps, i.e. not necessarily diffeomorphisms, it would produce a counterexample to the famous "Jacobian conjecture"; however, our approximation uses in an essential way the injectivity of the map that has to be approximated (we represent the map as a shift by the orbits of some smooth non-autonomous flow).

It should be mentioned that Theorem 1 does not hold true at $n = 1$. Indeed, if a map F on the interval B^1 has two fixed points (with the multipliers different from 1), then every close map $\hat{F}$ has a pair of fixed points $P_{1,2}$ as well. If such $\hat{F}$ is a renormalized iteration of a diffeomorphism g, i.e. if $\hat{F} = \psi^{-1} \circ g^m \circ \psi$, then g will also have a pair of fixed points, $\psi(P_1)$ and $\psi(P_2)$ (at $n > 1$ this is not true). The interval between P_1 and P_2 will therefore be invariant with respect to $\psi^{-1} \circ g \circ \psi$, hence $\psi^{-1} \circ g \circ \psi$ will be a root of degree $m > 1$ of the map $\hat{F}$ on this interval. Now note that the maps of the interval that have a root are not dense in C^2, according to [8]. Thus we obtain that renormalized iterations are not dense either.

One can check through the proof of Theorem 1 that it holds true for finite-parameter families of orientation-preserving diffeomorphisms:

in the space of k-parameter families F_ε, $\varepsilon \in B^k$, of C^r-smooth orientation-preserving diffeomorphisms of B^n into R^n ($n \geq 2$) there is a residual set $\mathcal{S}_{kr}$ such that for every $F_\varepsilon \in \mathcal{S}_{kr}$ and for every $\delta > 0$ there exists $g_\varepsilon : D \times B^k \to D$ such that $||g_\varepsilon - id||_{C^r} < \delta$ and $F_\varepsilon = \psi_\varepsilon^{-1} \circ g_\varepsilon^m \circ \psi_\varepsilon|_{B^n}$ for some $m > 0$ and some family ψ_ε of C^r-diffeomorphisms of R^n.

Thus, any dynamical phenomenon which occurs generically in finite-parameter families of dynamical systems can be encountered in maps arbitrarily close to identity (with the same dimension of the phase space).

To put the result into a general perspective, we recall that one of the main sources of difficulties in the theory of dynamical systems is that structurally stable systems are not dense in the space of all systems [9, 10, 11], moreover most natural examples of chaotic dynamics are indeed structurally unstable (like e.g. the famous Lorenz attractor [16]). Understanding the dynamics of systems from the open regions of structural instability (i.e. the regions where arbitrarily close to every system there is a system which is not topologically conjugate to it) has been the subject of active research for the past four decades. It often happens, and helps a lot, that structurally unstable systems may possess certain robust properties, i.e. dynamical properties which are not destroyed by small perturbations. For example, systems with Lorenz attractor are pseudohyperbolic (or volume-hyperbolic) [12, 13, 14], and this is, in fact, the very property which allowed for a very detailed description of them [15, 16, 17]. Another robust property is uniform partial hyperbolicity, a rich theory of systems possessing it is actively developing [18, 19]. In fact, not so much of robust properties are known, it could even happen that beyond the mentioned partial hyperbolicity and volume-hyperbolicity no other robust dynamical properties exist. This claim can be demonstrated for various examples of homoclinic bifurcations (see [20]), and can be used as a guiding principle in the study of bifurcations of systems with a non-trivial dynamics:

given an n-dimensional system with a compact invariant set that is neither partially- nor volume-hyperbolic, every dynamics that is possible in B^n should be expected to occur at the bifurcations of this particular system.

The last statement is not a theorem and it might be not true in some situations, still it gives a useful view on global bifurcations. In particular, it was explicitly applied in [21] to Galerkin approximations of damped nonlinear wave equations in order to obtain estimates from below on the dimension of attractors in the situation where classical methods [22] do not work.

Theorem 1 gives one more example to the above stated principle: the identity map has no kind of hyperbolic structure, neither it contracts nor expands volumes, so it should not be surprising that its bifurcations provide an ultimately rich dynamics.

The same idea can be expressed in somewhat different terms. Let us call the set of all renormalized iterations of a map $g : D \to D$ its *dynamical conjugacy class*. The map will be called C^r-*universal* (cf. [3]) if its dynamical conjugacy class is C^r-dense among all orientation-preserving C^r-diffeomorphisms of the closed unit ball B^n into R^n. By the definition, the dynamics of any single universal map is ultimately complicated and rich, and the detailed understanding of it is not simpler than understanding of all diffeomorphisms $B^n \to R^n$ altogether.

At the first glance, the mere existence of C^r-univesal maps of a closed ball is not obvious for sufficiently large r. However, Theorem 1 immediately implies the following

Theorem 2. *For every $r \geq 1$, C^r-universal diffeomorphisms of a given closed ball D exist arbitrarily close, in the C^r-metric, to the identity map.*

Proof. Take an arbitrary sequence of pairwise disjoint closed balls $D_j \subset D$, a sequence of maps F_j which is C^r-dense in space of orientation-preserving C^r-diffeomorphisms $B^n \to R^n$, and a sequence $\varepsilon_j \to +0$ as $j \to +\infty$.

By Theorem 1, given any δ, there exist maps g_j such that g_j is identity outside D_j, some renormalized iteration of g_j is ε_j-close to F_j, and $\|g_j - id\|_{C^r} \leq \delta$. By construction, the map $g(x) \equiv g_j(x)$ at $x \in D_j$ $(j = 1, 2, \dots)$ and $g(x) \equiv x$ at $x \in D \setminus \cup_{j=1}^{\infty} D_j$ is C^r-universal and δ-close to identity. □

As an immediate consequence of Theorem 2, we note that
arbitrarily close to identity map in the space of C^r-diffeomorphisms $B^n \to B^n$ (any $r \geq 1$, any $n \geq 2$) there exist maps with infinitely many coexisting uniformly-hyperbolic attractors of all possible topological types.
This is true because a hyperbolic attractor is a structurally stable object: given a map with a uniformly-hyperbolic attractor, any C^1-close map has a hyperbolic attractor topologically conjugate to the original one. For every $n \geq 2$ there exists a C^r-diffeomorphism $B^n \to R^n$ with a hyperbolic attractor. Hence, infinitely many of the maps F_j in the proof of Theorem 2 have a hyperbolic attractor as well. It follows that each universal map constructed in Theorem 2 has infinitely many hyperbolic attractors (and repellers, for that matter).

Thus, we can further pursue the approach of [1] and claim that hyperbolic attractors can be born at the third Andronov-Hopf bifurcation. Recall that at the primary Andronov-Hopf bifurcation a limit cycle is born from an equilibrium state, and at the secondary Andronov-Hopf bifurcation a two-dimensional invariant torus is born from the limit cycle. The third Andronov-Hopf bifurcation occurs when a three-dimensional invariant torus is born from the two-dimensional one (filled by quasiperiodic orbits). The so-called Landau-Hopf scenario of the onset of turbulence envisioned a chain of further Andronov-Hopf bifurcations which would lead to a creation of an invariant torus of a sufficiently high dimension, i.e. to a quasiperiodic regime with a high number of rationally independent frequencies (see more discussion in [23]). However, as Ruelle and Takens pointed out in [1], the dynamics on the invariant torus is not necessarily quasiperiodic: at the moment the torus is born, the system can be perturbed in such a way that every orbit on the torus will become periodic, hence the Poincaré map will be the identity, hence, as it follows from our results above, a further small perturbation may lead to a chaotic dynamics for a flow on the torus of dimension 3 or higher.

2 Universal maps in the Newhouse domain

In general, it follows from Theorem 2 that every time we have a periodic orbit for which the corresponding Poincaré map is, locally, identity:

$$\bar{x} = x,$$

or coincides with identity up to flat (i.e. sufficiently high order) terms:

$$\bar{x} = x + o(\|x\|^r),$$

a C^r-small perturbation of the system can make the Poincaré map universal, i.e. bifurcations of this orbit can produce dynamics as complicated as it only possible for the given dimension of the phase space. Thus, arbitrarily complicated dynamical phenomena can be uncovered by the study of bifurcations of periodic orbits alone.

In order to show how powerful this observation can be, let us apply it to the analysis of the dynamics of two-dimensional diffeomorphisms from the Newhouse domain. In the space of C^r-smooth dynamical systems on any smooth manifold, Newhouse domain is the interior of the closure of the set of systems that have a homoclinic tangency (a tangency between the stable and unstable manifolds of a saddle periodic orbit; for a saddle periodic orbit to exist, the dimension of the phase space should be at least 3 in the case of continuous time and at least 2 for discrete dynamical systems). A non-trivial fact [10, 11, 24, 25] is that the Newhouse domain at $r \geq 2$ is always non-empty and adjoins to every system with a homoclinic tangency. Importantly, most of known global bifurcations which lead to the emergence of chaotic dynamics or happen within the class of systems with complex (chaotic) behavior are accompanied by a creation of homoclinic tangencies. Therefore, Newhouse regions in the space of parameters can be detected for virtually every dynamical model with chaos (see more discussion in [26, 27, 3, 6, 7]). In my opinion, the ubiquitous presence of homoclinic tangencies in the dynamical models of a natural origin makes the study of maps from the Newhouse domain the most important problem of chaotic dynamics. It should also be mentioned that a commonly shared believe (with no hope to prove yet) is that the space of two-dimensional C^r-diffeomorphisms with $r \geq 2$ is the closure of the union of just three open sets: Morse-Smale systems (i.e. systems with simple dynamics), axiom A systems, and the Newhouse domain, i.e. unless a two-dimensional map with a chaotic dynamics is uniformly hyperbolic, it most probably lies in the Newhouse domain.

Typically, a map from the Newhouse domain possesses an invariant basic hyperbolic set which is wild – the term meaning that for the map itself, and for every C^r-close map ($r \geq 2$), there exists a pair of orbits within the hyperbolic set such that the unstable manifold of one orbit has a quadratic tangency with the stable manifold of the other [10, 11]. For a fixed pair of orbits, the corresponding tangency is a codimension-1 bifurcation phenomenon, so it can always be removed by a small smooth perturbation; the wildness, nevertheless, means that once the original tangency is removed a new tangency always appears, corresponding to some other pair of orbits from the same hyperbolic set.

A wild hyperbolic set of a C^r-diffeomorphism of a two-dimensional smooth manifold will be called *ultimately wild* if it contains a pair of periodic orbits such that the saddle value at one periodic orbit is less than 1 and at the other periodic orbit it is greater than 1. The open subset of the Newhouse domain which corresponds to maps with ultimately wild hyperbolic sets will be called the *absolute Newhouse domain*. The saddle value is, by definition, the absolute value of the product of multipliers of the periodic orbit. If Q is a period q point of a map f (i.e. $f^q Q = Q$), its multipliers are the eigenvalues of the matrix of derivatives of f^q at the point Q. Therefore, if the saddle value is greater than 1 in the absolute value, then the map f expands area near Q, and it is area-contracting near Q if the saddle value is less than 1. So, no map from the absolute Newhouse domain is area-contracting, nor area-expanding. Moreover, the persistent tangencies between the stable and unstable manifolds of the wild hyperbolic set mean that none of these maps is uniformly hyperbolic, nor uniformly partially-hyperbolic. Thus, there is no obvious restrictions on the dynamics of two-dimensional diffeomorphisms from the absolute Newhouse domain, and the following theorem is, therefore, in agreement with the general "guiding principle" formulated in Section 1.

Theorem 3. *For every $r \geq 2$, a C^r-generic diffeomorphism from the absolute Newhouse domain is C^r-universal.*

Proof. Fix any $r \geq 2$ and let a C^r-diffeomorphism of a smooth two-dimensional manifold possess an ultimately wild basic hyperbolic set Λ. Let P and Q be two saddle periodic points in Λ ($f^p P = P$ and $f^q Q = Q$) with the saddle values σ_P and σ_Q such that $\sigma_P < 1$ and $\sigma_Q > 1$. As the periodic points P and Q are hyperbolic, they are preserved at all sufficiently small perturbations. Recall also that the unstable manifold of every point in the basic hyperbolic set has a transverse intersection with the stable manifold of every other orbit in this set. Therefore, the invariant unstable manifold $W^u(P)$ intersects the invariant stable manifold $W^s(Q)$ transversely at the points of some heteroclinic orbit Γ_{PQ}. By the transversality, this orbit is preserved for all maps sufficiently close to f. According to [6], given every $m \geq 1$, in any neighborhood of f in the C^r-topology, there is a C^∞-diffeomoprphism for which the unstable manifold $W^u(Q)$ has a tangency of order m to the stable manifold $W^s(P)$ at the points of some heteroclinic orbit Γ_{QP} (Fig.3). This is a non-trivial statement: while the possibility to obtain a quadratic tangency (i.e. $m = 1$) by an arbitrarily small perturbation follows immediately from the wildness of Λ and from the fact that the stable and unstable manifold of any given periodic orbit in the basic hyperbolic set are dense within the union of stable and, respectively, unstable manifolds of all orbits in this set [10, 11], the tangencies of higher orders are due to the existence of moduli of local Ω-conjugacy [26, 28, 29].

Let $\tilde{f} \in C^\infty$ be a C^r-close to f diffeomorphism for which the above described heteroclinic cycle exists. This cycle is a closed set that consists of 4 orbits: two periodic orbits (the orbits of P and Q) with $\sigma_P < 1$ and $\sigma_Q > 1$, a transverse heteroclinic Γ_{PQ} and the orbit Γ_{QP} of heteroclinic tangency of a sufficiently high order m. We will call an open region filled by periodic orbits of the same period *a periodic spot*. In Section 5 (see Lemma 3 and Remark after it) we show that

for any neighborhood U of the heteroclinic cycle, arbitrarily C^r-close to $\tilde{f}$ there is a diffeomorphism $\hat{f}$ which has a periodic spot whose all iterations lie in U.

For every orbit in the periodic spot the corresponding Poincaré map (the map $\hat{f}^k$ where k is the period of the spot) is, locally, identity $\bar{x} = x$. Hence, as we mentioned in the beginning of this Section, it follows from Theorem 2 that in any C^r-neighborhood of $\hat{f}$ there exist C^r-universal maps.

We have just shown (modulo Lemma 3 which is proved in Section 5) that universal maps are dense in the absolute Newhouse domain. This immediately implies the genericity of the universal maps. Indeed, given an orientation-preserving C^r-diffeomorphism g of the unit disk into R^2, denote as $\mathcal{V}(g,\delta)$ the set of all C^r-diffeomorphisms from the absolute Newhouse domain whose dynamical conjugacy class intersects the open δ-neighborhood of g (i.e. whose certain renormalized iteration is at a C^r-distance smaller than δ from g). This set, by definition, contains all universal maps — hence, it is dense in the absolute Newhouse domain. This set is also open by definition. Take a countable sequence of maps g_i which is dense in the space of orientation-preserving C^r-diffeomorphisms of the unit disk into R^2, and a sequence δ_j of positive reals converging to zero. By construction, the countable intersection $\cap_{i,j}\mathcal{V}(g_i,\delta_j)$ is a residual subset of the absolute Newhouse domain, and every map that belongs to this set is universal. $\Box$

In essence, this theorem gives an exhaustive characterization of the richness of dynamical behavior in the absolute Newhouse domain: every two-dimensional dynamics can be approximated by iterations of any generic map from this domain. In a simpler case of the Newhouse domain in the space of area-preserving maps, a similar statement is contained in [7]: iterations of a generic area-preserving map from the Newhouse domain approximate all symplectic dynamics in a two-dimensional disc. For *area-contracting* maps, it follows from [29] that the closure of the dynamical conjugacy class of a generic map from the Newhouse domain contains all *one-dimensional* maps (we cannot have truly two-dimensional dynamics here, as the areas are contracted). As we mentioned, any two-dimensional map which is not in the class of maps with a uniformly-hyperbolic structure, nor on the boundary of this class, falls, hypothetically, in one of the three types of the Newhouse domain: the first is filled by area-contracting maps, the second by area-expanding maps (i.e. the maps inverse to the maps of the first type), and the third is the absolute Newhouse domain. We see that our Theorem 3 somehow completes the description of two-dimensional dynamics.

The three types of Newhouse domains of two-dimensional maps were introduced in [30]. It has been known since [10] that a generic area-contracting map from the Newhouse domain has an infinite set of stable periodic orbits, and the closure of this set contains a (wild) hyperbolic set. The latter fact is especially important: while chaotic dynamics is usually associated with hyperbolic sets, i.e. with saddle orbits, the Newhouse result shows that stable periodic motions can imitate chaos arbitrarily well, and they indeed do it generically. In [30], for the Newhouse domain of the third type, it was shown that a generic map has both an infinite set of stable periodic orbits and an infinite set of repelling periodic orbits, moreover the intersection of these sets is non-empty and contains an ultimately-wild hyperbolic set. Thus, not only the Newhouse phenomenon holds, we also have a new effect here: a generic inseparability of attractors from repellers. In fact, these attractors and repellers can be more complicated than just periodic orbits: in [31], the coexistence of infinitely many closed invariant curves was established for two-dimensional maps from the Newhouse domains of the third type. Our

Theorem 3 strengthens these observations: it implies the coexistence of infinitely many hyperbolic attractors and repellers for a generic map from the absolute Newhouse domain (see remarks to Theorem 2). As we obtain the hyperbolic attractors and repellers by a perturbation of periodic spots, and the periodic spots are found in an arbitrarily small neighborhood of any heteroclinic cycle of the type we consider in Theorem 3, it follows that the closures of the set of hyperbolic attractors and of the set of hyperbolic repellers that we construct here contain any transverse heteroclinic orbit connecting the points P and Q. Such heteroclinic orbits are dense in the basic hyperbolic set Λ. Thus, it follows from the proof of the theorem that
a C^r-generic two-dimensional map from the absolute Newhouse domain has an infinite set of uniformly-hyperbolic attractors and an infinite set of uniformly-hyperbolic repellers, and the intersection of the closures of these sets contains a non-trivial hyperbolic set.

In what follows we prove Theorem 1 and finish the proof of Theorem 3.

3 An approximation theorem.

Let F be an orientation-preserving C^r-diffeomorphism ($r \geq 3$) which maps the ball $B^n : \{\sum_{i=1}^n x_i^2 \leq 1\}$ into R^n. Without loss of generality we may assume that F is extended onto the whole R^n, i.e. it becomes a C^r-diffeomorphism $R^n \to R^n$, and it is identical (i.e. $F(x) = x$) at $\|x\|$ sufficiently large; such extension is always possible. Let K be a constant such that

$$\sup_{x \in R^n} \frac{\|\nabla J(x)\|}{J(x)} < K, \tag{3.1}$$

where $J(x)$ is the Jacobian of F. Denote $R^n_+ := \{x_n > 0\}$.

Lemma 1 . *There exists a pair of volume-preserving, orientation-preserving C^{r-2}-diffeomorphisms $\Phi_1 : R^n_+ \to R^n_+$ and $\Phi_2 : R^n \to R^n$ such that*

$$F = \Phi_2 \circ \Psi_2 \circ \Phi_1 \circ \Psi_1, \tag{3.2}$$

where

$$\Psi_j := (x_1, \dots, x_{n-1}, x_n) \mapsto (x_1, \dots, x_{n-1}, \psi_j(x_n)) \quad (j = 1, 2), \tag{3.3}$$

with

$$\psi_1(x_n) = e^{Kx_n}, \qquad \psi_2(x_n) = \ln x_n. \tag{3.4}$$

Proof. We need to construct a volume-preserving diffeomorphism $\Phi_1 : (x_1, \dots, x_n \geq 0) \mapsto (\bar{x}_1, \dots, \bar{x}_n \geq 0)$ in such a way that

$$\det \frac{\partial}{\partial x} \Psi_2 \circ \Phi_1 \circ \Psi_1(x) \equiv J(x) \tag{3.5}$$

(then the Jacobian of $\Phi_2 = F \circ (\Psi_2 \circ \Phi_1 \circ \Psi_1)^{-1}$ will be equal to 1 automatically). By (3.3),(3.4), condition (3.5) is equivalent to

$$\bar{x}_n = \phi(x_1, \dots, x_n) \equiv \frac{Kx_n}{J(x_1, \dots, x_{n-1}, \frac{1}{K} \ln x_n)}.$$

It follows from (3.1) that

$$\partial\phi / \partial x_n > 0 \tag{3.6}$$

everywhere. Moreover, as F is the identity map outside a bounded region of R^n, we have that

$$\phi(x) = Kx_n \tag{3.7}$$

outside a compact subregion of R^n_+. Therefore, every trajectory of the vector field

$$\dot{x}_j = 0 \ \ (j \leq n-2), \quad \dot{x}_{n-1} = \frac{\partial \phi}{\partial x_n}, \quad \dot{x}_n = -\frac{\partial \phi}{\partial x_{n-1}} \tag{3.8}$$

extends for all $x_{n-1} \in (-\infty, +\infty)$, and the time $\tau(x)$ that the trajectory of the point x needs to get to $x_{n-1} = 0$ is a C^{r-2}-function of x, well defined everywhere in R^n_+. Moreover, as it follows from (3.7),(3.8)

$$\tau(x) = -\frac{1}{K} x_{n-1} + \tau_0(x), \tag{3.9}$$

where τ_0 is a uniformly bounded function, vanishing identically at x_n close to zero and at sufficiently large x_n. Thus, for every fixed values of x_j $(j \leq n-2)$, given any $C \in (-\infty, +\infty)$ the level line $\tau(x) = C$ in the (x_{n-1}, x_n)-plane coincides with the straight line $x_{n-1} = -KC$ at x_n close to zero and at sufficiently large x_n. Every such level line is a connected set (as it is the image of the line $\{x_{n-1} = 0, x_n \geq 0\}$ by the time-$(-C)$ map of the flow of (3.8)). Thus, as x_n runs from 0 to $+\infty$, the value of ϕ on this line runs all the values from 0 to $+\infty$ (see (3.7)). It follows that the map $R^n_+ \to R^n_+$ defined by

$$\bar{x}_j = x_j \ \ (j \leq n-2), \quad \bar{x}_{n-1} = -\tau(x), \quad \bar{x}_n = \phi(x) \tag{3.10}$$

is surjective.

By (3.8),

$$\frac{\partial \tau}{\partial x_{n-1}} \frac{\partial \phi}{\partial x_n} - \frac{\partial \tau}{\partial x_n} \frac{\partial \phi}{\partial x_{n-1}} = -1. \tag{3.11}$$

It follows that for every fixed values of x_j $(j \leq n-2)$, the function ϕ changes monotonically along every level line of τ, which implies the injectivity of map (3.10). Thus, map (3.10) is a C^{r-2}-diffeomorphism $R^n_+ \to R^n_+$. By (3.11), it is volume-preserving and orientation-preserving, i.e. it is the sought map Φ_1. □

The maps $x \mapsto \bar{x}$ of the following form:

$$\bar{x}_1 = x_2, \ \ldots, \ \bar{x}_{n-1} = x_n, \quad \bar{x}_n = (-1)^{n+1} x_1 + h(x_2, \ldots, x_n) \tag{3.12}$$

(note no dependence on x_1 in h), will be called *Hénon-like volume-preserving maps.* Note that such maps are always one-to-one, and the inverse map is also Hénon-like.

Theorem 4. *Every orientation-preserving C^r-diffeomorphism $F : B^n \to R^n$ can be arbitrarily closely approximated, in the C^r-norm on B^n, by a map of the following form:*

$$H_{2q_2} \circ \ldots \circ H_{21} \circ \Psi_2 \circ H_{1q_1} \circ \ldots \circ H_{11} \circ \Psi_1, \tag{3.13}$$

where the maps $\Psi_{1,2}$ are given by (3.3), and H_{js} $(j = 1, 2; s = 1, \ldots q_j)$ are certain polynomial Hénon-like volume-preserving maps.

Proof. First, take a C^{r+2}-diffeomorphism $\hat{F}$ which approximates F sufficiently closely in C^r. For the map $\hat{F}$ construct the decomposition $\hat{F} = \Phi_2 \circ \Psi_2 \circ \Phi_1 \circ \Psi_1$ given by Lemma 1; all the maps in the decomposition are at least C^r. The map Φ_1 can be extended onto $x_n \leq 0$ by the rule $\bar{x}_n = K x_n$, $\bar{x}_{n-1} = x_{n-1}/K$ (see (3.10),(3.7),(3.9)), so it becomes a volume-preserving, orientation-preserving C^r-diffeomorphism $R^n \to R^n$. Then the theorem follows immediately from Lemma 2 below. □

Lemma 2 . *Every volume-preserving, orientation-preserving C^r-diffeomorphism $\Phi : R^n \to R^n$ can be arbitrarily closely approximated, in the C^r-norm on any given compact, by a composition of polynomial Hénon-like volume-preserving maps.*

Proof. At $n = 2$ this result is immediately given by Theorem 1 in [3], so we proceed to the case $n \geq 3$. It is well known that Φ can be imbedded in a smooth in t family $\mathcal{F}_t$ of volume-preserving C^r-diffeomorphisms $R^n \to R^n$ such that $\mathcal{F}_0 \equiv id$ and $\mathcal{F}_1 = \Phi$. The derivative $\frac{d}{dt}\mathcal{F}_t$ defines a divergence-free vector field $X(t,x)$, i.e. the diffeomorphism $\mathcal{F}_t$ is the time-t shift by the flow generated by the field X. One can approximate X arbitrarily closely on any given compact by a C^∞-smooth divergence-free vector field which is defined and bounded for all $(x,t) \in R^n \times [0,1]$. Therefore, it is enough to prove the lemma only for those Φ which can be obtained as the time-1 shift by the flow generated by such a vector field, i.e. we may assume that $X \in C_b^\infty$ with no loss of generality.

Let $T_{\tau t} = \mathcal{F}_{t+\tau} \circ \mathcal{F}_t^{-1}$, i.e. it is the shift by the flow of X from the time t to $t+\tau$. This map is $O(\tau)$-close to identity, in the C^r-norm on any compact subset of $R^n \times [0,1]$. By construction, given any arbitrarily large integer N,

$$\Phi = T_{\tau,(N-1)\tau} \circ \ldots \circ T_{\tau,m\tau} \circ \ldots \circ T_{\tau,0} \tag{3.14}$$

where $\tau = 1/N$, and $m = 0, \ldots, N-1$.

Note that the vector field X admits the following representation:

$$X = \sum_{i=1}^{n-1} X^{(i)} \tag{3.15}$$

where $X^{(i)}$ is a C^∞-smooth divergence-free vector field such that

$$\dot{x}_j \equiv 0 \quad \text{at} \quad j \neq i, i+1. \tag{3.16}$$

Indeed, if we write the field X as

$$\dot{x}_i = \xi_i(x,t), \quad i = 1, \ldots, n,$$

where $\displaystyle\sum_{i=1}^{n} \frac{\partial \xi_i}{\partial x_i} \equiv 0$ (the zero divergence condition), then the fields $X^{(i)}$ are defined as

$$\dot{x}_i = \eta_i(x,t), \quad \dot{x}_{i+1} = \zeta_i(x,t)$$

with

$$\eta_1 \equiv \xi_1, \quad \eta_i \equiv \xi_i - \zeta_{i-1} \;\; (i = 2, \ldots, n-1),$$

$$\zeta_i = -\int_0^{x_{i+1}} \frac{\partial}{\partial x_i} \eta_i(x_1, \ldots, x_i, s, x_{i+2}, \ldots, x_n, t) ds \;\; (i = 1, \ldots, n-2), \quad \zeta_{n-1} \equiv \xi_n.$$

By construction, the fields $X^{(1)}, \ldots, X^{(n-2)}$ are divergence-free, and $X^{(n-1)} = X - X^{(1)} - \ldots - X^{(n-2)}$, so $X^{(n-1)}$ is also divergence-free, as X is.

It follows from (3.15) that

$$T_{\tau t} = T_{\tau t}^{(n-1)} \circ \ldots \circ T_{\tau t}^{(1)} + O(\tau^2), \tag{3.17}$$

where $T_{\tau t}^{(i)}$ is the shift by the flow generated by the vector field $X^{(i)}$. Recall that the maps $T_{\tau,i\tau}$ in (3.14) are $O(1/N)$-close to identity. Therefore, it follows from (3.17),(3.14) that

$$\Phi = T_{\tau,(N-1)\tau}^{(n-1)} \circ \ldots \circ T_{\tau,(N-1)\tau}^{(1)} \circ \ldots \circ T_{\tau,m\tau}^{(n-1)} \circ \ldots \circ T_{\tau,m\tau}^{(1)} \circ \ldots \circ T_{\tau,0}^{(n-1)} \circ \ldots \circ T_{\tau,0}^{(1)} + O(\tau), \tag{3.18}$$

uniformly on compacta.

As τ can be taken arbitrarily small, it follows that in order to prove the lemma, it is enough to prove that every of the maps $T^{(i)}_{\tau t}$ in the right-hand side of (3.18) can be approximated arbitrarily well by a composition of Hénon-like volume-preserving maps. The maps $T^{(i)}_{\tau t}$ are volume-preserving and satisfy

$$\bar{x}_j = x_j \quad \text{at} \quad j \neq i, i+1 \tag{3.19}$$

(see (3.16)). Therefore, if we denote

$$\bar{x}_i = p(x), \quad \bar{x}_{i+1} = q(x), \tag{3.20}$$

then

$$\frac{\partial(p,q)}{\partial(x_i, x_{i+1})} = 1. \tag{3.21}$$

Thus, we can view (3.20) as an $(n-2)$-parameter family of symplectic two-dimensional maps $(x_i, x_{i+1}) \mapsto (\bar{x}_i, \bar{x}_{i+1})$ parametrized by $(x_1, \ldots, x_{i-1}, x_{i+2}, \ldots, x_n)$.

According to [3], every finite-parameter family of symplectic maps can be approximated (on any compact) by a composition of families of Hénon-like maps, i.e., in our case, maps of the form

$$\bar{x}_i = x_{i+1}, \quad \bar{x}_{i+1} = -x_i + h(x_{i+1}; x_1, \ldots, x_{i-1}, x_{i+2}, \ldots, x_n).$$

It follows that every map of the form (3.19),(3.20),(3.21) can be approximated arbitrarily closely by a composition of the maps of the form

$$\begin{array}{l} \bar{x}_j = x_j \quad \text{at} \quad j \neq i, i+1, \\ \bar{x}_i = x_{i+1}, \\ \bar{x}_{i+1} = -x_i + h(x_{i+1}; x_1, \ldots, x_{i-1}; x_{i+2}, \ldots, x_n). \end{array} \tag{3.22}$$

It just remains to note that every map of form (3.22) is a composition of volume-preserving Hénon-like maps; namely, it equals to

$$S^{n-i-1} \circ H \circ S \circ Q^{n-1} \circ S^{i+1},$$

where

$$S := (x_1, \ldots, x_n) \mapsto (x_2, \ldots, x_n, (-1)^{n+1} x_1),$$

$$Q := (x_1, \ldots, x_n) \mapsto (x_2, \ldots, x_n, \sum_{j=1}^{n} (-1)^{n+j} x_j),$$

$$\begin{aligned} H := \{ & \bar{x}_1 = x_2, \ldots, \bar{x}_{n-1} = x_n, \ \bar{x}_n = \\ = & \sum_{j=1}^{n-1} (-1)^{n+j} x_j - x_n + h(x_n; x_{n-i+1}, \ldots, x_{n-1}; (-1)^{n+1} x_2, \ldots, (-1)^{n+1} x_{n-i}) \}. \end{aligned}$$

End of the proof. □

Remark. Consider the map

$$\Phi_0 := (x_1, \ldots, x_{n-2}, x_{n-1}, x_n) \mapsto (x_1, \ldots, x_{n-2}, x_n, -x_{n-1}). \tag{3.23}$$

This is an orientation- and volume- preserving diffeomorphism of R^n, therefore we may rewrite (3.2) as follows:

$$F = \Phi_0 \circ \tilde{\Phi}_2 \circ \Psi_2 \circ \Phi_0 \circ \tilde{\Phi}_1 \circ \Psi_1,$$

where $\tilde{\Phi}_{1,2}$ are orientation-preserving, volume-preserving C^{r-2}-diffeomorphisms ($\tilde{\Phi}_j = \Phi_0^{-1} \circ \Phi_j$; we assume that Φ_1 is extended onto the whole of R^n, like in Theorem 4). Now, by Lemma 2, we obtain the following, more convenient for us, analog of Theorem 4:
the map F can be arbitrarily closely approximated by a map of the following form:

$$\Phi_0 \circ \tilde{H}_{2q_2} \circ \ldots \circ \tilde{H}_{21} \circ \Psi_2 \circ \Phi_0 \circ \tilde{H}_{1q_1} \circ \ldots \circ \tilde{H}_{11} \circ \Psi_1, \tag{3.24}$$

with polynomial Hénon-like volume-preserving maps $\tilde{H}_{js}$.

4 Proof of Theorem 1.

Given a diffeomorphism $F : B^n \to R^n$ we will take its sufficiently close approximation $\hat{F}$ in the form of (3.24). Then we will construct a close to identity map (which we denote $\tilde{Y}_\delta$ below) whose some renormalized iteration is a close (as close as we want) approximation to $\hat{F}$.

The map $\tilde{Y}_\delta$ is a small perturbation (as small as we want, in the C^r-norm for any aforehand given r) of the time-δ map Y_δ of a certain C^∞ flow Y in R^n; the constant δ can be chosen as small as we need (i.e. Y_δ is indeed a small perturbation of the identity). In our construction, the vector field of Y vanishes identically outside some ball D that does not depend on the choice of the approximation $\hat{F}$, it does not depend on δ either. The small perturbations which we will apply to Y_δ will also be localized in D. Hence, our close to identity maps $\tilde{Y}_\delta$ are all equal to identity outside the same ball D. Since their renormilzed iterations approximate F arbitrarily well, this gives us Theorem 1, as we explained it in the remark after the theorem.

We define the flow Y by means of the following procedure: we give explicit formulas for the vector field inside certain blocks $U_{1\pm}, U_{2\pm}, V_{1,2}$ described below, while between the blocks we specify only the transition time from the boundary of one block to another and the corresponding Poincaré map. The existence of a C^∞ flow with arbitrary (of class C^∞) transition times and orientation-preserving Poincaré maps between block boundaries is a routine fact (at least for the given geometry of the blocks, see Fig.2).

The idea of the construction is as follows. Given a diffeomorphism $F : B^n \to R^n$, we may always approximate it by a C^∞-diffeomorphism arbitrarily well, so we assume that $F \in C^\infty$ from the beginning. Let $\Phi_{1,2}$ and $\Psi_{1,2}$ be the maps defined by decomposition (3.2) of F. We define the vector field inside the blocks $U_{1\pm}, U_{2\pm}$ in such a way that a kind of saddle-node bifurcation is created inside each of the blocks (see (4.1.1),(4.2.1); we build very degenerate saddle-nodes in order to make formulas for the time-t map simpler – see (4.1.3),(4.2.4)). The Poincaré maps from the boundary Σ^{out}_{j+} of U_{j+} to the boundary Σ^{in}_{j-} of U_{j-} ($j = 1, 2$) are chosen in such a way that the resulting flow map from entering U_{j+} till exiting U_{j-} is linearly conjugate to the map Ψ_j (see (4.1.5),(4.2.5),(3.3),(3.4)).

We make the flow in $V_{1,2}$ volume-preserving and linear (see (4.1.6), (4.2.2),(4.2.3)). Moreover, we put saddle equilibria into $V_{1,2}$. We define the Poincaré map between the boundaries Π^{out}_{j+} and Π^{in}_{j-} of V_j (see Fig.2) in such a way that a homoclinic loop to the saddle is created. For the time-δ map Y_δ of the flow Y the saddle equilibrium is a saddle fixed point, and the homoclinic loop is a continuous family of homoclinic orbits. We perturb the map Y_δ in such a way that this family splits into a finite set of orbits of homoclinic tangency of sufficiently high orders and unfold these tangencies then. The exact form of the perturbation (see (4.1.17),(4.2.16)) may be chosen such that the iteration of the perturbed map $\tilde{Y}_\delta$ that corresponds to one round near the homoclinic loop is a close approximation (in appropriately scaled coordinates) to any given polynomial conservative Hénon-like map (see (4.1.22),(4.2.22)). Hence, a multi-round iteration of $\tilde{Y}_\delta$ can be made arbitrarily close to a superposition of a finite number of any given polynomial Hénon-like maps. By Lemma 2 such superpositions approximate any given volume-preserving maps, e.g. maps Φ_j from the decomposition (3.2). In this way an iteration of the map

$\tilde{Y}_\delta$ that, after a large number of rounds near homoclinic orbits, takes points entering V_j to the points entering U_{3-j} is made as close as we want to the map Φ_j $(j = 1, 2)$, in some rescaled coordinates.

The parameters of the flow and perturbations are chosen in such a way that the coordinate scalings we make in the blocks $U_{1,2}$ and $V_{1,2}$ match each other. Thus, by construction, the (renormalized) iteration of $\tilde{Y}_\delta$ that corresponds to passing from the entrance to U_{1+} through U_{1-} into V_1, then to many rounds near the homoclinic loop, then to exiting V_1 and entering U_{2+}, passing through U_{2-} to V_2, a number of near-homoclinic rounds and return to U_{1+}, is a close approximation to $\Phi_2 \circ \Psi_2 \circ \Phi_1 \circ \Psi_1$, i.e. to the original diffeomorphism F indeed.

In the two-dimensional case, the key fact that every area-preserving diffeomorphism can be approximated by some multi-round iteration of a map with a homoclinic tangency was proven in [6]. To deal with dimensions higher than 2, we construct a very degenerate homoclinic tangency (in terms of [20], both critical and Lyapunov dimensions for this tangency are equal to n – this is necessary in order not to lose dimension at the rescaling of the first-return map). We do not undertake an analysis of the corresponding bifurcation; instead, we make explicit computations of the rescaled return maps for our particular example only.

4.1 Two-dimensional case

To make computations more transparent we start with the case $n = 2$. Let $\Phi_{1,2}$ and $\Psi_{1,2}$ be the maps defined by (3.2). Let $I_{1\pm}$ and $I_{2\pm}$ be intervals of values of x_2 such that $x_2 \in I_{1+}$ at $(x_1, x_2) \in B^2$, $x_2 \in I_{1-}$ at $(x_1, x_2) \in \Psi_1(B^2)$, $x_2 \in I_{2+}$ at $(x_1, x_2) \in \Phi_1 \circ \Psi_1(B^2)$ and $x_2 \in I_{2-}$ at $(x_1, x_2) \in \Psi_2 \circ \Phi_1 \circ \Psi_1(B^2)$. Let R be such that all the intervals $I_{j\pm}$ lie within $\{|x_2| \leq R\}$. Choose numbers $a_{1+} = a_{1-} + 3 = b_1 + 6 = a_{2+} + 9 = a_{2-} + 12 = b_2 + 15$. Let the vector field of Y in the regions $U_{j\sigma} : \{|x_1 - a_{j\sigma}\| \leq 1, |x_2| \leq R\}$, $j = 1, 2$, $\sigma = \pm 1$, be equal to

$$\begin{array}{l} \dot{x}_1 = -\mu_j - (1 - \mu_j)(1 - \xi(x_1 - a_{j\sigma})), \\ \dot{x}_2 = \sigma\gamma_j \; x_2 \; \xi(x_1 - a_{j\sigma}), \end{array} \tag{4.1.1}$$

where $\mu_{1,2} > 0$ are small (see (4.1.31)), $\gamma_{1,2} \in [0, 1]$ (see (4.1.32)), and ξ is a C^∞ function such that

$$0 \leq \xi \leq 1, \quad \xi(0) = 1, \quad \xi(z) \equiv 0 \;\text{ at }\; |z| \geq \frac{1}{2}. \tag{4.1.2}$$

As $\dot{x}_1 < 0$ in $U_{j\sigma}$, every orbit of Y that starts in $U_{j\sigma}$ near $x_1 = a_{j\sigma} + 1$ must come in the vicinity of $x_1 = a_{j\sigma} - 1$ as time grows. For the corresponding time-t map, we have

$$x_1(t) = x_1(0) - t + \frac{1}{2}\beta(\mu_j), \qquad x_2(t) = e^{\sigma\gamma_j\alpha(\mu_j)} x_2(0), \tag{4.1.3}$$

where (see (4.1.1))

$$\begin{array}{l} \alpha(\mu) = \int_{x_{n-1}(t)-a_{j\sigma}}^{x_{n-1}(0)-a_{j\sigma}} \frac{\xi(z)dz}{\mu+(1-\mu)(1-\xi(z))} = \int_{-1/2}^{1/2} \frac{\xi(z)dz}{\mu+(1-\mu)(1-\xi(z))}, \\ \beta(\mu) = 2\left(\int_{-1/2}^{1/2} \frac{dz}{\mu+(1-\mu)(1-\xi(z))} - 1\right). \end{array} \tag{4.1.4}$$

Note that $\alpha(\mu)$ is positive, independent of $x_1(0)$ and t (because we assume that the integration limits $(x_1(0)-a_{j\sigma})$ and $(x_1(t) - a_{j\sigma})$ are close to 1 in the absolute value, i.e. they fall in the region where $\xi(z) \equiv 0$; see (4.1.2)), and both $\alpha(\mu)$ and $\beta(\mu)$ tend to infinity as $\mu \to +0$ (the integrals diverge at $\mu = 0$ because $\xi(0) = 1$).

Denote $\Sigma^{in}_{j+} := \{x_1 = a_{j+} + 1, |x_2| \leq 1\}$, $\Sigma^{out}_{j+} := \{x_1 = a_{j+} - 1, |x_2| \leq R\}$, $\Sigma^{in}_{j-} := \{x_1 = a_{j-} + 1, |x_2| \leq R\}$, $\Sigma^{out}_{j-} := \{x_1 = a_{j-} - 1, |x_2| \leq 1\}$. Every orbit of Y that intersects Σ^{in}_{j+} at x_2 sufficiently small leaves U_{j+} by crossing Σ^{out}_{j+}, and the orbits that intersect Σ^{in}_{j-} leave U_{j-} by crossing Σ^{out}_{j-} (see (4.1.3)). Define the vector field

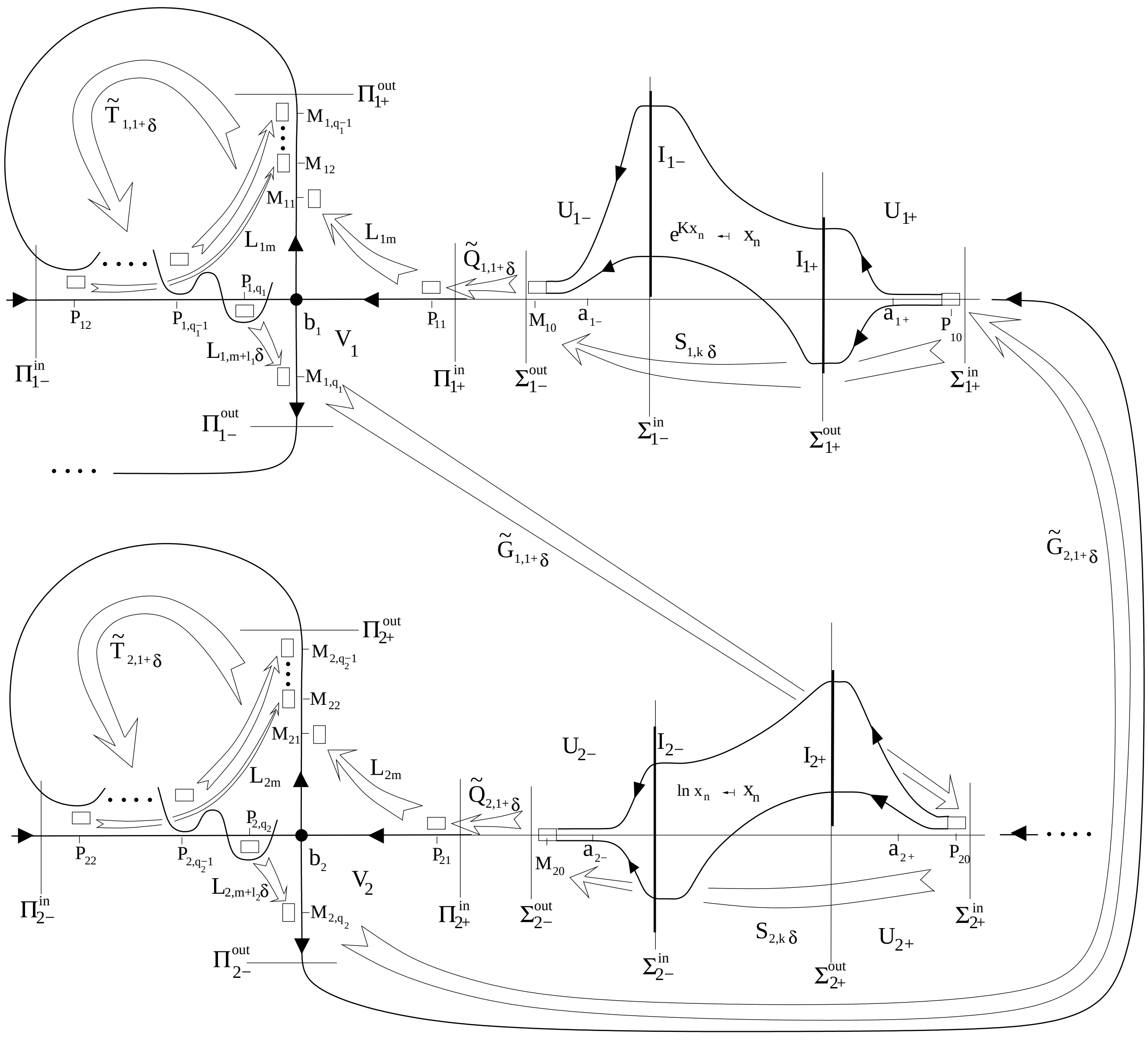

Figure 2: An illustration to the proof of Theorem 1.

Y in the region between Σ^{out}_{j+} and Σ^{in}_{j-} in such a way that the orbits starting in Σ^{out}_{+} reach Σ^{in}_{-} at time 1, and the corresponding Poincaré map $\Sigma^{out}_{j+} \to \Sigma^{in}_{j-}$ is

$$x_2 \mapsto \psi_j(x_2),$$

where $\psi_1(x_2) \equiv e^{Kx_2}$ at $x_2 \in I_{1+}$ and $\psi_2(x_2) \equiv \ln x_2$ at $x_2 \in I_{2+}$ (see (3.4)). Then, the flow takes the points from the vicinity of $x_1 = a_{j+} + 1$ in U_{j+} into the vicinity of $x_1 = a_{j-} - 1$ in U_{j-}. By (4.1.3), the corresponding time-t map S_{jt} is

$$x_1(t) = x_1(0) - t + \beta(\mu_j), \qquad x_2(t) = e^{-\gamma_j \alpha(\mu_j)}\ \psi_j\left(e^{\gamma_j\alpha(\mu_j)} x_2(0)\right). \tag{4.1.5}$$

In the regions $V_j : \{|x_1 - b_j| \leq 1, |x_2| \leq 1\}$ $(j = 1, 2)$ we put the vector field of Y to be equal to

$$\dot{x}_1 = -(x_1 - b_j), \qquad \dot{x}_2 = x_2. \tag{4.1.6}$$

Thus, in V_j, the point $O_j : \{x_1 = b_j, x_2 = 0\}$ is a linear saddle. Its local stable manifold W^s_j is $x_2 = 0$, and the local unstable manifold W^u_j is $x_1 = b_j$. The time-t map L_{jt} within V_j is given by

$$x_1(t) = b_j + e^{-t}(x_1(0) - b_j), \qquad x_2(t) = e^t x_2(0). \tag{4.1.7}$$

Let us define Y in the region between Σ^{out}_{j-} and $\Pi^{in}_{j+} := \{x_1 = b_j + 1, |x_2| \leq 1\}$ in such a way that all the orbits starting in a small neighborhood of $x_2 = 0$ in Σ^{out}_{j-} intersect Π^{in}_{j+} at time 1, and the corresponding Poincaré map is the identity:

$$x_2 \mapsto x_2.$$

Then the time-t map Q_{jt} from a small neighborhood of $x_1 = a_{j-} - 1, x_2 = 0$ in U_{j-} into a small neighborhood of $x_1 = b_j + 1$ in V_j is given by

$$x_1(t) - b_j = e^{-(t - x_1(0) - 2 + a_{j-})},\ \ x_2(t) = e^{t - x_1(0) - 2 + a_{j-}} x_2(0) \tag{4.1.8}$$

In order to see this, we recall that the vector field in U_{j-} near $x_1 = a_{j-} - 1$ is given by

$$\dot{x}_1 = -1, \quad \dot{x}_2 = 0$$

(see (4.1.1)). Therefore, the term $x_1(0) + 2 - a_{j-}$ in (4.1.8) is the time the orbit spends in order to get from $x(0)$ to Π^{in}_{j+}).

Every orbit that enters V_j at $x_2 > 0$ leaves V_j by crossing the cross-section $\Pi^{out}_{j+} := \{x_2 = 1, |x_1 - b_j| \leq 1\}$, and every orbit that enters V_j at $x_2 < 0$ leaves it by crossing the cross-section $\Pi^{out}_{j-} := \{x_2 = -1, |x_1 - b_j| \leq 1\}$. We assume that the orbits that start at Π^{out}_{j+} close to the point $W^u_j \cap \Pi^{out}_{j+} = (b_j, 1)$ return to W_j at time 1 and cross $\Pi^{in}_{j-} := \{x_1 = b_j - 1, |x_2| \leq 1\}$; we also assume that the corresponding Poincaré map $x_1 \mapsto \bar{x}_2$ is given by

$$\bar{x}_2 = -(x_1 - b_j)$$

(the minus sign stands to ensure the orientability of the flow map). It follows that the time-t map T_{jt} from a small neighborhood of $W^u_j \cap \Pi^{out}_{j+}$ in V_j into a small neighborhood of $x_1 = b_j - 1$ in V_j is given by

$$x_1(t) = b_j - e^{-(t-1)}/x_2(0), \qquad x_2(t) = -e^{t-1} x_2(0)^2 (x_1(0) - b_j). \tag{4.1.9}$$

For the orbits that start at Π_{j-}^{out} close to the point $W_j^u \cap \Pi_{j-}^{out} = (b_j, -1)$ we assume that they cross $\Sigma_{3-j,+}^{in}$ at time 1, and the corresponding Poincaré map is given by $\bar{x}_2 = -(x_1 - b_j)$. Thus (see (4.1.1),(4.1.2),(4.1.6)), the time-t map G_{jt} from a small neighborhood of $W_j^u \cap \Pi_{j-}^{out}$ in V_j into a small neighborhood of $x_1 = a_{3-j,+} + 1$ in $U_{3-j,+}$ is

$$x_1(t) = a_{3-j,+} + 2 - t - \ln|x_2(0)|, \qquad x_2(t) = -(x_1(0) - b_j)|x_2(0)|. \tag{4.1.10}$$

Every C^∞ flow Y, which satisfies (4.1.5),(4.1.8),(4.1.7),(4.1.9),(4.1.10), is good for our purposes. We may therefore assume that the vector field of Y is identically zero outside some sufficiently large ball D. For small δ, the time-δ map Y_δ of the flow is $O(\delta)$-close to identity in the C^r-norm, for any given r. It also equals to identity outside D. Let us fix a certain r, and take a sufficiently small δ (for convenience, we assume that $N := \delta^{-1}$ is an integer). Below we construct an arbitrarily small (in the C^r-norm), localized in D perturbation of Y_δ as follows.

For the given diffeomorphism F, take its sufficiently close approximation in the form of (3.24); in the two-dimensional case the map Φ_0 is given by

$$\Phi_0 := (x_1, x_2) \mapsto (x_2, -x_1). \tag{4.1.11}$$

As there is only a finite number $(q_1 + q_2)$ of the polynomial maps $\tilde{H}_{js}$ in (3.24), one can find some finite $d \geq 1$ (common for all $\tilde{H}_{js}$, $s = 1, \dots, q_j, j = 1, 2$) such that the maps $\tilde{H}_{js}$ are written as follows:

$$\bar{x}_1 = x_2, \qquad \bar{x}_2 = -x_1 + \sum_{0 \leq \nu \leq d} h_{js\nu} x_2^\nu. \tag{4.1.12}$$

In the segment $I_j^{out} := \{e^{-\delta} \leq x_2 < 1\}$ of W_j^u $(j = 1, 2)$, we choose $q_j - 1$ different points $M_{j1}, \dots, M_{j,q_j-1}$, and one point $M_{jq_j} \in W_j^u$ will be chosen in the segment $-e^{-\delta} \geq x_2 > -1$. Let u_{js} denote the coordinate x_2 of M_{js} $(s = 1, \dots, q_j)$. As $N\delta = 1 =$flight time from Π_{j+}^{out} to Π_{j-}^{in}, near the segment I_j^{out} the $(N+1)$-th iteration of the time-δ map Y_δ is the map $T_{j,1+\delta}$ from (4.1.9), i.e. it is given by

$$\bar{x}_1 = b_j - e^{-\delta}/x_2, \qquad \bar{x}_2 = -e^\delta x_2^2 (x_1 - b_j). \tag{4.1.13}$$

This map takes the segment I_j^{out} onto the segment $\{b_j - 1 < x_1 < b_j - e^{-\delta}, x_2 = 0\} \in W_j^s$. Let $P_{j,s+1} = T_{j,1+\delta} M_{js}$ $(s = 1, \dots, q_j - 1)$, and let P_{j1} be a point from $\{b_j + e^{-\delta} < x_1 < b_j + 1, x_2 = 0\} \in W_j^s$. We denote the coordinate x_1 of $P_{j,s+1}$ as z_{js}. By (4.1.13),

$$z_{j,s+1} = b_j - e^{-\delta}/u_{js}. \tag{4.1.14}$$

We also assume

$$z_{j1} = b_j + e^{-\delta/2}. \tag{4.1.15}$$

We stress that u_{js} and $(b_j - z_{js})$ are bounded away from zero.

Take a sufficiently large integer m and choose some points $P'_{js} = (z_{js}, z'_{js})$ and $M'_{js} = (u'_{js}, u_{js})$, sufficiently close to P_{js} and M_{js} respectively $(j = 1, 2; s = 1, \dots, q_j)$. We define the coordinates of P'_{js} and M'_{js} by the following rule:

$$\begin{array}{ll} z'_{js} = e^{-m} u_{js}), \quad u'_{js} = b_j + e^{-m}(z_{js} - b_j) \quad \text{at} \quad s \leq q_j - 1 \\ z'_{jq_j} = e^{-(m + l_j\delta)} u_{jq_j}, \quad u'_{jq_j} = b_j + e^{-(m + l_j\delta)}(z_{jq_j} - b_j), \end{array} \tag{4.1.16}$$

where l_j is an integer to be defined later (see (4.1.24); note that $l_j\delta$ remains uniformly bounded for arbitrarily small δ). As m is assumed to be large and $l_j\delta$ is bounded, such defined points P'_{js} and M'_{js} are closed to respective points P_{js} and M_{js} indeed. It follows from (4.1.16) that $M'_{js} = L_{jm} P'_{js}$ at $s \leq q_j - 1$, where L_{jt} is the map (4.1.7). At $s = q_j$ we have $M'_{jq_j} = L_{j,m+l_j\delta} P'_{jq_j}$.

Let us add to the map Y_δ a small perturbation, which is localized in a small neighborhood of the points $Y_\delta^{-1}(P_{j,s+1})$ (so outside these small neighborhoods Y_δ remains unchanged). We require that these localized perturbations are such that in a sufficiently small neighborhood of $M_{js} = Y_\delta^{-N}(Y_\delta^{-1}P_{j,s+1})$ the map $\tilde{Y}_\delta^{N+1}$ (where $\tilde{Y}_\delta$ denotes the perturbed map) is given by the following perturbation of (4.1.13):

$$\begin{aligned}\bar{x}_1 &= b_j - e^{-\delta}/x_2,\\ \bar{x}_2 &= z'_{j,s+1} - e^{\delta}x_2^2(x_1 - u'_{js}) + \sum_{0\le\nu\le d}\varepsilon_{js\nu}(x_2 - u_{js})^\nu,\end{aligned}\tag{4.1.17}$$

where $\varepsilon_{js\nu}$ are small coefficients to be determined later (see (4.1.21)). By (4.1.16), $z'_{j,s+1}$ and $(u'_{js} - b_j)$ are small as well, so (4.1.17) is a small perturbation of (4.1.13) indeed. It is easy to see from (4.1.17),(4.1.14) that $P'_{j,s+1} = \tilde{Y}_\delta^{N+1}M'_{js}$ at $\varepsilon = 0$.

At $\varepsilon = 0$ (hence at all small ε) the map $\tilde{T}_{j,1+\delta}\circ L_{jm} \equiv \tilde{Y}_\delta^{mN+N+1}$ (where $\tilde{T}$ stands for the perturbed map T) takes a small neighborhood of $P'_{js} = (z_{js}, z'_{js})$ into a small neighborhood of $P'_{j,s+1} = (z_{j,s+1}, z'_{j,s+1})$. By (4.1.17),(4.1.7),(4.1.16), this map is written as

$$\begin{aligned}\bar{x}_1 &= b_j - e^{-(m+\delta)}/x_2,\\ \bar{x}_2 &= z'_{j,s+1} - e^{(m+\delta)}x_2^2(x_1 - z_{js}) + \sum_{0\le\nu\le d}\varepsilon_{js\nu}e^{\nu m}(x_2 - z'_{js})^\nu,\end{aligned}\tag{4.1.18}$$

We choose some $\eta(m)$ that tends to zero as $m\to+\infty$ and introduce rescaled coordinates (v_1, v_2) near P'_{js} by the rule

$$x_1 = z_{js} + C_{js}\,\eta\, v_1, \qquad x_2 = z'_{js} + \frac{1}{C_{js}}\,\eta\, e^{-m}v_2,\tag{4.1.19}$$

where the independent of m positive coefficients C_{js} are determined later (see (4.1.20); note that C_{js} are bounded away from zero and infinity). Since η tends to zero as $m\to+\infty$, any bounded region of values of v corresponds to a small neighborhood of P'_{js}.

After the rescaling, map (4.1.18) takes the following form (see (4.1.16),(4.1.14)):

$$\begin{aligned}C_{j,s+1}\bar{v}_1 &= \frac{1}{e^\delta u_{js}^2 C_{js}}v_2 + O(\eta),\\ \frac{1}{C_{j,s+1}}\bar{v}_2 &= -e^\delta u_{js}^2 C_{js}v_1 + O(\eta) + \sum_{0\le\nu\le d}\varepsilon_{js\nu}e^m\eta^{\nu-1}C_{js2}^\nu v_2^\nu,\end{aligned}$$

As we see, by putting

$$C_{j,s+1} = \frac{1}{e^\delta u_{js}^2 C_{js}},\tag{4.1.20}$$

and

$$\varepsilon_{js\nu} = h_{js\nu}e^{-m}\eta^{1-\nu}C_{j,s+1}^{-1}C_{js}^\nu,\tag{4.1.21}$$

the map $\tilde{T}_{j,1+\delta}\circ L_{jm}$ near P'_{js} takes the form

$$\begin{aligned}\bar{v}_1 &= v_2 + O(\eta),\\ \bar{v}_2 &= -v_1 + O(\eta) + \textstyle\sum_{0\le\nu\le d}h_{js\nu}v_2^\nu,\end{aligned}\tag{4.1.22}$$

i.e. it can be made as close as we want to the map $\tilde{H}_{js}$, provided m is taken large enough (recall that $\eta\to 0$ as $m\to+\infty$). We take η tending to zero sufficiently slowly, so all $\varepsilon_{js\nu}\to 0$ (see (4.1.21); recall that $\nu\le d$ where d is independent of m and δ). Thus, our perturbation to Y_δ is arbitrarily small indeed.

It follows that in the rescaled coordinates the map $\left(\tilde{T}_{j,1+\delta}\circ L_{jm}\right)^{q_j}\equiv\tilde{Y}_\delta^{q_j(mN+N+1)}$ from a small neighborhood of P'_{j1} into a small neighborhood of P'_{jq_j} can be made as close as we want to the map

$$\tilde{H}_{jq_j}\circ\ldots\circ\tilde{H}_{j1},$$

provided m is large enough. The rescaled coordinates near P'_{j1} and P'_{jq_j} are given by formulas (4.1.19), where the coefficients $C_{ji}>0$ may be taken arbitrary, and the coefficients C_{jq_j} are then recovered from the recursive formula (4.1.20). Further it is convenient to put

$$C_{j1}=e^{-\delta/2}. \tag{4.1.23}$$

Note that m does not enter (4.1.20),(4.1.23), hence C_{jq_j} stay bounded away from zero and infinity as $m\to+\infty$.

Let us also fix the choice of the integer l_j that enters the definition of the points P'_{jq_j} and M'_{jq_j} (see (4.1.16)). Namely, we require that

$$\ln|u_{jq_j}|+\ln C_{jq_j}=l_j\delta. \tag{4.1.24}$$

Recall that u_{jqj} is the coordinate x_2 of the point M_{jq_j}, and it can be arbitrarily taken within the interval

$$-\,e^{-\delta}\geq u_{jqj}>-1. \tag{4.1.25}$$

As C_{jq_j} does not depend on the choice of u_{jqj} (see (4.1.20)), condition (4.1.24) uniquely defines both the integer l_j and the value of u_{jq_j}. The claimed before uniform boundedness of $l_j\delta$ follows from the uniform boundedness of $\ln C_{jq_j}$.

Now, from (4.1.7),(4.1.19),(4.1.16),(4.1.24) we obtain that the map $L_{j,m+l_j\delta}$ from a small neighborhood of P'_{jq_j} into a small neighborhood of M'_{jq_j} is identity (i.e. $\bar{v}_1=v_1,\ \bar{v}_2=v_2$), provided the rescaled coordinates (v_1,v_2) near M'_{jq_j} are introduced as follows:

$$\begin{array}{l} x_1=u'_{jq_j}+|u_{jq_j}|^{-1}\,\eta\,e^{-m}v_1,\\ x_2=u_{jq_j}-u_{jq_j}\,\eta\,v_2. \end{array} \tag{4.1.26}$$

Therefore, for the given choice of the coordinates, by taking $m\to+\infty$, the map $L_{j,m+l_j\delta}\circ\left(\tilde{T}_{j,1+\delta}\circ L_{jm}\right)^{q_j}\equiv\tilde{Y}_\delta^{l_j+mN+q_j(mN+N+1)}$ from a small neighborhood of P'_{j1} into a small neighborhood of M'_{jq_j} can be made as close as we want to $\tilde{H}_{jq_j}\circ\ldots\circ\tilde{H}_{j1}$ (we have already proved the same for the map $\left(\tilde{T}_{j,1+\delta}\circ L_{jm}\right)^{q_j}$, and the map $L_{j,m+l_j\delta}$ is identity in the chosen coordinates).

Recall that, by construction, the point $Y_\delta^{N+1}M'_{jq_j}$ lies in $U_{3-j,+}$ in the region $a_{3-j,+}+1-\delta<x_1<a_{3-j,+}+1$. We add to the map Y_δ a perturbation, localized near the point $Y_\delta^N M'_{jq_j}$, such that the corresponding map $\tilde{G}_{j,1+\delta}\equiv\tilde{Y}_\delta^{N+1}$ will have the following form near M'_{jq_j}:

$$\bar{x}_1=a_{3-j,+}+1-\delta-\ln|x_2|,\qquad \bar{x}_2=-|x_2|(x_1-u'_{jq_j}); \tag{4.1.27}$$

since $u'_{jq_j}\to b_j$ as $m\to+\infty$ (see (4.1.16)), these fomulas define a small perturbation of the map $G_{j,1+\delta}$ given by (4.1.10) indeed.

Denote $P'_{3-j,0}=\tilde{G}_{j,1+\delta}M'_{jq_j}$. By (4.1.27), the coordinates of $P'_{3-j,0}$ are given by $\{x_1=a_{3-j,+}+1-\delta-\ln|u_{jq_j}|,\ x_2=0\}$. Introduce rescaled coordinates near the points $P'_{3-j,0}$ $(j=1,2)$ by the rule

$$x_1=a_{3-j,+}+1-\delta-\ln|u_{jq_j}|+\eta\,v_1,\qquad x_2=\eta\,e^{-m}v_2. \tag{4.1.28}$$

When coordinates are rescaled by rule (4.1.26) near M'_{jq_j} and by rule (4.1.28) near $P'_{3-j,0}$, map (4.1.27) takes the form $(v_1, v_2) \mapsto (v_2, -v_1) + O(\eta)$, i.e. it becomes arbitrarily close to the map Φ_0 (see (4.1.11)) as $m_{\rightarrow} + \infty$. Thus, in the coordinates rescaled by rule (4.1.19) near P'_{j1} and by rule (4.1.28) near $P'_{3-j,0}$, the map $\tilde{G}_{j,1+\delta} \circ L_{j,m+l_j\delta} \circ \left(\tilde{T}_{j,1+\delta} \circ L_{jm}\right)^{q_j} \equiv \tilde{Y}_\delta^{l_j+(1+q_j)(mN+N+1)}$ from a small neighborhood of P'_{j1} into a small neighborhood of $P'_{3-j,0}$, can be made as close as we want to the map $\Phi_0 \circ \tilde{H}_{jq_j} \circ \ldots \circ \tilde{H}_{j1}$ as m grows.

Analogously, we take the point $M'_{j0} : \{x_1 = a_{j-} - 1 + \delta/2,\ x_2 = 0\} \in U_{j-}$ and perturb the map Y_δ near $Y_\delta^N P'_{j0}$ in such a way that the map $\tilde{Q}_{j,1+\delta} \equiv \tilde{Y}_\delta^{N+1}$ near M'_{j0} will be given by

$$\bar{x}_1 - b_j = e^{-(\delta - x_1 - 1 + a_{j-})}, \ \ \bar{x}_2 = e^{\delta - x_1 - 1 + a_{j-}} x_2 + e^{-m} u_{j1}. \tag{4.1.29}$$

It is a small perturbation of the map $Q_{j,1+\delta}$ given by (4.1.8), and it takes M'_{j0} to P'_{j1} (see (4.1.15),(4.1.16)). When we introduce rescaled variables near M'_{j0} by the rule

$$x_1 = a_{j-} - 1 + \delta/2 + \eta\, v_1, \qquad x_2 = \eta\, e^{-m} v_2, \tag{4.1.30}$$

and near P'_{j1} by rule (4.1.19),(4.1.23), map (4.1.29) will take the form $\bar{v} = v + O(\eta)$, i.e. it is close to the identity map. Thus, in the rescaled coordinates given by (4.1.30),(4.1.28), the map $\tilde{G}_{j,1+\delta} \circ L_{j,m+l_j\delta} \circ \left(\tilde{T}_{j,1+\delta} \circ L_{jm}\right)^{q_j} \circ \tilde{Q}_{j,1+\delta} \equiv \tilde{Y}_\delta^{l_j+(1+q_j)(mN+N+1)+N+1}$ from a small neighborhood of M'_{j0} into a small neighborhood of $P'_{3-j,0}$, is as close as we want to the map $\Phi_0 \circ \tilde{H}_{jq_j} \circ \ldots \circ \tilde{H}_{j1}$ at m large enough, i.e. it is a close approximation of the map Φ_j.

Let us now determine the form of the map $S_{jt} : v \mapsto \bar{v}$ from a small neighborhood of P'_{j0} into a small neighborhood of M'_{j0} in the coordinates rescaled by the same rules (4.1.28) and (4.1.30) (note that when rescaling near P'_{j0} one should change j to $3-j$ in the right-hand side of (4.1.28)). We will choose the coefficients μ_j in (4.1.1) such that

$$\beta(\mu_j) = k - 5 + 3/2\delta + \ln|u_{3-j,q_{3-j}}| \tag{4.1.31}$$

where k is some sufficiently large integer. Since $\beta \to +\infty$ as $\mu \to +0$ (see (4.1.4)), and $\ln|u_{3-j,q_{3-j}}|$ is bounded (see (4.1.25)), equation (4.1.31) has a solution $\mu_j(k)$ for every sufficiently large k, and $\mu_j(k) \to +0$ as $k \to +\infty$. It follows that $\alpha(\mu_j(k)) \to +\infty$. Thus, for any sufficiently large m we can find $\gamma_j \in (0,1]$ and large k such that

$$\eta(m)\, e^{-m} = e^{-\gamma_j \alpha(\mu_j(k))}. \tag{4.1.32}$$

From (4.1.5) we immediately obtain that with this choice of μ_j, γ_j and k the map $S_{j,k\delta} \equiv Y_\delta^k$ from a small neighborhood the point P'_{j0} into a small neighborhood of M'_{j0} takes the following form in the coordinates rescaled by rules, respectively, (4.1.28) (with $(3-j)$ changed to j) and (4.1.30):

$$\bar{v}_1 = v_1, \qquad \bar{v}_2 = \psi_j(v_2).$$

As wee see, the map $S_{j,k\delta}$ in the rescaled coordinates coincides with the map Ψ_j for v from some open neighborhood of D (if $j=1$) or of $\Phi_1 \circ \Psi_1(D)$ (if $j=2$).

Summarizing, we obtain that the map

$$\begin{aligned} &\tilde{G}_{2,1+\delta} \circ L_{2,m+l_2\delta} \circ \left(\tilde{T}_{2,1+\delta} \circ L_{2m}\right)^{q_2} \circ \tilde{Q}_{2,1+\delta} \circ S_{2,k\delta} \circ \\ &\qquad \circ \tilde{G}_{1,1+\delta} \circ L_{1,m+l_1\delta} \circ \left(\tilde{T}_{1,1+\delta} \circ L_{1m}\right)^{q_1} \circ \tilde{Q}_{1,1+\delta} \circ S_{1,k\delta} \equiv \\ &\equiv\ \tilde{Y}_\delta^{2k+l_1+l_2+(2+q_1+q_2)(mN+N+1)+2(N+1)} \end{aligned}$$

is a close approximation to the map $\Phi_2 \circ \Psi_2 \circ \Phi_1 \circ \Psi_1$ (i.e. to the original map F), provided $\tilde{H}_{jq_j} \circ \ldots \circ \tilde{H}_{j1}$ are sufficiently close approximations to $\tilde{\Phi}_j$ ($j = 1, 2$) and m is large enough. This completes the proof of the theorem in the two-dimensional case.

4.2 Higher-dimensional case

In the case of arbitrary $n > 2$, the construction follows the same line as in the two-dimensional case. As above, $\Phi_{1,2}$ and $\Psi_{1,2}$ are the maps defined by (3.2), and $I_{1\pm}$ and $I_{2\pm}$ are intervals of values of x_n such that $x_n \in I_{1+}$ at $x \in B^n$, $x_n \in I_{1-}$ at $x \in \Psi_1(B^n)$, $x_n \in I_{2+}$ at $x \in \Phi_1 \circ \Psi_1(B^n)$ and $x_n \in I_{2-}$ at $x \in \Psi_2 \circ \Phi_1 \circ \Psi_1(B^n)$. A value R is chosen such that all the intervals $I_{j\pm}$ lie within $\{|x_n| \leq R\}$. Choose numbers $a_{1+} = a_{1-} + 3 = b_1 + 6 = a_{2+} + 9 = a_{2-} + 12 = b_2 + 15$. Define the regions $U_{j\sigma} : \{|x_{n-1} - a_{j\sigma}\| \leq 1, |x_n| \leq R, |x_i| \leq 1 \ (i \leq n-2)\}$, and $V_j : \{|x_{n-1} - b_j| \leq 1, |x_i| \leq 1 \ \ (i \neq n-1)\}$, $j = 1, 2$, $\sigma = \pm 1$. Let the vector field of a C^∞ flow Y in $U_{j\sigma}$ be equal to

$$\begin{aligned} \dot{x}_{n-1} &= -\mu_j - (1 - \mu_j)(1 - \xi(x_{n-1} - a_{j\sigma})), \\ \dot{x}_i &= \sigma\gamma_{i\sigma}\, x_i\, \xi(x_{n-1} - a_{j\sigma}) \quad (i \neq n-1), \end{aligned} \tag{4.2.1}$$

where $\mu_{1,2} > 0$ are small (see (4.2.28),(4.2.29)), $\gamma_{i\pm} \in [0, 1]$ (see (4.2.29),(4.2.30)), and the C^∞ function ξ satisfies (4.1.2). In the regions V_j we make Y equal to

$$\dot{x}_i = -\lambda_i x_i \ \ (i = 1, \ldots, n-2), \ \ \dot{x}_{n-1} = -\lambda_{n-1}(x_{n-1} - b_j), \quad \dot{x}_n = x_n; \tag{4.2.2}$$

here $\lambda_i > 0$ are such that

$$\lambda_2 = \ldots = \lambda_{n-1} = \lambda, \qquad \lambda_1 = 1 - (n-2)\lambda, \tag{4.2.3}$$

where the positive number λ is specified below (see (4.2.11)).

As $\dot{x}_{n-1} < 0$ in $U_{j\sigma}$, every orbit of Y that starts in $U_{j\sigma}$ near $x_{n-1} = a_{j\sigma} + 1$ must come in the vicinity of $x_{n-1} = a_{j\sigma} - 1$ as time grows. For the corresponding time-t map, we have

$$x_i(t) = e^{\sigma\gamma_{i\sigma}\alpha(\mu_j)} x_i(0) \quad (i \leq n-2), \qquad x_n(t) = x_n(0) - t + \frac{1}{2}\beta(\mu_j), \tag{4.2.4}$$

where the tending to infinity, as $\mu \to +0$, functions $\alpha(\mu)$ and $\beta(\mu)$ are defined by (4.1.4).

Denote $\Sigma^{in}_{j+} := \{x_{n-1} = a_{j+} + 1, |x_n| \leq 1\}$, $\Sigma^{out}_{j+} := \{x_{n-1} = a_{j+} - 1, |x_n| \leq R\}$, $\Sigma^{in}_{j-} := \{x_{n-1} = a_{j-} + 1, |x_n| \leq R\}$, $\Sigma^{out}_{j-} := \{x_{n-1} = a_{j-} - 1, |x_n| \leq 1\}$ (we also assume that $|x_i| \leq 1$ for $i \leq n-2$ on $\Sigma^{in,out}_{j\pm}$). Every orbit of Y that intersects Σ^{in}_{j+} at $x_n, x_1, \ldots, x_{n-2}$ sufficiently small leaves U_{j+} by crossing Σ^{out}_{j+}, and the orbits that intersect Σ^{in}_{j-} leave U_{j-} by crossing Σ^{out}_{j-} (see (4.2.4)). We define Y in the region between Σ^{out}_{j+} and Σ^{in}_{j-} in such a way that the orbits starting in Σ^{out}_{+} reach Σ^{in}_{-} at time 1, and the corresponding Poincaré map $\Sigma^{out}_{j+} \to \Sigma^{in}_{j-}$ is $(x_1, \ldots, x_{n-2}, x_n) \mapsto (x_1, \ldots, x_{n-2}, \psi_j(x_n))$, where the functions ψ_j are defined by (3.4). Then, the flow takes the points from the vicinity of $x_{n-1} = a_{j+} + 1$ in U_{j+} into the vicinity of $x_{n-1} = a_{j-} - 1$ in U_{j-}. By (4.2.4), the corresponding time-t map S_{jt} is

$$\begin{aligned} &x_i(t) = e^{(\gamma_{i+} - \gamma_{i-})\alpha(\mu_j)} x_i(0) \qquad (i \leq n-2), \\ &x_{n-1}(t) = x_{n-1}(0) - t + \beta(\mu_j), \qquad x_n(t) = e^{-\gamma_{n-}\alpha(\mu_j)} \psi_j(e^{\gamma_{n+}\alpha(\mu_j)} x_n(0)). \end{aligned} \tag{4.2.5}$$

In the region between Σ^{out}_{j-} and $\Pi^{in}_{j+} := \{x_{n-1} = b_j + 1, |x_i| \leq 1\}$, we define Y in such a way that all the orbits starting in a small neighborhood of $x_n = x_1 = \ldots = x_{n-2} = 0$ in Σ^{out}_{j-} intersect Π^{in}_{j+} at time 1, and the corresponding Poincaré map is the identity: $(x_1, \ldots, x_{n-2}, x_n) \mapsto (x_1, \ldots, x_{n-2}, x_n)$. Then the time-$t$ map Q_{jt} from a small neighborhood of $x_{n-1} = a_{j-} - 1, x_n = x_1 = \ldots = x_{n-2} = 0$ in U_{j-} into a small neighborhood of $x_{n-1} = b_j + 1$ in V_j is given by

$$\begin{array}{l} x_i(t) = e^{-\lambda_i(t - x_{n-1}(0) - 2 + a_{j-})} x_i(0) \qquad (i \leq n-2), \\ x_{n-1}(t) - b_j = e^{-\lambda(t - x_{n-1}(0) - 2 + a_{j-})}, \quad x_n(t) = e^{t - x_{n-1}(0) - 2 + a_{j-}} x_n(0) \end{array} \tag{4.2.6}$$

(see (4.2.2)).

In V_j $(j = 1, 2)$, the local stable manifold W_j^s of the linear saddle equilibrium state $O_j : \{x_{n-1} = b_j, x_i = 0 \ (i \neq n-1)\}$ is $x_n = 0$, and the local unstable manifold W_j^u is $x_{n-1} = b_j, x_1 = \ldots = x_{n-2} = 0$. The time-$t$ map L_{jt} within V_j is given by

$$\begin{array}{l} x_i(t) = e^{-\lambda_i t} x_i(0) \qquad (i \leq n-2), \\ x_{n-1}(t) - b_j = e^{-\lambda t}(x_{n-1}(0) - b_j), \qquad x_n(t) = e^t x_n(0). \end{array} \tag{4.2.7}$$

Every orbit that enters V_j at $x_n > 0$ leaves V_j by crossing the cross-section $\Pi_{j+}^{out} := \{x_n = 1, |x_{n-1} - b_j| \leq 1, |x_i| \leq 1 \ (i \leq n-2)\}$, and every orbit that enters V_j at $x_n < 0$ leaves it by crossing the cross-section $\Pi_{j-}^{out} := \{x_n = -1, |x_{n-1} - b_j| \leq 1, |x_i| \leq 1 \ (i \leq n-2)\}$. We assume that the orbits that start at Π_{j+}^{out} close to the point $W_j^u \cap \Pi_{j+}^{out} = (x_1 = \ldots = x_{n-2} = 0, x_{n-1} = b_j)$ return to W_j at time 1 and cross $\Pi_{j-}^{in} := \{x_{n-1} = b_j - 1, |x_i| \leq 1 (i \neq n-1)\}$; we also assume that the corresponding Poincaré map $(x_1, \ldots, x_{n-1}) \mapsto (\bar{x}_1, \ldots, \bar{x}_{n-2}, \bar{x}_n)$ is given by

$$\bar{x}_i = x_{i+1} \quad (i \leq n-3), \qquad \bar{x}_{n-2} = x_{n-1} - b_j, \qquad \bar{x}_n = (-1)^{n+1} x_1$$

(the factor $(-1)^{n+1}$ stands to ensure the orientability). It follows that the time-t map T_{jt} from a small neighborhood of $W_j^u \cap \Pi_{j+}^{out}$ in V_j into a small neighborhood of $x_{n-1} = b_j - 1$ in V_j is given by

$$\begin{array}{l} x_i(t) = e^{-\lambda_i(t-1)} x_n(0)^{\lambda_{i+1} - \lambda_i} x_{i+1}(0) \qquad (i \leq n-3), \\ x_{n-2}(t) = e^{-\lambda_{n-2}(t-1)} x_n(0)^{\lambda_{n-1} - \lambda_{n-2}} (x_{n-1}(0) - b_j), \\ x_{n-1}(t) = b_j - e^{-\lambda_{n-1}(t-1)} x_n(0)^{-\lambda_{n-1}}, \\ x_n(t) = (-1)^{n+1} e^{t-1} x_n(0)^{1+\lambda_1} x_1(0). \end{array} \tag{4.2.8}$$

For the orbits that leave V_j by crossing Π_{j-}^{out}, we assume that the orbits that start at Π_{j-}^{out} close to the point $W_j^u \cap \Pi_{j-}^{out} = (x_1 = \ldots = x_{n-2} = 0, x_{n-1} = b_j)$ cross $\Sigma_{3-j,+}^{in}$ at time 1, and the corresponding Poincaré map $(x_1, \ldots, x_{n-1}) \mapsto (\bar{x}_1, \ldots, \bar{x}_{n-2}, \bar{x}_n)$ is given by $\bar{x}_i = x_i$ at $i = 1, \ldots, n-2$ and $\bar{x}_n = -(x_{n-1} - b_j)$. Thus (see (4.2.1),(4.1.2),(4.2.2)), the time-t map G_{jt} from a small neighborhood of $W_j^u \cap \Pi_{j-}^{out}$ in V_j into a small neighborhood of $x_{n-1} = a_{3-j,+} + 1$ in $U_{3-j,+}$ is

$$\begin{array}{l} x_i(t) = |x_n(0)|^{\lambda_i} x_i(0) \qquad (i \leq n-2), \\ x_{n-1}(t) = a_{3-j,+} + 2 - t - \ln|x_n(0)|, \\ x_n(t) = -|x_n(0)|^{\lambda_{n-1}} (x_{n-1}(0) - b_j). \end{array} \tag{4.2.9}$$

These conditions define the flow Y (we also assume that the vector field of Y is identically zero outside some sufficiently large ball D). Let us take a sufficiently small δ such that $N := \delta^{-1}$ is an integer, and proceed to the construction of a small (arbitrarily small in the C^r-norm, with any chosen in advance r) perturbation of Y_δ, localized in D.

Take a sufficiently close approximation of the given diffeomorphism F by the product (3.24). There exists some finite $d \geq 1$, common for all $\tilde{H}_{js}$ $(s = 1, \ldots, q_j, j = 1, 2)$ such that the polynomial maps $\tilde{H}_{js}$ in (3.24) are written as follows:

$$\bar{x}_i = x_{i+1} \quad (i \leq n-1), \qquad \bar{x}_n = (-1)^{n+1} x_1 + \sum_{\substack{\nu_2 \geq 0, \ldots, \nu_n \geq 0 \\ \nu_2 + \ldots + \nu_n \leq d}} h_{js\nu} \prod_{2 \leq p \leq n} x_p^{\nu_p}. \tag{4.2.10}$$

We will now fix the choice of the positive λ in (4.2.3) such that

$$\lambda < \frac{1}{(n-1)d + r}. \tag{4.2.11}$$

In the segment $I_j^{out} := \{e^{-\delta} \leq x_n < 1\}$ of W_j^u $(j = 1, 2)$, we choose $q_j - 1$ different points $M_{j1}, \ldots, M_{j,q_j-1}$, and one point $M_{jq_j} \in W_j^u$ will be chosen in the segment $-e^{-\delta} \geq x_n > -1$. Let u_{js} denote the coordinate x_n of M_{js} $(s = 1, \ldots, q_j)$. As $N\delta = 1$, near the segment I_j^{out} the $(N+1)$-th iteration of the time-δ map Y_δ is the map $T_{j,1+\delta}$ from (4.2.8). Thus, the map Y_δ^{N+1} near I_j^{out} will be given by

$$\begin{aligned} &\bar{x}_1 = e^{((n-2)\lambda - 1)\delta} x_n^{(n-1)\lambda - 1} \hat{x}_2, \qquad \bar{x}_i = e^{-\lambda\delta} \hat{x}_{i+1} \quad (2 \leq i \leq n-2), \\ &\bar{x}_{n-1} = b_j - e^{-\lambda\delta} x_n^{-\lambda}, \qquad \bar{x}_n = (-1)^{n+1} e^{\delta} x_n^{2-(n-2)\lambda} x_1 \end{aligned} \tag{4.2.12}$$

(where we denote $\hat{x}_i = x_i$ at $i \neq n-1$ and $\hat{x}_{n-1} = x_{n-1} - b_j$). This map takes the segment I_j^{out} onto the segment $\{b_j - 1 < x_{n-1} < b_j - e^{-\lambda\delta}, x_1 = \ldots = x_{n-2} = x_n = 0\} \in W_j^s$. Let $P_{j,s+1} = T_{j,1+\delta} M_{js}$ $(s = 1, \ldots, q_j - 1)$, and let P_{j1} be a point from $\{b_j + e^{-\lambda\delta} < x_{n-1} < b_j + 1, x_1 = \ldots = x_{n-2} = x_n = 0\} \in W_j^s$. By (4.2.12), the coordinate x_{n-1} of $P_{j,s+1}$ equals to $b_j - e^{-\lambda\delta} u_{js}^{-\lambda}$.

We take sufficiently large integer m and choose some points P'_{js} and M'_{js}, sufficiently close to P_{js} and M_{js} respectively $(j = 1, 2; s = 1, \ldots, q_j)$, such that at $s \leq q_j - 1$ we have $M'_{js} = L_{jm} P'_{js}$ (where L_{jt} is the map (4.2.7)). At $s = q_j$ we assume $M'_{jq_j} = L_{j,m+l_j\delta} P'_{jq_j}$ where l_j is an integer to be defined later (see (4.2.31); note that $l_j\delta$ is uniformly bounded). Denote the coordinates of P'_{js} and M'_{js} as $(z'_{js1}, \ldots, z'_{js,n-2}, b_j + z'_{js,n-1}, z'_{jsn})$ and $(u'_{js1}, \ldots, u'_{js,n-2}, b_j + u'_{js,n-1}, u'_{jsn})$ respectively. By (4.2.7),

$$u'_{jsi} = e^{-\lambda_i m} z'_{jsi} \ \ (i = 1, \ldots, n-1), \ \ u'_{jsn} = e^m z'_{jsn} \tag{4.2.13}$$

at $s \leq q_j - 1$. At $s = q_j$ we have

$$u'_{jq_ji} = e^{-\lambda_i(m + l_j\delta)} z'_{jq_ji} \quad (i = 1, \ldots, n-1), \quad u'_{jq_jn} = e^{m + l_j\delta} z'_{jq_jn}. \tag{4.2.14}$$

Note that u'_{jsi} are small at $i \leq n-1$, as m is assumed to be large, and $l_j\delta$ is bounded. The values of z'_{jsi} with $i \neq n-1$ will be taken sufficiently small as well, and we will keep

$$u'_{jsn} = u_{js} \quad \text{and} \quad z'_{j,s+1,n-1} = -e^{-\lambda\delta} u_{js}^{-\lambda}, \tag{4.2.15}$$

in order to ensure the closeness of P'_{js} to P_{js} and M'_{js} to M_{js}.

The first of the small perturbations which we add to the map Y_δ is localized in a small neighborhood of the points $Y_\delta^{-1}(P_{j,s+1})$ (so outside these small neighborhoods Y_δ remains unchanged). We take these localized perturbations such that in a sufficiently small neighborhood of $M_{js} = Y_\delta^{-N}(Y_\delta^{-1} P_{j,s+1})$ the map $\tilde{Y}_\delta^{N+1}$ (where

$\tilde{Y}_\delta$ denotes the perturbed map) is given by (4.2.12) with the following correction term

$$z'_{j,s+1,n} - (-1)^{n+1} e^{\delta} x_n^{2-(n-2)\lambda} \left| \frac{x_{n-1} - b_j}{z'_{js,n-1}} \right|^{\frac{1}{\lambda}-(n-1)} e^{-\lambda m} z'_{js1} + \\ + \sum_{\substack{\nu_2 \geq 0, \ldots, \nu_n \geq 0 \\ \nu_2 + \ldots + \nu_n \leq d}} \varepsilon_{js\nu} \prod_{2 \leq p \leq n} (\hat{x}_p - u'_{jsp})^{\nu_p} \tag{4.2.16}$$

added into the equation for $\bar{x}_n$, where $\varepsilon_{js\nu}$ are small coefficients to be determined later (see (4.2.21)). The first term in (4.2.16) is small as well (see (4.2.13),(4.2.14)); in the second term the values of x_n and $z'_{j,s,n-1}$ are bounded away from zero, and the exponent $\left(\frac{1}{\lambda} - (n-1)\right)$ is larger than r (see (4.2.11)), hence the second term is also small with the derivatives up to the order r at least. The first two terms in (4.2.16) ensure, in particular, that at $\varepsilon = 0$ the coordinate x_n of $\tilde{Y}_\delta^{N+1} M'_{js}$ coincides with that of $P'_{j,s+1}$ (see (4.2.13),(4.2.15),(4.2.3)). We want $P'_{j,s+1} = \tilde{Y}_\delta^{N+1} M'_{js}$ at $\varepsilon = 0$, so we put

$$z'_{j,s+1,1} = e^{-(1-(n-2)\lambda)\delta - \lambda m} u_{js}^{(n-1)\lambda - 1} z'_{js2}, \\ z'_{j,s+1,i} = e^{-\lambda(m+\delta)} z'_{js,i+1} \quad (2 \leq i \leq n-2), \tag{4.2.17}$$

(see (4.2.12),(4.2.13),(4.2.15)). At $s = 1$ we assume

$$z'_{j1i} = 0 \ \text{ at } \ i \leq n-2, \quad z'_{j1,n-1} = e^{-\lambda\delta/2}. \tag{4.2.18}$$

Now, the values of z'_{jsi}, u'_{jsi} are defined by (4.2.13),(4.2.14),(4.2.15),(4.2.17) for all j, s, i. As one can see, z'_{jsi} at $i \neq n-1$ and u'_{jsi} at $i \neq n$ tend to zero as $m \to +\infty$, i.e. $P'_{js} \to P_{js}$ and $M'_{js} \to M_{js}$ indeed.

At all small ε the map $\tilde{T}_{j,1+\delta} \circ L_{jm} \equiv \tilde{Y}_\delta^{mN+N+1}$ takes a small neighborhood of P'_{js} into a small neighborhood of $P'_{j,s+1}$. We choose some $\eta(m)$ that tends to zero as $m \to +\infty$ and some, independent of m, coefficients $C_{jsi} > 0$, and introduce rescaled coordinates $v_1, \ldots, v_n$ near P'_{js} by the rule

$$x_1 |b_j - x_{n-1}|^{n-1-1/\lambda} = z'_{js1} |z'_{js,n-1}|^{n-1-1/\lambda} + C_{js1}\, \eta\, e^{-\lambda m(n-2)} v_1 \\ \hat{x}_i = z'_{jsi} + C_{jsi}\, \eta\, e^{-\lambda m(n-i-1)} v_i \quad (2 \leq i \leq n-1), \qquad x_n = z'_{jsn} + C_{jsn}\, \eta\, e^{-m} v_n \tag{4.2.19}$$

(recall that $|b_j - x_{n-1}|$ is close to 1 near P_{js}, hence (4.2.19) is a smooth coordinate transformation). Since η tends to zero as $m \to +\infty$, any bounded region of values of v corresponds to a small neighborhood of P'_{js}.

After the rescaling, the map $\tilde{T}_{j,1+\delta} \circ L_{jm} \equiv \tilde{Y}_\delta^{mN+N+1}$ from a small neighborhood of P'_{js} into a small neighborhood of $P'_{j,s+1}$ takes the following form (see (4.2.12),(4.2.7),(4.2.3),(4.2.13),(4.2.15),(4.2.17),(4.2.19)):

$$C_{j,s+1,i} \bar{v}_i = e^{-\lambda\delta} C_{j,s,i+1} v_{i+1} \quad (i \leq n-2), \\ C_{j,s+1,n-1} \bar{v}_{n-1} = e^{-\lambda\delta}\, (u_{js}^{-\lambda} - (u_{js} + C_{jsn}\, \eta\, v_n)^{-\lambda})/\eta, \\ C_{j,s+1,n} \bar{v}_n = (-1)^{n+1} \phi_{js} C_{js1} v_1 + \sum_{\substack{\nu_2 \geq 0, \ldots, \nu_n \geq 0 \\ \nu_2 + \ldots + \nu_n \leq d}} \varepsilon_{js\nu} E_{js\nu} \prod_{2 \leq p \leq n} v_p^{\nu_p}$$

where we denote

$$\phi_{js} = e^{\delta}(u_{js} + \eta C_{jsn} v_n)^{2-(n-2)\lambda} |z'_{js,n-1} + \eta C_{j,s,n-1} v_{n-1}|^{\frac{1}{\lambda}-(n-1)},$$

$$E_{js\nu} = e^{m(1-\lambda\sum_{2\le p\le n-1}(n-p)\nu_p)}\ \eta^{(-1+\sum_{2\le p\le n}\nu_p)} \prod_{2\le p\le n} C_{jsp}^{\nu_p}.$$

Note that $\sum_{2\le p\le n-1}(n-p)\nu_p \le (n-2)d$, hence $1-\lambda\sum_{2\le p\le n-1}(n-p)\nu_p > 0$ (see (4.2.11)). Therefore, all the coefficients $E_{js\nu}$ tend to infinity as $m \to +\infty$ (provided η tends to zero sufficiently slowly).

As we see, by putting

$$\begin{aligned} &C_{j,s+1,i} = e^{-\lambda\delta} C_{j,s,i+1} \quad (i \le n-2), \qquad C_{j,s+1,n-1} = \lambda e^{-\lambda\delta} u_{js}^{-\lambda-1} C_{jsn}, \\ &C_{j,s+1,n} = e^{\delta} u_{js}^{2-(n-2)\lambda} |z'_{js,n-1}|^{\frac{1}{\lambda}-(n-1)} C_{js1}, \end{aligned} \tag{4.2.20}$$

and

$$\varepsilon_{js\nu} = h_{jsi} \frac{C_{j,s+1,n}}{E_{js\nu}}, \tag{4.2.21}$$

the map $\tilde{T}_{j,1+\delta} \circ L_{jm}$ near P'_{js} takes the form

$$\begin{aligned} &\bar{v}_i = v_{i+1} \quad (i \le n-2), \qquad \bar{v}_{n-1} = v_n + O(\eta), \\ &\bar{v}_n = (-1)^{n+1} v_1 + O(\eta) + \sum_{\substack{\nu_2\ge 0,\ldots,\nu_n\ge 0 \\ \nu_2+\ldots+\nu_n\le d}} h_{js\nu} \prod_{2\le p\le n} v_p^{\nu_p}, \end{aligned} \tag{4.2.22}$$

i.e. it can be made as close as we want to the map $\tilde{H}_{js}$, provided m is taken large enough (recall that $\eta \to 0$ as $m \to +\infty$). We take η tending to zero sufficiently slowly, so, as we mentioned, $E_{ij\nu} \to \infty$ as $m \to +\infty$, which implies that all $\varepsilon_{js\nu} \to 0$ (see (4.2.21)), i.e. our perturbation to Y_δ is arbitrarily small indeed.

It follows that in the rescaled coordinates the map $\left(\tilde{T}_{j,1+\delta} \circ L_{jm}\right)^{q_j} \equiv \tilde{Y}_\delta^{q_j(mN+N+1)}$ from a small neighborhood of P'_{j1} into a small neighborhood of P'_{jq_j} can be made as close as we want to the map $\tilde{H}_{jq_j} \circ \ldots \circ \tilde{H}_{j1}$, provided m is large enough (the rescaled coordinates near P'_{j1} and P'_{jq_j} are given by formulas (4.2.19), where the coefficients $C_{j1i} > 0$ are taken arbitrary, and the coefficients C_{jq_ji} are recovered from the recursive formula (4.2.20); since m does not enter (4.2.20), it follows that C_{jq_ji} stay bounded away from zero and infinity as $m \to +\infty$).

Now, from (4.2.7) we obtain that the same holds true for the map $L_{j,m+l_j\delta} \circ \left(\tilde{T}_{j,1+\delta} \circ L_{jm}\right)^{q_j} \equiv \tilde{Y}_\delta^{l_j+mN+q_j(mN+N+1)}$ from a small neighborhood of P'_{j1} into a small neighborhood of M'_{jq_j}, where the rescaled coordinates $(v_1, \ldots, v_n)$ are introduced as follows:

$$\begin{aligned} &x_1|x_{n-1} - b_j|^{n-1-1/\lambda} = u'_{jq_j1} |u'_{jq_j,n-1}|^{n-1-1/\lambda} + C_{jq_j1}\ \eta\ e^{-\lambda(n-1)m-\lambda l_j\delta} v_1, \\ &\hat{x}_i = u'_{jq_ji} + C_{jq_ji}\ \eta\ e^{-\lambda m(n-i)-\lambda l_j\delta} v_i \ \ (2 \le i \le n-1), \qquad x_n = u_{jq_j} + C_{jq_jn}\ \eta\ e^{l_j\delta} v_n, \end{aligned} \tag{4.2.23}$$

with the same constants C_{jq_ji} as above.

Recall that, by construction, the point $Y_\delta^{N+1} M'_{jq_j}$ lies in $U_{3-j,+}$ in the region $a_{3-j,+} + 1 - \delta < x_{n-1} < a_{3-j,+} + 1$. We add to the map Y_δ an additional perturbation, localized near the point $Y_\delta^N M'_{jq_j}$, such that the corresponding map $\tilde{G}_{j,1+\delta} \equiv \tilde{Y}_\delta^{N+1}$ will have the following form near M'_{jq_j}:

$$\begin{aligned} &\bar{x}_1 = |x_n|^{1-(n-2)\lambda}(x_1 - e^{-\lambda(m+l_j\delta)} z'_{jq_j1} \left|(x_{n-1} - b_j)/z'_{jq_j,n-1}\right|^{\frac{1}{\lambda}-(n-1)}), \\ &\bar{x}_i = |x_n|^\lambda (x_i - u'_{jq_ji}) \ \ (2 \le i \le n-2), \\ &\bar{x}_{n-1} = a_{3-j,+} + 1 - \delta - \ln|x_n|, \qquad \bar{x}_n = -|x_n|^\lambda (x_{n-1} - b_j - u'_{jq_j,n-1}). \end{aligned} \tag{4.2.24}$$

Note that u'_{jq_ji} at $i \le n-1$ tend to zero as $m \to +\infty$ (see (4.2.14)), while the values of x_n near M'_{jq_j} and $z'_{j,q_j,n-1}$ are bounded away from zero; the exponent $\frac{1}{\lambda} - (n-1)$ in the first line is larger than r (see (4.2.11)). Thus, for sufficiently large m, map (4.2.24) is indeed a small perturbation of the map $G_{j,1+\delta}$ given by (4.2.9).

Denote $P'_{3-j,0} = \tilde{G}_{j,1+\delta} M'_{jq_j}$. By (4.2.24), this is the point with the coordinates $x_i = 0$ at $i \ne n-1$, and $x_{n-1} = a_{3-j,+} + 1 + (\kappa_j - 1)\delta$ (we assume that the coordinate x_n of M'_{jq_j} is $u'_{jq_jn} = u_{jq_j} = -e^{-\kappa_j\delta}$ where $\kappa_j \in (0,1]$ is defined by (4.2.31)). Introduce rescaled coordinates near $P'_{3-j,0}$ by the rule

$$\begin{aligned} &x_i = e^{-(l_j+\kappa_j)\delta\lambda_i} C_{jq_ji}\, \eta\, e^{-\lambda m(n-i)} v_i \ \ (i \le n-2), \\ &x_{n-1} = a_{3-j,+} + 1 + (1-\kappa_j)\delta + e^{(l_j+\kappa_j)\delta} C_{jq_jn}\, \eta\, v_{n-1}, \\ &x_n = e^{-(l_j+\kappa_j)\delta\lambda} C_{j,q_j,n-1}\, \eta\, e^{-\lambda m} v_n \end{aligned} \tag{4.2.25}$$

(with the same constants C_{jq_ji} as above). In coordinates (4.2.23),(4.2.25), map (4.2.24) takes the form $(v_1, \ldots, v_{n-1}, v_n) \mapsto (v_1, \ldots, v_n, -v_{n-1}) + O(\eta)$, i.e. it becomes arbitrarily close to the map Φ_0 (see (3.23)) as $m \to +\infty$. Thus, in the rescaled coordinates, the map $\tilde{G}_{j,1+\delta} \circ L_{j,m+l_j\delta} \circ \left(\tilde{T}_{j,1+\delta} \circ L_{jm}\right)^{q_j} \equiv \tilde{Y}_\delta^{l_j+(1+q_j)(mN+N+1)}$ from a small neighborhood of P'_{j1} into a small neighborhood of $P'_{3-j,0}$, can be made as close as we want to the map $\Phi_0 \circ \tilde{H}_{jq_j} \circ \ldots \circ \tilde{H}_{j1}$ as m grows.

Analogously, we take the point $M'_{j0} : \{x_{n-1} = a_{j-} - 1 + \delta/2, x_i = 0 \ (i \ne n-1)\} \in U_{j-}$, and perturb the map Y_δ near $Y_\delta^N P'_{j0}$ in such a way that the map $\tilde{Q}_{j,1+\delta} \equiv \tilde{Y}_\delta^{N+1}$ near M'_{j0} will be given by

$$\begin{aligned} &\bar{x}_i = e^{-\lambda_i(\delta - x_{n-1} - 1 + a_{j-})} x_i \qquad (i \le n-2), \\ &\bar{x}_{n-1} - b_j = e^{-\lambda(\delta - x_{n-1} - 1 + a_{j-})}, \ \ \bar{x}_n = e^{\delta - x_{n-1} - 1 + a_{j-}} x_n + e^{-m} u_{j1}. \end{aligned} \tag{4.2.26}$$

It is a small perturbation of the map $Q_{j,1+\delta}$ from (4.2.6), and it takes M'_{j0} to P'_{j1} (see (4.2.13),(4.2.15)). When we introduce rescaled variables near M'_{j0} by the rule

$$\begin{aligned} &x_i = e^{\delta\lambda_i/2} C_{j1i}\, \eta\, e^{-\lambda m(n-i-1)} v_i \ \ (i \le n-2), \\ &x_{n-1} = a_{j-} - 1 + \delta/2 + \frac{1}{\lambda} e^{\lambda\delta/2} C_{j,1,n-1}\, \eta\, v_{n-1}, \\ &x_n = e^{-\delta/2} C_{j1n}\, \eta\, e^{-m} v_n, \end{aligned} \tag{4.2.27}$$

map (4.2.26) will take the form $\bar{v} = v + O(\eta)$, i.e. it is close to the identity map. Thus, in the rescaled coordinates given by (4.2.27),(4.2.25), the map $\tilde{G}_{j,1+\delta} \circ L_{j,m+l_j\delta} \circ \left(\tilde{T}_{j,1+\delta} \circ L_{jm}\right)^{q_j} \circ \tilde{Q}_{j,1+\delta} \equiv \tilde{Y}_\delta^{l_j+(1+q_j)(mN+N+1)+N+1}$

from a small neighborhood of M'_{j0} into a small neighborhood of $P'_{3-j,0}$, is as close as we want to the map $\Phi_0 \circ \tilde{H}_{jq_j} \circ \ldots \circ \tilde{H}_{j1}$ at m large enough, i.e. it is a close approximation of the map Φ_j.

Let us now determine the form of the map $S_{jt} : v \mapsto \bar{v}$ from a small neighborhood of P'_{j0} into a small neighborhood of M'_{j0} in the rescaled coordinates (4.2.27),(4.2.25). By (4.2.5), for an integer $k > 0$, the map $S_{j,k\delta} \equiv Y^k_\delta$ takes the point P'_{j0} into M'_{j0} if

$$\beta(\mu_j) = (k + \kappa_{3-j} - 1/2)\delta - 5 \tag{4.2.28}$$

(see (4.2.27),(4.2.25)). Since $\beta \to +\infty$ as $\mu \to +0$ (see (4.1.4)), for every sufficiently large k equation (4.2.28) has a solution $\mu_j(k)$, and $\mu_j(k) \to +0$ as $k \to +\infty$. It follows that $\alpha(\mu_j(k)) \to +\infty$. Thus, for any sufficiently large m we can find $\gamma_{n\pm} \in (0, 1]$ and k such that

$$\begin{aligned} e^{-\gamma_{n+}\alpha(\mu_j(k))} &= e^{-(l_{3-j}+\kappa_{3-j})\delta\lambda} C_{3-j,q_{3-j},n-1}\, \eta\, e^{-\lambda m}, \\ e^{-\gamma_{n-}\alpha(\mu_j(k))} &= e^{-\delta/2} C_{j1n}\, \eta\, e^{-m}. \end{aligned} \tag{4.2.29}$$

This guarantees that $\bar{v}_n = \psi_j(v_n)$ (see (4.2.27),(4.2.25),(4.2.5)).

We also obtain $\bar{v}_i = v_i$ at $i \le n-2$ by choosing $\gamma_{i\pm} \in (0, 1]$ such that

$$\begin{aligned} e^{-\gamma_{i+}\alpha(\mu_j(k))} &= e^{\delta\lambda_i/2} C_{j1i}\, \eta\, e^{-\lambda m}, \\ e^{-\gamma_{i-}\alpha(\mu_j(k))} &= e^{-(l_j+\kappa_j)\delta\lambda_i} C_{3-j,q_{3-j},i}\eta. \end{aligned} \tag{4.2.30}$$

Finally, we fix the choice of the integer l_j and $\kappa_j \in (0, 1]$ as follows:

$$e^{(l_j+\kappa_j)\delta} = \frac{1}{\lambda} e^{\lambda\delta/2} C_{3-j,1,n-1}/C_{jq_jn}. \tag{4.2.31}$$

This (along with (4.2.28)) gives us $\bar{v}_{n-1} = v_{n-1}$ for the map $S_{j,k\delta}$ in the coordinates (4.2.27),(4.2.25). As wee see, the map $S_{j,k\delta}$ in the rescaled coordinates coincides with the map Ψ_j for v from some open neighborhood of D (if $j = 1$) or of $\Phi_1 \circ \Psi_1(D)$ (if $j = 2$).

Thus, we see that the map

$$\begin{aligned} &\tilde{G}_{2,1+\delta} \circ L_{2,m+l_2\delta} \circ \left(\tilde{T}_{2,1+\delta} \circ L_{2m}\right)^{q_2} \circ \tilde{Q}_{2,1+\delta} \circ S_{2,k\delta} \circ \\ &\qquad \circ \tilde{G}_{1,1+\delta} \circ L_{1,m+l_1\delta} \circ \left(\tilde{T}_{1,1+\delta} \circ L_{1m}\right)^{q_1} \circ \tilde{Q}_{1,1+\delta} \circ S_{1,k\delta} \equiv \\ &\equiv \tilde{Y}_\delta^{2k+l_1+l_2+(2+q_1+q_2)(mN+N+1)+2(N+1)} \end{aligned}$$

is a close approximation to the map $F = \Phi_2 \circ \Psi_2 \circ \Phi_1 \circ \Psi_1$, provided $\tilde{H}_{jq_j} \circ \ldots \circ \tilde{H}_{j1}$ are sufficiently close approximations to $\tilde{\Phi}_j$ $(j = 1, 2)$ and m is large enough. This completes the proof of the theorem.

5 Birth of periodic spots from a heteroclinic cycle

In this Section we finish the proof of Theorem 3. Let a C^ρ-diffeomorphism f of a smooth two-dimensional manifold have a pair of saddle periodic points P and Q of periods p and, respectively, q. Denote as T_{01} the map f^p restricted onto a small neighborhood of P, and denote as T_{02} the map f^q restricted onto a small neighborhood

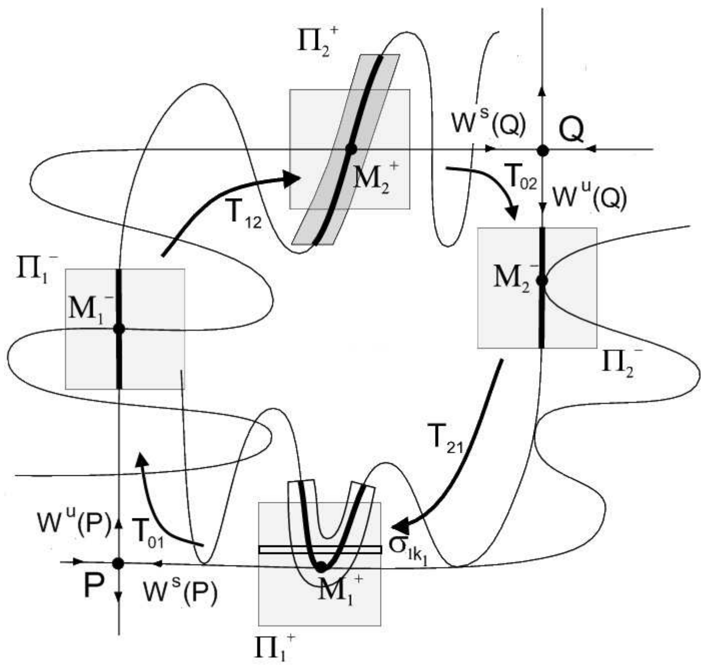

Figure 3: A non-transverse heteroclinic cycle

of Q. By definition, $T_{01}P = P$ and $T_{02}Q = Q$. One can introduce coordinates (x_1, y_1) in the neighborhood of P and (x_2, y_2) in the neighborhood of Q such that the maps T_{0j} will have the form

$$\bar{x}_j = \lambda_j x_j + \dots, \qquad \bar{y}_j = \gamma_j y_j + \dots,$$

where $|\lambda_j| < 1$, $|\gamma_j| > 1$; the dots stand for nonlinearities. The numbers λ_1, γ_1 and λ_2, γ_2 are the multipliers of the periodic points P and Q, respectively. We assume

$$J_1 := |\lambda_1 \gamma_1| < 1 \quad \text{and} \quad J_2 := |\lambda_2 \gamma_2| > 1. \tag{5.1}$$

Every saddle periodic point lies in the intersection of two C^ρ-smooth invariant manifolds: the points in the stable invariant manifold W^s tend to the periodic orbit at forward iterations of the map, and the points in the unstable invariant manifold W^u tend to the periodic orbit at backward iterations. With our choice of the coordinates, the unstable manifold is tangent at the periodic point to the y-axis, and the stable manifold is tangent to the x-axis. One can locally straighten W^u and W^s, i.e. the coordinates (x_j, y_j) can be chosen in such a way that $W^s_{loc}(P) = \{y_1 = 0\}$, $W^u_{loc}(P) = \{x_1 = 0\}$, $W^s_{loc}(Q) = \{y_2 = 0\}$, $W^u_{loc}(Q) = \{x_2 = 0\}$.

Assume that $W^u(P)$ has an orbit of a transverse intersection with $W^s(Q)$. This means that $M_2^+ := f^{k_{12}} M_1^- \in W^s_{loc}(Q)$ for some positive integer k_{12} and some point $M_1^-(0, y_1^-) \in W^u_{loc}(Q)$, and the curve $f^{k_{12}}(W^u_{loc}(P))$ intersects $W^s_{loc}(Q)$ at the point $M_2^+(x_2^+, 0)$ transversely. We denote the map $f^{k_{12}}$ restricted onto a small neighborhood of M_1^- as T_{12}. It can be written as

$$\bar{x}_2 - x_2^+ = a_1 x_1 + b_1(y_1 - y_1^-) + \dots, \qquad \bar{y}_2 = c_1 x_1 + d_1(y_1 - y_1^-) + \dots, \tag{5.2}$$

where $d_1 \neq 0$ because of the transversality of $T_{12}(W^u_{loc}(P))$ to $W^s_{loc}(Q)$. As an orbit of intersection of $W^u(P)$ and $W^s(Q)$, the orbit Γ_{PQ} of the point M_1^- is heteroclinic: it tends to the orbit of P at backward iterations of f and to the orbit of Q at forward iterations. Note that since $d_1 \neq 0$, we may rewrite (5.2) in the so-called cross-form:

$$d_1(\bar{x}_2 - x_2^+) = Dx_1 + b_1\bar{y}_2 + \dots, \qquad d_1(y_1 - y_1^-) = \bar{y}_2 - c_1x_1 + \dots, \tag{5.3}$$

where $D := a_1d_1 - b_1c_1$.

Another assumption is that f has a heteroclinic orbit Γ_{QP} at the points of which $W^u(Q)$ has a tangency of order m with $W^s(P)$. This means that there exists a pair of points, $M_2^-(0, y_2^-) \in W^u_{loc}(Q)$ and $M_1^+(x_1^+, 0) \in W^s_{loc}(P)$, such that $M_1^+ = f^{k_{21}}M_2^-$ for some positive integer k_{21}, and the curve $f^{k_{21}}(W^u_{loc}(Q))$ has a tangency of order m with $W^s_{loc}(P)$ at M_1^+ (see Fig.3). We denote the map $f^{k_{21}}$ restricted onto a small neighborhood of M_2^- as T_{21}. It can be written as

$$\bar{x}_1 - x_1^+ = a_2x_2 + b_2(y_2 - y_2^-) + \dots, \qquad \bar{y}_1 = c_2x_2 + d_2(y_2 - y_2^-)^{m+1} + \dots, \tag{5.4}$$

where $d_2 \neq 0$; the dots stand for higher order terms. Obviously, to speak about the tangency of order m, the smoothness of f has to be sufficiently high, i.e. $\rho \geq m + 1$.

We will fix the orientation in the neighborhood of Q by requiring that the determinant D of the derivative matrix $\partial T_{12}/\partial(x, y)$ is positive at the point M_1^-. Then, we require that the determinant of $\partial T_{21}/\partial(x, y)$ at the point M_2^- is also positive, i.e. $b_2c_2 < 0$. This is always the case if the manifold on which f is defined is orientable and f is orientation-preserving. In Remark after Lemma 3 we discuss the case $b_2c_2 > 0$ too.

In Lemma 3 we will also need a technical assumption $\ln|\gamma_1|\ln|\gamma_2| \neq \ln|\lambda_1|\ln|\lambda_2|$ (one can always achieve this by an arbitrarily small perturbation of f without destroying the order m tangency between $W^u(Q)$ and $W^s(P)$). In fact, we can always assume

$$\ln|\gamma_1|\ln|\gamma_2| < \ln|\lambda_1|\ln|\lambda_2|; \tag{5.5}$$

the case $\ln|\gamma_1|\ln|\gamma_2| > \ln|\lambda_1|\ln|\lambda_2|$ reduces to the given one by considering the map f^{-1} instead of f (and interchanging P with Q).

Let us imbed f in any $(m+1)$-parameter family f_ν of maps, of class C^ρ with respect to coordinate and parameters. The corresponding map T_{21} will also depend smoothly on parameters. Since it has form (5.4) at $\nu = 0$, i.e. all the derivatives of $\bar{y}_1$ with respect to $(y_2 - y_2^-)$ vanish up to the order m, it follows that at non-zero ν the map T_{21} can be written in the form

$$\bar{x}_1 - x_1^+ = a_2x_2 + b_2(y_2 - y_2^-) + \dots, \qquad \bar{y}_1 = c_2x_2 + \sum_{s=0}^{m-1} \mu_s(y_2 - y_2^-)^s + d_2(y_2 - y_2^-)^{m+1} + \dots, \tag{5.6}$$

where μ_s are smooth functions of ν (note that x_1^+ and y_2^- also depend on ν now: the value of y_2^- is fixed by the condition $\partial^m\bar{y}_1/\partial y_2^m = 0$ at $y_2 = y_2^-$; thus y_2^- is $C^{\rho-m}$-function of ν, and μ_s are also $C^{\rho-m}$; the high-order terms that are denoted by dots in (5.6) and (5.3) depend now also on ν, $C^{\rho-m}$-smoothly).

Denote $\theta = |\ln J_2/\ln J_1|$. The value of θ is a $C^{\rho-1}$-smooth function of ν. We may always put a family f_ν in general position, i.e. we may further assume

$$\frac{\partial(\mu_0, \dots, \mu_{m-1}, \theta)}{\partial(\nu_1, \dots, \nu_{m+1})} \neq 0. \tag{5.7}$$

This, in particular, means that by changing ν we may change the values of any of the parameters θ and μ_s while keeping the other parameters constant. For example, one may change the value of θ and keep $\mu = 0$, i.e. keep the order m tangency between $W^u(Q)$ and $W^s(P)$.

The two periodic orbits (the orbit of P and the orbit of Q) and two heteroclinic orbits, Γ_{PQ} and Γ_{QP} comprise a heteroclinic cycle. Let U be a small neighborhood of the heteroclinic cycle. Varying the parameters ν can lead to the destruction of the heteroclinic cycle and to bifurcations of other orbits in U. We will further focus on one instance of such bifurcations. We call a periodic orbit of the map f_ν *single-round* if it stays entirely in U and visits a small neighborhood of each of the points $M^{\pm}_{1,2}$ only once, i.e. the point of intersection of the single-round orbit with a small neighborhood of M_1^+ is a fixed point of the map $T_{21}T_{02}^{k_2}T_{12}T_{01}^{k_1}$ for some positive integers k_1 and k_2; the image of this point by the map $T_{01}^{k_1}$ lies in a small neighborhood of M_1^+, the image by $T_{12}T_{01}^{k_1}$ lies in a small neighborhood of M_2^-, and the image by $T_{02}^{k_2}T_{12}T_{01}^{k_1}$ lies in a small neighborhood of M_2^+.

Lemma 3. *There exist a sequence $\nu_l \to 0$ and sequences of integers $k_{1l} \to +\infty$ and $k_{2l} \to +\infty$ such that at $\nu = \nu_l$ the map f_ν has a single-round periodic orbit which corresponds to a fixed point of the map $\mathcal{T}_l := T_{21}T_{02}^{k_{2l}}T_{12}T_{01}^{k_{1l}}$, and the map $\mathcal{T}_l$ near this point is, in some C^ρ-coordinates (u,v), given by*

$$(\bar{u},\bar{v}) = \Phi(u,v) + o(|u|^m + |v|^m), \tag{5.8}$$

where Φ denotes the time-1 map by the flow

$$\dot{u} = v, \qquad \dot{v} = -\Psi(u)(1+v) \tag{5.9}$$

near $(u,v) = (0,0)$; here Ψ is a polynomial such that $\Psi(0) = 0$ and $\Psi'(0) \geq 0$.

Remark. System (5.9) has an integral $H(u,v) = \int \Psi(u)du + v - \ln(1+v) = \frac{v^2}{2} + \Psi'(0)\frac{u^2}{2} + \ldots$. Thus, when $\Psi'(0) > 0$, the equilibrium at zero is a center: every orbit of (5.9) is in this case a closed curve surrounding the origin. The corresponding time-1 map Φ is, therefore, conservative (it preserves the area form $\frac{1}{1+v}\, du \wedge dv$) and has an elliptic point at the origin. Note that map (5.8) can, by an arbitrarily C^m-small perturbation, be made equal to $(\bar{u},\bar{v}) = \Phi(u,v)$ identically in a sufficiently small neighborhood of zero. Hence, Lemma 3 implies, that by an arbitrarily C^m-small perturbation of the given map f a periodic point can be born in a small neighborhood of the heteroclinic cycle such that the first-return map near this point will be area-preserving and the point will be elliptic. We recall that one of the conditions of the lemma is that the heteroclinic cycle is "orientable" in the sense that the determinant of $\partial T_{21}/\partial(x,y)$ at the point M_2^- is positive. However, it is easy to show (see Lemma 8 in [6]) that a non-orientable heteroclinic cycle can always be perturbed in such a way that a new (double-round) orbit of heteroclinic tangency of order $(m-1)$ between $W^u(Q)$ and $W^s(P)$ is born, and the corresponding cycle is now orientable; applying the lemma to the newly born cycle, we find that an elliptic periodic point can be born by a perturbation which is arbitrarily small in the C^{m-1}-metric. In any case, we have that for any fixed r, given a C^∞-map f which satisfies (5.1) and which has a heteroclinic cycle with a sufficiently high order of tangency between $W^u(Q)$ and $W^s(P)$, an arbitrarily C^r-small perturbation leads to the birth of an elliptic periodic point: a point whose both multipliers lie on the unit circle and, importantly, the corresponding first-return map is area-preserving. By [32], an arbitrarily C^r-small perturbation of the map in a neighborhood of such point creates an orbit of homoclinic tangency; the perturbation does not destroy the area-preserving property. By Theorem 5 of [6], an arbitrarily C^r-small perturbation of any area-preserving map with a homoclinic tangency leads to the birth of a periodic spot. In other words, once Lemma 3 provides us with a periodic orbit whose normal form is area-preserving up to a sufficiently high order, the birth of periodic spots from the heteroclinic cycles under consideration follows from the results of [6] on perturbations of area-preserving maps.

Proof of Lemma 3. At $m = 1$ an equivalent statement can be found in [30, 31], so we further focus on the case $m \geq 2$. Since the periodic points P and Q are saddle, it follows that given any small x_{j0} and y_{jk} (where

$j=1,2$) and any $k \geq 0$ there exist uniquely defined small x_{jk} and y_{j0} such that $(x_{jk}, y_{jk}) = T_{0j}^k(x_{j0}, y_{j0})$ and all the points in the orbit $\{(x_{j0)}, y_{j0}), T_{0j}(x_{j0}, y_{j0}), \ldots, T_{0j}^k(x_{j0}, y_{j0})\}$ lie in a small neighborhood of P (at $j=1$) or Q (at $j=2$), see [30, 33]. Denote

$$x_{jk} = \lambda_j^k x_{j0} + \xi_{jk}(x_{j0}, y_{jk}, \nu), \quad y_{j0} = \gamma_j^{-k} y_{jk} + \eta_{jk}(x_{j0}, y_{jk}, \nu). \tag{5.10}$$

By [33], one can introduce C^ρ-coordinates (x_j, y_j) near the saddle periodic points in such a way that

$$\xi_{jk} = o(\lambda_j^k), \quad \eta_{jk} = o(\gamma_j^{-k}), \tag{5.11}$$

i.e. the map T_{0j}^k written in the "cross-form" (5.10) is linear in the main order. By [7], the same $o(\lambda_j^k)$ and, resp., $o(\gamma_j^{-k})$ estimates hold for the derivatives of the functions ξ_{jk} and η_{jk} up to the order $(\rho-1)$ with respect to (x_{j0}, y_{jk}) and up to the order $(\rho-2)$ with respect to parameters; while for the higher order derivatives up to the order ρ for which the number of differentiations with respect to the parameters does not exceed $(\rho-2)$ we have that they uniformly tend to zero as $k \to \infty$.

Let $\delta > 0$ be sufficiently small. Consider rectangular neighborhoods $\Pi_j^+ : \{|x_j - x_j^+| < \delta, |y_j| < \delta\}$ and $\Pi_j^- : \{|y_j - y_j^-| < \delta, |x_j| < \delta\}$ of M_j^+ and M_j^- $(j=1,2)$, and take any sufficiently large integers k_1 and k_2. By (5.10),(5.11), the set $\sigma_{jk_j} = \Pi_j^+ \cap T_{0j}^{-k_j} \Pi_j^-$ is non-empty: it is a strip of the form $\{|\gamma_j^{k_j} y_j - y_j^-| < \delta + o(1)_{k \to +\infty}\}$. We introduce a new coordinate y_j on σ_{jk_j} such that

$$y_{j,old} = \gamma_j^{-k_j} y_{j,new} + \gamma_j^{-k_j} \eta_{jk_j}(x_j, y_{j,new}) \tag{5.12}$$

(i.e. $y_{j,new}$ equals to y_{jk_j} from (5.10)).

By construction, the map $T_{12} T_{01}^{k_1}$ is defined on σ_{1k_1}; in the new coordinates, when this map takes a point $(x_1, y_1) \in \sigma_{1k_1}$ into a point $(\bar{x}_2, \bar{y}_2) \in \sigma_{2k_2}$ it can be written in the following form (see (5.3),(5.12)):

$$\begin{aligned} d_1(\bar{x}_2 - x_2^+) &= D\lambda_1^{k_1} x_1 + \phi_1(x_1) + b_1 \gamma_2^{-k_2} \bar{y}_2 + o(\gamma_2^{-k_2}), \\ d_1(y_1 - y_1^-) &= -c_1 \lambda_1^{k_1} x_1 - \phi_2(x_1) + \gamma_2^{-k_2} \bar{y}_2 + o(\gamma_2^{-k_2}), \end{aligned} \tag{5.13}$$

where $\phi_{1,2}(x_1) = o(\lambda_1^{k_1})$. Analogously, the map $T_{21} T_{02}^{k_2}$ that takes a point $(x_2, y_2) \in \sigma_{2k_2}$ into a point $(\bar{x}_1, \bar{y}_1) \in \sigma_{1k_1}$ is given by

$$\begin{aligned} \bar{x}_1 - x_1^+ &= a_2 \lambda_2^{k_2} x_2 + b_2(y_2 - y_2^-) + o(y_2 - y_2^-) + o(\lambda_2^{k_2}), \\ \gamma_1^{-k_1} \bar{y}_1 &= c_2 \lambda_2^{k_2} x_2 + \sum_{s=0}^{m-1} \mu_s (y_2 - y_2^-)^s + d_2(y_2 - y_2^-)^{m+1} + o((y_2 - y_2^-)^{m+1}) + o(\lambda_2^{k_2}) + o(\gamma_1^{-k_1}) \end{aligned} \tag{5.14}$$

(see (5.6)). We will now change the variable y_1 to $y_1 + d_1^{-1}(c_1 \lambda_1^{k_1} x_1 + \phi_2(x_1))$. Then the second lines in (5.13) and (5.14) will change to

$$d_1(y_1 - y_1^-) = \gamma_2^{-k_2} \bar{y}_2 + o(\gamma_2^{-k_2}),$$

and, respectively,

$$\gamma_1^{-k_1} \bar{y}_1 = c_2 \lambda_2^{k_2} x_2 + \sum_{s=0}^{m} \hat{\mu}_s (y_2 - y_2^-)^s + d_2(y_2 - y_2^-)^{m+1} + o((y_2 - y_2^-)^{m+1}) + o(\lambda_2^{k_2}) + o(\gamma_1^{-k_1}),$$

where $\hat{\mu}_s = \mu_s + O(\lambda_1^{k_1})$ $(s = 0, \ldots, m-1)$ and $\hat{\mu}_m = O(\lambda_1^{k_1})$.

It is easy to see, that since $d_1 \neq 0$, $d \neq 0$, one can find constants $C_j = O(|\lambda_{3-j}|^{k_{3-j}} + |\gamma_j|^{-k_j})$ and $K_j = O(|\gamma_{3-j}|^{-k_{3-j}} + |\lambda_j|^{k_j})$ $(j = 1, 2)$ such that after the following shift of the origin:

$$x_{j,new} = x_j - x_j^+ + C_j, \qquad y_{j,new} = y_j - y_j^- + K_j, \tag{5.15}$$

we will have $(\bar{x}_{2,new}, y_{1,new}) = 0$ at $(x_{1,new}, \bar{y}_{2,new}) = 0$, and $\bar{x}_{1,new} = 0$, $\partial^m \bar{y}_{1,new}/\partial y_{1,new}^m = 0$ at $(x_{2,new}, y_{2,new}) = 0$. Thus, after this transformation, the maps $T_{12}T_{01}^{k_1}$ and $T_{21}T_{02}^{k_2}$ will be written as

$$\begin{aligned} d_1\bar{x}_2 &= D\lambda_1^{k_1} x_1 + O(\gamma_2^{-k_2}\bar{y}_2) + o((|\lambda_1|^{k_1} + |\gamma_2|^{-k_2})x_1), \\ d_1 y_1 &= \gamma_2^{-k_2}\bar{y}_2 + o(\gamma_2^{-k_2}\bar{y}_2) + o(|\gamma_2|^{-k_2}x_1), \end{aligned} \tag{5.16}$$

and, respectively,

$$\begin{aligned} \bar{x}_1 &= b_2 y_2 + o(y_2) + O(\lambda_2^{k_2} x_2), \\ \gamma_1^{-k_1}\bar{y}_1 &= c_2\lambda_2^{k_2} x_2 + \sum_{s=0}^{m-1} \tilde{\mu}_s y_2^s + d_2 y_2^{m+1} + o(y_2^{m+1}) + o(\lambda_2^{k_2} x_2), \end{aligned} \tag{5.17}$$

where the modified parameters $\tilde{\mu}_s$ are such that $\tilde{\mu}_s = \mu_s + o(1)_{k_{1,2}\to+\infty}$.

By virtue of (5.7), by an arbitrarily small change of ν we can make $\theta := |\ln|\lambda_2\gamma_2| / \ln|\lambda_1\gamma_1||$ rational without changing the values of μ. Therefore, we may from the very beginning assume that at $\nu = 0$ the system has a heteroclinic cycle with a tangency of the order m, and the value $\theta_0 := \theta(0)$ is rational. Further we always assume that

$$k_1 = \theta_0 k_2 \tag{5.18}$$

in (5.16),(5.17). We will also assume that both k_1 and k_2 are even, so $\lambda_j^{k_j}$ and $\gamma_j^{k_j}$ are positive. It follows from (5.18),(5.5),(5.1) that

$$\gamma_2^{-k_2} \ll \lambda_1^{k_1} \ll \gamma_1^{-k_1} \ll \lambda_2^{k_2}. \tag{5.19}$$

Now, let us introduce new, rescaled coordinates (X_1, Y_1, X_2, Y_2) and parameters $(B, E_0, \dots, E_{m-1})$ by the following rule:

$$\begin{aligned} &x_1 = b_2 d_1 \gamma_1^{-k_1/m}\gamma_2^{-k_2/m} X_1, \qquad x_2 = D b_2 \lambda_1^{k_1}\gamma_1^{-k_1/m}\gamma_2^{-k_2/m} X_2, \\ &y_1 = \gamma_1^{-k_1/m}\gamma_2^{-k_2(1+1/m)} Y_1, \qquad y_2 = d_1\gamma_1^{-k_1/m}\gamma_2^{-k_2/m} Y_2, \\ &\tilde{\mu}_s = d_1^s \left(\gamma_1^{k_1/m}\gamma_2^{k_2/m)}\right)^{-(m+1-s)} E_s \quad (s = 0, \dots, m-1), \\ &\theta = \beta_0 + \frac{\ln(-B/(Db_2c_2)}{k_2|\ln|\lambda_1\gamma_1||}; \end{aligned} \tag{5.20}$$

recall that $Db_2c_2 < 0$ by our assumptions, so the new parameter B should be positive. After the rescaling, the maps $T_{12}T_{01}^{k_1}$ (given by (5.16)) and $T_{21}T_{02}^{k_2}$ (given by (5.17)) are rewritten as follows (we take into account (5.18),(5.19)):

$$\begin{aligned} \bar{X}_2 &= X_1 + o(1)_{k_{1,2}\to+\infty} \\ Y_1 &= \bar{Y}_2 + o(1)_{k_{1,2}\to+\infty}, \end{aligned} \quad \text{and} \quad \begin{aligned} \bar{X}_1 &= Y_2 + o(1)_{k_{1,2}\to+\infty}, \\ \bar{Y}_1 &= -BX_2 + \textstyle\sum_{s=0}^{m-1} E_s Y_2^s + d\,Y_2^{m+1} + o(1)_{k_{1,2}\to+\infty}, \end{aligned}$$

where $d := d_2 d_1^{m+1} \neq 0$. It follows that with our choice of $k_{1,2}$ the map $\mathcal{T} := T_{21}T_{02}^{k_2}T_{12}T_{01}^{k_1}$ on σ_{1k_1} can be written as

$$\bar{X} = Y + o(1)_{k_{1,2}\to+\infty}, \qquad \bar{Y} = -BX + \sum_{s=0}^{m-1} E_s Y^s + d\,Y^{m+1} + o(1)_{k_{1,2}\to+\infty},$$

where (X,Y) are the coordinates which were previously denoted as (X_1,Y_1). By denoting the right-hand side of the equation for $\bar{X}$ as the new Y-variable, we finally write the map $\mathcal{T}$ in the following form

$$\bar{X} = Y, \qquad \bar{Y} = -BX + \sum_{s=0}^{m-1} E_s Y^s + d\, Y^{m+1} + o(1)_{k_{1,2}\to+\infty}. \tag{5.21}$$

The transformation to coordinates (X,Y) was of class C^ρ with respect to the original coordinates and $C^{\rho-m}$ with respect to the parameters (more precisely: the m-th derivatives with respect to the coordinates are $C^{\rho-m}$ with respect to both the coordinates and parameters). The $o(1)$-terms in (5.21) are functions of (X,Y) and $(B,E_0,\dots,E_{m-1})$ which tend to zero as $k_{1,2}\to+\infty$ along such derivatives up to the order ρ, for which the number of differentiations with respect to the parameters does not exceed $(\rho-m)$. In any case, since $\rho\geq m+1$, we have that their derivatives with respect to (X,Y) up to the order m tend to zero, all with at least one derivative with respect to $(B,E_0,\dots,E_{m-1})$.

We mention that a similar form for the rescaled first-return map was obtained in [31] for the case $m=1$. Map (5.21) was also obtained in [6] as the rescaled first-return map near a homoclinic tangency of order m. Let us show that if the $o(1)$-terms are sufficiently small, then a map of form (5.21) has, at certain uniformly bounded values of E and $B>0$ and in a bounded region of (X,Y), a fixed point near which the map is smoothly conjugate to (5.8). This will give us the lemma: by (5.20),(5.15), any bounded values of (X,Y) correspond to $(x_1,y_1)\to(x_1^+,y_1^-)$ as $k_{1,2}\to+\infty$, i.e. if the values of (X,Y) are bounded at the fixed point, then the latter belongs to the domain of definition of the first-return map $T_{21}T_{02}^{k_2}T_{12}T_{01}^{k_1}$ at all sufficiently large $k_{1,2}$, and any bounded values of $(E_0,\dots,E_{m-1})$ and $B>0$ correspond to $\nu_k\to0$ as $k_{1,2}\to+\infty$.

We may take any Y^* as the coordinate $X=Y=Y^*$ of the fixed point of (5.21); then the second equation defines the corresponding value of $E_0 = Y^*(1+B) - \sum_{s=1}^{m-1} E_s (Y^*)^s - d\,(Y^*)^{m+1} + o(1)$. After shifting the origin to the fixed point, the map takes the form

$$\begin{aligned}&\bar{X} = Y,\\ &\bar{Y} = -\hat{B}X + 2Y + \textstyle\sum_{s=1}^{m} \hat{E}_s Y^s + O(Y^{m+1}) + O(|XY| + X^2)\cdot o(1)_{k\to+\infty},\end{aligned} \tag{5.22}$$

where the coefficients $\hat{E}_s$, $s=1,\dots,m-1$, are related to $E_1,\dots,E_{m-1}$ by a diffeomorphism:

$$\hat{E}_1 = E_1 - 2 + \sum_{j=2}^{m-1} jE_j (Y^*)^{j-1} + (m+1)d\,(Y^*)^m + o(1),$$

$$\hat{E}_s = E_s + \sum_{j=s+1}^{m-1} E_j C_j^s (Y^*)^{j-s} + C_{m+1}^s d\,(Y^*)^{m+1-s} + o(1)\ (s=2,\dots,m-1),$$

and

$$\hat{E}_m = (m+1)d\,Y^* + o(1), \qquad \hat{B} = B + o(1).$$

As we see, bounded values of $\hat{B},\hat{E}_1,\dots,\hat{E}_m$ give us bounded values of $B,E_0,\dots,E_{m-1}$ as well.

We will further fix $\hat{B}=1$ and will take $(\hat{E}_1,\dots,\hat{E}_m) = o(1)_{k\to+\infty}$, so the map will limit, as $k\to+\infty$, to

$$\bar{X} = Y, \qquad \bar{Y} = -X + 2Y.$$

The fixed point of this map at $(X, Y) = 0$ has a double multiplier $(+1)$. Let us recall the normal form theory for maps with such a fixed point. In general, we can always write such map in the form

$$\begin{aligned}\bar{u} &= u + v + \sum_{2\le i+j\le m} \alpha_{ij} u^i v^j + o(|u|^m + |v|^m),\\ \bar{v} &= v + \sum_{2\le i+j\le m} \beta_{ij} u^i v^j + o(|u|^m + |v|^m).\end{aligned} \tag{5.23}$$

By denoting $z = v + \sum_{2\le i+j\le m} \alpha_{ij} u^i v^j + o(|u|^m + |v|^m) \equiv \bar{u} - u$, the map takes the form

$$\bar{u} = u + z, \qquad \bar{z} = z + \sum_{2\le i+j\le m} \phi_{ij} u^i z^j + o(|u|^m + |z|^m). \tag{5.24}$$

Here

$$\phi_{ij} = \beta_{ij} + \sum_{i<k\le i+j} C_k^i \alpha_{k,i+j-k} + G_{ij}, \tag{5.25}$$

where G_{ij} stand for polynomial functions of the coefficients $\alpha_{i'j'}$ and $\beta_{i'j'}$ with $i' + j' < i + j$; each monomial in G_{ij} has degree 2 or higher. After the polynomial coordinate transformation $(u, z) \mapsto (U, Z)$, where

$$U = u + \sum_{0\le j\le n-2} A_j u^{n-j} z^j, \qquad Z = z + \sum_{0\le j\le n-2} A_j (\bar{u}^{n-j} \bar{z}^j - u^{n-j} z^j),$$

the map will retain its form (5.24) with the coefficients ϕ_{ij} unchanged for $i + j < n$, and $\phi_{ij,new} = \phi_{ij} + \sum_{0\le l\le j-2} (2^{j-l} - 2) C_{n-l}^{j-l} A_l$ at $i + j = n$. As we see, by choosing A_l appropriately, one can make ϕ_{ij} with $j \ge 2, i = n - j$ assume any given values. Thus, by making such transformations consecutively, with n running from 2 to m, one can make all the coefficients ϕ_{ij} with $j \ge 2$ vanish; the new coefficients ϕ_{ij} $(j = 0, 1)$ differ from the old ones by nonlinear (i.e. quadratic and higher order) expressions which depend only on the coefficients $\phi_{i'j'}$ with $i' + j' < i + j$, i.e. they are still given by (5.25) with some new G_{ij}. Obviously, for any fixed m, we arrive to a similar conclusion for any map (with a fixed point at zero) whose linear part is sufficiently close to that of (5.23). Specifically, for any sufficiently small $\varepsilon_{1,2}$, a map of the form

$$\begin{aligned}\bar{u} &= u + v + \sum_{2\le i+j\le m} \alpha_{ij} u^i v^j + o(|u|^m + |v|^m),\\ \bar{v} &= (1 + \varepsilon_1) v + \varepsilon_2 u + \sum_{2\le i+j\le m} \beta_{ij} u^i v^j + o(|u|^m + |v|^m),\end{aligned} \tag{5.26}$$

can be brought by a smooth coordinate transformation to the normal form

$$\bar{u} = u + z, \qquad \bar{z} = (1 + \varepsilon_1) z + \varepsilon_2 u + \sum_{i=2}^{m} (\phi_{i0} u^i + \phi_{i-1,1} u^i z) + o(|u|^m + |z|^m), \tag{5.27}$$

where

$$\phi_{i0} = \beta_{i0} + G_{i0} + H_{i0}, \qquad \phi_{i1} = \beta_{i1} + (i + 1)\alpha_{i+1,0} + G_{i1} + H_{i1}. \tag{5.28}$$

Here G_{ij} is a polynomial of $\alpha_{i'j'}$ and $\beta_{i'j'}$ with $i' + j' < i + j$, all monomials of G_{ij} are of degree at least 2; the function H_{ij} is a linear function of $\alpha_{i'j'}$ and $\beta_{i'j'}$ with $i' + j' = i + j$; the coefficients of G_{ij} and H_{ij} depend on $\varepsilon_{1,2}$ as well, and H_{ij} vanish at $\varepsilon_1 = \varepsilon_2 = 0$.

In fact, given any m, for any sufficiently small $\varepsilon_{1,2}$, there exists a pair of functions $\psi_{0,1}(u)$ such that after an appropriate choice of the coordinates (u, v), map (5.26) can be made $o(|u|^m + |v|^m)$-close to the time-1 map by a flow of the form

$$\dot{u} = v, \qquad \dot{v} = \psi_0(u) + v\psi_1(u). \tag{5.29}$$

Obviously, the flow has to have an equilibrium at zero and the eigenvalues of the corresponding linearization matrix have to be logarithms of the eigenvalues of $\begin{pmatrix} 1 & 1 \\ \varepsilon_2 & 1+\varepsilon_1 \end{pmatrix}$, the linearization matrix of (5.26). Thus, we assume

$$\begin{aligned} &\psi_0(0) = 0, \qquad \psi_1(0) = \ln(1 + \varepsilon_1 - \varepsilon_2), \\ &\psi_0'(0) = -\ln(1 + \frac{\varepsilon_1}{2} + \sqrt{\varepsilon_2 + \frac{\varepsilon_1^2}{4}}) \ln(1 + \frac{\varepsilon_1}{2} - \sqrt{\varepsilon_2 + \frac{\varepsilon_1^2}{4}}). \end{aligned} \tag{5.30}$$

The time-t map of (5.29) can be found by expanding it in powers of the initial conditions $(u(0), v(0))$. This gives us the time-1 map in the form which is brought, by an $O(\varepsilon_{1,2})$-close to identity linear transformation, to form (5.26) with

$$\begin{aligned} \alpha_{ij} &= \frac{1}{j+1} \left(\psi_{i+j,0} \left[\frac{C^i_{i+j}}{j+2} + O(\varepsilon_{1,2}) \right] + \psi_{i+j-1,1} \left[\frac{C^i_{i+j-1}}{j} + O(\varepsilon_{1,2}) \right] \right) + \tilde{\alpha}_{ij}, \\ \beta_{ij} &= \psi_{i+j,0} \left[\frac{C^i_{i+j}}{j+1} + O(\varepsilon_{1,2}) \right] + \psi_{i+j-1,1} \left[\frac{C^i_{i+j-1}}{j} + O(\varepsilon_{1,2}) \right] + \tilde{\beta}_{ij}, \end{aligned} \tag{5.31}$$

where $\tilde{\alpha}_{ij}$ and $\tilde{\beta}_{ij}$ are nonlinear functions of the coefficients $\psi_{i'j'}$ with $i'+j' < i+j$ (we denote $\psi_j(u) = \sum_i \psi_{ij} u^i$). Two maps of form (5.26) can be made $o(|u|^m + |v|^m)$-close by means of a smooth coordinate transformation if their normal forms (5.27) coincide up to the order m. Thus, we find from formulas (5.28),(5.30),(5.31) that a map of form (5.26) can indeed, after an appropriate coordinate transformation, be made $o(|u|^m + |v|^m)$-close to the time-1 map by a flow of form (5.29) with the coefficients defined by

$$\psi_{i0} = \beta_{i0} + \tilde{\psi}_{i0}, \qquad \psi_{i1} = \beta_{i1} + (i+1)(\alpha_{i+1,0} - \beta_{i+1,0}) + \tilde{\psi}_{i1}, \tag{5.32}$$

where $\tilde{\psi}_{ij}$ is a polynomial function of $\alpha_{i'j'}$ and $\beta_{i'j'}$ with $i'+j' \leq i+j$; the monomials that include $\alpha_{i'j'}$ or $\beta_{i'j'}$ with $i'+j' < i+j$ are of degree 2 or higher, while the terms that include $\alpha_{i'j'}$ or $\beta_{i'j'}$ with $i'+j' = i+j$ are linear and the corresponding coefficients are small of order $O(\varepsilon_{1,2})$. It follows from (5.30) that formulas (5.32) remain valid at $j = 1$, if we formally put $\beta_{10} = \varepsilon_2$, $\beta_{01} = \varepsilon_1$, $\alpha_{10} = \alpha_{01} = 0$.

Returning to map (5.22), by putting $X = u$ and $Y = u + v$, we bring the map at $\hat{B} = 1$ to form

$$\bar{u} = u + v, \qquad \bar{v} = v + \beta_0(u) + v\beta_1(u) + O(v^2) + o(|u|^m + |v|^m),$$

where the coefficients of the polynomials $\beta_j = \sum_{i=1}^{m-j} \beta_{ij} u^i$ $(j = 0, 1)$ satisfy

$$\beta_{i0} = \hat{E}_i + o(1)_{k\to+\infty}, \qquad \beta_{i1} = (i+1)\hat{E}_{i+1} + o(1)_{k\to+\infty}.$$

Thus, by making normal form transformations described above, we can, at $\hat{E}_1 = \varepsilon$ sufficiently small, make the map $o(|u|^m + |v|^m)$-close to the time-1 map of the flow (5.29) with

$$\psi_{i0} = \hat{E}_i + o(1)_{k\to+\infty} + o(\hat{E}_1, \dots, \hat{E}_m), \qquad \psi_{i1} = o(1)_{k\to+\infty} + o(\hat{E}_1, \dots, \hat{E}_m);$$

see (5.32). It follows that at sufficiently large k we can always choose the values of $\hat{E}_1, \dots, \hat{E}_m$ in such a way that $\psi_0(u) \equiv s\psi_1(u)$, where $s = 1$ if $\psi_1'(0) \leq 0$, and $s = -1$ if $\psi_1'(0) > 0$.

As we see, at an appropriate choice of the parameters the map can be made $o(|u|^m + |v|^m)$-close to the time-1 map of the flow

$$\dot{u} = v, \qquad \dot{v} = \psi_1(u)(s + v),$$

which takes the desired form (5.9) after the change $(u, v) \to (su, sv)$; here $\Psi(u) = -\psi_1(su)$, so the requirement $\Psi'(0) \geq 0$ is ensured by our choice of s. $\square$

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