DECOMPOSITIONS OF MEASURES ON PSEUDO EFFECT ALGEBRAS

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Abstract. Recently in [Dvu3] it was shown that if a pseudo effect algebra satisfies a kind of the Riesz Decomposition Property ((RDP) for short), then its state space is either empty or a nonempty simplex. This will allow us to prove a Yosida–Hewitt type and a Lebesgue type decomposition for measures on pseudo effect algebra with (RDP). The simplex structure of the state space will entail not only the existence of such a decomposition but also its uniqueness.

1. Introduction

The classical decomposition theorems of or Yosida–Hewitt [YoHe] initiated in the last two decades interest of authors studying finitely additive measures on quantum structures like orthomodular lattices or posets to study an interesting problem of decomposition measures. There appeared a whole series of papers studying Lebesgue and Yosida-Hewitt type decompositions, see e.g. [DDP, DeMo, Rut1, Rut2]. They exhibited at least the existence of such a decomposition. To prove even the uniqueness of decompositions, some sufficient conditions are presented in [Rut2].

Quantum structures were inspired by the research of the mathematical foundations of quantum structures. An analogue of a probability measure is a *state*. One of the most important examples of orthomodular lattices or of the Hilbert space quantum mechanics is the system $\mathcal{L}(H)$ of all closed subspaces of a Hilbert space (real, complex or quaternionic) H. The Gleason Theorem, see e.g. [Dvu0], says that every σ-additive state on $\mathcal{L}(H)$ is uniquely expressible via a Hermitian trace operator on H if $3 \le \dim H \le \aleph_0$. The Aarnes Theorem, [Dvu0, Thm 3.2.28] says that every (finitely additive) state on $\mathcal{L}(H)$ is a unique convex combination of two states, s_1 and s_2, where s_1 is a completely additive state and s_2 is a finitely additive state that vanishes on each finite-dimensional subspace of H.

In the Nineties, Foulis and Bennett [FoBe] introduced effect algebras that are partial structures with a partially defined operation $+$ that models the join of mutually excluding events. They generalize orthomodular lattices and posets, orthoalgebras,

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and the basic example important for so-called POV-measures of quantum mechanics is the system, $\mathcal{E}(H)$, of all Hermitian operators of a Hilbert space H that are between the zero operator and the identity one.

These commutative structures were extended in [DvVe1, DvVe2] to so-called *pseudo effect algebras* where the partial addition, $+$, is not more assumed to be commutative. In many important examples they are intervals in po-groups (= partially ordered groups). E.g. $\mathcal{E}(H)$ is an interval in the po-group $\mathcal{B}(H)$ of all Hermitian operators on H.

A sufficient condition for a pseudo effect algebra to be an interval of a unital po-group is a variant of the Riesz Decomposition Property ((RDP) in abbreviation), see [Dvu2, DvVe2]. It is a weaker form of the distributivity that allows to do a joint refinement of two decompositions of the unit element 1. For example, (RDP) on an orthomodular poset entails that it has to be a Boolean algebra, and therefore, (RDP) fails to hold on $\mathcal{L}(H)$ or on $\mathcal{E}(H)$.

We recall that every effect algebra with (RDP) has at least one state, however the state space of a pseudo effect algebra with a stronger type of (RDP) can be empty, [Dvu1].

Recently in [Dvu3, Thm 5.1], it was shown that the state space of every pseudo effect algebra with (RDP) is a simplex, more precisely a Choquet simplex. The simplex is a special type of a convex set that generalizes the classical one in $\mathbb{R}^n$. For a comprehensive source on simplices see [Alf]. We note that the state space of $\mathcal{E}(H)$ is not a simplex, however the state space of any commutative C*-algebra is a simplex, see e.g. [AlSc, Thm 4.4, p. 7]. The simplex structure of the state spaces allows also to represent uniquely states as an integral [Dvu4] through a regular Borel probability measure.

The simplex structure of the state space of a pseudo effect algebra E satisfying (RDP) entails that the space of all Jordan measures on E is an Abelian Dedekind complete ℓ-group, [Dvu3, Thm 3.5, Thm 3.6]. This new fact is our basic tool to present a Yosida–Hewitt type and a Lebesgue type of decompositions of finitely additive measures on E. The property (RDP) as we show is a sufficient condition to prove not only the existence but also the uniqueness of such a decomposition that is a main goal of the present paper.

The paper is organized as follows.

Section 2 is an introduction to the theory of pseudo effect algebras gathering the necessary latest results. Section 3 describes the faces of the state space of pseudo effect algebras and it gives a general result on a decomposition and it will be applied in Section 4 to present the main body of the paper - the Yosida–Hewitt type of decomposition and the Lebesque type decomposition.

2. Elements of Pseudo Effect Algebras

Pseudo effect algebras were introduced in [DvVe1, DvVe2]. We say that a *pseudo effect algebra* is a partial algebra $(E; +, 0, 1)$, where $+$ is a partial binary operation and 0 and 1 are constants, such that for all $a, b, c \in E$, the following holds

(i) $a + b$ and $(a + b) + c$ exist if and only if $b + c$ and $a + (b + c)$ exist, and in this case $(a + b) + c = a + (b + c)$;
(ii) there is exactly one $d \in E$ and exactly one $e \in E$ such that $a+d = e+a = 1$;
(iii) if $a + b$ exists, there are elements $d, e \in E$ such that $a + b = d + a = b + e$;
(iv) if $1 + a$ or $a + 1$ exists, then $a = 0$.

If we define $a \le b$ if and only if there exists an element $c \in E$ such that $a+c=b$, then $\le$ is a partial ordering on E such that $0 \le a \le 1$ for any $a \in E$. It is possible to show that $a \le b$ if and only if $b = a+c = d+a$ for some $c,d \in E$. We write $c = a \,/\, b$ and $d = b \setminus a$. Then

$$(b \setminus a) + a = a + (a \,/\, b) = b,$$

and we write $a^- = 1 \setminus a$ and $a^\sim = a \,/\, 1$ for any $a \in E$.

For basic properties of pseudo effect algebras see [DvVe1, DvVe2]. We recall that if $+$ is commutative, E is said to be an *effect algebra*; for a guide overview on effect algebras we recommend e.g. [DvPu].

For example, let (G,u) be a unital po-group (= partially ordered group) with strong unit u that is not necessarily Abelian. We recall that a po-group (= partially ordered group) is a group with a partial ordering $\le$ such that if $a \le b$, then $x+a+y \le x+b+y$ for all $x,y \in G$; and an element $u \in G^+ := \{g \in G : g \ge 0\}$ is said to be a *strong unit* if given $G = \bigcup_n [-nu, nu]$.

Then for (G,u) we define $\Gamma(G,u) = [0,u]$ and we endow it with $+$ that is the restriction of the group addition to the set of all those $(x,y) \in \Gamma(G,u) \times \Gamma(G,u)$ that $x \le u - y$. Then $(\Gamma(G,u);+,0,u)$ is a pseudo effect algebra with possible two negations: $a^- = u - a$ and $a^\sim = -a + u$. Any pseudo effect algebra of the form $\Gamma(G,u)$ is said to be an *interval pseudo effect algebra*. In [DvVe1, DvVe2], we have some sufficient conditions posed to a pseudo effect algebra to be an interval. They are analogues of the Riesz Decomposition Properties. Roughly speaking they are a weaker form of distributivity that allows a joint refinement of two partitions of 1. This is a reason why they fail to hold for $\mathcal{L}(H)$ and $\mathcal{E}(H)$.

Now we introduce according to [DvVe1] the following types of the Riesz Decomposition properties for pseudo effect algebras that in the case of effect algebras may coincide, but not for pseudo effect algebras, in general.

(a) For $a,b \in E$, we write a **com** b to mean that for all $a_1 \le a$ and $b_1 \le b$, a_1 and b_1 commute.
(b) We say that E fulfils the *Riesz Interpolation Property*, (RIP) for short, if for any $a_1,a_2,b_1,b_2 \in E$ such that $a_1,a_2 \le b_1,b_2$ there is a $c \in E$ such that $a_1,a_2 \le c \le b_1,b_2$.
(c) We say that E fulfils the *weak Riesz Decomposition Property*, (RDP_0) for short, if for any $a,b_1,b_2 \in E$ such that $a \le b_1+b_2$ there are $d_1,d_2 \in E$ such that $d_1 \le b_1$, $d_2 \le b_2$ and $a = d_1+d_2$.
(d) We say that E fulfils the *Riesz Decomposition Property*, (RDP) for short, if for any $a_1,a_2,b_1,b_2 \in E$ such that $a_1+a_2 = b_1+b_2$ there are $d_1,d_2,d_3,d_4 \in E$ such that $d_1+d_2 = a_1$, $d_3+d_4 = a_2$, $d_1+d_3 = b_1$, $d_2+d_4 = b_2$.
(e) We say that E fulfils the *commutational Riesz Decomposition Property*, (RDP_1) for short, if for any $a_1,a_2,b_1,b_2 \in E$ such that $a_1+a_2 = b_1+b_2$ there are $d_1,d_2,d_3,d_4 \in E$ such that (i) $d_1+d_2 = a_1$, $d_3+d_4 = a_2$, $d_1+d_3 = b_1$, $d_2+d_4 = b_2$, and (ii) d_2 **com** d_3.
(f) We say that E fulfils the *strong Riesz Decomposition Property*, (RDP_2) for short, if for any $a_1,a_2,b_1,b_2 \in E$ such that $a_1+a_2 = b_1+b_2$ there are $d_1,d_2,d_3,d_4 \in E$ such that (i) $d_1+d_2 = a_1$, $d_3+d_4 = a_2$, $d_1+d_3 = b_1$, $d_2+d_4 = b_2$, and (ii) $d_2 \wedge d_3 = 0$.

We have the implications

$$(\mathrm{RDP}_2) \Rightarrow (\mathrm{RDP}_1) \Rightarrow (\mathrm{RDP}) \Rightarrow (\mathrm{RDP}_0) \Rightarrow (\mathrm{RIP}).$$

The converse of any of these implications does not hold. For commutative effect algebras we have

$$(\text{RDP}_2) \Rightarrow (\text{RDP}_1) \Leftrightarrow (\text{RDP}) \Leftrightarrow (\text{RDP}_0) \Rightarrow (\text{RIP}).$$

If in the above definitions of (RDP)'s we change E to G^+, we have po-groups with the corresponding forms of the Riesz Decomposition Properties. According to [DvVe2, Thm 5.7], every pseudo effect algebra with $(\text{RDP})_1$ is an interval in some unital po-group with $(\text{RDP})_1$, so is any effect algebra with (RDP) in an Abelian po-group with (RDP), see see [Rav] or [DvPu, Thm 1.7.17]. Any effect algebra with (RDP) that is a lattice, or equivalently, with $(\text{RDP})_2$ is an MV-algebra, for a definition see e.g. [DvPu]. Any pseudo effect algebra with $(\text{RDP})_2$ is a so-called pseudo MV-algebra, see [GeIo], and the group G is an ℓ-group (= lattice ordered group).

A *signed measure* on a pseudo effect algebra E is any mapping $m : E \to \mathbb{R}$ such that $m(a+b) = m(a) + m(b)$ provided $a+b$ is defined in E. We have $s(0) = 0$ and $s(a^-) = s(a^\sim)$. If a signed measure m is positive, i.e., $m(a) \geq 0$ for each $a \in E$, we call it a *measure*, and any normalized measure, i.e. a measure s such that $s(1) = 1$, is said to be a *state*. For any measure m, we have $m(a) \leq m(b)$ whenever $a \leq b$. If m_1 and m_2 are two measures on E, then the signed measure $m = m_1 - m_2$ is said to be a *Jordan signed measure*. We denote by $\mathcal{M}(E)$, $\mathcal{M}^+(E)$, $\mathcal{S}(E)$, and $\mathcal{J}(E)$ the sets of all signed measures, or measures, or states or Jordan signed measures on E. We recall that it can happen that $\mathcal{M}(E) = \{0\} = \mathcal{J}(E)$.

The set $\mathcal{S}(E)$ is convex, i.e., any convex combination $s = \lambda s_1 + (1-\lambda)s_2$, $\lambda \in [0,1]$, of two states s_1 and s_2 and $\lambda \in [0,1]$ is a state. If s cannot be expressed by a convex combination of two distinct states, it is called an *extremal state*. Let $\partial_e \mathcal{S}(E)$ denote the set of all extremal states. On $\mathcal{M}(E)$ we introduce a *weak topology*: We say that a net of measures $\{m_\alpha\}$ *converges weakly* to a measure m if $\lim_\alpha m_\alpha(a) = m(a)$. Then $\mathcal{S}(E)$ is a convex compact Hausdorff space, and due to the Krein–Mil'man Theorem, see [Goo, Thm 5.17], every state on E is a weak limit of a net of convex combinations of extremal states.

If E is a pseudo effect algebra with (RDP), then $\mathcal{S}(E)$ is either empty or a non-void simplex, [Dvu3, Thm 5.1], for definition of a simplex and its basic properties, see [Goo, Chap 10].

For two signed measures m_1 and m_2, we write $m_1 \leq^+ m_2$ if $m_1(a) \leq m_2(a)$ for each $a \in E$.

The following important statement was proved in [Dvu3, Thm 3.5, Thm 3.6]:

Theorem 2.1. *Let E be a pseudo effect algebra with* (RDP). *Then $\mathcal{J}(E)$ is an Abelian Dedekind complete ℓ-group such that if $\{m_i\}_{i\in I}$ is a nonempty system of $\mathcal{J}(E)$ that is bounded above, and if $d(x) = \bigvee_i m_i(x)$ for all $x \in E$, then*

$$\left(\bigvee_i m_i\right)(x) = \bigvee\{d(x_1) + \cdots + d(x_n) : x = x_1 + \cdots + x_n,\ x_1, \ldots, x_n \in E\}$$

for all $x \in E$.

And if $e(x) = \bigwedge_i f_i(x)$ for all $x \in E$, then

$$\left(\bigwedge_i m_i\right)(x) = \bigwedge\{e(x_1) + \cdots + e(x_n) : x = x_1 + \cdots + x_n,\ x_1, \ldots, x_n \in E\}$$

for all $x \in E$.

Given $m_1, \ldots, m_n \in \mathcal{J}(E)$,

$$\left(\bigvee_{i=1}^n m_i\right)(x) = \sup\{m_1(x_1)+\cdots+m_n(x_n) : x = x_1+\cdots+x_n,\ x_1,\dots,x_n \in E\},$$
$$\left(\bigwedge_{i=1}^n m_i\right)(x) = \inf\{m_1(x_1)+\cdots+m_n(x_n) : x = x_1+\cdots+x_n,\ x_1,\dots,x_n \in E\},$$

for all $x \in E$.

3. Faces of the State Space

The present section describes the faces of the state space of a pseudo effect algebra with (RDP). Since our state space is a simplex, we know that if F is a closed face, then every simplex is a direct convex sum of F and its complementary face, [Goo, Thm 11.28]. However, not every face is closed, we present a general decomposition, Theorem 3.4, where a weaker form of the closedness, $\vee$-closedness, allows to obtain a unique decomposition of measures. This result will apply in the next section to obtain Yosida–Hewitt and Lebesgue types of decomposition.

A *face* of a convex set K is a convex subset F of K such that if $x = \lambda x_1 + (1-\lambda)x_2 \in F$ for $\lambda \in (0,1)$, then also $x_1, x_2 \in F$. We note that if $x \in K$, then $\{x\}$ is a face iff $x \in \partial_e K$.

For any $X \subseteq K$, there is the face generated by X. Due to [Goo, Prop 5.7], the face F generated by X is the set of those points $x \in K$ for which there exists a positive convex combination $\lambda x + (1-\lambda)y = z$ with $y \in K$ and z belongs to the convex hull of X.

If $\mathcal{S}(E) \neq \emptyset$, then a state $s \in \mathcal{S}(E)$ belongs to the face generated by X if and only if $s \le^+ \alpha t$ for some positive constant α and some state t in the convex hull of X. In particular, the face of $\mathcal{S}(E)$ generated by a state s is the set of states $s' \in \mathcal{S}(E)$ such that $s' \le^+ \alpha s$ for some real number $\alpha > 0$.

Let s be a state on a pseudo effect algebra E. The *kernel* of s is the set

$$\mathrm{Ker}(s) := \{x \in E : s(x) = 0\}.$$

Then $\mathrm{Ker}(s)$ is a normal ideal of E. We note that a subset I of a pseudo effect algebra is an *ideal* if (i) $0 \in I$, (ii) $a, b \in I$ and $a + b \in E$ imply $a + b \in I$, and (iii) $a \in E$, $b \in I$ and $a \le b$ entail $a \in I$. An ideal I is *normal*, if $a + I := \{a + b \in E : b \in I\} = I + a := \{b + a \in E : b \in I\}$ for any $a \in E$.

Proposition 3.1. *Let E be a pseudo effect algebra and let X be a subset of E. Then the set*

$$F = \{s \in \mathcal{S}(E) : X \subseteq \mathrm{Ker}(s)\}$$

is a closed face of $\mathcal{S}(E)$.

Proof. If $F = \emptyset$, then F is trivially a closed face. Assume $F \neq \emptyset$. If $s = \lambda s_1 + (1-\lambda)s_2$ for $\lambda \in [0,1]$ and for two states s_1 and s_2, then $\mathrm{Ker}(s) \subseteq \mathrm{Ker}(s_1) \cap \mathrm{Ker}(s_2)$ which proves that F is a convex set. If $\{s_\alpha\}$ is a net of states from F that converges weakly to a state s on E, then for each $x \in X$, $0 = \lim_\alpha s_\alpha(x) = s(x)$ so that F is closed. If now $s = \lambda s_1 + (1-\lambda)s_2 \in F$ for $\lambda \in (0,1)$ and $s_1, s_2 \in F$, then for each $x \in X$ we have $0 = s(x) = \lambda s_1(x) + (1-\lambda)s_2(x)$ so that $s_1(x) = s_2(x) = 0$ and therefore, $s_1, s_2 \in F$. $\square$

Let F be a face of a simplex K, and let F' be the union of those faces of K that are disjoint from F. According to [Goo, Prop 10.12], F' is a face of K and it is the largest face of K that is disjoint from F. If a point $x \in K$ can be expressed as convex combinations

$$x = \alpha_1 x_1 + \alpha_2 x_2 = \beta_1 y_1 + \beta_2 y_2$$

with $x_1, y_1 \in F$ and $x_2, y_2 \in F'$, then $\alpha_i = \beta_i$ for $i = 1, 2$ and $x_i = y_i$ for those i such that $\alpha_i > 0$. The face F' is said to be a *complementary face* of F in K.

It is possible to show that if $F_1 \subseteq F_2$ are two faces of $\mathcal{S}(E)$, then $F_2' \subseteq F_1'$ and for a set $\{F_i\}$ of faces, we have $F = \bigcap_i F_i$ is a face, and

$$\left(\bigcap_i F_i\right)' = \bigcup_i F_i' \qquad (3.0).$$

Proposition 3.2. *Let F be a face of the state space of a pseudo effect algebra E with* (RDP) *and let $\mathcal{S}(E) \neq \emptyset$. Then its complementary face F' consists of all states $s' \in \mathcal{S}(E)$ such that $s' \wedge s = 0$ for every state $s \in F$.*

Equivalently, F' consists of all states $s' \in \mathcal{S}(E)$ such that if $s \in F$ is such that $\alpha s \le^+ s'$ for some constant $\alpha \ge 0$, then $\alpha = 0$.

Proof. We know that $\mathcal{S}(E)$ is a simplex. According to [Goo, Prop 10.12], F' is the union of all faces of $\mathcal{S}(E)$ that are disjoint from F. If $s' \in F'$ and $s \in F$, and if s' and s belong to mutually disjoint faces, then $s' \wedge s = 0$, see [Dvu3, Prop 4.2]. Conversely, let s' be a state on E such that $s' \wedge s = 0$ for each $s \in F$. Then by [Dvu3, Prop 4.2], s' and s belong to mutually disjoint faces. Let $F(s')$ and $F(s)$ be the faces generated by s' and s. Therefore, $F(s') \cap F(s) = \emptyset$. But $F = \bigcup_{s\in F} F(s)$, so that $F(s') \cap F = \bigcup_{s \in F} F(s') \cap F(s) = \emptyset$ that gives $s' \in F'$

Now let $s' \in F'$ and let $s \in F$ be such that $\alpha s \le s'$ for some $\alpha \ge 0$. If $\alpha > 0$, then $s \le s'/\alpha$ that s belongs to the face generated by s' that is impossible, whence $\alpha = 0$.

Assume that F_0 is the set of all those states s' on E such that if $\alpha s \le^+ s'$ for $s \in F$ and $\alpha \ge 0$, then $\alpha = 0$. We have seen that $F' \subseteq F_0$. Let $s' \in F_0$ and $s \in F$ If $s' \wedge s >^+ 0$, then s belongs to the face generated by s', hence, $s \le^+ ts'$ for some $t > 0$. This gives $s/t \le^+ s$ so that $1/t = 0$ that is absurd. Hence, $F_0 \subseteq F'$. □

Let $K_1, \dots, K_n$ be convex subsets of K. We say that K is the *direct convex sum* of $K_1, \dots, K_n$ if (i) K equals the convex hull of $\bigcup_{i=1}^n K_i$ and every element $x \in K$ can be uniquely expressed as a convex combination of some elements $x_i \in K_i$, $i = 1, \dots, n$. That is, if

$$\alpha_1 x_1 + \cdots + \alpha_n x_n = \beta_1 y_1 + \cdots + \beta_n y_n$$

are convex combinations of $x_i, y_i \in K_i$, for $i = 1, \dots, n$, then $\alpha_i = \beta_i$ for all i's and $x_i = y_j$ for those i such that $\alpha_i > 0$.

Theorem 3.3. *Let E be a pseudo effect algebra with* (RDP) *and let $\mathcal{S}(E) \neq \emptyset$. Let I be a normal ideal of E, and set*

$$F = \{s \in \mathcal{S}(E) : I \subseteq \mathrm{Ker}(s)\}.$$

Then F is a closed face of $\mathcal{S}(E)$, and $\mathcal{S}(E)$ is the direct convex sum of F and its complementary face.

Proof. By Proposition 3.1, F is a closed face of $\mathcal{S}(E)$. If F is empty, then $F' = \mathcal{S}(E)$.

Now assume F is non-void. Let $\mathcal{J}(E)$ be the set of all Jordan signed measures on E. By Theorem 2.1, $\mathcal{J}(E)$ is an Abelian Dedekind complete ℓ-group, and $\mathcal{S}(E)$ is a base for the positive cone $\mathcal{J}(E)^+ = \{\alpha s : \alpha \geq 0, s \in \mathcal{S}(E)\}$. We show that the set

$$C = \{\alpha s : \alpha \geq 0,\ s \in \mathcal{S}(E),\ s \in F\}$$

contains the supremum of any subset of C which is bounded above in $\mathcal{J}(E)$. Given a nonempty system of positive measures $\{m_i\}_i$ from C that is bounded in $\mathcal{J}(E)$, let $m = \bigvee_i m_i$. According to Theorem 2.1,

$$m(x) = \sup\{d(x_1) + \cdots + d(x_n) : x = x_1 + \cdots + x_n\}, \quad x \in E,$$

where $d(x) = \sup_i m_i(x)$, $x \in E$.

If $x \in I$, then $x = x_1 + \cdots + x_n$ entails $x_1, \ldots, x_n \in I$ so that $d(x_j) = \sup_i m_i(x_j) = 0$ for each $j = 1, \ldots, n$. Hence, $m(x) = 0$ and $m \in C$.

According to [Goo, Prop 10.14], $\mathcal{S}(E)$ is the direct convex sum of F and its complementary face. $\square$

Let F be a face of $\mathcal{S}(E)$, where E is a pseudo effect algebra with (RDP). We set

$$V(F) := \{\alpha s : \alpha \geq 0, s \in F\}.$$

It is a cone that is a subcone of $\mathcal{M}^+(S)$, i.e., if (i) $m_1, m_2 \in V(F)$, then $m_1 + m_2 \in V(F)$, (ii) $\mathbb{R}^+(F) \subseteq V(F)$ (ii) $-V(F) \cap V(F) = \{0\}$. We say that $V(F)$ is $\vee$-*closed* if, for any chain $\{m_i\}$ from $V(F)$ bounded in $\mathcal{M}^+(E)$, $\bigvee_i m_i \in V(F)$.

If F' is the complementary face of F, according to Proposition 3.2, $m \in V(F')$ iff for any $t \in V(F)$ with $t \leq^+ m$, we have $t = 0$.

For an arbitrary cone V of $\mathcal{S}(E)$, we denote by $V^\sharp$ the set of all those measures $t \in \mathcal{M}^+(E)$ that $m \leq^+ t$ for $m \in V$ entails $m = 0$. The elements of $V^\sharp$ are said to be V-*singular measures*. Hence, $V(F)^\sharp = V(F')$ for any face F of $\mathcal{S}(E)$ of a pseudo effect algebra E with (RDP).

Theorem 3.4. *Let F be a face of the state space $\mathcal{S}(E)$ of a pseudo effect algebra E with* (RDP)*. If $V(F)$ is $\vee$-closed, then $\mathcal{S}(E)$ equals the direct convex sum of F and its complementary face F'.*

In addition, every measure m on E can be uniquely decomposed as a sum

$$m = m_1 + m_1 \tag{3.1}$$

of two measures such that $m_1 \in V(F)$ and $m_2 \in V(F')$.

Proof. Existence: Let m be a measure on E, and let $\Gamma(m) = \{m_1 \in V(F) : m_1 \leq^+ m\}$. Since the zero measure belongs to $\Gamma(m)$, $\Gamma(m)$ is non-void. Let $\{m_i\}$ be a chain of elements from $\Gamma(m)$. The measure m is an upper bound for $\{m_i\}$. By Theorem 2.1, $m_0 := \bigvee_i m_i$ is a measure on E and by the hypotheses, m_0 belongs to $V(F)$. It follows from the Zorn's Lemma that $\Gamma(m)$ contains a maximal element m_1 such that $m_1 \leq^+ m$.

If we set $m_2 = m - m_1$, we show that $m_2 \in V(F')$. Let $t \in V(F)$ and let $t \leq^+ m_2$. Then $m_1 + t \leq^+ m_1 + m_2 = m$. Since $m_1 + t \in V(F)$, the maximality of m_1 in $\Gamma(m)$ implies $t = 0$, and therefore, $m_2 \in V(F)$.

Uniqueness: Now let s be an arbitrary state on E. According to the first part of the present proof, we see that s can be decomposed in the convex form

$$s = \lambda_1 s_1 + \lambda_2 s_2, \tag{3.2}$$

where $s_1 \in F$ and $s_2 \in F'$. This implies that $\mathcal{S}(E)$ is a direct convex sum of F and F'. In particular, according to [Goo, Prop 10.12], this yields that the decomposition (3.2) is unique, i.e. if $s = \alpha_1 s_1' + \alpha_2 s_2'$ is another convex combination of $s_1' \in F$ and $s_2' \in F'$, then $\alpha_i = \lambda_i$ for $i = 1, 2$, and if $\alpha_i > 0$ implies $s_i = s_i'$. In particular, this implies that the decomposition in (3.1) is unique. □

4. Decomposition of States on Pseudo Effect Algebras

This section presents the main results of the paper: Yosida–Hewitt type and Lebesgue type of decomposition of finitely additive measures and states.

Using closed faces F, we can decompose any state s on a pseudo effect algebra with (RDP) in the unique form: $s = \lambda s_1 + (1-\lambda)s_2$, where $s_1 \in F$ and $s_2 \in F'$, where F' is the complementary face of F, see [Goo, Thm 11.28]. But not every face of $\mathcal{S}(E)$ is closed. E.g. the set of all σ-additive states on E is a face that is not necessarily closed. However, also for some such situations we show that $\mathcal{S}(E)$ can be the direct convex sum of the face F and its complementary face.

A non-empty set X of a poset E is *directed downwards* (*directed upwards*), and we write $D \downarrow$ ($D \uparrow$), if for any $x, y \in X$ there exists $z \in D$ such that $z \le x$, $z \le y$ ($z \ge x$, $z \ge y$). Two downwards directed sets $\{x_t : t \in T\}$ and $\{y_t : t \in T\}$ indexed by the same index set T are called *downwards equidirected* if, for any $s, t \in T$, there exists $v \in T$ such that $x_v \le x_s$ and $x_v \le x_t$ as well as $y_v \le y_s$ and $y_v \le y_t$. A similar definition holds for upwards directed sets.

Let $x \in E$ and $D \subseteq E$. We say that $D \uparrow x$ if $D \uparrow$ and $x = \bigvee D$. Dually we define $D \downarrow x$, i.e. $D \downarrow$ and $x = \bigwedge D$.

Let $\{a_n\}$ be a sequence of elements of a pseudo effect E such that $b_n = a_1 + \cdots + a_n$ exists in E for each $n \ge 1$ and if $a = \bigvee_n b_n$ exists in E, we write $a = \sum_n a_n$.

A signed measure m on a pseudo effect algebra E is σ-*additive* if, $\{a_n\} \nearrow a$, i.e. $a_n \le a_{n+1}$ for each $n \ge 1$ and $\bigvee_n a_n = a$, then $m(a) = \lim_n m(a_n)$. A signed measure m is σ-additive iff $\{a_n\} \searrow 0$ entails $\lim_n m(a_n) = 0$.

Now let E be an effect algebra (not a pseudo effect algebra). We say that a system $\{a_t\}_{t\in T}$ of elements of E is *summable* if, for each finite subset $F \subseteq T$, the element $a_F = \sum_{t\in F} a_t$ is defined in E. If there exists the element $a = \bigvee\{a_F : F \subseteq T\}$, we called it the *sum* of the summable system $\{a_t\}_{t\in T}$, and write $a = \sum_{t\in T} a_t$.

A signed measure m on an effect algebra is said to be *completely additive* if $m(a) = \sum_{t\in T} m(a_t)$ whenever $a = \sum_{t\in T} a_t$.

We denote by $\mathcal{J}(E)_\sigma, \mathcal{J}(E)_{\rm ca}$ the sets of all σ-additive Jordan signed measures and completely additive Jordan signed measures on E, respectively. In the same way we define $\mathcal{M}^+(E)_\sigma$, $\mathcal{M}^+(E)_{\rm ca}$ and similarly for the states: $\mathcal{S}(E)_\sigma$ and $\mathcal{S}(E)_{\rm ca}$ denote the systems of all σ-additive and completely additive states on E.

Then $\mathcal{S}(E)_{\rm ca} \subseteq \mathcal{S}(E)_\sigma \subseteq \mathcal{S}(E)$. Each of these sets can be empty. Moreover, $\mathcal{S}(E)_\sigma$ and $\mathcal{S}(E)_{\rm ca}$ are also faces of $\mathcal{S}(E)$, and

$$V(\mathcal{S}(E)_\sigma) = \mathcal{M}^+(E)_\sigma \quad \text{and} \quad V(\mathcal{S}(E)_{\rm ca}) = \mathcal{M}^+(E)_{\rm ca}. \tag{4.1}$$

Proposition 4.1. *Let $m_1, \ldots, m_n$ be completely additive measures on an effect algebra E that satisfies* (RDP)*. Then $m = m_1 \vee \cdots \vee m_n$ is also a completely additive measure on E.*

The same is true if $m_1, \dots, m_n$ *are* σ*-additive measures on a pseudo effect algebra* E *with* (RDP).

Proof. Without loss of generality, we can assume that $n = 2$. Let $a = \sum_{t\in T} a_t$ and let F be any finite subset of T and let $a_F = \sum_{t\in F} a_t$. Given $\epsilon > 0$, there is F_0 such that, for each finite $F \supseteq F_0$, $m_1(a - a_F) < \epsilon/2$ and $m_2(a - a_F) < \epsilon/2$.

By Theorem 2.1, $m(a - a_F) = \sup\{m_1(x) + m_2(y) : x + y = a - a_F\}$. Since $x \le a - a_F$ and $y \le a - a_F$, we have $m_1(x) \le m_1(a - a_F) < \epsilon/2$ and $m_2(y) \le m_2(a - a_F) < \epsilon/2$. Then $m_1(x) + m_2(y) < \epsilon$, consequently $m(a - a_F) \le \epsilon$. Hence, $m(a) - m(a_F) \le \epsilon$ and m is completely additive. □

Lemma 4.2. *If* $\{f_t\}_{t\in T} \uparrow f$ *and* $\{g_t\}_{t\in T} \uparrow g$*, where* $f_t, g_t, f, g \in \mathbb{R}^+$ *for all* $t \in T$*. Then*

$$\bigvee_{s,t} (f_s + g_t) = f + g, \tag{4.2}$$

If, in addition, $\{f_t\}_{t\in T}$ *and* $\{g_t\}_{t\in T}$ *are upwards equidirected, then*

$$\{f_t + g_t\}_{t\in T} \uparrow f + g. \tag{4.3}$$

Proof. We have $f_s + g_t \le f + g$ for all $s, t \in T$. If $f_s + g_t \le x$ for some $x \in \mathbb{R}$, then $f_s \le x - g_t$ so that $f \le x - g_t$ and hence, $g_t \le x - f$ so that $g \le x - f$ and $f + g \le x$ that gives, (4.2).

Now assume $\{f_t\}$ and $\{g_t\}$ are upwards equidirected. It is clear that $f_t + g_t \le f + g$. The equidirectness entails that, for all indices $s_0, t_0 \in T$, there exists an index t such that $f_{s_0} \le f_t$ and $f_{t_0} \le g_t$. Therefore, for all indices s_0 and t_0, $f_{s_0} + g_{t_0} \le f_t + g_t \le f + g$ which by (4.2) gives (4.3). □

Lemma 4.3. *Let* E *be a pseudo effect algebra with* (RDP)*. If* $\{m_i\}$ *is a chain of measures in* $\mathcal{M}^+(E)$ *that is bounded above, then for* $m_0 = \bigvee_i m_i$ *we have*

$$m_0(a) = \sup_i m_i(a), \tag{4.4}$$

for each $a \in E$.

Proof. We assert that if $d(x) = \bigvee_i m_i(x)$, $x \in E$, then d is additive, i.e., $d(x + y) = d(x) + d(y)$ whenever $x + y$ is defined in E. This follows from the fact that $\{m_i(x)\} \uparrow d(x)$ and $\{m_i(y)\} \uparrow d(y)$, are upwards equidirected because $\{m_i\}$ is a chain, and $\{m_i(x) + m_i(y)\} = \{m_i(x + y)\} \uparrow (d(x) + d(y))$ by (4.3).

Since, d is a measure such that $m_i \le^+ d \le^+ m_0$, we conclude $d = m_0$. □

Now we present a Yosida-Hewitt type of decomposition for measures on E.

Theorem 4.4. *Let* E *be an effect algebra with* (RDP)*. Then every measure* m *on* E *can be uniquely expressed in the form*

$$m = m_1 + m_2, \tag{4.5}$$

where $m_1 \in \mathcal{M}^+(E)_{\rm ca}$ *and* m_2 *is a finitely additive measure on* E *such that if* $t \in \mathcal{M}^+(E)_{\rm ca}$*, such* $t \le^+ m_2$*, then* $t = 0$.

In particular, every state s *on* E *can be uniquely expressed as a convex combination*

$$s = \lambda_1 s_1 + \lambda_2 s_2, \tag{4.6}$$

where s_1 *is a completely additive state and* s_2 *is a finitely additive state such that if* $\alpha s' \le^+ s_2$ *for some completely additive state* s' *on* E *and for some constant* $\alpha \ge 0$*, then* $\alpha = 0$.

Proof. First we show that $\mathcal{M}^+(E)_{\rm ca}$ is a $\vee$-closed cone in $\mathcal{M}^+(E)$. Thus let $\{m_i\}$ be a chain of completely additive measures on E that is bounded above by a finitely additive measure m'. By Theorem 2.1, there is $m_0 = \bigvee_i m_i$ that is finitely additive and $m_0 \le^+ m'$.

We assert that m_0 is completely additive. Let $a = \sum_{t\in T} a_t$ exist in L. Then from the monotonicity of m_0 we have $m_0(a_F) \le m_0(a)$, where $a_F := \sum_{t\in F} a_t$ for any F finite subset of the index set T.

Assume that $m_0(a_F) \le x$ for some $x \in \mathbb{R}^+$ and for any finite F. Then $m_i(a_F) \le x$ for any i and any F. The complete additivity of m_i entails that $m_i(a) \le x$ for any i, so that by (4.4), $m_0(a) = \sup_i m_i(a) \le x$. This gives

$$m_0(a) = \sup_F \sum_{t\in F} m_0(a_t),$$

consequently $m_0 \in \mathcal{M}^+(E)_{\rm ca}$, and $\mathcal{M}^+(E)_{\rm ca}$ is a $\vee$-closed cone of $\mathcal{J}(E)$.

Now let m be an arbitrary finitely additive measure on E. Since all the conditions of Theorem 3.4 are satisfied, we obtain the unique decomposition $m = m_1 + m_2$ in question. Similarly we have (4.6). ☐

Theorem 4.5. *Let E be a pseudo effect algebra with* (RDP)*. Then every measure m on E can be uniquely expressed in the form*

$$m = m_1 + m_2, \tag{4.7}$$

where $m_1 \in \mathcal{M}^+(E)_\sigma$ and m_2 is a finitely additive measure on E such that if $t \in \mathcal{M}^+(E)_\sigma$, $t \ge 0$, such $t \le^+ m_2$, then $t = 0$.

In particular, every state s on E can be uniquely expressed as a convex combination

$$s = \lambda_1 s_1 + \lambda_2 s_2, \tag{4.8}$$

where s_1 is a σ-additive state and s_2 is a finitely additive state such that if $\alpha s' \le^+ s_2$ for some completely additive state s' on E and for some constant $\alpha \ge 0$, then $\alpha = 0$.

Proof. It follows the same steps as the proof of Theorem 4.4. ☐

Remark 4.6. Theorems 4.5 and 4.5 have been proved in [11, 12, 9, 17]. They are analogues of the classical Yosida–Hewitt decomposition from [YoHe]. In [DeMo], the component m_2 from Theorem 4.4 is said to be a *weakly purely additive measure* and that from Theorem 4.5 a *purely additive measure*.

Now we present another Yosida–Hewitt type decomposition for an analogue of complete additivity of measures for pseudo effect algebra. We say that a measure m on E is *upwards continuous* if $\{a_t\} \uparrow a$ entails $\{m(a_t)\} \uparrow m(a)$. A measure m is upwards continuous iff $\{a_t\} \downarrow 0$ implies $\{m(a_t)\} \downarrow 0$.

For example, if E is an effect algebra, then m is completely additive whenever m is upwards continuous. Indeed, let m be an upwards continuous measure and let $a = \sum_{t\in T} a_t$. Given any finite subset F of indices we define $a_F = \sum_{t\in F} a_t$. Then $\{a_F\}_F$ is upwards directed and $\{a_F\} \uparrow a$, so that $m(a) = \sum_t m(a_t)$.

Theorem 4.7. *Let E be a pseudo effect algebra with* (RDP)*. Then every measure m on E can be uniquely expressed in the form*

$$m = m_1 + m_2,$$

where m_1 is an upwards continuous measure and m_2 is a finitely additive measure on E such that if t is an upwards continuous measure with $t \le^+ m_2$, then $t = 0$.

In particular, every state s on E can be uniquely expressed as a convex combination

$$s = \lambda_1 s_1 + \lambda_2 s_2,$$

where s_1 is an upwards continuous state and s_2 is a finitely additive state such that if $\alpha s' \le^+ s_2$ for some upwards continuous state s' on E and for some constant $\alpha \ge 0$, then $\alpha = 0$.

Proof. Let $\mathcal{M}^+(E)_{\rm uc}$ and $\mathcal{S}(E)_{\rm uc}$ be the sets of upwards continuous measures and states, respectively, on E. Then $\mathcal{S}(E)_{\rm uc}$ is a face and $V(\mathcal{S}(E)_{\rm uc}) = \mathcal{M}^+(E)$. We show that $\mathcal{M}^+(E)_{\rm uc}$ is a $\vee$-closed cone. Indeed, let $\{m_i\}$ be a chain in $\mathcal{M}^+(E)_{\rm uc}$. Then $\sup_t m(a_t) \le m(a)$. By (4.4), for $m = \bigvee_i m_i$, we have $m(a) = \sup_i m_i(a)$ for each $a \in E$. Assume $\{a_t\} \uparrow a$. Given $\epsilon > 0$, there is i such that $m_i(a) > m(a) - \epsilon/2$. Since m_i is upwards continuous, there is a_t such that $m_i(a_t) > m_i(a) - \epsilon/2$. Then $m_i(a_t) \ge m(a) - \epsilon$ and by (4.4), $m(a_t) > m(a) - \epsilon$. Therefore, $m(a) = \sup_t m(a_t)$.
For the final desired result, we apply Theorem 3.4. $\square$

In what follows, we present two types of the Lebesgue decomposition.

Let m_1 and m_2 be measures on a pseudo effect algebra E. We say that (i) m_1 is *absolutely continuous with* respect to m_2, and we write $m_1 \ll m_2$ if $m_2(a) = 0$ implies $m_1(a) = 0$ for $a \in E$. (ii) m_1 is m_2-*continuous*, and we write $m_1 \ll_\epsilon m_2$ provided given $\epsilon > 0$, there is a $\delta > 0$ such that $m_2(a) < \delta$ yields $m_1(a) < \epsilon$. (iii) $m_1 \perp m_2$ if there is an element $a \in E$ such that $m_2(a) = 0 = m_1(a^-)$.

It is clear that $m_1 \ll_\epsilon m_2$ entails $m_1 \ll m_2$.

Theorem 4.8. *Let E be a pseudo effect algebra with* (RDP)*. Let t be a fixed measure on E. Then every measure m on E can be uniquely expressed in the form*

$$m = m_1 + m_2, \tag{4.9}$$

where m_1 and m_2 are finitely additive measures on E such that $m_1 \ll_\epsilon t$ and if m' is any measure such that $m' \ll_\epsilon t$ and $m_1 \le^+ m_2$, then $m' = 0$. Moreover, $m_2 \wedge t = 0$.

In particular, every state s on E can be uniquely expressed as a convex combination of two states s_1 and s_1 on E,

$$s = \lambda_1 s_1 + \lambda_2 s_2, \tag{4.10}$$

where $s_1 \ll_\epsilon t$ and if s' is any state such that $s' \ll_\epsilon t$ and $\alpha s' \le^+ s_2$, then $\alpha = 0$.

Proof. If $t = 0$, the statement is trivial. Suppose that $t(1) > 0$ and let $F(t) = \{s \in \mathcal{S}(E) : s \ll_\epsilon t\}$. Then $t_0 = t/t(1) \in F(t)$ and $F(t)$ is a non-empty face of $\mathcal{S}(E)$.

Let $C(t) := \{s \in \mathcal{M}^+(E) : s \ll_\epsilon t\}$. Then $C(t)$ is a nonempty cone that is a subcone of $\mathcal{M}^+(E)$, and $C(t) = V(F(t))$. We claim that $C(t)$ is $\vee$-closed. Let $\{m_i\}$ be a chain from $C(t)$ that is bounded above by $m' \in \mathcal{M}^+(E)$. If we define $d(x) = \bigvee_i m_i(x)$, according to Lemma 4.3, we have $d = m_0 := \bigvee_i m_i$ and $\{m_i(1)\} \uparrow d(1)$. Since $\mathcal{J}(E)$ is an Abelian Dedekind complete ℓ-group, we have $\{m_0 - m_i\} \downarrow 0$. Therefore, $\{(m_0 - m_i)(1)\} = \{m_0(1) - m_i(1)\} \downarrow 0$. Thus given $\epsilon > 0$, there is an index i_0 such that, for each m_i with $m_i \ge^+ m_{i_0}$, we have $m_0(1) - m_i(1) < \epsilon/2$.

Fix the index i_0. Given $\epsilon > 0$, there is $\delta > 0$ such that $t(a) < \delta$ implies $m_{i_0}(a) < \epsilon/2$.

Calculate,

$$m_0(a) = m_{i_0}(a)+(m_0-m_{i_0})(a) \le m_{i_0}(a)+(m_0-m_{i_0})(1) \le \epsilon/2+(m_0-m_{i_0})(1) < \epsilon$$

that gives $m_0 \ll_\epsilon t$ and $m_0 \in C(t)$.

Applying the general result from Theorem 3.4, we have the existence and uniqueness of (4.9).

Now we show that $m_2 \wedge t = 0$. Let k be a measure on E such that $k \le^+ m_2$ and $k \le^+ t$. Then $k \ll_\epsilon t$, so that $k = 0$ and whence $m_2 \wedge t = 0$. □

Theorem 4.9. *Let E be a pseudo effect algebra with* (RDP)*. Let t be a fixed measure on E. Then every measure m on E can be uniquely expressed in the form*

$$m = m_1 + m_2, \tag{4.11}$$

where m_1 and m_2 are finitely additive measures on E such $m_1 \ll t$ and if m' is any measure on E such that $m' \ll t$ and $m' \le^+ m_2$, then $m' = 0$.

Proof. Let us define $V(t) = \{m \in \mathcal{M}^+(E) : m \ll t\}$ and $C(t) := \{s \in \mathcal{S}(E) : s \ll t\}$. Then $F(t)$ is a face of $\mathcal{S}(E)$ and it generates $V(t) = V(F(t))$. Assume that $\{m_i\}$ is a chain from $V(t)$ that is bounded in $\mathcal{J}(E)$. If we set $d(x) = \bigvee_i m_i(x)$ for each $x \in E$, then according to (4.4), $d = m_0 := \bigvee_i m_i$. Therefore, if $t(a) = 0$, then $m_i(a) = 0$ for each i so that $m(a) = d(a) = 0$ and $m_0 \in V(t)$.

The existence and uniqueness of (4.11) follows from Theorem 3.4. □

Finally, we show a relation among two types of continuity of measures.

Proposition 4.10. *Let s_1 and s_s be two σ-additive measures on a σ-complete MV-algebra E. Then $s_1 \ll s_2$ if and only if $s_1 \ll_\epsilon s_2$.*

Proof. It is clear that $s_1 \ll_\epsilon s_2$ entails $s_1 \ll s_2$. Now let us suppose $s_1 \ll s_2$ and let (ad absurdum) $s_1 \not\ll_\epsilon s_2$. Then there is an $\epsilon > 0$ such that for each $n \ge 1$, there is an $a_n \in E$ such that $s_2(a_n) < 1/2^n$ and $s_1(a_n) \ge \epsilon$. Set $a = \bigwedge_{n=1}^\infty \bigvee_{n=k}^\infty a_k$. Then

$$s_2(a) \le s_2(a_n \vee a_{n+1} \vee \cdots) \le \sum_{k=n}^\infty s_2(a_k) < 1/2^{n-1},$$

so that $s_2(a) = 0$.

On the other hand, $s_1(a) = \lim_n s_1(a_n \vee a_{n+1} \vee \cdots) \ge \limsup_n s_1(a_n) \ge \epsilon$ that contradicts $s_1 \ll s_2$. □

We say that a measure t on E is *Jauch-Piron* if $t(a) = t(b) = 0$ entails there is an element $c \in E$ such that $a, b \le c$ and $m(c)$. Every measure on an MV-algebra is Jauch-Piron.

Let E be a pseudo effect with (RDP). According to [Dvu, Thm 3.2], we say that an element $a \in E$ is said to be *central* or *Boolean*, if $a \wedge a^- = 0$. Then also $a \wedge a^\sim = 0$ and $a^- = a^\sim$. Moreover, for any $x \in E$, $x \wedge a$ is defined in E, and

$$x = (x \wedge a) + (x \wedge a^-).$$

Let $C(E)$ be the set of all central elements of E. Then it is a Boolean algebra that is a subalgebra of E. If, in addition, E is monotone σ-complete, then $C(E)$ is a Boolean σ-algebra, [Dvu, Thm 5.11]. We recall that a pseudo effect algebra E is said to be *monotone σ-complete* provided, for any sequence $\{a_n\}$ from E such that $a_n \le a_{n+1}$ for each $n \ge 1$, $a = \bigvee_n a_n$ is defined in E.

If a,b are central elements and t is a measure, then $t(a)+t(b) = t(a\wedge b)+t(a\vee b)$, so that if, in addition, $t(a) = t(b) = 0$, then $t(a \vee b) = 0$.

Now we present another Lebesgue type of decomposition for measures. For two measures m and t on E we write $m \ll_C t$ provided $t(a) = 0$ for $a \in C(E)$ implies $m(a) = 0$. It is a weaker form of $m \ll t$.

Theorem 4.11. *Let E be a monotone σ-complete pseudo effect algebra with* (RDP) *and let t be a σ-additive measure on E. Every σ-additive measure m on E can be uniquely decomposed in the form*

$$m = m_1 + m_2 \tag{4.12}$$

such that m_1, m_2 are σ-additive measures, $m_1 \ll_C t$, and $m_2 \perp t$.

In particular, every σ-additive measure s can be uniquely decompose in the form

$$s = \lambda s_1 + (1-\lambda)s_2, \tag{4.13}$$

where s_1 and s_2 are σ-additive measures on E such that $s_1 \ll t$ and $s_2 \perp t$.

Proof. Existence: Let $\mathrm{Ker}(t)_C = \{a \in C(E) : t(a) = 0\}$. The zero element 0 is central, so that $0 \in \mathrm{Ker}(t)_C$. We order the elements of $\mathrm{Ker}(t)_C$ by $a \preceq b$ iff $a,b \in \mathrm{Ker}(t)_C$ and $m(a) \le m(b)$. Then $\preceq$ is a partial order and now let $\{a_i\}$ be a chain of elements from $\mathrm{Ker}(t)_C$ with respect to $\preceq$, and let $\delta = \sup_i m(a_i)$. Then either there is an upper bound a of $\{a_i\}$ itself or there is a sequence $\{a_n\}$ in $\{a_i\}$ such that $m(a_n) < m(a_{n+1}) \nearrow \delta$. Set $a = \bigvee_n a_n$; then $a \in \mathrm{Ker}(t)_C$ and $m(a) = \lim_n m(a_n) = \delta$, so that a is an upper bound in $\mathrm{Ker}(t)_C$ for the chain $\{a_i\}$. Applying the Zorn Lemma, $\mathrm{Ker}(t)_C$ contains a maximal element, say a_0.

Set $m_1(x) = m(x \wedge a_0^-)$ and $m_2(x) = m(x \wedge a_0)$ for each $x \in E$. Since $C(E)$ is a Boolean σ-algebra, and $x = (x \wedge a_0^-) + (x \wedge a_0)$ for each $x \in E$, we see that m_1 and m_2 are σ-additive measures on E such that $m = m_1 + m_2$. Since $t(a_0) = 0$, then $m_2(a_0^-) = m(a_0^- \wedge a_0) = 0$ so that $m_2 \perp t$. We assert that $m_1 \ll_C t$. If not, there is an element $a \in \mathrm{Ker}(t)_C$ such that $m_1(a) = m(a \wedge a_0^-) > 0$. Since $a_0 \le a_0 \vee (a \wedge a_0^-) \in \mathrm{Ker}(t)_C$ which contradicts the maximality of a_0. This proves that $m_1 \ll_C t$.

Uniqueness: Let $F(t)_\sigma$ be the set of all σ-additive states s on E such that $s \ll_C t$. Then $F(t)_\sigma$ is a face and due to Theorems 4.4–4.5, if $\{m_i\}$ is a bounded chain from $V(F(t)_\sigma)$, then $m_0 = \bigvee_i m_i$ is a σ-additive measure, and in view of (4.4), $m_0 \ll_C t$. This gives that $V(F(t)_\sigma)$ is a $\vee$-closed cone. Now let m' be an arbitrary σ-additive measure from $V(F(t)_\sigma)$ such that $m' \le^+ m_2$. Then $m'(a_0) = 0$ while $t(a_0)$ and $m'(a_0^-) \le m_1(a_0^-) = 0$ so that $m' = 0$. Therefore, $m_1 \in V(F(t)'_\sigma)$ so that by Theorem 3.4 we have that the decomposition (4.12) is unique. □

Corollary 4.12. *Let X be any subset of a pseudo effect algebra with* (RDP)*, and let $F = \{s \in \mathcal{S}(E) : X \subseteq \mathrm{Ker}(s)\}$. Every measure m on E can be uniquely decomposed in the form*

$$m = m_1 + m_2,$$

where $m_1 \in V(F)$ and m_2 is $V(F)$-singular.

In particular, every state s on E can be uniquely expressed as a convex combination

$$m = \lambda s_1 + (1-\lambda)s_2,$$

where $s_1 \in F$ and $s_2 \in F'$.

Proof. Since $V(F) = \{m \in \mathcal{M}^+(E) : X \subseteq \mathrm{Ker}(m)\}$ is by (4.4) $\vee$-closed, the statements follow from Theorem 3.4. □

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