WELL-POSEDNESS AND STABILITY OF THE PERIODIC NONLINEAR WAVES INTERACTIONS FOR THE BENNEY SYSTEM

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Abstract. We establish local well-posedness results in weak periodic function spaces for the Cauchy problem of the Benney system. The Sobolev space $H^{1/2}\times L^2$ is the lowest regularity attained and also we cover the energy space $H^1\times L^2$, where global well-posedness follows from the conservation laws of the system. Moreover, we show the existence of smooth explicit family of periodic travelling waves of *dnoidal* type and we prove, under certain conditions, that this family is orbitally stable in the energy space.

1. Introduction

In this paper we consider the system introduced by Benney in [9] which models the interaction between short and long waves, for example in the theory of resonant water wave interaction in nonlinear medium:

$$\begin{cases} iu_t + u_{xx} = uv + \beta |u|^2 u, & (x,t)\in \mathcal{M}\times \triangle T \\ v_t = (|u|^2)_x, \\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), \end{cases} \tag{1.1}$$

where $u=u(x,t)$ is a complex valued function representing the enveloped of short waves, and $v=v(x,t)$ is a real valued function representing the long wave. Here β is a real parameter, $\triangle T$ is the time interval $[0,T]$ and $\mathcal{M}$ is the real line $\mathbb{R}$ or the one dimensional torus $\mathbb{T}=\mathbb{R}/\mathbb{Z}$.

We let $H^s(\mathcal{M})$ by denoting the classical Sobolev space with the norm

$$\|f\|_s = \left(\int_{\mathcal{M}} (1+|x|)^{2s} |\hat{f}(x)|^2 dx\right)^{\frac{1}{2}}$$

where $\hat{f}$ denotes the Fourier transform operator and we consider the initial data (u_0,v_0) in the space $H^r(\mathcal{M})\times H^s(\mathcal{M})$ with the induced norm

$$\|(f,g)\|_{r\times s} := \|f\|_r + \|g\|_s.$$

The system (1.1) has the following conservation laws:

$$E_1[u(.,t)] = \int_{\mathcal{M}} |u(x,t)|^2 dx, \tag{1.2}$$

$$E_2[u(.,t),v(.,t)] = \int_{\mathcal{M}} \left[v(x,t)|u(x,t)|^2 + |u_x(x,t)|^2 + \tfrac{\beta}{2}|u(x,t)|^4 \right] dx \tag{1.3}$$

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and

$$E_3[u(.,t),v(.,t)] = \int_{\mathcal{M}} \Big[|v(x,t)|^2 + 2\text{Im}\,(u(x,t)\bar{u}_x(x,t))\Big]dx, \tag{1.4}$$

being the natural energy space $H^1(\mathcal{M}) \times L^2(\mathcal{M})$.

We are interesting in two important aspects concerning the Cauchy problem (1.1), which are:

- well-posedness results in the spaces $H^r(\mathcal{M}) \times H^s(\mathcal{M})$ with low regularities;
- existence and stability of periodic traveling waves.

As follows we define the concepts of well-posedness and the stability that will be use in this work.

Definition 1.1 (Well-posedness and Ill-posedness)**.** *We say that the system (1.1) is locally well-posed, in time, in the space $H^r(\mathcal{M}) \times H^s(\mathcal{M})$ if the following conditions hold:*

(a) *for every (u_0, v_0) in the space $H^r(\mathcal{M}) \times H^s(\mathcal{M})$ there exists a positive time $T = T(\|u_0\|_r, \|v_0\|_s)$ and a distributional solution $(u, v) : \mathcal{M} \times \triangle T \longrightarrow \mathbb{C} \times \mathbb{R}$ which is in the space $C(\triangle T;\ H^r(\mathcal{M}) \times H^s(\mathcal{M}))$;*

(b) *the data-solution mapping $(u_0, v_0) \longmapsto (u, v)$ is uniformly continuous from $H^{r\times s}(\mathcal{M})$ to $C(\triangle T;\ H^r(\mathcal{M}) \times H^s(\mathcal{M}))$;*

(c) *there is an additional Banach space $\mathcal{X}$ such that (u, v) is the unique solution to the Cauchy problem in $\mathcal{X} \cap C(\triangle T;\ H^r(\mathcal{M}) \times H^s(\mathcal{M}))$.*

Moreover, we say that the problem is ill-posed if, at least, one of the above conditions fails.

When $\mathcal{M} = \mathbb{R}$ the local well-posedness for (1.1) for data $(u_0, v_0) \in H^{(s+1/2)}(\mathbb{R}) \times H^s(\mathbb{R})$ for indices $s \geq 0$ was established in the works [7], [15] and [23]. Furthermore, in [23] also was proved global well-posedness in $H^{(s+1/2)}(\mathbb{R}) \times H^s(\mathbb{R})$ for $s = 0$ if $\beta = 0$ and for $s \in \mathbb{Z}^+$ and any real β by using the conservation laws

Recently, Corcho [14] showed that for $\beta < 0$ (focusing case) and for data $(u_0, v_0) \in H^r(\mathbb{R}) \times H^s(\mathbb{R})$, with $0 \leq 3r + 1 < 1$ and $r(2s + 3) + 1 \geq 0$, this problem is ill-posed in the following sense: the data-solution mapping fails to be uniformly continuous on bounded sets of $H^r(\mathbb{R}) \times H^s(\mathbb{R})$.

Concerning to the existence and stability of solitary waves solutions for (1.1) of the general form

$$\begin{cases} u(x,t) = e^{i\omega t} e^{ic(x-ct)/2} \phi_s(x-ct), \\ v(x,t) = \psi_s(x-ct), \end{cases} \tag{1.5}$$

where $\phi_s, \psi_s : \mathbb{R} \to \mathbb{R}$ are smooth, $c > 0$, $\omega \in \mathbb{R}$, and $\phi_s(\xi), \psi_s(\xi) \to 0$ as $|\xi| \to \infty$, Laurençot in [20] studied for $\beta = 0$, the nonlinear stability of the orbit

$$\Omega_{(\Phi,\Psi)} = \{(e^{i\theta}\Phi(\cdot + x_0), \Psi(\cdot + x_0));\ (\theta, x_0) \in [0, 2\pi) \times \mathbb{R}\},$$

in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ by the flow generated by (1.1). Here we have that $\Phi(\xi) = e^{ic\xi/2}\phi_s(\xi)$, $\Psi(\xi) = \psi_s(\xi)$, and

$$\phi_s(\xi) = \sqrt{2c\sigma}\,\text{sech}(\sqrt{\sigma}\xi), \qquad \psi_s(\xi) = -\frac{1}{c}\phi_s^2(\xi) \tag{1.6}$$

$\sigma = \omega - \frac{c^2}{4} > 0$.

In this work we focus the attention on the problems of well-posedness and existence and nonlinear stability of periodic travelling waves. For the periodic initial values problem the data (u_0, v_0) will belong to the space $H^r(\mathbb{T}) \times H^s(\mathbb{T})$, also denoted by $H^r_{per} \times H^s_{per}$. Before stating our well and ill posedness results we will give some useful notations. Let η be a function in $C_0^\infty(\mathbb{R})$ such that $0 \leq \eta(t) \leq 1$,

$$\eta(t) = \begin{cases} 1 & \text{if } |t| \leq 1, \\ 0 & \text{if } |t| \geq 2, \end{cases}$$

and $\eta_\delta(t) = \eta(\frac{t}{\delta})$. We denote by $\lambda\pm$ a number slightly larger, respectively smaller, than λ and by $\langle\cdot\rangle$, $\langle\xi\rangle = 1+|\xi|$. The characteristic function on the set A is denoted by χ_A. Furthermore, we will work with the auxiliary periodic Bourgain space $X^{s,b}_{per}$ defined as follows: first we denote by $\mathcal{X}$ the space of functions $f : \mathbb{T}\times\mathbb{R} \to \mathbb{C}$ such that

(i) $f(x,\cdot) \in \mathcal{S}(\mathbb{R})$ for each $x \in \mathbb{T}$;
(ii) $f(\cdot,t) \in C^\infty(\mathbb{T})$ for each $t \in \mathbb{R}$.

For $s, b \in \mathbb{R}$, the spaces $H^b_t H^s_{per}$ and $X^{s,b}_{per}$ are the completion of $\mathcal{X}$ with respect to the norms

$$\|f\|_{H^b_t H^s_{per}} = \left(\sum_{n\in\mathbb{Z}} \int_{-\infty}^{+\infty} (1+|n|)^{2s}(1+|\tau|)^{2b}|\widehat{f}(n,\tau)|^2 d\tau \right)^{\frac{1}{2}} \tag{1.7}$$

and

$$\begin{aligned} \|f\|_{X^{s,b}_{per}} &= \|S(-t)f\|_{H^b_t H^s_{per}} \\ &= \left(\sum_{n\in\mathbb{Z}} \int_{-\infty}^{+\infty} (1+|n|)^{2s}(1+|\tau+n^2|)^{2b}|\widehat{f}(n,\tau)|^2 d\tau \right)^{\frac{1}{2}}, \end{aligned} \tag{1.8}$$

respectively, where $S(t) := e^{it\partial_x^2}$ is the corresponding Schrödinger generator (unitary group) associated to the linear problem,

$$\begin{cases} iu_t + u_{xx} = 0 \\ u(x,0) = g_(x). \end{cases} \tag{1.9}$$

For any $r, s \in \mathbb{R}$ and $b_1, b_2 > 1/2$, we have the embedding $X^{r,b_1}_{per} \hookrightarrow C\left(\mathbb{R}; H^r_{per}\right)$ and $H^{b_2}_t H^s_{per} \hookrightarrow C\left(\mathbb{R}; H^s_{per}\right)$. For the case $b = 1/2$ the embedding can be guaranteed by considering the following slightly modifications of the Bourgain spaces:

$$\|f\|_{X^r_{per}} := \|f\|_{X^{r,1/2}_{per}} + \|\langle n\rangle^r \widehat{f}(n,\tau)\|_{\ell^2_n L^1_\tau} \tag{1.10}$$

and

$$\|f\|_{Y^s_{per}} := \|f\|_{H^{1/2}_t H^s_{per}} + \|\langle n\rangle^s \widehat{f}(n,\tau)\|_{\ell^2_n L^1_\tau} \tag{1.11}$$

Concerning local well-posedness we obtain the following result:

Theorem 1.2 (Local Well-Posedness)**.** *For any $(u_0, v_0) \in H^r_{per} \times H^s_{per}$ provided the conditions:*

$$\max\{0,\, r-1\} \leq s \leq \min\{r,\, 2r-1\}, \tag{1.12}$$

there exist a positive time $T = T\left(\|u_0\|_r, \|v_0\|_s\right)$ and a unique solution $(u(t), v(t))$ of the initial value problem (1.1), satisfying

(a) $(\eta_T(t)u, \eta_T(t)v) \in X^r_{per} \times Y^s_{per}$;

(b) $(u, v) \in C\left(\triangle T;\ H^r_{per} \times H^s_{per}\right)$.

Moreover, the map $(u_0, v_0) \longmapsto (u(t), v(t))$ *is locally uniformly continuous from* $H^r_{per} \times H^s_{per}$ *into* $C\left(\triangle T;\ H^r_{per} \times H^s_{per}\right)$.

The proof of Theorem 1.2 is based on the Banach fixed point theorem applied on the integral formulation of the system combined with new sharp periodic bilinear estimates, in adequate mixed Bourgain spaces $X^{r,b_1}_{per} \times H^{b_2}_t H^s_{per}$, for the coupling terms uv and $\partial_x(|u|^2)$.

Also we find a region which the Cauchy problem is not locally well-posed, more precisely we prove the following theorem:

Theorem 1.3. *Let $\beta \neq 0$. Then for any $r < 0$ and $s \in \mathbb{R}$, the initial value problem (1.1) is locally ill-posed for data in $H^r_{per} \times H^s_{per}$.*

Regarding the stability of periodic travelling waves, namely, solutions for (1.1) of the form

$$\begin{cases} u(t,x) = e^{-i\omega t} e^{ic(x-ct)/2} \varphi_{\omega,c}(x - ct) \\ v(x,t) = n_{\omega,c}(x - ct) \end{cases} \tag{1.13}$$

where $\varphi_{\omega,c}$, $n_{\omega,c}$ are real smooth, L-periodic functions, $c > 0$, and $\omega < 0$, we have the following definition.

Definition 1.4 (Non-Linear Stability). *The periodic traveling wave $\Phi(\xi) = e^{ic\xi/2}\varphi_{\omega,c}(\xi)$, $\Psi(\xi) = n_{\omega,c}(\xi)$, is orbitally stable in $H^1_{per}([0, L]) \times L^2_{per}([0, L])$ if for all $\varepsilon > 0$, there exists $\delta > 0$, such that if $||(u_0, v_0) - (\Phi, \Psi)||_{H^1_{per} \times L^2_{per}} < \delta$ and $(u(t), v(t))$ is the solution of (1.1) with $(u(0), v(0)) = (u_0, v_0)$, then*

$$\inf_{s\in[0,2\pi)} \inf_{r\in\mathbb{R}} ||(u(t), v(t)) - (e^{is}\Phi(\cdot + r), \Psi(\cdot + r))||_{H^1_{per} \times L^2_{per}} < \varepsilon, \qquad t \in \mathbb{R}.$$

Otherwise (Φ, Ψ) is called orbitally unstable.

We will show below that there exist a smooth explicit family of profiles solutions of minimal period L,

$$(\omega, c) \in \mathcal{A}_\beta \to (\varphi_{\omega,c}, n_{\omega,c}) \in H^n_{per}([0, L]) \times H^m_{per}([0, L]),$$

where $\mathcal{A}_\beta = \{(x, y) : y > 0, 1 > \beta y, \ \ and \ \ x < -\frac{2\pi^2}{L^2} - \frac{y^2}{4}\}$ and which depends of the Jacobian elliptic function dn called *dnoidal*, more precisely,

$$\begin{cases} \varphi_{\omega,c}(\xi) = \sqrt{\frac{c}{1-\beta c}}\, \eta_1 dn\left(\frac{\eta_1}{\sqrt{2}}\xi; \kappa\right) \\ \\ n_{\omega,c}(\xi) = -\frac{\eta_1^2}{1-\beta c} dn^2\left(\frac{\eta_1}{\sqrt{2}}\xi; \kappa\right) \end{cases} \tag{1.14}$$

with $\eta_1 = \eta_1(\omega, c)$ and $\kappa = \kappa(\omega, c)$, being smooth functions of ω and c.

So, by following Angulo [4] and Grillakis *et al.* [16], [17], we obtain the following stability theorem.

Theorem 1.5 (Stability Theory). *Let $(\omega, c) \in \mathcal{A}_\beta$ such that for $c > 0$ there is $q \in \mathbb{N}$ satisfying $4\pi q/c = L$. Define $\sigma \equiv -\omega - \frac{c^2}{4}$. Then $\Phi(\xi) = e^{ic\xi/2}\varphi_{\omega,c}(\xi)$, $\Psi(\xi) = n_{\omega,c}(\xi)$, with $\varphi_{\omega,c}, n_{\omega,c}$ given in (1.14), is orbitally stable in $H^1_{per}([0, L]) \times L^2_{per}([0, L])$ by the periodic flow generated by (1.1):*

(a) *for* $\beta \leq 0$,
(b) *for* $\beta > 0$ *and* $8\beta\sigma - 3c(1-\beta c)^2 \leq 0$.

2. Local theory

We prove Theorem 1.2 using the standard technique, that is: we use the Duhamel integral formulation for the system (1.1) combined with the Banach fixed point theorem in adequate Bourgain spaces $X^r_{per} \times Y^s_{per}$ with the objective to get the desired solution. The main difficulty is the necessity to prove two news mixed periodic bilinear estimates, which we will prove in the following sections.

2.1. **Sharp Periodic Bilinear Estimates.**

We begin recalling the following elementary inequalities, which will be used in the proof of the next main estimates.

Lemma 2.1. *Let $\theta_1, \theta_2 > 0$ with $\theta_1 + \theta_2 > 1$ and $\lambda > 1/2$. Then, there are a positive constants C_1 and C_2 such that*

(a) $\int_{-\infty}^{+\infty} \frac{dx}{\langle x-a\rangle^{\theta_1}\langle x-b\rangle^{\theta_2}} \leq \frac{C_1}{\langle a-b\rangle^{\mu}}$, *where* $\mu := \min\{\theta_1, \theta_2, \theta_1+\theta_2-1\}$*;*

(b) $\sum_{n\in\mathbb{Z}} \frac{1}{\langle n^2+an+b\rangle^{\lambda}} \leq C_2$, *with* a, $b \in \mathbb{R}$.

Proof. For details of the proof we can see, for instance, the works [18] and [6]. □

Lemma 2.2. *Let $0 < \theta < 1/4$. Then, the following estimates*

$$\|uv\|_{X^{r,-1/2}_{per}} \lesssim \|u\|_{X^{r,1/2-\theta}_{per}} \|v\|_{H^{1/2}_t H^s_{per}} + \|u\|_{X^{r,1/2}_{per}} \|v\|_{H^{1/2-\theta}_t H^s_{per}} \tag{2.15}$$

$$\left\| \langle n\rangle^r \frac{\widehat{uv}(n,\tau)}{\langle \tau+n^2\rangle} \right\|_{\ell^2_n L^1_\tau} \lesssim \|u\|_{X^{r,1/2-\theta}_{per}} \|v\|_{H^{1/2}_t H^s_{per}} + \|u\|_{X^{r,1/2}_{per}} \|v\|_{H^{1/2-\theta}_t H^s_{per}} \tag{2.16}$$

hold provided $r \geq 0$ *and* $\max\{0, r-1\} \leq s$.

Proof. First we prove (2.15). We define $f(n,\tau) := \langle \tau+n^2\rangle^{b_1}\langle n\rangle^r \widehat{u}(n,\tau)$ and $g(n,\tau) := \langle \tau\rangle^{b_2}\langle n\rangle^s \widehat{v}(n,\tau)$. Then, using duality arguments we obtain

$$\|uv\|_{X^{r,-1/2}_{per}} = \sup\big\{W(\varphi) : \|\varphi\|_{\ell^2_n L^2_\tau} \leq 1\big\},$$

where,

$$W(\varphi) = \sum_{(n,n_1)\in\mathbb{Z}^2} \int_{\mathbb{R}^2} \frac{\langle \tau+n^2\rangle^{-1/2}\langle n\rangle^r f(n_1,\tau_1) g(n-n_1,\tau-\tau_1)\varphi(n,\tau)}{\langle \tau_1+n_1^2\rangle^{b_1}\langle \tau-\tau_1\rangle^{b_2}\langle n_1\rangle^r\langle n-n_1\rangle^s} d\tau d\tau_1. \tag{2.17}$$

We will divide the space $\mathbb{Z}^2\times\mathbb{R}^2$ in three regions, namely $\mathbb{Z}^2\times\mathbb{R}^2 = A_0\cup A_1\cup A_2$ and we separate the integral W as follows:

$$W(\varphi) = W_0(\varphi) + W_1(\varphi) + W_2(\varphi), \tag{2.18}$$

where

$$W_j(\varphi) = \sum\int\sum\int_{(n,n_1,\tau,\tau_1)\in A_j} \frac{\langle \tau+n^2\rangle^{-1/2}\langle n\rangle^r f(n_1,\tau_1) g(n-n_1,\tau-\tau_1)\varphi(n,\tau)}{\langle \tau_1+n_1^2\rangle^{b_1}\langle \tau-\tau_1\rangle^{b_2}\langle n_1\rangle^r\langle n-n_1\rangle^s},$$

for $j = 0, 1, 2$. It is easy to see that to obtain (2.15) is suffices to prove that whenever $r,\ s \geq 0$ and $r - s \leq 1$ the estimate

$$W_j(\varphi) \lesssim \|f\|_{\ell^2_n L^2_\tau} \|g\|_{\ell^2_n L^2_\tau} \|\varphi\|_{\ell^2_n L^2_\tau} = \|u\|_{X^{b_1,r}_{per}} \|v\|_{H^{b_2}_t H^s_{per}} \|\varphi\|_{\ell^2_n L^2_\tau}, \tag{2.19}$$

holds with $b_1 = 1/2 - \theta$ and $b_2 = 1/2$ or with $b_1 = 1/2$ and $b_2 = 1/2 - \theta$. Indeed, next we will prove the following estimates:

$$\begin{aligned} &W_j(\varphi) \lesssim \|u\|_{X^{1/2}_{per}} \|v\|_{H^{1/2-\theta}_t H^s_{per}} \|\varphi\|_{\ell^2_n L^2_\tau}, \text{ for } j = 0, 1,\\ &W_2(\varphi) \lesssim \|u\|_{X^{1/2-\theta,r}_{per}} \|v\|_{H^{1/2}_t H^s_{per}} \|\varphi\|_{\ell^2_n L^2_\tau}. \end{aligned} \tag{2.20}$$

For this purpose, in region A_0 we integrate first over (n_1, τ_1), in region A_1 we integrate first over (n, τ) and in region A_2 we integrate first over $(n_2, \tau_2) = (n - n_1, \tau - \tau_1)$; then using Cauchy-Schwarz inequality we easily see that it remains only to uniformly bound the following three expressions:

$$\widetilde{W}_0 := \sup_{n,\tau} \frac{\langle n\rangle^{2r}}{\langle \tau + n^2\rangle} \sum_{n_1} \int\limits_{A_0} \frac{d\tau_1}{\langle \tau_1 + n_1^2\rangle \langle \tau_2\rangle^{1-2\theta} \langle n_1\rangle^{2r} \langle n_2\rangle^{2s}} \tag{2.21}$$

$$\widetilde{W}_1 := \sup_{n_1,\tau_1} \frac{1}{\langle n_1\rangle^{2r} \langle \tau_1 + n_1^2\rangle} \sum_{n} \int\limits_{A_1} \frac{\langle n\rangle^{2r} d\tau}{\langle \tau + n^2\rangle \langle \tau_2\rangle^{1-2\theta} \langle n_2\rangle^{2s}} \tag{2.22}$$

$$\widetilde{W}_2 := \sup_{n_2,\tau_2} \frac{1}{\langle n_2\rangle^{2s} \langle \tau_2\rangle} \sum_{n} \int\limits_{A_2} \frac{\langle n\rangle^{2r} d\tau}{\langle \tau + n^2\rangle \langle \tau_1 + n_1^2\rangle^{1-2\theta} \langle n_1\rangle^{2r}} \tag{2.23}$$

Now we define the regions A_0, A_1 and A_2. We use the notation

$$\mathcal{L} := \max\Big\{ \left|\tau + n^2\right|, \left|\tau_1 + n_1^2\right|, |\tau_2| \Big\}. \tag{2.24}$$

and first we introduce the subsets:

$$\begin{aligned} &A_{0,1} := \left\{ (n, n_1, \tau, \tau_1) \in \mathbb{Z}^2 \times \mathbb{R}^2 : |n| \leq 100 \right\},\\ &A_{0,2} := \left\{ (n, n_1, \tau, \tau_1) \in \mathbb{Z}^2 \times \mathbb{R}^2 : |n| > 100 \text{ and } |n| \leq 2|n_1| \right\},\\ &A_{0,3} := \Big\{ (n, n_1, \tau, \tau_1) \in \mathbb{Z}^2 \times \mathbb{R}^2 : |n| > 100, |n_1| < |n|/2 \text{ and } \mathcal{L} = \left|\tau + n^2\right| \Big\}. \end{aligned} \tag{2.25}$$

Then, we put

$$\begin{aligned} &A_0 := A_{0,1} \cup A_{0,2} \cup A_{0,3},\\ &A_1 := \Big\{ (n, n_1, \tau, \tau_1) \in \mathbb{Z}^2 \times \mathbb{R}^2 : |n| > 100, |n_1| < |n|/2 \text{ and } \mathcal{L} = \left|\tau_1 + n_1^2\right| \Big\},\\ &A_2 := \Big\{ (n, n_1, \tau, \tau_1) \in \mathbb{Z}^2 \times \mathbb{R}^2 : |n| > 100, |n_1| < |n|/2 \text{ and } \mathcal{L} = |\tau_2| \Big\}. \end{aligned} \tag{2.26}$$

For later use, we recall that the dispersive relation of this bilinear estimate is:

$$\tau + n^2 - (\tau_1 + n_1^2) - \tau_2 = n^2 - n_1^2, \tag{2.27}$$

where $\tau - \tau_1 = \tau_2 \quad$ and $\quad n - n_1 = n_2$.

We begin with the analysis of (2.21). In the region $A_{0,1}$, using that $|n| \lesssim 1$ and $r,\ s \geq 0$ we have

$$\begin{aligned}
\widetilde{W}_{0,1} &:= \sup_{n,\tau} \frac{\langle n\rangle^{2r}}{\langle \tau + n^2\rangle} \sum_{n_1} \int\limits_{A_{0,1}} \frac{d\tau_1}{\langle \tau_1 + n_1^2\rangle \langle \tau_2\rangle^{1-2\theta} \langle n_1\rangle^{2r} \langle n_2\rangle^{2s}} \\
&\lesssim \sup_{n,\tau} \frac{1}{\langle \tau + n^2\rangle} \sum_{n_1} \int\limits_{-\infty}^{+\infty} \frac{d\tau_1}{\langle \tau_1 + n_1^2\rangle \langle \tau_2\rangle^{1-2\theta} \langle n_1\rangle^{2r} \langle n_2\rangle^{2s}} \qquad (2.28) \\
&\lesssim \sup_{\tau} \sum_{n_1} \frac{1}{\langle \tau + n_1^2\rangle^{1-2\theta}} \lesssim 1,
\end{aligned}$$

where in the last inequality we have used that $0 < \theta < 1/4$ combined with Lemma 2.1.

In the region $A_{0,2}$, we have that $\langle n\rangle^{2r} \lesssim \langle n_1\rangle^{2r}$. Thus, similarly to the previous case, we get

$$\begin{aligned}
\widetilde{W}_{0,2} &:= \sup_{n,\tau} \frac{\langle n\rangle^{2r}}{\langle \tau + n^2\rangle} \sum_{n_1} \int\limits_{A_{0,2}} \frac{d\tau_1}{\langle \tau_1 + n_1^2\rangle \langle \tau_2\rangle^{1-2\theta} \langle n_1\rangle^{2r} \langle n_2\rangle^{2s}} \\
&\lesssim \sup_{n,\tau} \frac{1}{\langle \tau + n^2\rangle} \sum_{n_1} \int\limits_{-\infty}^{+\infty} \frac{d\tau_1}{\langle \tau_1 + n_1^2\rangle \langle \tau_2\rangle^{1-2\theta} \langle n_2\rangle^{2s}} \qquad (2.29) \\
&\lesssim \sup_{\tau} \sum_{n_1} \frac{1}{\langle \tau + n_1^2\rangle^{1-2\theta}} \lesssim 1.
\end{aligned}$$

In the region $A_{0,3}$ we have that $|n_1| < |n|/2$ and $|n| > 100$, which imply that $|n - n_1| \sim |n + n_1| \sim |n|$. Moreover, the dispersive relation (2.27) says that

$$\mathcal{L} = |\tau + n^2| \gtrsim |n^2 - n_1^2| = |n - n_1||n + n_1| \sim |n|^2.$$

Therefore,

$$\begin{aligned}
\widetilde{W}_{0,3} &:= \sup_{n,\tau} \frac{\langle n\rangle^{2r}}{\langle \tau + n^2\rangle} \sum_{n_1} \int\limits_{A_{0,3}} \frac{d\tau_1}{\langle \tau_1 + n_1^2\rangle \langle \tau_2\rangle^{1-2\theta} \langle n_1\rangle^{2r} \langle n_2\rangle^{2s}} \\
&\lesssim \sup_{n,\tau} \frac{\langle n\rangle^{2r-2s}}{\langle \tau + n^2\rangle} \sum_{n_1} \int\limits_{-\infty}^{+\infty} \frac{d\tau_1}{\langle \tau_1 + n_1^2\rangle \langle \tau_2\rangle^{1-2\theta}} \qquad (2.30) \\
&\lesssim \sup_{n,\tau} \frac{\langle n\rangle^{2r-2s}}{\langle n\rangle^{2}} \sum_{n_1} \frac{1}{\langle \tau + n_1^2\rangle^{1-2\theta}} \lesssim 1,
\end{aligned}$$

since $r \geq 0$, $r - s \leq 1$ and $0 < \theta < 1/4$.

Putting together the estimates (2.28), (2.29) and (2.30) we conclude that

$$|\widetilde{W}_0| \leq |\widetilde{W}_{0,1}| + |\widetilde{W}_{0,2}| + |\widetilde{W}_{0,3}| \lesssim 1,$$

obtaining the desired bounded for (2.21).

Next we estimate the contribution of (2.22). In the region A_1, we know that $|n_1| < |n|/2$, $|n| > 100$ and $\mathcal{L} = |\tau_1 + n_1^2|$. So, $|n_2| \sim |n|$ and the dispersive relation (2.27) implies that $|\tau_1+n_1^2|\gtrsim n^2$. Thus,

$$\begin{aligned}\widetilde{W}_1 &= \sup_{n_1,\tau_1}\frac{1}{\langle n_1\rangle^{2r}\langle\tau_1+n_1^2\rangle}\sum_n\int_{A_1}\frac{\langle n\rangle^{2r}}{\langle\tau+n^2\rangle\langle\tau_2\rangle^{1-2\theta}\langle n_2\rangle^{2s}}d\tau\\ &\lesssim \sup_{\tau_1}\sum_n\int_{-\infty}^{+\infty}\frac{\langle n\rangle^{2r-2s-2}}{\langle\tau+n^2\rangle\langle\tau_2\rangle^{1-2\theta}}d\tau\\ &\lesssim \sup_{\tau_1}\sum_n\frac{1}{\langle\tau_1+n^2\rangle^{1-2\theta}}\lesssim 1,\end{aligned}$$

since $r\geq 0$, $r-s\leq 1$ and $0<\theta<1/4$.

Finally, we bound (2.23) by noting that, in the region A_2 it holds $|n|>100$, $|n_1|<|n|/2$ and $\mathcal{L}=|\tau_2|$. Then, $|n_2|\sim|n|$ and the dispersive relation (2.27) yield that $|\tau_2|\gtrsim n^2$. Using these conditions and that $r\geq 0$, $r-s\leq 1$ we obtain

$$\begin{aligned}\widetilde{W}_2 &= \sup_{n_2,\tau_2}\frac{1}{\langle n_2\rangle^{2s}\langle\tau_2\rangle}\sum_n\int_{A_2}\frac{\langle n\rangle^{2r}}{\langle\tau+n^2\rangle\langle\tau_1+n_1^2\rangle^{1-2\theta}\langle n_1\rangle^{2r}}d\tau\\ &\lesssim \sup_{n_2,\tau_2}\sum_n\int_{-\infty}^{+\infty}\frac{\langle n\rangle^{2r-2s-2}}{\langle\tau+n^2\rangle\langle\tau_1+n_1^2\rangle^{1-2\theta}}d\tau\\ &\lesssim \sup_{n_2,\tau_2}\sum_n\frac{1}{\langle 2n_2(n+\frac{\tau_2}{2n_2}-\frac{n_2}{2})\rangle^{1-2\theta}}\\ &= \sup_{n_2,\tau_2}\left\{\sum_{n\in H_1}\frac{1}{\langle 2n_2(n+\frac{\tau_2}{2n_2}-\frac{n_2}{2})\rangle^{1-2\theta}}+\sum_{n\in H_2}\frac{1}{\langle 2n_2(n+\frac{\tau_2}{2n_2}-\frac{n_2}{2})\rangle^{1-2\theta}}\right\},\end{aligned}$$

where

$$H_1:=\left\{n\in\mathbb{Z}:\ \left|n+\frac{\tau_2}{2n_2}-\frac{n_2}{2}\right|<2\right\}\ \text{ and }\ H_2:=\left\{n\in\mathbb{Z}:\ \left|n+\frac{\tau_2}{2n_2}-\frac{n_2}{2}\right|\geq 2\right\}.$$

Now we note that $\#H_1\leq 4$ and for any $n\in H_2$ we have

$$\langle 2n_2(n+\tfrac{\tau_2}{2n_2}-\tfrac{n_2}{2})\rangle\gtrsim\langle n\rangle\langle n+\tfrac{\tau_2}{2n_2}-\tfrac{n_2}{2}\rangle,$$

since $|n_2|\sim|n|$. Then, by Hölder's inequality

$$\begin{aligned}&\sum_{n\in H_1}\frac{1}{\langle 2n_2(n+\frac{\tau_2}{2n_2}-\frac{n_2}{2})\rangle^{1-2\theta}}+\sum_{n\in H_2}\frac{1}{\langle 2n_2(n+\frac{\tau_2}{2n_2}-\frac{n_2}{2})\rangle^{1-2\theta}}\\ &\quad\leq 4+\sum_{n\in H_2}\frac{1}{\langle n\rangle^{1-2\theta}\langle n+\frac{\tau_2}{2n_2}-\frac{n_2}{2}\rangle^{1-2\theta}}\\ &\quad 4+\left(\sum_n\frac{1}{\langle n\rangle^{2(1-2\theta)}}\right)^{1/2}\left(\sum_n\frac{1}{\langle n+\frac{\tau_2}{2n_2}-\frac{n_2}{2}\rangle^{2(1-2\theta)}}\right)^{1/2}\lesssim 1,\end{aligned}$$

since and $0<\theta<1/4$. This completes the proof of (2.15).

Next, we prove (2.16). We let $a\in(1/2,\ 3/4-\theta)$. By using Cauchy-Schwarz inequality, we have that

$$\left\|\langle n\rangle^r\frac{\widehat{uv}(n,\tau)}{\langle\tau+n^2\rangle}\right\|^2_{\ell^2_nL^1_\tau}\leq\sum_n\langle n\rangle^{2r}\Big\{\int_{-\infty}^{+\infty}\frac{|\widehat{uv}(n,\tau)|^2}{\langle\tau+n^2\rangle^{2(1-a)}}d\tau\int_{-\infty}^{+\infty}\frac{d\tau}{\langle\tau+n^2\rangle^{2a}}\Big\}. \tag{2.31}$$

Now, we separate $\mathbb{Z}^2 \times \mathbb{R}^2$ in the same regions used to estimate (2.15) and we note that, except in the region $A_{0,3}$, the right-hand of (2.1) can be estimated in the same way that (2.15). To see this, we observe that the integral $\int_{-\infty}^{+\infty} \frac{d\tau}{\langle \tau+n^2\rangle^{2a}}$ is convergent and we replace the term $\langle \tau + n^2\rangle$ by $\langle \tau + n^2\rangle^{2(1-a)}$ in (2.21), (2.22) and (2.23), then we follows the same steps to bound the corresponding expressions in each region, using that the condition $2(1-a) + (1-2\theta) - 1 > 1/2$ holds for $a \in (1/2,\ 3/4 - \theta)$.

Now we proceed with the estimate of the right-hand of (2.1) in $A_{0,3}$. Here, by using the fact that $|\tau + n^2| \gtrsim |n|^2$ we have that

$$\int_{A_{0,3}} \frac{d\tau}{\langle \tau + n^2\rangle^{2a}} \lesssim \langle n\rangle^{2(1-2a)}. \tag{2.32}$$

Then, using (2.32), we have

$$\left\| \langle n\rangle^r \frac{\widehat{uv}(n,\tau)}{\langle \tau + n^2\rangle} \chi_{A_{0,3}} \right\|^2_{\ell^2_n L^1_\tau} \lesssim \widetilde{W}_{0,3} \|u\|^2_{X^{r,1/2}_{per}} \|v\|^2_{H_t^{1/2-\theta} H^s_{per}}, \tag{2.33}$$

where

$$\widetilde{W}_{0,3} = \sup_{n,\tau} \frac{\langle n\rangle^{2r} n^{2(1-2a)}}{\langle \tau + n^2\rangle^{2(1-a)}} \sum_{n_1} \int_{A_{0,3}} \frac{d\tau_1}{\langle \tau_1 + n_1^2\rangle \langle \tau_2\rangle^{1-2\theta} \langle n_1\rangle^{2r} \langle n_2\rangle^{2s}}. \tag{2.34}$$

Similarly to the estimate make in (2.30) we obtain

$$\begin{aligned} \widetilde{W}_{0,3} &\lesssim \sup_{n,\tau} \frac{\langle n\rangle^{2r-2s+2-4a}}{\langle \tau + n^2\rangle^{2(1-a)}} \sum_{n_1} \int_{-\infty}^{+\infty} \frac{d\tau_1}{\langle \tau_1 + n_1^2\rangle \langle \tau_2\rangle^{1-2\theta}} \\ &\lesssim \sup_{n,\tau} \frac{\langle n\rangle^{2r-2s+2-4a}}{\langle n\rangle^{4(1-a)}} \sum_{n_1} \frac{1}{\langle \tau + n_1^2\rangle^{1-2\theta}} \lesssim 1, \end{aligned} \tag{2.35}$$

since $0 < \theta < 1/4$ and $r - s \le 1$. Finally, combining (2.32) and (2.35) we get

$$\left\| \langle n\rangle^r \frac{\widehat{uv}(n,\tau)}{\langle \tau + n^2\rangle} \chi_{A_{0,3}} \right\|_{\ell^2_n L^1_\tau} \lesssim \|u\|_{X^{r,1/2}_{per}} \|v\|_{H_t^{1/2-\theta} H^s_{per}},$$

as we desired. Then, we finished the proof of Lemma 2.2. □

The next result shows that the conditions obtained above for indices r and s are necessary.

Proposition 2.3. *For any real numbers b_1 and b_2, the veracity of the inequality*

$$\|uv\|_{X^{r,-1/2}} \lesssim \|u\|_{X^{r,b_1}} \|v\|_{H_t^{b_2} H_x^s}$$

implies that $\max\{0, r-1\} \le s$.

Proof. Firstly, we fix $N \gg 1$ a large integer and define de sequences

$$\alpha_1(n) = \begin{cases} 1 & \text{if } n = N, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \beta_1(n) = \begin{cases} 1 & \text{if } n = -2N, \\ 0 & \text{otherwise.} \end{cases}$$

Let $u_{1_N}(x,t)$ and $v_{1_N}(x,t)$ be given by $\widehat{u}_{1_N}(n,\tau) = \alpha_1(n)\chi_{[-1,1]}(\tau + n^2)$ and $\widehat{v}_{1_N}(n,\tau) = \beta_1(n)\chi_{[-1,1]}(\tau)$. Taking into account the dispersive relation

$$\tau + n^2 - (\tau_1 + n_1^2) - \tau_2 = n^2 - n_1^2,$$

we can easily compute that

$$\|u_{1_N}v_{1_N}\|_{X^{r,-1/2}} \sim N^r, \quad \|u_{1_N}\|_{X^{r,b_1}} \sim N^r \quad \text{and} \quad \|v_{1_N}\|_{H_t^{b_2}H_x^s} \sim N^s$$

Hence, from the bound $\|u_{1_N}v_{1_N}\|_{X^{r,-1/2}} \lesssim \|u_{1_N}\|_{X^{r,b_1}}\|v_{1_N}\|_{H_t^{b_2}H_x^s}$ we must have $N^r \lesssim N^{r+s}$ for $N \gg 1$, which implies that $s \geq 0$.

Secondly, we define the sequences

$$\alpha_2(n) = \begin{cases} 1 & \text{if } n = N, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \beta_2(n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\widehat{u}_{2_N}(n,\tau) = \alpha_2(n)\chi_{[-1,1]}(\tau + n^2)$ and $\widehat{v}_{2_N}(n,\tau) = \beta_2(n)\chi_{[-1,1]}(\tau)$. Again, it is easy to see that

$$\|u_{2_N}v_{2_N}\|_{X^{r,-1/2}} \sim N^{r-1}, \quad \|u_{2_N}\|_{X^{r,b_1}} \sim 1 \quad \text{and} \quad \|v_{2_N}\|_{H_t^{b_2}H_x^s} \sim N^s$$

Hence, the bound $\|u_{2_N}v_{2_N}\|_{X^{r,-1/2}} \lesssim \|u_{2_N}\|_{X^{r,b_1}}\|v_{2_N}\|_{H_t^{b_2}H_x^s}$ implies $N^{r-1} \lesssim N^s$ for $N \gg 1$, so we must have $r - 1 \leq s$.

□

Lemma 2.4. *Let $0 < \theta < 1/4$. Then, the following estimates*

$$\|\partial_x(u\bar{w})\|_{H_t^{-1/2}H_{per}^s} \lesssim \|u\|_{X_{per}^{r,1/2-\theta}}\|w\|_{X_{per}^{r,1/2}} + \|u\|_{X_{per}^{r,1/2}}\|w\|_{X_{per}^{r,1/2-\theta}} \tag{2.36}$$

$$\left\| \langle n\rangle^s \frac{\widehat{\partial_x(u\bar{w})}(n,\tau)}{\langle\tau\rangle} \right\|_{\ell_n^2 L_\tau^1} \lesssim \|u\|_{X_{per}^{r,1/2-\theta}}\|w\|_{X_{per}^{r,1/2}} + \|u\|_{X_{per}^{r,1/2}}\|w\|_{X_{per}^{r,1/2-\theta}} \tag{2.37}$$

hold provided $0 \leq s \leq \min\{2r-1,\ r\}$.

Proof. The proof is similar to Lemma (2.2). Here, the relevant dispersive relation is given by

$$(\tau_1 + n_1^2) + (\tau_2 - n_2^2) - \tau = n_1^2 - n_2^2, \tag{2.38}$$

where $\tau_2 = \tau - \tau_1$ and $n_2 = n - n_1$.

To prove (2.36), by duality arguments, it suffices to bound the following expressions:

$$Z_0 = \sup_{n_1,\tau_1} \frac{1}{\langle n_1\rangle^{2r}\langle\tau_1 + n_1^2\rangle} \sum_n \int_{C_0} \frac{|n|^2\langle n\rangle^{2s}}{\langle\tau\rangle\langle\tau_2 - n_2^2\rangle^{1-2\theta}\langle n_2\rangle^{2r}} d\tau, \tag{2.39}$$

$$Z_1 = \sup_{n,\tau} \frac{|n|^2\langle n\rangle^{2s}}{\langle\tau\rangle} \sum_{n_1} \int_{C_1} \frac{d\tau_1}{\langle\tau_1 + n_1^2\rangle\langle\tau_2 - n_2^2\rangle^{1-2\theta}\langle n_1\rangle^{2r}\langle n_2\rangle^{2r}}, \tag{2.40}$$

$$Z_2 = \sup_{n_2,\tau_2} \frac{1}{\langle n_2\rangle^{2r}\langle\tau_2 - n_2^2\rangle} \sum_n \int_{C_2} \frac{|n|^2\langle n\rangle^{2s}}{\langle\tau\rangle\langle\tau_1 + n_1^2\rangle^{1-2\theta}\langle n_1\rangle^{2r}} d\tau, \tag{2.41}$$

where C_0, C_1 and C_2 are defined as follows. We denote by

$$\mathcal{L} := \max\{|\tau|, |\tau_1 + n_1^2|, |\tau_2 - n_2^2|\}$$

and then we define de following sets:

$C_{0,1} := \{(n,\tau,n_1,\tau_1) : |n| \le 100\},$

$C_{0,2} := \left\{(n,\tau,n_1,\tau_1) : |n| > 100, \frac{|n_2|}{2} \le |n_1| \le 2|n_2|\right\},$

$C_{0,3} := \left\{(n,\tau,n_1,\tau_1) : |n| > 100, |n_1| < \frac{|n_2|}{2} \text{ or } |n_2| < \frac{|n_1|}{2} \text{ and } \mathcal{L} = |\tau_1 + n_1^2|\right\}.$

Now we put

$C_0 := C_{0,1} \cup C_{0,2} \cup C_{0,3},$

$C_1 := \left\{(n,\tau,n_1,\tau_1) : |n| > 100, |n_1| < \frac{|n_2|}{2} \text{ or } |n_2| < \frac{|n_1|}{2} \text{ and } \mathcal{L} = |\tau|\right\},$

$C_2 := \left\{(n,\tau,n_1,\tau_1) : |n| > 100, |n_1| < \frac{|n_2|}{2} \text{ or } |n_2| < \frac{|n_1|}{2} \text{ and } \mathcal{L} = |\tau_2 - n_2^2|\right\}.$

Now, we bound (2.39). In the region $C_{0,1}$, it holds $|n| \le 100$. Hence,

$$
\begin{aligned}
Z_{0,1} &:= \sup_{n_1,\tau_1} \frac{1}{\langle n_1\rangle^{2r}\langle \tau_1 + n_1^2\rangle} \sum_n \int_{C_{0,1}} \frac{|n|^2\langle n\rangle^{2s}}{\langle\tau\rangle\langle\tau_2 - n_2^2\rangle^{1-2\theta}\langle n_2\rangle^{2r}} d\tau \\
&\lesssim \sup_{n_1,\tau_1} \sum_{|n|\le 100} \int_{-\infty}^{+\infty} \frac{d\tau}{\langle\tau\rangle\langle\tau_2 - n_2^2\rangle^{1-2\theta}} \\
&\lesssim \sup_{n_1,\tau_1} \sum_{|n|\le 100} \frac{1}{\langle\tau_1 + (n-n_1)^2\rangle^{1-2\theta}} \lesssim 1,
\end{aligned}
$$

since $r \ge 0$ and $1 - 2\theta > 0$.

In the region $C_{0,2}$, we have that $|n_1| \sim |n_2|$. Hence,

$$
\begin{aligned}
Z_{0,2} &:= \sup_{n_1,\tau_1} \frac{1}{\langle n_1\rangle^{2r}\langle \tau_1 + n_1^2\rangle} \sum_n \int_{C_{0,2}} \frac{|n|^2\langle n\rangle^{2s}}{\langle\tau\rangle\langle\tau_2 - n_2^2\rangle^{1-2\theta}\langle n_2\rangle^{2r}} d\tau \\
&\lesssim \sup_{n_1,\tau_1} \frac{\langle n_1\rangle^{2s-4r+2}}{\langle\tau_1 + n_1^2\rangle} \sum_n \int_{-\infty}^{+\infty} \frac{d\tau}{\langle\tau\rangle\langle\tau_2 - n_2^2\rangle^{1-2\theta}} \\
&\lesssim \sup_{n_1,\tau_1} \sum_n \frac{1}{\langle\tau_1 + (n-n_1)^2\rangle^{1-2\theta}} \lesssim 1,
\end{aligned}
$$

for $0 \le s \le 2r - 1$ and $0 < \theta < 1/4$.

In the region $C_{0,3}$, the dispersion relation (2.38)and the assumptions $|n_1| \not\sim |n_2|$, $|n| \ge 100$ and $\mathcal{L} = |\tau_1 + n_1^2|$ imply that $|\tau_1 + n_1^2| \gtrsim (\max\{|n_1|, |n_2|\})^2$. Then,

$$
\begin{aligned}
Z_{0,3} &:= \sup_{n_1,\tau_1} \frac{1}{\langle n_1\rangle^{2r}\langle \tau_1 + n_1^2\rangle} \sum_n \int_{C_{0,3}} \frac{|n|^2\langle n\rangle^{2s}}{\langle\tau\rangle\langle\tau_2 - n_2^2\rangle^{1-2\theta}\langle n_2\rangle^{2r}} d\tau \\
&\lesssim \sup_{n_1,\tau_1} \sum_n \int_{-\infty}^{+\infty} \frac{\big\langle \max\{|n_1|, |n_2|\}\big\rangle^{2s-2r}}{\langle\tau\rangle\langle\tau_2 - n_2^2\rangle^{1-2\theta}} d\tau \\
&\lesssim \sup_{n_1,\tau_1} \sum_n \frac{1}{\langle\tau_1 + (n-n_1)^2\rangle^{1-2\theta}} \lesssim 1,
\end{aligned}
$$

for $0 \le s \le r$ and $0 < \theta < 1/4$. Then, the inequality $|Z_0| \le |Z_{0,1}| + |Z_{0,2}| + |Z_{0,3}| \lesssim 1$ yields the desired estimate for Z_0.

The contribution of (2.40) can be estimated as follows. In the region C_1, we have that $|n| \sim \max\{|n_1|,|n_2|\}$ and $|\tau| \geq (\max\{|n_1|,|n_2|\})^2$. Thus,

$$\begin{aligned}
Z_1 &\leq \sup_{n,\tau} \frac{\langle n\rangle^{2s+2}}{\langle\tau\rangle} \sum_{n_1} \int_{C_1} \frac{d\tau_1}{\langle\tau_1+n_1^2\rangle\langle\tau_2-n_2^2\rangle^{1-2\theta}\langle n_1\rangle^{2r}\langle n_2\rangle^{2r}} \\
&\lesssim \sup_{n,\tau} \sum_{n_1} \int_{-\infty}^{\infty} \frac{\big\langle \max\{|n_1|,|n_2|\}\big\rangle^{2s-2r}}{\langle\tau_1+n_1^2\rangle\langle\tau_2-n_2^2\rangle^{1-2\theta}} d\tau_1 \\
&\lesssim \sup_{n,\tau} \sum_{n_1} \frac{1}{\langle\tau+n_1^2-n_2^2\rangle^{1-2\theta}} \\
&\lesssim \sup_{n,\tau} \sum_{n_1} \frac{1}{\langle 2nn_1+\tau-n^2\rangle^{1-2\theta}} \lesssim 1,
\end{aligned}$$

for $0 \leq s \leq r$ and $0 < \theta < 1/4$, using the same arguments to estimate $\widetilde{W}_2$ in Lemma 2.2.

On the other hand, the expression (2.41) can be controlled by using that in the region C_2 hold $|n| \sim \max\{|n_1|,|n_2|\}$ and $|\tau_2 - n_2^2| \gtrsim (\max\{|n_1|,|n_2|\})^2$. Then,

$$\begin{aligned}
Z_2 &= \sup_{n_2,\tau_2} \frac{1}{\langle n_2\rangle^{2r}\langle\tau_2-n_2^2\rangle} \sum_{n} \int_{C_2} \frac{|n|^2\langle n\rangle^{2s}}{\langle\tau\rangle\langle\tau_1+n_1^2\rangle^{1-2\theta}\langle n_1\rangle^{2r}} d\tau \\
&\lesssim \sup_{n_2,\tau_2} \sum_{n} \int_{-\infty}^{+\infty} \frac{\big\langle \max\{|n_1|,|n_2|\}\big\rangle^{2s-2r}}{\langle\tau\rangle\langle\tau_1+n_1^2\rangle^{1-2\theta}} d\tau \\
&\lesssim \sup_{n_2,\tau_2} \sum_{n} \frac{1}{\langle (n+n_2)^2-\tau_2\rangle^{1-2\theta}} \lesssim 1,
\end{aligned}$$

for $s \leq r$ and $0 < \theta < 1/4$. Collecting all the estimates above we obtain the claimed estimate (2.36).

The prove of (2.37) follows from a similar way to the proof of (2.16). $\square$

Now we exhibit examples showing the necessity of the conditions for r and s used in Lemma 2.4.

Proposition 2.5. *For any real numbers b_1 and b_2 the veracity of the inequality*

$$\|\partial_x(u\bar{w})\|_{H_t^{-1/2}H_{per}^s} \lesssim \|u\|_{X^{r,b_1}}\|w\|_{X^{r,b_2}}$$

implies that $s \leq \min\{2r-1,\ r\}$.

Proof. For a fixed large integer $N \gg 1$, we define de following sequences:

$$\alpha_1(n) = \begin{cases} 1 & \text{if } n = N, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \beta_1(n) = \begin{cases} 1 & \text{if } n = -N, \\ 0 & \text{otherwise.} \end{cases}$$

Putting $\widehat{u}_{1_N}(n,\tau) = \alpha_1(n)\chi_{[-1,1]}(\tau+n^2)$ and $\widehat{w}_{1_N}(n,\tau) = \beta_1(n)\chi_{[-1,1]}(\tau+n^2)$, a simple calculation using the dispersive relation (2.38) gives that

$$\|(u_1\bar{w}_1)_x\|_{H_t^{-1/2}H_{per}^s} \sim N^{s+1} \quad \text{and} \quad \|u_1\|_{X^{r,b_1}} \sim N^r \sim \|w_1\|_{X^{r,b_2}}.$$

Hence, the inequality $\|(u_1\bar{w}_1)_x\|_{H_t^{-1/2}H_{per}^s} \lesssim \|u_1\|_{X^{r,b_1}}\|w_1\|_{X^{r,b_2}}$ implies

$$N^{s+1} \leq N^{2r}, \text{ for } N \gg 1 \Longleftrightarrow s \leq 2r-1.$$

Finally, we define

$$\alpha_2(n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \beta_2(n) = \begin{cases} 1 & \text{if } n = N, \\ 0 & \text{otherwise.} \end{cases}$$

and we put $\widehat{u}_{2_N}(n,\tau) = \alpha_2(n)\chi_{[-1,1]}(\tau+n^2)$ and $\widehat{w}_{2_N}(n,\tau) = \beta_2(n)\chi_{[-1,1]}(\tau+n^2)$. Then, by similar calculations as in the previous case we obtain

$$\|(u_2\bar{w}_2)_x\|_{H_t^{-1/2}H^s_{per}} \sim N^s, \quad \|u_2\|_{X^{r,b_1}} \sim 1 \quad \text{and} \sim \|w_2\|_{X^{r,b_2}} \sim N^r.$$

Again, the inequality $\|(u_2\bar{w}_2)_x\|_{H_t^{-1/2}H^s_{per}} \lesssim \|u_2\|_{X^{r,b_1}}\|w_2\|_{X^{r,b_2}}$ implies

$$N^s \le N^r, \text{ for } N \gg 1 \iff s \le r.$$

Thus, we finished the proof. □

2.2. **Proof of Local Theorem.** The next lemmas will be useful in the proof of Theorem 1.2.

Lemma 2.6. *For any $s \in \mathbb{R}$, $\delta \in (0,1]$, $0 < \mu < 1/2$ and $-1/2 < b_1 \le b_2 < 1/2$ we have*

(a) $\|\eta_\delta(\cdot)F\|_{X^{s,1/2}_{per}} \le C\delta^{-\mu}\|F\|_{X^{s,1/2}_{per}}$ *and* $\|\eta_\delta(\cdot)F\|_{H_t^{1/2}H^s_{per}} \le C\delta^{-\mu}\|F\|_{H_t^{1/2}H^s_{per}}$;

(b) $\|\eta_\delta(\cdot)F\|_{X^{s,b_1}_{per}} \le C\delta^{b_2-b_1}\|F\|_{X^{s,b_2}_{per}}$ *and* $\|\eta_\delta(\cdot)F\|_{H_t^{b_1}H^s_{per}} \le C\delta^{b_2-b_1}\|F\|_{H_t^{b_2}H^s_{per}}$.

Proof. The proof of this result can be found, for instance, in [22] and [6]. □

Lemma 2.7 (Trilinear Estimate)**.** *For any $s \ge 0$, we have*

$$\|uv\bar{w}\|_{X^r_{per}} \lesssim \|u\|_{X^{s,3/8}_{per}}\|v\|_{X^{s,3/8}_{per}}\|w\|_{X^{s,3/8}_{per}}$$

Proof. See [12] and [6]. □

Now we give the sketch of the proof of local theorem. First, we let $(u_0, v_0) \in H^r_{per} \times H^s_{per}$ where r and s satisfying

$$\max\{0,\, r-1\} \le s \le \min\{r,\, 2r-1\}$$

and we consider the operator $\Phi = (\Phi_1,\, \Phi_2)$, with

$$\begin{aligned} \Phi_1(u,v) &= \eta(t)u_0 - i\eta(t)\int_0^t e^{i(t-t')\partial_x^2}\left((\eta_\delta u\eta_\delta v)(t') + \eta_\delta u|\eta_\delta u|^2(t')\right)dt', \\ \Phi_2(u,v) &= \eta(t)v_0 + \eta(t)\int_0^t \partial_x(|\eta_\delta u|^2)(t')dt', \end{aligned} \tag{2.42}$$

defined on the ball

$$\mathcal{B}[a,b] = \left\{(u,v) \in X^r_{per} \times Y^s_{per} :\ \|u\|_{X^r_{per}} \le a \text{ and } \|v\|_{Y^s_{per}} \le b\right\}.$$

Then, by Lemmas 2.2, 2.4, 2.6 and 2.7 we have

$$\begin{aligned} \|\Phi_1(u,v)\|_{X^r_{per}} &\le C_0\|u_0\|_{H^r_{per}} + C\Big(\|\eta_\delta u\|_{X^{r,1/2-\theta}_{per}}\|\eta_\delta v\|_{H_t^{1/2}H^s_{per}} + \\ &\qquad + \|\eta_\delta u\|_{X^{r,1/2}_{per}}\|\eta_\delta v\|_{H_t^{1/2-\theta}H^s_{per}} + \|\eta_\delta u\|^3_{X^{r,3/8}_{per}}\Big) \\ &\le C_0\|u_0\|_{H^r_{per}} + C\delta^\epsilon(ab+a^3) \end{aligned} \tag{2.43}$$

and

$$\begin{aligned}\|\Phi_2(u,v)\|_{Y^s_{per}} &\leq C_0\|v_0\|_{H^s_{per}} + C\left(\|\eta_\delta u\|_{X^{r,1/2-\theta}_{per}}\|\eta_\delta u\|_{X^{r,1/2}_{per}}\right)\\ &\leq C_0\|v_0\|_{H^s_{per}} + C\delta^\epsilon a^2,\end{aligned} \tag{2.44}$$

with ϵ enough small.

Now we put $a = 2C_0\|u_0\|_{H^r_{per}}$ and $b = 2C_0\|v_0\|_{H^s_{per}}$ and then we let δ such that $\delta^\epsilon \leq \min\left\{\frac{1}{2C(ab+a^3)}, \frac{1}{2Ca^2}\right\}$. Thus, we have that $\Phi(\mathcal{B}[a,b]) \subset \mathcal{B}[a,b]$. The contraction condition

$$\|\Phi(u,v) - \Phi(\tilde{u},\tilde{v})\|^{r\times s}_{per} \leq C(a,b)\delta^\theta\|(u-\tilde{u}, v-\tilde{v})\|^{r\times s}_{per},$$

where $\|(f,g)\|^{r\times s}_{per} := \|f\|_{X^r_{per}} + \|g\|_{Y^s_{per}}$ and $C(a,b)$ is a positive constant depending only on a and b, follows similarly. This shows that the map Φ is a contraction on $\mathcal{B}[a,b]$. There we obtain a unique fixed point which solves the system for $T < \delta$ and we finish the proof.

Remark 2.8. *We note that global well-posedness in $H^1_{per} \times L^2_{per}$ follows directly of the local theorem for $(r,s) = (1,0)$ combined with the conservation laws (1.2), (1.3) and (1.4).*

3. Ill-posedness

In this section we will show that the solution of (1.1) cannot depend uniformly continuously on its initial data for $r < 0$ and $s \in \mathbb{R}$. We will use the same argument given in [13].

3.1. Proof of theorem 1.3.

It is easy to check that

$$\begin{aligned} &u_{N,a}(t,x) = a\exp(iNx)\exp(-it(N^2 + (\gamma+\beta)a^2))\\ &v_{N,a}(t,x) = \gamma a^2,\end{aligned} \tag{3.45}$$

where $a \in \mathbb{R}$ and N is any positive integer, solves (1.1) with initial data $u_0(x) = a\exp(iNx)$ and $v_0(x) = \gamma a^2$. Moreover, for $a = \alpha(1+N^2)^{\frac{r}{2}}$, where α is a real constant, and $|\gamma| = (1+N^2)^r$ we have

$$||u_0(x)||^2_{H^r} \leq c\alpha^2$$

and

$$||v_0(x)||^2_{H^s} \leq c\alpha^4,$$

where c is a constant.

Let $a_1 = \alpha_1(1+N^2)^{\frac{r}{2}}$ and $a_2 = \alpha_2(1+N^2)^{\frac{r}{2}}$. For the Sobolev norm of the difference of two initial data, we have

$$||u_{N,a_1}(0) - u_{N,a_2}(0)||^2_{H^r} = c|\alpha_1 - \alpha_2|^2 \to 0, \text{as } \alpha_1 \to \alpha_2$$

and

$$||v_{a_1}(0) - v_{a_2}(0)||^2_{H^s} = |\gamma|^2|\alpha_1^2 - \alpha_2^2|^2(1+N^2)^{-2r} = |\alpha_1^2 - \alpha_2^2|^2, \text{as } \alpha_1 \to \alpha_2$$

On the other hand we have

$$\begin{aligned} ||u_{N,a_1}(t,x) - u_{N,a_2}(t,x)||^2_{H^r} &= \sum_{\xi=-\infty}^{+\infty} (1+|\xi|^2)^r |\hat{u}_{N,\alpha_1}(\xi) - \hat{u}_{N,\alpha_2}(\xi)|^2 \\ &= (1+N^2)^r |a_1 e^{-it(N^2+(\gamma+\beta)a_1^2)} - a_2 e^{-it(N^2+(\gamma+\beta)a_2^2)}|^2 \\ &= |\alpha_1 - \alpha_2 e^{it(\gamma+\beta)(\alpha_1^2-\alpha_2^2)(1+N^2)^{-r}}|^2 \end{aligned}$$

Let $r<0$, and α_1 and α_2 are such that

$$\beta(\alpha_1^2-\alpha_2^2)(1+N^2)^{-r} = \delta(1+N^2)^{\nu},$$

where $\nu>0$, and $\nu+r<0$. Then for $t=\frac{\pi}{2}(\delta^{-1}(1+N^2)^{-\nu})$ we have

$$||u_{N,a_1}(t,x) - u_{N,a_2}(t,x)||^2_{H^r} \geq c(\alpha_1^2+\alpha_2^2)$$

Note that t can made arbitrary small, by choosing N sufficiently large.

4. Existence of periodic travelling wave solutions

We are interesting in this section in finding explicit solutions for (1.1) in the form

$$\begin{cases} u(t,x) = e^{-i\omega t} e^{i\frac{c}{2}(x-ct)} \varphi_{\omega,c}(x-ct) \\ \\ v(t,x) = n_{\omega,c}(x-ct) \end{cases} \tag{4.46}$$

where $\varphi_{\omega,c}$, $n_{\omega,c}$ are smooth, L-periodic functions, $c>0$, $\omega\in\mathbb{R}$ and suppose that there is a $q\in\mathbb{N}$ such that $\frac{4\pi q}{c}=L$. So, putting (4.46) into (1.1) we obtain

$$\begin{cases} \varphi^{''}_{\omega,c} + (\omega + \frac{c^2}{4})\varphi_{\omega,c} = \varphi_{\omega,c} n_{\omega,c} + \beta\varphi^3_{\omega,c} \\ \\ -cn^{'}_{\omega,c} = 2\varphi_{\omega,c}\varphi^{'}_{\omega,c} \end{cases} \tag{4.47}$$

If $n_{\omega,c}=\gamma\varphi^2_{\omega,c}$, then from the second equation in (4.47) we have $\gamma=-\frac{1}{c}$. Substituting $n_{\omega,c}$ in the first equation in (4.47), it follows that $\varphi_{\omega,c}$ satisfies

$$\varphi^{''}_{\omega,c} + \left(\omega+\frac{c^2}{4}\right)\varphi_{\omega,c} = \left(\beta - \frac{1}{c}\right)\varphi^3_{\omega,c} \tag{4.48}$$

If $1-\beta c>0$ and $\varphi_{\omega,c}=\left(\frac{c}{1-\beta c}\right)^{\frac{1}{2}}\phi_{\omega,c}$, then $\phi_{\omega,c}$ satisfies the equation

$$\phi^{''}_{\omega,c} - \sigma\phi_{\omega,c} + \phi^3_{\omega,c} = 0 \tag{4.49}$$

where $\sigma=-\omega-\frac{c^2}{4}$. So, by following Angulo in ([3], [4]) we have from (4.49) that $\phi_{\omega,c}$ satisfies the first-order equation

$$[\phi'_{\omega,c}]^2 = \frac{1}{2}P_\phi(\phi) \tag{4.50}$$

where $P_\phi(t)=-t^4+2\sigma t^2+2B_\phi$ and B_ϕ is an integration constant. Let $-\eta_1 < -\eta_2 < \eta_2 < \eta_1$ are the zeros of the polynomial $P_\phi(t)$. Then

$$[\phi^{'}_{\omega,c}]^2 = \frac{1}{2}(\eta_1^2 - \phi^2_{\omega,c})(\phi^2_{\omega,c} - \eta_2^2) \tag{4.51}$$

The solution of (4.51) is

$$\phi_{\omega,c} = \eta_1 dn\left(\frac{\eta_1}{\sqrt{2}}\xi;\kappa\right) \tag{4.52}$$

where

$$\begin{cases} \eta_1^2+\eta_2^2 = 2\sigma \\ \\ \kappa^2 = \frac{\eta_1^2-\eta_2^2}{\eta_1^2} \\ \\ 0<\eta_2<\eta_1. \end{cases} \tag{4.53}$$

Define the function in variable κ, $0<\kappa<1$

$$K = K(\kappa) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-\kappa^2t^2)}}$$

called the complete elliptic integral of the first kind. Since dn has fundamental period $2K(\kappa)$, it follows that $\phi_{\omega,c}$ has fundamental period

$$T_{\phi_{\omega,c}} = \frac{2\sqrt{2}}{\eta_1}K(\kappa)$$

Analogously as in [3] we obtain the following.

Theorem 4.1. *Let L be fixed but arbitrary positive constant and $1-\beta c>0$, and $-\omega-\frac{c^2}{4}>0$. Let $\sigma_0>\frac{2\pi^2}{L^2}$ and $\eta_{2,0}=\eta_2(\sigma_0)\in(0,\sqrt{\frac{\sigma_0}{3}})$ is the unique such that $T_\phi = L$. Then*

(1) There exists an interval $I(\sigma_0)$ around of σ_0, an interval $B(\eta_{2,0})$ around $\eta_{2,0}$, and a unique smooth function $\Lambda : I(\sigma_0)\to B(\eta_{2,0})$, such that

$$\Lambda(\sigma_0)=\eta_{2,0} \ \text{ and } \ \frac{2\sqrt{2}}{\sqrt{2\sigma-\eta_2^2}}\mathrm{K}(\kappa)=\mathrm{L}$$

where $\sigma\in I(\sigma_0), \eta_2=\Lambda(\sigma)$

(2) Solutions $(\varphi_{\omega,c}, n_{\omega,c})$ of (4.47) given by

$$\begin{cases} \varphi_{\omega,c} = \sqrt{\frac{c}{1-\beta c}}\eta_1 dn\left(\frac{\eta_1}{\sqrt{2}}\xi;\kappa\right) \\ \\ n_{\omega,c} = -\frac{\eta_1^2}{1-\beta c}dn^2\left(\frac{\eta_1}{\sqrt{2}}\xi;\kappa\right) \end{cases} \tag{4.54}$$

with $\eta_1=\eta_1(\sigma)$, $\eta_2=\eta_2(\sigma)$, $\eta_1^2+\eta_2^2=2\sigma$, have the fundamental period L and satisfies (4.47). Moreover, the mapping

$$\sigma\in I(\sigma_0)\to(\varphi_{\omega,c}, n_{\omega,c})$$

is a smooth function

(3) $I(\sigma_0)$ can be chosen as $(\frac{2\pi^2}{L^2},+\infty)$

(4) The mapping $\sigma\to\kappa(\sigma)$ is a strictly increasing function

5. Stability of travelling waves

In this section we consider the stability of the orbit

$$\Omega_{(\Phi,\Psi)} = \{(e^{i\theta}\Phi(\cdot + x_0), \Psi(\cdot + x_0));\ (\theta, x_0) \in [0, 2\pi) \times \mathbb{R}\},$$

in $H^1_{per}([0, L]) \times L^2_{per}([0, L])$ by the periodic flow generated by (1.1), where we have that $\Phi(\xi) = e^{ic\xi/2}\varphi_{\omega,c}(\xi)$, $\Psi(\xi) = n_{\omega,c}(\xi)$, with $\varphi_{\omega,c}, n_{\omega,c}$ given in (4.54). Let X be the space $X = H^1_{complex}([0, L]) \times L^2_{real}([0, L])$, with real inner product

$$(\vec{u_1}, \vec{u_2}) = \Re \int_0^L (\varepsilon_1 \overline{\eta}_1 + \varepsilon_{1x}\overline{\eta}_{1x} + \varepsilon_2 \overline{\eta_2})dx.$$

Let $T_1,\ T_2$ be one-parameter groups of unitary operators on X defined by

$$T_1(s)\vec{u}(\cdot) = \vec{u}(\cdot + s)$$
$$T_2(r)\vec{u}(\cdot) = (e^{-ir}\varepsilon(\cdot), n(\cdot))$$

for $\vec{u} \in X,\ \ s,\ r \in \mathbb{R}$. Obviously

$$T_1^{'}(0) = \begin{pmatrix} -\partial_x & \\ & -\partial_x \end{pmatrix}, \quad T_2^{'}(0) = \begin{pmatrix} -i & \\ & 0 \end{pmatrix}.$$

Note that the equation (1.1) is invariant under T_1 and T_2. If

$$\Phi_{\omega,c}(x) = (\varepsilon_{\omega,c}(x), n_{\omega,c}(x))$$

where $\varepsilon_{\omega,c}(x) = e^{i\frac{c}{2}x}\varphi_{\omega,c}(x)$, then from Theorem 4.1 we obtain that

$$T_1(ct)T_2(\omega t)\Phi_{\omega,c}(x)$$

is a travelling wave solution of (4.47) with $\varphi_{\omega,c}(x), n_{\omega,c}(x)$ defined by (4.54).

Now, it is easy to verify that $E_2(\vec{u})$ is invariant under T_1 and T_2

$$E(T_1(s)T_2(r)\vec{u}) = E(\vec{u}). \tag{5.55}$$

We also have

$$E(\vec{u}(t)) = E(\vec{u}_0). \tag{5.56}$$

Note that equation (1.1) can be written as the following Hamiltonian system

$$\frac{d\vec{u}}{dt} = JE'(\vec{u}) \tag{5.57}$$

where $\vec{u} = (u, v)$ and J is a skew-symmetric linear operator defined by

$$J = \begin{pmatrix} -i & 0 \\ 0 & 2\partial_x \end{pmatrix}$$

and

$$E'(u, v) = \begin{pmatrix} -u_{xx} + uv + \beta|u|^2 u \\ \frac{1}{2}|u|^2 \end{pmatrix}$$

is the Frechet derivative of E.

Define B_1 and B_2 such that $T_1'(0) = JB_1,\ \ T_2'(0) = JB_2$. Then

$$B_1 = \begin{pmatrix} -i\partial_x & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$Q_1(\vec{u}) = \frac{1}{2}\langle B_1\vec{u}, \vec{u}\rangle = -\frac{1}{4}\int_0^L v^2 dx + \frac{1}{2} Im \int_0^L u_x \overline{u} dx$$

$$Q_2(\vec{u}) = \frac{1}{2}\langle B_2\vec{u}, \vec{u}\rangle = \frac{1}{2}\int_0^L |u|^2 dx.$$

It is easy to verify that

$$Q_1(T_1(s)T_2(r)\vec{u}) = Q_1(\vec{u}), \;\; Q_2(T_1(s)T_2(r)\vec{u}) = Q_2(\vec{u}) \tag{5.58}$$

$$Q_1(\vec{u}(t)) = Q_1(\vec{u}(0)), \;\; Q_2(\vec{u}(t)) = Q_2(\vec{u}(0)) \tag{5.59}$$

and

$$Q_1^{'}(u,v) = \begin{pmatrix} -iu_x \\ -\frac{1}{2}v \end{pmatrix}, \quad {Q_2}'(u,v) = \begin{pmatrix} u \\ 0 \end{pmatrix}.$$

From (4.47) we have

$$E'(\Phi_{\omega,c}) - cQ_1'(\Phi_{\omega,c}) - \omega Q_2'(\Phi_{\omega,c}) = 0. \tag{5.60}$$

Define an operator from X to X^*

$$H_{\omega,c} = E''(\Phi_{\omega,c}) - cQ_1''(\Phi_{\omega,c}) - \omega Q_2''(\Phi_{\omega,c}) \tag{5.61}$$

and the function $d(\omega, c) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ by

$$d(\omega, c) = E(\Phi_{\omega,c}) - cQ_1(\Phi_{\omega,c}) - \omega Q_2(\Phi_{\omega,c}). \tag{5.62}$$

The operator $H_{\omega,c}$ is self-adjoint. The spectrum of $H_{\omega,c}$ consists of the real numbers λ such that $H_{\omega,c} - \lambda I$ is not invertible.

From (4.47) we have

$$T_1^{'}(0)\Phi_{\omega,c} \in KerH_{\omega,c}, \;\; T_2^{'}(0)\Phi_{\omega,c} \in KerH_{\omega,c}. \tag{5.63}$$

Let $Z = \{k_1T_1^{'}(0)\Phi_{\omega,c} + k_2T_2^{'}(0)\Phi_{\omega,c}\ , \;\; k_1, k_2 \in \mathbb{R}\}$. By (5.63), Z is in the kernel of $H_{\omega,c}$.

Assumption *(Spectral decomposition of $H_{\omega,c}$)* : The space X is decomposed as a direct sum

$$X = N \oplus Z \oplus P$$

where Z is defined above, N is a finite-dimensional subspace such that

$$\langle H_{\omega,c}\vec{u}, \vec{u}\rangle < 0 \;\text{ for }\; \tilde{\mathrm{u}} \in \mathrm{N}$$

and P is a closed subspace such that

$$\langle H_{\omega,c}\vec{u}, \vec{u}\rangle \geq \delta||\vec{u}||_X^2$$

for $\vec{u} \in P$ with some constant $\delta > 0$ independent of $\vec{u}$.

Our stability results is based in the following general theorem in [17],

Theorem 5.1. *(Abstract Stability Theorem) Assume that there exists three functionals E, Q_1, Q_2 satisfying (5.55)-(5.59). Let $n(H_{\omega,c})$ be the number of negative eigenvalues of $H_{\omega,c}$. Assume $d(\omega, c)$ is non-degenerated at (ω, c) and let $p(d'')$ be the number of positive eigenvalues of d''. If $p(d'') = n(H_{\omega,c})$, then the periodic travelling wave $\Phi_{\omega,c}(x)$ is orbitally stable.*

The idea of the proof of Theorem 1.5 is to apply the general Theorem 5.1. Initially we identify the quadratic form associated to $H_{\omega,c}$. Let $\vec{z} = (e^{i\frac{c}{2}x}z_1, z_2)$, with $z_1 = y_1 + iy_2$, $y_1 = Rez_1, y_2 = Imz_1$. By direct computation, we get

$$\langle H_{\omega,c}\vec{z_1}, \vec{z_1}\rangle = \langle L_1y_1, y_1\rangle + \langle L_2y_2, y_2\rangle + \frac{c}{2}\int_0^L \left(z_2 + \frac{2}{c}\varphi_{\omega,c}y_1\right)^2 dx$$

where

$$L_1 = -\partial_x^2 - \left(\frac{c^2}{4} + \omega\right) + 3\left(\beta - \frac{1}{c}\right)\varphi_{\omega,c}^2$$
$$L_2 = -\partial_x^2 - \left(\frac{c^2}{4} + \omega\right) + \left(\beta - \frac{1}{c}\right)\varphi_{\omega,c}^2.$$

From (4.47) we also have $L_1(\partial_x\varphi_{\omega,c}) = 0$ and $L_2\varphi_{\omega,c} = 0$. Consider the following periodic eigenvalue problems

$$\begin{cases} L_1 f = \lambda f \\ f(0) = f(L), \quad f'(0) = f'(L), \end{cases} \tag{5.64}$$

$$\begin{cases} L_2 g = \lambda g \\ g(0) = g(L), \quad g'(0) = g'(L). \end{cases} \tag{5.65}$$

The problem (5.64) determines a countable infinite set of eigenvalues $\{\lambda_n\}$ with $\lambda_n \to \infty$. We shall denote by χ_n the eigenfunction associated to the eigenvalue λ_n. For the periodic eigenvalue problem (5.64) there is an associated semi-periodic eigenvalue problem in $[0, L]$, namely,

$$\begin{cases} L_1 h = \lambda h \\ h(0) = -h(L), \quad h'(0) = -h'(L). \end{cases} \tag{5.66}$$

As in the periodic case, there is a countable infinity set of eigenvalues $\{\mu_n\}$. Denote by ξ_n the eigenfunction associated to the eigenvalue μ_n. From the Oscillation Theorem [21] we have that

$$\lambda_0 < \mu_0 \le \mu_1 < \lambda_1 \le \lambda_2 < \mu_2 \le \mu_3, ...$$

λ_0 is simple and

(a) χ_0 has no zeros on $[0, L]$

(b) χ_{2n+1} and χ_{2n+2} have exactly $2n+2$ zeros on $[0, L)$

(c) ξ_{2n} and ξ_{2n+1} have exactly $2n+1$ zeros on $[0, L)$.

The intervals $(\lambda_0, \mu_0), (\mu_1, \lambda_1), ...$ are called intervals of stability and the intervals $(-\infty, \lambda_0), (\mu_0, \mu_1), (\lambda_1, \lambda_2), ...$ are called intervals of instability.

For the eigenvalue problem (5.65) we have the same results.

Theorem 5.2. *Let $\sigma \in [\frac{2\pi^2}{L^2}, +\infty)$ and $(\varphi_{\omega,c}, n_{\omega,c})$ be the travelling wave solutions of (4.54). Then the first three eigenvalues of operator L_1 are simple, 0 is the second eigenvalue of L_1 with eigenfunction $\partial_x\varphi_{\omega,c}$. The first eigenvalue of the operator L_2 is 0, which is simple.*

Proof. Since $L_2\varphi_{\omega,c} = 0$ and $\varphi_{\omega,c}$ has no zeros on $[0, L]$, then form (a) it follows that zero is the first eigenvalue of L_2.

Now since $L_1\partial_x\varphi_{\omega,c} = 0$ and $\partial\varphi_{\omega,c}$ has two zeros on $[0, L)$, then it follows that eigenvalue zero of L_1 is either λ_1 or λ_2. Let $\psi = f(\theta x)$, where $\theta^2 = \frac{2}{\eta_1^2}$. From equality $\kappa^2 sn^2(x) + dn^2(x) = 1$ and (5.64), we obtain that ψ satisfies the equation

$$\psi'' + (\rho - 6\kappa^2 sn^2(x))\psi = 0, \tag{5.67}$$

where

$$\rho = 6 - \frac{2}{\eta_1^2}(\sigma - \lambda). \tag{5.68}$$

From Floquet theory, it follows that $(-\infty, \rho_0), (\mu_0, \mu_1)$ and (ρ_1, ρ_2) are instability intervals associated to the Lame's equation. Therefore the eigenvalues ρ_0, ρ_1 and ρ_2 of (5.68) are simple and the rest of eigenvalues $\rho_3 \leq \rho_4, ...$ satisfies $\rho_3 = \rho_4, \rho_5 = \rho_6, ...$ The eigenvalues ρ_0, ρ_1, ρ_2 and its corresponding eigenfunctions ψ_0, ψ_1, ψ_2 are

$$\rho_0 = 2(1 + \kappa^2 - \sqrt{1 - \kappa^2 + \kappa^4}, \quad \psi_0 = 1 - (1 + \kappa^2 - \sqrt{1 - \kappa^2 + \kappa^4})sn^2(x)$$

$$\rho_1 = 4 + \kappa^2, \qquad \psi_2 = sn(x)cn(x)$$

$$\rho_2 = 2(1 + \kappa^2 + \sqrt{1 - \kappa^2 + \kappa^4}, \quad \psi_2 = 1 - (1 + \kappa^2 + \sqrt{1 - \kappa^2 + \kappa^4})sn^2(x)$$

Since $\rho_0 < \rho_1$ for every $\kappa^2 \in (0, 1)$, then from (5.68) we have

$$3\lambda_0 = \frac{\eta_1^2}{2}(\kappa^2 - 2 - 2\sqrt{1 - \kappa^2 + \kappa^4}) < 0$$

Therefore λ_0 is negative eigenvalue of L_1 with eigenfunction $\chi_0(x) = \psi_0(\frac{x}{\theta})$. Similarly

$$3\lambda_2 = \frac{\eta_1^2}{2}(\kappa^2 - 2 + 2\sqrt{1 - \kappa^2 + \kappa^4}) > 0$$

and λ_2 is the positive eigenvalue of L_1 with eigenfunction $\chi_2(x) = \psi_2(\frac{x}{\theta})$. Thus

$$\lambda_1 = \frac{\eta_1^2(\rho_1 - 6) + 2\sigma}{6} = \frac{\eta_1^2}{6}(4 + \kappa^2 - 6 + 2 - \kappa^2) = 0$$

is the second eigenvalue of L_1. This complete the proof of the theorem. □

Remark 5.3. *The main properties of the spectrum of L_1, namely, there is exactly a negative eigenvalue and zero is simple, it can also be obtained via positive properties of the Fourier transform of the solution $\varphi_{\omega,c}$ (see Angulo & Natali* [5]*).*

So, from Theorem 5.2 we obtain immediately the following two results.

Lemma 5.4. *For any real function $y_1 \in H^1$ satisfying $\langle y_1, \chi_0\rangle = \langle y_1, \partial_x\varphi_{\omega,c}\rangle = 0$ there exists a positive constant $\delta_1 > 0$ such that $\langle L_1y_1, y_1\rangle \geq \delta_1||y_1||^2_{H^1}$.*

Lemma 5.5. *For any real function $y_2 \in H^1$ satisfying $\langle y_2, \varphi_{\omega,c}\rangle = 0$ there exists a positive constant δ_2 such that $\langle L_2y_2, y_2\rangle \geq \delta_2||y_2||^2_{H^1}$.*

Proof. [**Theorem 1.5**] Choose $y_1^- = \chi_0, y_2^- = 0, z_2^- = -\frac{2}{c}\varphi_{\omega,c}\chi_0$ and $\Psi^- = (y_1^-, y_2^-, z_2^-)$ then

$$\langle H_{\omega,c}\Psi^-, \Psi^-\rangle = \lambda_0\langle\chi_0, \chi_0\rangle < 0.$$

So $H_{\omega,c}$ has a negative eigenvalue. Note that the following vectors

$$\Psi_{0,1} = (\partial_x\varphi_{\omega,c}, 0, -\frac{2}{c}\varphi_{\omega,c}\partial_x\varphi_{\omega,c}), \quad \Psi_{0,2} = (0, \varphi_{\omega,c}, 0)$$

are in the kernel of operator $H_{\omega,c}$.

Define the following subspaces associated to $H_{\omega,c}$:

$$Z = \{k_1\Psi_{0,1} + k_1\Psi_{0,2} : k_1, k_2 \in \mathbb{R}\}$$

$$N = \{k\Psi^- : k \in \mathbb{R}\}$$

$$P = \{\vec{p} \in X : \vec{p} = (p_1, p_2, p_3), \langle p_1, \chi_1\rangle = \langle p_1, \partial_x\varphi_{\omega,c}\rangle = \langle p_2, \varphi_{\omega,c}\rangle = 0\}.$$

For any $\vec{u} \in X, \vec{u} = (y_1, y_2, y_2)$ choose

$$a = \frac{\langle y_1, \chi_0\rangle}{\langle\chi_0, \chi_0\rangle}, \ b_1 = \frac{\langle\partial_x\varphi_{\omega,c}, y_1\rangle}{\langle\partial_x\varphi_{\omega,c}, \partial_x\varphi_{\omega,c}\rangle}, \ b_2 = \frac{\langle\varphi_{\omega,c}, y_2\rangle}{\langle\varphi_{\omega,c}, \varphi_{\omega,c}\rangle},$$

then $\vec{u}$ uniquely can be represented by

$$\vec{u} = a\Psi^- + b_1\Psi_{0,1} + b_2\Psi_{0,2} + \vec{p},$$

where $\vec{p} \in P$.

For any $\vec{p} \in P$, by Lemmas 5.4 and 5.5, we have

$$\langle H_{\omega,c}\vec{p}, \vec{p}\rangle \geq \delta_1||p_1||^2_{H^1} + \delta_1||p_2||^2_{H^1} + \frac{c}{2}\int_0^L \left(p_3 + \frac{2}{c}\varphi p_1\right)^2 dx$$

Next we consider the following two cases:

(1) If $||p_3||_{L^2} \geq \frac{8||\varphi_{\omega,c}||_{L^\infty}}{c}||p_1||_{L^2}$, then

$$\frac{c}{2}\int_0^L \left(p_3 + \frac{2}{c}\varphi_{\omega,c}p_1\right)^2 dx \geq \frac{c}{2}\left[||p_3||^2_{L^2} - \frac{4}{c}||\varphi_{\omega,c}||_{L^\infty}||p_1||_{L^2}||p_3||_{L^2}\right] = \frac{c}{4}||p_3||^2_{L^2}$$

(2) If $||p_3||_{L^2} \leq \frac{8||\varphi_{\omega,c}||_{L^\infty}}{c}||p_1||_{L^2}$, then

$$\delta_1||p_1||^2_{H^1} \geq \frac{\delta_1}{2}||p_1||^2_{H^1} + \frac{\delta_1}{2}\frac{c}{8||\varphi_{\omega,c}||_{L^\infty}}||p_3||^2_{L^2}$$

Thus, for any $\vec{p} \in P$, it follows that

$$\langle H_{\omega,c}\vec{p}, \vec{p}\rangle \geq \delta_3||p_3||^2_{L^2} + \frac{\delta_1}{2}||p_1||^2_{H^1} + \delta_2||p_2||^2_{H^1},$$

where $\delta_3 = \min\{\frac{\delta_1 c}{16||\varphi_{\omega,c}||_{L^\infty}}, \frac{c}{4}\}$. Finally, we have

$$\langle H_{\omega,c}\vec{p}, \vec{p}\rangle \geq \delta||\vec{p}||^2_X,$$

where $\delta > 0$ is independent of $\vec{p}$. This proved that Assumption above is holds, and $n(H_{\omega,c}) = 1$.

Now we shall verify that $p(d'') = 1$. We have

$$d_c(\omega, c) = -Q_1(\Phi_{\omega,c}) = \frac{1}{4(1-\beta c)^2}\int_0^L \varphi^4_{\omega,c}dx - \frac{c^2}{4(1-\beta c)}\int_0^L \varphi_{\omega,c}dx$$

$$d_\omega(\omega, c) = -Q_2(\Phi_{\omega,c}) = -\frac{c}{2(1-\beta c)}\int_0^L \varphi^2_{\omega,c}dx.$$

From equalities

$$\int_0^L \varphi^2_{\omega,c}dx = \frac{8KE}{L}, \quad \int_0^L \varphi^4_{\omega,c}dx = \frac{64}{L^3}V(\kappa)$$

where $E = E(\kappa) = \int_0^1 \sqrt{\frac{1-\kappa^2 t^2}{1-t^2}}dt$ is the complete elliptic integral of the second kind and $V(\kappa) = \frac{\kappa^2-1}{3}K^4 + \frac{2}{L}(2-\kappa^2)K^2E$, we obtain

$$\begin{aligned}
d_{\omega\omega} = &\quad \tfrac{4c}{L(1-\beta c)}(K'(\kappa)E(\kappa)+K(\kappa)E'(\kappa))\kappa'(\sigma) \\
d_{\omega c} = &\quad -\tfrac{4}{L(1-\beta c)^2}K(\kappa)E(\kappa)+\tfrac{c}{2}d_{\omega\omega} \\
d_{c\omega} = &\quad -\tfrac{16}{L^3(1-\beta c)^2}V'(\kappa)\kappa'(\sigma)+\tfrac{c}{2}d_{\omega\omega} \\
d_{cc} = &\quad \tfrac{32\beta}{L^3(1-\beta c)^3}V(\kappa)-\tfrac{8c}{L^3(1-\beta c)^2}V'(\kappa)\kappa'(\sigma)- \\
&\quad \tfrac{2c(2-\beta c)}{L(1-\beta c)^2}K(\kappa)E(\kappa)+\tfrac{c^2}{4}d_{\omega\omega}.
\end{aligned} \tag{5.69}$$

Thus

$$\begin{aligned}
d_{cc}d_{\omega\omega}-d_{c\omega}d_{\omega c} = &\quad -\tfrac{64}{L^4(1-\beta c)^4}V'(\kappa)\kappa'(\sigma)K(\kappa)E(\kappa)+ \\
&\quad \tfrac{1}{L(1-\beta c)}\left[\tfrac{32\alpha}{L^2(1-\beta c)^2}V(\kappa)-2cK(\kappa)E(\kappa)\right]d_{\omega\omega}.
\end{aligned}$$

We have

$$V'(\kappa) = \frac{2K^2E}{\kappa(1-\kappa^2)}\left[(2-\kappa^2)E-(1-\kappa^2)K\right].$$

and

$$\frac{V}{L^2} = \frac{\sigma(\kappa^2-1)}{12(2-\kappa^2)}K^2+\frac{\sigma}{6}KE.$$

Using the above estimates, we obtain

$$\begin{aligned}
d_{cc}d_{\omega\omega}-d_{c\omega}d_{\omega c} &= \tfrac{4\kappa'}{L^2(1-\beta c)^2}\Big\{-32\tfrac{K^2}{L^2}KE^2\left[(2-\kappa^2)E-2(1-\kappa^2)K\right]\Big\} \\
&+\tfrac{c}{3}\left[\tfrac{8\beta\sigma(\kappa^2-1)}{2-\kappa^2}K^2+(16\beta\sigma-6c(1-\beta c)^2)KE\right]\left[\tfrac{E^2}{\kappa(1-\kappa^2)}-\tfrac{K^2}{\kappa}\right]\Big\} \\
&= \tfrac{4K\kappa'}{L^2(1-\beta c)^2}\Big\{-\tfrac{8\sigma}{2-\kappa^2}\left[(2-\kappa^2)E^3-2(1-\kappa^2)KE^2\right]\Big\} \\
&+\left[\tfrac{8\beta\sigma c(\kappa^2-1)}{3(2-\kappa^2)}K+\tfrac{c(16\beta\sigma-6c(1-\beta c)^2)}{3}E\right]\left[\tfrac{E^2}{\kappa(1-\kappa^2)}-\tfrac{K^2}{\kappa}\right]\Big\}
\end{aligned}$$

From Theorem 4.1-(4), we have that $\kappa' > 0$. Therefore the sign of $det(d'') = d_{cc}d_{\omega\omega}-d_{c\omega}d_{\omega c}$ depends on the sign of

$$\begin{aligned}
B(c,\omega,\kappa,\beta) = &\quad \Big\{-\tfrac{8\sigma}{2-\kappa^2}\left[(2-\kappa^2)E^3-2(1-\kappa^2)KE^2\right] \\
&\quad +\left[\tfrac{8\beta\sigma c(\kappa^2-1)}{3(2-\kappa^2)}K+\tfrac{c(16\beta\sigma-6c(1-\beta c)^2)}{3}E\right]\left[\tfrac{E^2}{\kappa(1-\kappa^2)}-\tfrac{K^2}{\kappa}\right]\Big\}.
\end{aligned}$$

From the relation

$$0 < \frac{(1-\kappa^2)K}{(2-\kappa^2)E} < \frac{1}{2} \tag{5.70}$$

we get that the first term of $B(c,\omega,\kappa,\beta)$ is negative. Now we consider three cases for β.

(1) Obviously if $\beta = 0$, then $det(d'') < 0$.

(2) For $\beta < 0$, using (5.70), we get

$$\frac{8\beta\sigma(\kappa^2-1)K}{3(2-\kappa^2)E} + \frac{c(16\beta\sigma-6c(1-\beta c)^2)}{3}E =$$

$$= -\frac{8c\beta\sigma E}{3}\left[\frac{(1-\kappa^2)K}{(2-\kappa^2)E} - 2 + \frac{6c(1-\beta c)^2}{8c\beta}\right] < 0$$

and $det(d^{''}) < 0$.

(3) If $\beta > 0$ and $8\beta\sigma - 3c(1-\beta c)^2 \leq 0$, then all terms of $B(c, \omega, \kappa, \beta)$ are negatives and $det(d'') < 0$.

Thus under above three conditions, $d''(\omega, c)$ has exactly one positive and one negative eigenvalues and $p(d'') = 1$. This finishes the proof of the Theorem. □

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