ASYMPTOTIC BEHAVIOUR OF THE LIE MODULE

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Abstract. We obtain a projective submodule of Lie(n) in prime characteristic p for n not a p-power, and show that the ratio of its dimension to that of Lie(n) approaches 1 as n tends to infinity.

1. Introduction

The Lie module Lie(n) of the symmetric group $\mathfrak{S}_n$ appears in many contexts; in particular it is closely related to the free Lie algebra. It may be defined as the left ideal of the group algebra $F\mathfrak{S}_n$ generated by the 'Dynkin-Specht-Wever element'

$$\upsilon_n := (1 - c_2)(1 - c_3) \cdots (1 - c_n),$$

where c_k is the descending k-cycle $(k, k-1, \dots, 1)$. (We use the convention of composing the elements of $\mathfrak{S}_n$ from right to left.)

Our main motivation comes from the work of Selick and Wu [SW1]. Their problem is to find natural homotopy decompositions of the loop suspension of a p-torsion suspension where p is a prime. In [SW1] it is proved that this problem is equivalent to the algebraic problem of finding natural coalgebra decompositions of the primitively generated tensor algebras over the field with p elements. They determine the finest coalgebra decomposition of a tensor algebra (over arbitrary fields), which can be described as a functorial Poincaré-Birkhoff-Witt theorem [SW1, Theorem 6.5]. In order to compute the factors in this decomposition, one must know a maximal projective submodule, called $\mathrm{Lie}^{\max}(n)$, of the Lie module Lie(n).

The projective modules for the symmetric groups over fields of positive characteristic are not known. Their structure depends on the decomposition matrices for symmetric groups, and the determination of the latter is a famous open problem, for which a complete solution does not seem forthcoming in the near future. However, according to [SW2], it would be interesting to know, even if the modules cannot be computed precisely, how quickly the dimensions grow, and whether or not the growth rate is exponential. Determination of $\mathrm{Lie}^{\max}(6)$ and $\mathrm{Lie}^{\max}(8)$ in characteristic 2 in [SW2] suggests that $\mathrm{Lie}^{\max}(n)$ is relatively large compared with Lie(n). If this is

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true in general, it would have the desirable consequence that the factors in the functorial PBW theorem are relatively small.

Some progress has been made in the understanding of $\mathrm{Lie}^{\max}(n)$ for the case where $n = pk$ with $p \nmid k$. Among other things, Erdmann and Schocker [ES] established a one-to-one correspondence between the indecomposable non-projective summands of $\mathrm{Lie}(pk)$ and the indecomposable summands of $\mathrm{Lie}(k)$. Erdmann and the second author [ET] provided an upper bound for the dimension of $\mathrm{Lie}^{\max}(pk)$ and showed that the ratio of this upper bound to the dimension of $\mathrm{Lie}(pk)$ approaches 1 as k tends to infinity, and conjectured the same to hold for the ratio of the dimension of $\mathrm{Lie}^{\max}(pk)$ to that of $\mathrm{Lie}(pk)$.

The main result of this paper proves this conjecture and more:

Theorem 1. *Let $A = \mathbb{Z}^+ \setminus \{p^m \mid m \in \mathbb{Z}_{\geq 0}\}$. Then*

$$\frac{\dim(\mathrm{Lie}^{\max}(n))}{\dim(\mathrm{Lie}(n))} \to 1$$

as $n \to \infty$ in A.

This theorem in particular shows that $\dim(\mathrm{Lie}^{\max}(n))$ grows exponentially with n for n not a p-power.

Our approach is to exploit the decomposition theorem of Bryant and Schocker ([BS]) via the Schur functor. This provides a projective submodule of $\mathrm{Lie}(n)$ when n is not a p-power. We describe its dimension via a recurrence relation and show that its ratio to the dimension of $\mathrm{Lie}(n)$ approaches 1 as n tends to infinity. The second author thanks Roger Bryant for his suggestion of this approach.

Our paper is organised as follows: we introduce the notations and background theory, as well as obtain some simple preliminary results, in the next section, while in Section 3, we describe the dimension of a projective submodule of $\mathrm{Lie}(n)$ for n not a p-power via a recurrence relation, and show that its ratio to $(n-1)!$, the dimension of $\mathrm{Lie}(n)$, approaches 1 as n tends to infinity.

Throughout this paper, we fix a field F of prime characteristic p. All our algebras and vector spaces are defined over F. All our tensor products are taken over F too.

2. Preliminaries

In this section, we give an account of the background theory and prove some preliminary results which we shall require.

2.1. **Schur algebra.** Let $n, r \in \mathbb{Z}^+$. Let $I(n,r)$ be the set of r-tuples of positive integers between 1 and n (both inclusive) and let $\Lambda(n,r)$ be the set of compositions of r with n parts, i.e.

$$I(n,r) = \{(i_1,\dots,i_r) \in (\mathbb{Z}^+)^r \mid 1 \le i_1,\dots,i_r \le n\},$$

$$\Lambda(n,r) = \{(\lambda_1,\dots,\lambda_n) \in (\mathbb{Z}_{\ge 0})^n \mid \sum_{i=1}^n \lambda_i = r\}.$$

To each $(i_1,\dots,i_r) \in I(n,r)$, we associate a $(\lambda_1,\dots,\lambda_n) \in \Lambda(n,r)$ by $\lambda_j = |\{k \mid i_k = j\}|$. The symmetric group $\mathfrak{S}_r$ acts on the right of $I(n,r)$ by place permutations, i.e. $(i_1,\dots,i_r)\cdot\sigma = (i_{\sigma(1)},\dots,i_{\sigma(r)})$, and the orbits under this action are precisely indexed by $\Lambda(n,r)$, i.e. two r-tuples in $I(n,r)$ lie in the same $\mathfrak{S}_r$-orbit if and only if they are associated to the same composition of r. This induces a right action of $\mathfrak{S}_r$ on $I^2 = (I(n,r))^2$ via $(\mathbf{i},\mathbf{j})\cdot\sigma = (\mathbf{i}\cdot\sigma,\mathbf{j}\cdot\sigma)$. We write $I^2/\sim$ for a set of orbit representatives under the action.

We note also that $\mathfrak{S}_n$ acts on the right of $\Lambda(n,r)$ by place permutations.

The Schur algebra $S(n,r)$ has a distinguished basis $\{\xi_{\mathbf{i},\mathbf{j}} \mid (\mathbf{i},\mathbf{j}) \in I^2/\sim\}$. We write ξ_α for $\xi_{\mathbf{i},\mathbf{i}}$ where $\alpha \in \Lambda(n,r)$ is the composition of r associated to $\mathbf{i}$. These ξ_α's are mutually orthogonal idempotents of $S(n,r)$, and $1_{S(n,r)} = \sum_{\alpha\in\Lambda(n,r)} \xi_\alpha$. They thus provide a (vector space) decomposition of any left $S(n,r)$-module $M = \bigoplus_{\alpha\in\Lambda(n,r)} M^\alpha$, where $M^\alpha := \xi_\alpha M$ is the *weight space* of M associated to α.

Let $s \in \mathbb{Z}^+$. Given a left $S(n,r)$-module M and a left $S(n,s)$-module N, one may endow a natural left $S(n,r+s)$-module structure on $M \otimes N$.

We collate together some results relating to weight spaces.

Lemma 2 ([G, (3.3a),(3.3c)]). *Let $n,r,s \in \mathbb{Z}^+$, and let M and N be finite-dimensional left $S(n,r)$- and $S(n,s)$-modules respectively.*

(1) *If $\sigma \in \mathfrak{S}_n$ and $\alpha \in \Lambda(n,r)$, then $M^\alpha \cong M^{\alpha\cdot\sigma}$ as vector spaces.*
(2) *If $\gamma \in \Lambda(n,r+s)$, then*

$$(M\otimes N)^\gamma = \bigoplus_{\substack{\alpha\in\Lambda(n,r)\\ \beta\in\Lambda(n,s)\\ \alpha+\beta=\gamma}} M^\alpha \otimes N^\beta.$$

(Here, and hereafter, if $\alpha = (\alpha_1,\dots,\alpha_n) \in \Lambda(n,r)$ and $\beta = (\beta_1,\dots,\beta_n) \in \Lambda(n,s)$, then $\alpha+\beta = (\alpha_1+\beta_1,\dots,\alpha_n+\beta_n) \in \Lambda(n,r+s)$.)

In the case where $n \ge r$, let $\omega_r = (\overbrace{1,\dots,1}^{r \text{ times}}, \overbrace{0,\dots,0}^{n-r \text{ times}}) \in \Lambda(n,r)$. It is well-known that $\xi_{\omega_r} S(n,r) \xi_{\omega_r} \cong F\mathfrak{S}_r$ as F-algebras. This induces the Schur functor $f_r : {}_{S(n,r)}\mathbf{mod} \to {}_{F\mathfrak{S}_r}\mathbf{mod}$ which sends a left $S(n,r)$-module M to its weight space M^{ω_r}.

2.2. **Lie module.** Let $n \in \mathbb{Z}^+$, and let $\mathfrak{S}_n$ denote the symmetric group on n letters. We use the convention of composing the elements of $\mathfrak{S}_n$ from right to left. The Lie module $\mathrm{Lie}(n)$ may be defined as the left ideal of $F\mathfrak{S}_n$ generated by the Dynkin-Specht-Wever element

$$\upsilon_n = (1-c_2)(1-c_3)\dots(1-c_n),$$

where $c_i \in \mathfrak{S}_n$ is the descending i-cycle $(i, i-1, \ldots, 1)$.

It is well-known that $\upsilon_n^2 = n\upsilon_n$. Thus when $p \nmid n$, $\frac{1}{n}\upsilon_n$ is an idempotent, and hence $\mathrm{Lie}(n)$ is projective, while $\mathrm{Lie}(n)$ is non-projective when $p \mid n$. We also have the following identity.

Lemma 3. *Let $r, n \in \mathbb{Z}^+$ with $2 \leq r \leq n$. Then $\upsilon_n = -\upsilon_{r-1}c_r\upsilon_n$ (where $\upsilon_1 = 1$).*

Proof. We note first that

$$\upsilon_r = \upsilon_{r-1}(1 - c_r) = \upsilon_{r-1} - \upsilon_{r-1}c_r;$$
$$\upsilon_s\upsilon_n = \upsilon_s\upsilon_s(1 - c_{s+1})\cdots(1 - c_n) = s\upsilon_s(1 - c_{s+1})\cdots(1 - c_n) = s\upsilon_n$$

for all $s = 1, \ldots, n$. Thus

$$(1 + \upsilon_{r-1}c_r)\upsilon_n = (1 + \upsilon_{r-1} - \upsilon_r)\upsilon_n = \upsilon_n + (r-1)\upsilon_n - r\upsilon_n = 0.$$

□

For $\rho \in \mathfrak{S}_n \setminus \mathfrak{S}_{\{2,\ldots,r\}}$ with $\rho(r) = 1$, we note that $\rho\upsilon_{r-1}c_r \in F\mathfrak{S}_{\{2,\ldots,n\}}$, where here, and hereafter, $\mathfrak{S}_{\{2,\ldots,n\}} = \{\sigma \in \mathfrak{S}_n \mid \sigma(1) = 1\}$. Thus, using Lemma 3, we see that $\mathrm{Lie}(n)$ is spanned by $\{\sigma\upsilon_n \mid \sigma \in \mathfrak{S}_{\{2,\ldots,n\}}\}$ as a vector space. It is well-known that this spanning set is in fact a basis, so that $\mathrm{Lie}(n)$ has dimension $(n-1)!$.

2.3. **Modular Lie powers.** Let M be a left $S(n,r)$-module, and let $s \in \mathbb{Z}^+$. The s-fold tensor product $M^{\otimes s}$ is then a left $S(n, rs)$-module, and it also admits another commuting right action of $\mathfrak{S}_s$ by place permutations, i.e. $(m_1 \otimes \cdots \otimes m_s) \cdot \sigma = m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(s)}$ $(m_1, \ldots, m_s \in M,\ \sigma \in \mathfrak{S}_s)$.

Let $N_1, \ldots, N_s$ be vector subspaces of M. Then $(N_1 \otimes \cdots \otimes N_s)\sigma$ is a vector subspace of $M^{\otimes s}$ for all $\sigma \in \mathfrak{S}_s$. We write $[N_1, \ldots, N_s]$ for $(N_1 \otimes \cdots \otimes N_s)\upsilon_s$, where υ_s is the Dynkin-Specht-Wever element generating the Lie module $\mathrm{Lie}(s)$. This is a vector subspace of $\sum_{\sigma \in \mathfrak{S}_s}(N_1 \otimes \cdots \otimes N_s)\sigma$.

Lemma 4. *Let $N_1, \ldots, N_s$ be vector subspaces of M whose sum is direct. For each i, let $\mathcal{B}_i$ be a basis for N_i. Then $\sum_{\sigma \in \mathfrak{S}_s}[N_{\sigma(1)}, \ldots, N_{\sigma(s)}]$ has a basis*

$$\{(b_1 \otimes b_2 \otimes \cdots \otimes b_s)\sigma\upsilon_s \mid b_i \in \mathcal{B}_i,\ \sigma \in \mathfrak{S}_{\{2,\ldots,s\}}\}.$$

Proof. Let $Y = \{(b_1 \otimes b_2 \otimes \cdots \otimes b_s)\sigma\upsilon_s \mid b_i \in \mathcal{B}_i,\ \sigma \in \mathfrak{S}_{\{2,\ldots,s\}}\}$. Note first that $[N_{\sigma(1)}, \ldots, N_{\sigma(s)}] = (N_1 \otimes \cdots \otimes N_s)\sigma\upsilon_s$. If $\sigma \notin \mathfrak{S}_{\{2,\ldots,s\}}$, we may choose t so that $\sigma(t) = 1$. Then $\sigma\upsilon_s = \sigma\upsilon_{t-1}c_t\upsilon_s$, and $\sigma\upsilon_{t-1}c_t \in F\mathfrak{S}_{\{2,\ldots,s\}}$. Thus Y spans $\sum_{\sigma \in \mathfrak{S}_s}[N_{\sigma(1)}, \ldots, N_{\sigma(s)}]$.

To show that Y is linearly independent, let $\mathbf{B} = \{b_1 \otimes \cdots \otimes b_s \mid b_i \in \mathcal{B}_i\ \forall i\}$, and suppose that $\sum_{\sigma \in \mathfrak{S}_{\{2,\ldots,s\}}}\sum_{\mathbf{b} \in \mathbf{B}}\lambda_{\mathbf{b},\sigma}\mathbf{b}\sigma\upsilon_s = 0$. Since the sum of the N_i's is direct, there exists a basis $\mathcal{C}$ for M such that $\mathcal{B}_i \subseteq \mathcal{C}$ for all i. Then $\mathbf{C} = \{c_1 \otimes \cdots \otimes c_s \mid c_1, \ldots, c_s \in \mathcal{C}\}$ is a basis for $M^{\otimes s}$, and $\mathbf{C} \supseteq \mathbf{B}$. Fix a $\mathbf{b}_0 \in \mathbf{B}$; then $\mathbf{b}_0\mathfrak{S}_s \subseteq \mathbf{C}$. Let ϕ be the projection of $M^{\otimes s}$ onto the vector subspace spanned by $\mathbf{b}_0\mathfrak{S}_s$, with $\phi(\mathbf{c}) = 0$ for all $\mathbf{c} \in \mathbf{C} \setminus \mathbf{b}_0\mathfrak{S}_s$. Then for $\mathbf{b} \in \mathbf{B}$ and $\sigma \in \mathfrak{S}_{\{2,\dots,s\}}$, we have $\phi(\mathbf{b}\sigma\upsilon_s) = \delta_{\mathbf{b},\mathbf{b}_0}\mathbf{b}_0\sigma\upsilon_s$. Thus,

$$0 = \phi\left(\sum_{\sigma\in\mathfrak{S}_{\{2,\dots,s\}}}\sum_{\mathbf{b}\in\mathbf{B}}\lambda_{\mathbf{b},\sigma}\mathbf{b}\sigma\upsilon_s\right) = \sum_{\sigma\in\mathfrak{S}_{\{2,\dots,s\}}}\lambda_{\mathbf{b}_0,\sigma}\mathbf{b}_0\sigma\upsilon_s.$$

This is a linear relation on $\{\mathbf{b}_0\sigma\upsilon_s \mid \sigma \in \mathfrak{S}_{\{2,\dots,s\}}\}$. The span of this set is isomorphic to Lie(s) via the correspondence $\mathbf{b}_0\sigma\upsilon_s \leftrightarrow \sigma\upsilon_s$. Since $\{\sigma\upsilon_s \mid \sigma \in \mathfrak{S}_{\{2,\dots,s\}}\}$ is a well-known basis for Lie(s), the relation on $\{\mathbf{b}_0\sigma\upsilon_s \mid \sigma \in \mathfrak{S}_{\{2,\dots,s\}}\}$ must be trivial. Hence Y is linearly independent. □

This immediately gives the following corollary.

Corollary 5. *Let $N_1, \dots, N_s$ be vector subspaces of M whose sum is direct. Then*

$$\sum_{\sigma\in\mathfrak{S}_s}[N_{\sigma(1)}, \dots, N_{\sigma(s)}] = \bigoplus_{\sigma\in\mathfrak{S}_{\{2,\dots,n\}}}[N_{\sigma(1)}, \dots, N_{\sigma(s)}].$$

Furthermore, if each N_i is finite-dimensional, then

$$\dim\left(\sum_{\sigma\in\mathfrak{S}_s}[N_{\sigma(1)}, \dots, N_{\sigma(s)}]\right) = (s-1)!\prod_{i=1}^{s}\dim(N_i).$$

When $N_1, \dots, N_s$ are $S(n,r)$-submodules of M, then $(N_1 \otimes \cdots \otimes N_s)\sigma$ ($\sigma \in \mathfrak{S}_s$) and $[N_1, \dots, N_s]$ are all $S(n, rs)$-submodules of $M^{\otimes s}$. We have the following lemma relating the weight spaces of $[N_1, \dots, N_s]$ of those of $N_1, N_2, \dots, N_s$.

Lemma 6. *Let M be a left $S(n,r)$-module, with submodules $N_1, \dots, N_s$, and let $\alpha \in \Lambda(n, rs)$. Then*

$$[N_1, \dots, N_s]^\alpha = \sum_{\substack{\beta^{(1)},\dots,\beta^{(s)}\in\Lambda(n,r)\\ \beta^{(1)}+\cdots+\beta^{(s)}=\alpha}}[N_1^{\beta^{(1)}}, \dots, N_s^{\beta^{(s)}}].$$

Proof. We have

$$[N_1, \dots, N_s] = \left[\bigoplus_{\beta^{(1)}\in\Lambda(n,r)}N_1^{\beta^{(1)}}, \dots, \bigoplus_{\beta^{(s)}\in\Lambda(n,r)}N_s^{\beta^{(s)}}\right]$$
$$= \sum_{\beta^{(1)},\dots,\beta^{(s)}\in\Lambda(n,r)}[N_1^{\beta^{(1)}}, \dots, N_s^{\beta^{(s)}}].$$

Since $(N_1^{\beta^{(1)}} \otimes \cdots \otimes N_s^{\beta^{(s)}})\sigma \subseteq (M^{\beta^{(1)}} \otimes \cdots \otimes M^{\beta^{(s)}})\sigma \subseteq (M^{\otimes s})^{\sum_{i=1}^s \beta^{(i)}}$ for all $\sigma \in \mathfrak{S}_s$ by Lemma 2(2), we have

$$[N_1^{\beta^{(1)}}, \dots, N_s^{\beta^{(s)}}] = (N_1^{\beta^{(1)}} \otimes \cdots \otimes N_s^{\beta^{(s)}})\upsilon_s \subseteq (M^{\otimes s})^{\sum_{i=1}^s \beta^{(i)}}.$$

Thus, $\xi_\gamma[N_1^{\beta^{(1)}},\dots,N_s^{\beta^{(s)}}]=0$ for all $\gamma\in\Lambda(n,rs)\setminus\{\sum_{i=1}^s\beta^{(i)}\}$, so that

$$\begin{aligned}[N_1,\dots,N_s]^\alpha &= \xi_\alpha[N_1,\dots,N_s]\\ &= \xi_\alpha\left(\sum_{\beta^{(1)},\dots,\beta^{(s)}\in\Lambda(n,r)}[N_1^{\beta^{(1)}},\dots,N_s^{\beta^{(s)}}]\right)\\ &= \sum_{\substack{\beta^{(1)},\dots,\beta^{(s)}\in\Lambda(n,r)\\ \beta^{(1)}+\cdots+\beta^{(s)}=\alpha}}[N_1^{\beta^{(1)}},\dots,N_s^{\beta^{(s)}}].\end{aligned}$$

□

An n-dimensional vector space V has a natural left $S(n,1)$-module structure. Thus $V^{\otimes r}$ and $L^r(V):=[\overbrace{V,\dots,V}^{r\text{ times}}]$ are naturally left $S(n,r)$-modules. When $n\geq r$, it is well-known that the Schur functor f_r, sends $V^{\otimes r}$ and $L_r(V)$ to $F\mathfrak{S}_r$ and $\mathrm{Lie}(r)$ respectively.

3. Main results

In this section, we describe the dimension of a projective submodule of $\mathrm{Lie}(n)$ when n is not a p-power via a recurrence relation, and show that its ratio to $\dim(\mathrm{Lie}(n))=(n-1)!$ approaches 1 as n tends to infinity.

We begin with the decomposition theorem of Bryant and Schocker.

Theorem 7 ([BS, Theorem 4.4]). *Let V be an n-dimensional vector space, and let k be a positive integer not divisible by p. Then for each $r\in\mathbb{Z}_{\geq 0}$, there exists a direct summand B_{p^rk} of $V^{\otimes p^rk}$ (as an $S(n,p^rk)$-submodule) such that*

$$L^{p^mk}(V)=L^{p^m}(B_k)\oplus L^{p^{m-1}}(B_{pk})\oplus\cdots\oplus L^1(B_{p^mk})$$

for all $m\in\mathbb{Z}_{\geq 0}$.

We may assume that n is 'large' for all our purposes; then under the direct-sum-preserving Schur functor f_{p^mk}, we have a decomposition of

$$(*)\qquad \mathrm{Lie}(p^mk)=f_{p^mk}L^{p^m}(B_k)\oplus f_{p^mk}L^{p^{m-1}}(B_{pk})\oplus\cdots\oplus f_{p^mk}L^1(B_{p^mk}).$$

Since B_{p^mk} is a direct summand of $V^{\otimes p^mk}$, we see that $f_{p^mk}L^1(B_{p^mk})=f_{p^mk}B_{p^mk}$ is a projective submodule of $\mathrm{Lie}(p^mk)$. Its dimension is then a lower bound for $\dim(\mathrm{Lie}^{\max}(p^mk))$. Unfortunately, the abstract description of B_{p^mk} given in [BS] makes its image under the Schur functor difficult to describe. We get around this by using $(*)$ to provide a recurrence relation on $\dim(f_{p^mk}B_{p^mk})$.

We have the following description of $f_{rs}(L^s(M))$, in which we use the following notations. Let $n,r,s\in\mathbb{Z}^+$ with $n\geq rs$. We write $[rs]$ for $\{1,2,\dots,rs\}$. Let $\mathcal{A}$ denote the set of r-element subsets of $[rs]$, and let $\geq$ be any total ordering on $\mathcal{A}$. If $A\in\mathcal{A}$, we write $\omega_A\in\Lambda(n,rs)$ for the composition $(I_A(1), I_A(2), \ldots, I_A(n))$, where $I_A(x) = 1$ if $x \in A$, and $= 0$ otherwise.

Proposition 8. *Let $n, r, s \in \mathbb{Z}^+$ with $n \geq rs$, and let M be an $S(n,r)$-module. Then*

$$f_{rs}L^s(M) = \bigoplus_{\substack{A_1,\ldots,A_s \in \mathcal{A} \\ A_1 > \cdots > A_s \\ \bigcup_{i=1}^s A_i = [rs]}} \bigoplus_{\sigma \in \mathfrak{S}_{\{2,\ldots,s\}}} [M^{\omega_{A_{\sigma(1)}}}, \ldots, M^{\omega_{A_{\sigma(s)}}}].$$

In particular, if M is finite-dimensional, then

$$\dim(f_{rs}L^s(M)) = \frac{(rs)!}{s}\left(\frac{\dim(f_r M)}{r!}\right)^s.$$

Proof. We have

$$f_{rs}L^s(M) = (L^s(M))^{\omega_{rs}} = \sum_{\substack{\alpha^{(1)},\ldots,\alpha^{(s)} \in \Lambda(n,r) \\ \alpha^{(1)} + \cdots + \alpha^{(s)} = \omega_{rs}}} [M^{\alpha^{(1)}}, \ldots, M^{\alpha^{(s)}}]$$

by Lemma 6. Note that if $\alpha^{(1)}, \ldots, \alpha^{(s)} \in \Lambda(n,r)$ such that $\sum_{i=1}^s \alpha^{(i)} = \omega_{rs}$, then for each i, $\alpha^{(i)} = \omega_{A_i}$ for some $A_i \in \mathcal{A}$, and $\bigcup_{i=1}^s A_i = [rs]$. Thus,

$$f_{rs}L^s(M) = \sum_{\substack{A_1,\ldots,A_s \in \mathcal{A} \\ A_1 > \cdots > A_s \\ \bigcup_{i=1}^s A_i = [rs]}} \sum_{\sigma \in \mathfrak{S}_s} [M^{\omega_{A_{\sigma(1)}}}, \ldots, M^{\omega_{A_{\sigma(s)}}}].$$

The outer sum is direct by Lemma 2(2), while the inner sum, by Corollary 5, equals

$$\bigoplus_{\sigma \in \mathfrak{S}_{\{2,\ldots,s\}}} [M^{\omega_{A_{\sigma(1)}}}, \ldots, M^{\omega_{A_{\sigma(s)}}}],$$

and has dimension $(s-1)! \prod_{i=1}^s \dim(M^{\omega_{A_{\sigma(i)}}}) = (s-1)!(\dim(M^{\omega_r}))^s$ by Lemma 2(1) when M is finite-dimensional. Thus

$$\dim(f_{rs}L^s(M)) = \frac{(rs)!}{(r!)^s s!}(s-1)! \dim(f_r M)^s,$$

and the proposition follows. □

Definition 9. For each $m \in \mathbb{Z}_{\geq 0}$ and each $k \geq 2$ be an integer not divisible by p, let $z_{m,k} = \dim(f_{p^m k} B_{p^m k})/(p^m k - 1)!$.

In other words, $z_{m,k}$ is the ratio of the dimension of the projective submodule $f_{p^m k} B_{p^m k}$ to that of the Lie module $\mathrm{Lie}(p^m k)$.

Theorem 10. *Let $k \geq 2$ be an integer not divisible by p. Then $z_{m,k}$ satisfies the following recurrence relation:*

$$z_{0,k} = 1, \qquad \text{and} \qquad z_{m,k} = 1 - \sum_{i=0}^{m-1} \frac{z_{i,k}^{p^{m-i}}}{(p^i k)^{p^{m-i}-1}} \quad (m > 0).$$

Proof. From (∗) with $m = 0$, we see that $\mathrm{Lie}(k) = B_k$, so that $z_{0,k} = 1$. By Proposition 8, we have

$$\dim(f_{p^m k} L^{p^{m-i}}(B_{p^i k})) = \frac{(p^m k)!}{p^{m-i}} \left(\frac{\dim(f_{p^i k} B_{p^i k})}{(p^i k)!} \right)^{p^{m-i}} = \frac{(p^m k - 1)!}{(p^i k)^{p^{m-i}-1}} z_{i,k}^{p^{m-i}}.$$

The theorem now follows from (∗). □

Theorem 11. *Let $A = \mathbb{Z}^+ \setminus \{p^m \mid m \in \mathbb{Z}_{\geq 0}\}$. For $n \in A$, let $z_n = z_{m,k}$ where $n = p^m k$ with $p \nmid k$. Then $z_n \to 1$ as $n \to \infty$ in A.*

Proof. Let $a_i = (p^i k)^{p^{m-i}-1}$. For $1 \leq i \leq m-2$, we have

$$\frac{a_i}{a_{i+1}} = \frac{(p^i k)^{p^{m-i}-1}}{(p^{i+1}k)^{p^{m-i-1}-1}} = \frac{(p^i k)^{p^{m-i}-p^{m-i-1}}}{p^{p^{m-i-1}-1}} = \frac{(p^i k)^{p^{m-i-1}(p-1)}}{p^{p^{m-i-1}-1}} \geq \left(\tfrac{p^i k}{p}\right)^{p^{m-i-1}} = (p^{i-1}k)^{p^{m-i-1}} \geq 1.$$

Thus

$$1 \geq z_{m,k} = 1 - \sum_{i=0}^{m-1} \frac{z_{i,k}^{p^{m-i}}}{a_i} \geq 1 - \frac{1}{a_0} - \frac{m-1}{a_{m-1}} = 1 - \frac{1}{k^{p^m-1}} - \frac{m-1}{(p^{m-1}k)^{p-1}}.$$

The theorem now follows. □

We end the paper by noting that Theorem 1 follows immediately from Theorem 11.

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