EIGENFUNCTION LOCALIZATION FOR THE $2D$ PERIODIC SCHRÖDINGER OPERATOR

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Abstract. We prove that for any *fixed* trigonometric polynomial potential satisfying a genericity condition, the spectrum of the two dimension periodic Schrödinger operator has finite multiplicity and the Fourier series of the eigenfunctions are uniformly exponentially localized about a finite number of frequencies. As a corollary, the L^p norms of the eigenfunctions are bounded for all $p > 0$, which answers a question of Toth and Zelditch [TZ].

Contents

1. Introduction and statement of the theorem

We consider the Schrödinger operator on the square 2-torus $\mathbb{T}^2$:

$$H = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + V \tag{1.1}$$

on $L^2([-\pi, \pi]^2)$ with periodic boundary condition, where V is real and as a function on $\mathbb{R}^2$ is $2\pi \times 2\pi$ periodic.

Let V be a trigonometric polynomial of degree k. Assume V is *generic* satisfying the genericity conditions (i, ii) at the end of this section. Here it suffices to remark that the genericity condition is explicit, for example $\cos x \cos y$ is generic. Moreover the non generic set is of codimension at least 1. We postpone the discussion of generic potentials until then, where we will also show that for V of the form $V(x, y) = V_1(x) + V_2(y)$, H always has uniformly bounded multiplicity.

Our main result is

Theorem. *Let V be a generic trigonometric polynomial of degree k. The spectrum of H is of multiplicity at most Ck^4 and the Fourier series $\hat{\phi}$ of the eigenfunctions ϕ with eigenvalues E satisfy*

$$|\hat{\phi}(j)| \leq C \sum_{|\ell| \leq Ck^4} e^{-|j-j_\ell|}, \tag{1.2}$$

where C is uniform in E, while $\{j_\ell\}$ depends on E:

$$\{j_\ell\} \subset \{(m, n) \in \mathbb{Z}^2 | \, |m^2 + n^2 - E| \leq \|V\|_\infty + 1\}. \tag{1.3}$$

The above Theorem has the following consequences. Using (1.2),

$$\|\hat{\phi}\|_{\ell^1} \leq C', \tag{1.4}$$

and we have

$$\|\phi\|_{L^\infty} \leq C'. \tag{1.5}$$

So we obtain

Corollary. *The eigenfunctions ϕ have bounded L^p norms for all $p > 0$:*

$$\|\phi\|_{L^p(\mathbb{T}^2)} < C_p, \qquad \forall p > 0. \tag{1.6}$$

Motivation for the Theorem.

Our motivation is threefold. The first comes from spectral theory. Consider the Laplacian on the d-torus. When $d = 1$, the periodic Schrödinger operator is also called the Hill operator. Its spectral properties are well known. There is an extensive literature on the subject starting from the 1946 paper of Borg on Sturm-Liouville problems [Bor]. The main point here is that the equation $n^2 = E$ ($E \neq 0$) has only two solutions. The spectrum is therefore of multiplicity at most two. When $d > 1$, the number of solutions to

$$n_1^2 + n_2^2 + \cdots + n_d^2 = E \tag{1.7}$$

grows with E. The spectrum of the Laplacian has unbounded multiplicity. The problem here is therefore basic, namely how to do perturbation theory when there is *unbounded degeneracy.*

Using separation properties of integer solutions to (1.7) and more generally to the inequality:

$$|n_1^2 + n_2^2 + \cdots + n_d^2 - E| \le A,$$

we prove that when $d = 2$ for generic trigonometric polynomial potentials, the spectrum of the periodic Schrödinger operator in (1.1) has finite multiplicity and the Fourier series of the eigenfunctions are uniformly exponentially localized about a finite number of frequencies, hence solving a basic problem in spectral theory.

There are previous results on some related problems. For the integrated density of states of the corresponding Schrödinger operator on $L^2(\mathbb{R}^2)$, see the recent paper [PS], cf. also [So]. There are results on the Schrödinger operators when $\mathbb{T}^2$ is replaced by $\mathbb{R}^d/\Gamma$, where Γ is a generic lattice. Hence the multiplicity of the spectrum of the Laplacian is typically finite [FKT1, 2].

A related motivation is the L^p bounds of eigenfunctions on compact manifolds X. Let λ be an eigenvalue of a self-adjoint operator H on X. Define

$$M_p \overset{\text{def}}{=} \sup_{\substack{\phi \\ H\phi=\lambda\phi}} \frac{\|\phi\|_{L^p}}{\|\phi\|_{L^2}}. \tag{1.8}$$

Assume λ has multiplicity $\mu(\lambda)$. Taking $p = \infty$, it is easy to see that

$$M_\infty \ge \sqrt{\frac{\mu}{\text{vol } X}}. \tag{1.9}$$

by taking the eigenfunction $\psi(x) = \sum_{j=1}^{\mu} \bar{\phi}_j(x_0)\phi_j(x)$, where $\{\phi_j\}_{j=1}^{\mu}$ is an orthonormal basis for the eigenspace corresponding to λ and x_0 is the point where $\sum_{j=1}^{\mu} |\phi_j(x_0)|^2 \ge \mu/\text{vol } X$. Such an x_0 always exists, since $\int \sum_{j=1}^{\mu} |\phi_j(x)|^2 = \mu$. On the other hand, there is the general upper bound from [H, SS]:

$$M_\infty \le \lambda^{\frac{d-1}{4}}. \tag{1.10}$$

On the sphere (1.9, 1.10) are of the same order, where there is maximal eigenfunction growth.

On the flat torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, (1.10) is far from optimal. For example, when $d = 2$ simple number theory consideration gives

$$M_\infty \ll \lambda^\epsilon, \qquad \forall \epsilon > 0, \tag{1.11}$$

and there are λ, where $M_\infty(\lambda)$ are at least logarithmic in λ. When $p = 4$, Zygmund [Zy] proved however that $M_4(\lambda) \leq 5^{1/4}$.

With the addition of a generic polynomial potential V to the Laplacian, the theorem says that on $\mathbb{T}^2$, the Schrödinger operator H has finite multiplicity and

$$M_\infty \leq C. \tag{1.12}$$

The corollary answers a question in [TZ, conj. 4.4], where it is further stipulated that minimal growth criterion similar to (1.12) characterize flat manifolds under classical integrability conditions. In this context, see [Bou] for an example where (1.11) is violated by a change of metric. For a general survey on the subject with connections to number theory and quantum chaos, see [Sa].

Our motivation for the present problem also comes from parameter dependent situations, e.g., time dependent or nonlinear perturbations of linear Schrödinger equations, where typically the frequency (of the perturbation in the linear case and of the quasi-periodic solutions in the nonlinear case) is an essential parameter in order to exclude resonances.

The situation in (1.1) roughly corresponds to the resonant case, where the Theorem shows that there is *uniform* Fourier restriction and (1.2) hold. (cf. [W] for a related result in the time dependent case.) The small divisors are overcome *deterministically* using the separation property of integer solutions to $|m^2 + n^2 - E| \leq \|V\|_\infty + 1$.

Method of the proof and genericity

Using the Fourier basis, H is unitarily transformed to a matrix operator $\hat{H}$:

$$\hat{H} = \text{diag }(m^2 + n^2) + \hat{V}*$$

on $\ell^2(\mathbb{Z}^2)$, where $\hat{V}$ is the Fourier series of V. To prove the Theorem, it suffices to control local eigenvalue spacing. For a given E in the spectrum of H: $\sigma(H)$, we only need to consider the level set $L = \{(m,n) | \, |m^2 + n^2 - E| \leq \|V\|_\infty + 1\}$, which is the resonant set. Using the separation property of L over $\mathbb{Z}^2$, the local Hamiltonians can be reduced to effective matrices $\mathcal{M}$ of rank at most κ, where κ is uniform in E.

To investigate $\mathcal{M}$, we first exclude a geometric singular set:

$$\{(m,n) | \, |m\alpha + n\beta| < K, \, (0,0) \neq (\alpha, \beta) \in \mathbb{Z}^2, \, |\alpha|, \, |\beta| \leq ck, \, \mathbb{Z} \ni c, \, K > 1\},$$

which includes rays determined by the Fourier support of V: supp $\hat{V}$. For $\mathcal{M}$ which do not involve resonant sites in the geometric singular set, the sites are at least at a distance ck apart. So we can approximate $\mathcal{M}$ by a direct sum of scalar (1×1 matrices) functions. These scalars M correspond to the same function, but at different angles θ, which in turn enable us to make 1 variable polynomial approximations of $\det \mathcal{M}$ leading to the genericity conditions on V.

We define the geometric support of $\hat{V}$ to be

$$\text{gsupp } \hat{V} \stackrel{\text{def}}{=} \{(a,b) \in \mathbb{Z}^2 \backslash \{0\} | \exists s \geq 1 \text{ such that } (sa, sb) \in \text{supp } \hat{V}\},$$

and

$$v_{a,b} \stackrel{\text{def}}{=} \sum_{1\le s\le k} |\hat{V}(sa,sb)|^2(a^2+b^2), \quad (a,b)\in \text{gsupp } \hat{V},$$

$$g_{a,b;c,d} \stackrel{\text{def}}{=} \sum_{\substack{-k\le s,s'\le k\\ s,s'\neq 0}} \frac{\hat{V}(sa,sb)\hat{V}(s'c-sa,s'd-sb)\hat{V}(s'c,s'd)}{4s^2s'}, \quad (a,b),(c,d)\in \text{gsupp } \hat{V}.$$

Let

$$f=(1+x^2)\Big(\sum_{(a,b)\in\text{gsupp }\hat{V}} v_{a,b}\frac{ax-b}{(a+bx)^3}+\sum_{(a,b),(c,d)\in\text{gsupp }\hat{V}} g_{a,b;c,d}\frac{ax-b}{(a+bx)^2(c+dx)}\Big),$$
$$\text{where } a+bx\neq 0,\ c+dx\neq 0; \tag{1.13}$$

or

$$f=(1+x^2)\Big(\sum_{(a,b)\in\text{gsupp }\hat{V}} v_{a,b}\frac{a-bx}{(ax+b)^3}+\sum_{(a,b),(c,d)\in\text{gsupp }\hat{V}} g_{a,b;c,d}\frac{a-bx}{(ax+b)^2(cx+d)}\Big),$$
$$\text{where } ax+b\neq 0,\ cx+d\neq 0. \tag{1.14}$$

Both f are rational functions and can be written as

$$f=\frac{P_1}{P_2} \tag{1.15}$$

with P_1, P_2 polynomials in x of degrees at most $\mathcal{O}(k^4)$ and whose coefficients only depend on $\hat{V}$ and supp $\hat{V}$.

Definition. V is generic, if

(i)

$$\sum_{1\le s\le k}|\hat{V}(sa,sb)|^2(a^2+b^2)-\sum_{1\le |s|,|s'|\le k}\frac{\hat{V}(sa,sb)\hat{V}(s'a,s'b)\hat{V}((s-s')a,(s-s')b))}{4s^2s'}\neq 0.$$

(ii) Resultant $(P_1,P_1')\neq 0$ for both P_1 defined from (1.13-1.15).

(i, ii) show that $\cos x\cos y$ are indeed generic as claimed earlier.

The analysis of the derivative of the scalar function M uses the resolvent expansion. The generic condition (i) ensures that when θ is close to the angle of a ray in the geometric support of $\hat{V}$, the first two terms dominate and the derivative is away from zero, cf. (4.4, 4.5).

When θ is otherwise, we make polynomial approximations. The genericity condition (ii) comes from requiring both P_1 to have only simple zeroes so that the excised set contains at most $\mathcal{O}(k^4)$ sites. In studying these polynomials, we also used a second separation property, namely, if v_1 and v_2 are two non colinear vectors in the Fourier support of V, then the angle between them is of order 1. (For more details, see sect. 4.)

Consequently we show that for generic V, there are at most $\mathcal{O}(k^4)$ local eigenvalues which are "close" to any given E. Using again the separation property of the resonant sites in the level sets L over $\mathbb{Z}^2$ mentioned above, we prove the Theorem.

We note that when V has separation of variables, i.e., $V(x,y) = V_1(x) + V_2(y)$, the geometric support of $\hat{V}$ is of dimension 2 and the relevant polynomials can be written in terms of $x = m^2$ only, where m is the horizintal coordinate, and are of uniformly bounded degree independent of k, cf., (4.13, 4.19- 4.21). It is easy to show that the multiplicity of the eigenvalues are uniformly bounded in R and hence agree with the known results.

The scheme presented here to localize an individual eigenfunction is essentially the general one. Moreover it is intrinsically independent of self-adjointness. Instead it relies on the geometry of the Fourier support of V, which is more intrinsic. In higher dimensions, there are counterparts to the techniques used here, which might be worth pursuing.

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2. Partition of the annuli and singular set

Let

$$H = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + V \tag{2.1}$$

on $L^2([-\pi,\pi]^2)$ with periodic boundary condition as in section 1. Since V is a real trigonometric polynomial of degree k, the Fourier series satisfies

$$\text{supp } \hat{V} \subset \{(a,b) \in \mathbb{Z}^2 \| a|, |b| \le k\}.$$

Without loss, we may assume $\hat{V}(0,0) = 0$. Otherwise it attributes an overall constant. So

$$\text{supp } \hat{V} \subset \{(a,b) \in \mathbb{Z}^2 \backslash \{(0,0)\} \| a|, |b| \le k\}. \tag{2.2}$$

H is unitarily equivalent to

$$\hat{H} = \text{diag } (m^2 + n^2) + \hat{V} * \tag{2.3}$$

on $\ell^2(\mathbb{Z}^2)$, which is the operator that we will work with in the rest of the paper. From now on we write H for $\hat{H}$ and $\hat{V}$ for $\hat{V}*$.

Assume E is an eigenvalue of H,

$$E \in R + [-1/2, 1/2] \tag{2.4}$$

for some $\mathbb{Z} \ni R \ge -\|V\|_\infty$. To deduce localization properties of the eigenfunctions, we only need to be concerned with the annulus:

$$S \stackrel{\text{def}}{=} \{(m,n) \in \mathbb{Z}^2 \| m^2 + n^2 - R| \le \|V\|_\infty + 1\}, \tag{2.5}$$

as

$$\|(H_{\mathbb{Z}^2\setminus S}-E)^{-1}\|\leq 2, \tag{2.6}$$

where for any set A, $A\subset\mathbb{Z}^2$, H_A denotes the restricted operator:

$$\begin{aligned} H_A(i,j) &= H(i,j) \quad (i,j)\in A\times A, && (2.7)\\ &= 0, \qquad\qquad \text{otherwise.} && (2.8)\end{aligned}$$

The following separation property plays a crucial role:

Lemma 2.1. *Let S' be the annulus over $\mathbb{R}^2$: $|x^2+y^2-R|\leq\|V\|_\infty+1$, $R\in\mathbb{N}$. There exist $\mathbb{N}\ni\kappa>0$ (uniform in R) and Π a partition of S' such that if $\mathbb{R}^2\supset p\in\Pi$, then*

- $|p\cap S'|=\mathcal{O}(R^{1/6})$, (2.9)
- $\#\{p\cap S\}\leq\kappa$, (2.10)
- *dist* $(\{p\cap S\},\partial p)=\mathcal{O}(R^{1/6})$, (2.11)

where $||$ in (2.9) denotes the length.

Remark. It follows from (2.9, 2.10) that $\#S\leq\mathcal{O}(R^{1/3})$. Estimates on the divisor function give a better bound $\#S\ll R^\epsilon$ for all $\epsilon>0$ (leading in particular to (1.11)), but with no geometric information on the integers (m,n).

Proof. We use the argument of Janick [J], which extends to all stritly convex annuli, cf. [CW]. For completeness we reproduce the proof for the circular annuli.

We first let A_1, A_2 and A_3 be 3 integers (in this order) on the circle $\tilde{S}$ over $\mathbb{R}^2$ centered at $O=(0,0)$ of radius $R^{1/2}$, $A_1\neq A_2\neq A_3$. In view of (2.9), it suffices to assume that max $(|A_1A_2|,|A_2A_3|)\leq\mathcal{O}(R^{1/6})$. Since they are not colinear, the area S_1 of the triangle formed by A_1, A_2 and A_3 satisfy

$$\mathbb{N}/2\ni S_1\geq 1/2. \tag{2.12}$$

From convexity the area S_2 formed by the arc $A_1A_2A_3$ and the straight segment A_1A_3 satisfies

$$S_2\geq S_1\geq 1/2.$$

But

$$S_2=\frac{\theta}{2\pi}\cdot\pi R-\frac{1}{2}R\sin\theta\asymp R\theta^3\geq\frac{1}{2}, \tag{2.13}$$

where θ is the angle formed by OA_1 and OA_3. So $\theta\geq\mathcal{O}(1/R^{1/3})$ and

$$|A_1A_3|\geq\mathcal{O}(R^{1/6}) \tag{2.14}$$

using that the radius is $R^{1/2}$.

We now let A_1, A_2 and A_3 be any 3 non colinear integers in S', $A_1\neq A_2\neq A_3$ and max $(|A_1A_2|,|A_2A_3|)\leq\mathcal{O}(R^{1/6})$. (2.12) holds. Let $A_j'=OA_j\cap\tilde{S}$, $j=1,2,3$. The area S_1' formed by A_j' satisfies: $S_1' \geq 1/2 - \mathcal{O}(1)R^{1/6} \cdot R^{-1/2} > 1/4$ using bilinearity. So (2.14) holds. Since the number of colinear integers in S' is bounded (uniformly in R), (2.9-2.11) follow by choosing the $\mathcal{O}(R^{1/6})$ smaller than that in (2.14). []

Assume $p \in \Pi$ is such that $p \cap S \neq \emptyset$. Let H_p be defined as in (2.8), where for simplicity we also used p to denote $p \cap \mathbb{Z}^2$. In section 3, we reduce the study of $\sigma(H_p) \cap R + [-1/2, 1/2]$ to that of an effective matrix $\mathcal{M}$, where $\mathcal{M}$ is at most a $\kappa \times \kappa$ matrix.

To further the analysis, we need to examine the sets $p \in \Pi$.

Lemma 2.2. *Let $(x, y) \in \mathbb{R}^2$ satisfy*

$$|x\alpha + y\beta| \leq K, \qquad (K > 0, \text{ independent of } R) \tag{2.15}$$

for some $(\alpha, \beta) \in \mathbb{Z}^2 \backslash \{0\}$, $|\alpha|, |\beta| \leq ck$, ($\mathbb{N} \ni c > 1$) and k is the degree of the polynomial V. Let

$$\Pi' = \{p \in \Pi | (2.15) \text{ is violated on } p \cap S'\}, \tag{2.16}$$

where S' is as defined in Lemma 2.1. Then

$$|\Pi \backslash \Pi'| \leq 17c^2k^2. \tag{2.17}$$

Assume $K > c^2k^2 + \|V\|_\infty + 1$, we have more over that for $p \in \Pi'$, if (m, n), $(m', n') \in p \cap S'$ satisfying $(m - m', n - n') \in \mathbb{Z}^2 \backslash \{0\}$, then

$$\text{sup } (|m - m'|, |n - n'|) > ck. \tag{2.18}$$

Proof. Since α and β are integers, (2.15) contains at most $(2ck + 1)^2$ tubes T bounded by the straight lines

$$x\alpha + y\beta = \pm K. \tag{2.19}$$

Since for each T, $T \cap S'$ contains 2 "arcs" of length $\mathcal{O}(2K + 1) \ll \mathcal{O}(R^{1/6})$, it can intersect at most 4 $p \in \Pi$ in view of (2.9), which leads to (2.17).

To prove (2.18), write $m' - m = \alpha$, $n' - n = \beta$, $(\alpha, \beta) \neq (0, 0)$. We have

$$\begin{cases} m^2 + n^2 = R'', \\ (m + \alpha)^2 + (n + \beta)^2 = R', \end{cases} \tag{2.20}$$

with $|R' - R''| \leq 2(\|V\|_\infty + 1)$. So

$$|m\alpha + n\beta| = \frac{1}{2}|(R' - R'') - (\alpha^2 + \beta^2)|. \tag{2.21}$$

On the other hand, since $p \in \Pi'$, for all $\mathbb{Z}^2 \ni (\alpha, \beta) \neq (0, 0)$, $|\alpha|, |\beta| \leq ck$,

$$|m\alpha + n\beta| > K > c^2k^2 + \|V\|_\infty + 1. \tag{2.22}$$

(2.21, 2.22) imply that

$$\sup (|\alpha|, |\beta|) = \sup (|m - m'|, |n - n'|) > ck.$$

[]

From now on, Π' is to denote the set satisfying (2.16) with $K > c^2k^2 + \|V\|_\infty + 1$.

Remark. It is important to note that (2.15, 2.17) are *independent* of R. They only depend on the degree k of the trigonometric polynomial V. $\Pi \backslash \Pi'$ contains the "singular" set. The effective Hamiltonian reduction will only be used in Π', where the resonant sites are at least at a distance ck apart.

3. Effective Hamiltonian and reduction to scalar

We now assume $p \in \Pi'$ and use the Schur complement reduction [Sc1, 2] to investigate $\sigma(H_p) \cap [R-1/2, R+1/2]$, $R \in \mathbb{N}$, where H_p is as defined in (2.8). Let $S = \{(m,n) \in \mathbb{Z}^2 | \, |m^2+n^2-R| \le \|V\|_\infty + 1\}$. Assume $p \cap S \neq \emptyset$. Let P be the projection onto $p \cap S$ and P_c onto $p \backslash S$.

We have the following equivalence relation:

$$E \in \sigma(H_p) \cap [R-1/2, R+1/2] \iff 0 \in \sigma(\mathcal{M}), \tag{3.1}$$

where

$$\mathcal{M} = E - PH_pP + PH_pP_c(E - P_cH_pP_c)^{-1}P_cH_pP, \tag{3.2}$$

cf. [Sect. 2.3, SZ]. Since Rank $P \le \kappa$, $\mathcal{M}$ is at most rank κ, i.e., a $\kappa \times \kappa$ matrix. Moreover $\mathcal{M}$ is analytic in E for $E \in (R-1/2, R+1/2)$. Since $p \in \Pi'$, in view of (2.18), the first two terms in (3.2) are diagonal. In the following, we view E as a parameter.

Assume $c > 8$ in (2.18). For all $i \in p \cap S$, define:

$$\begin{aligned}
\Lambda_0 &= i + [-4k-1, 4k+1]^2, \qquad \text{(So } \Lambda_0 \cap S = \{i\}.\text{)} && (3.3)\\
M_0 &= H_{\Lambda_0 \backslash \{i\}}, && (3.4)\\
M'_{ii} &= E - |i|^2 + [\hat{V}(E-M_0)^{-1}\hat{V}](i,i), && (3.5)\\
M' &= M'_{ii}, \qquad\qquad \text{if } |p \cap S| = 1,\\
&= \oplus_i M'_{ii}, \qquad\quad \text{if } |p \cap S| \ge 2. && (3.6)
\end{aligned}$$

M' is analytic in E for $E \in (R-1/2, R+1/2)$.

Proposition 3.1. *For $p \in \Pi'$,*

$$\begin{aligned}
\|(\mathcal{M} - M')\pi_i\|_{\ell^2 \to \ell^2} &\le \mathcal{O}(1) \sum_{(a_\ell, b_\ell) \in \text{supp } \hat{V}} \frac{1}{\prod_{\ell=1}^{8} |ma_\ell + nb_\ell + \frac{a_\ell^2 + b_\ell^2}{2} - \frac{\lambda}{2}|} && (3.7)\\
&< \mathcal{O}(\frac{k^{16}}{K^8}), && (3.8)
\end{aligned}$$

for all $E \in [R-1/2, R+1/2]$, where $i \in p \cap S$, $i = (m,n)$, $\lambda = E - |i|^2$ and π_i is the projection onto δ_i, provided $K > c^2k^2 + c\|V\|_\infty + 1$ ($\mathbb{N} \ni c > 8$).

Remark. It is important to note that the right side of (3.7) only depends on $i = (m,n)$ and supp $\hat{V}$.

The following lemma is crucial to prove the proposition, in fact to all subsequent analysis. We first define a few notions. For $(m,n) \in S' = \{(x,y) | \, |x^2+y^2-R| \le \|V\|_\infty + 1\}$, write $\bar{m} = (m,n)$. Let

$$\mathbb{Z}^2_{\bar{m}} \overset{\text{def}}{=} \bar{m} + \mathbb{Z}^2. \tag{3.9}$$

For $j \in \mathbb{Z}^2_{\bar{m}} \backslash \{\bar{m}\}$, define

$$D_{jj} = E - |j|^2 \tag{3.10}$$
$$D = \text{diag } D_{jj}. \tag{3.11}$$

Assume D^{-1} exists and define

$$\begin{aligned} F(\bar{a}) =& \hat{V} D^{-1}(\bar{m}, \bar{m} + \bar{a}) \\ \stackrel{\text{def}}{=}& \hat{V} D^{-1}(\cdot, \cdot + \bar{a}), \qquad \bar{a} \in \text{supp } \hat{V}; \end{aligned} \tag{3.12}$$

$$\begin{aligned} F(\bar{a}_1, \bar{a}_2) = & \quad \hat{V} D^{-1}(\cdot, \cdot + \bar{a}_1) \hat{V} D^{-1}(\cdot + \bar{a}_1, \cdot + \bar{a}_1 + \bar{a}_2) \\ & + \hat{V} D^{-1}(\cdot, \cdot + \bar{a}_2) \hat{V} D^{-1}(\cdot + \bar{a}_1, \cdot + \bar{a}_1 + \bar{a}_2) \\ \stackrel{\text{def}}{=} & \sum_{\text{perm } (\bar{a}_1, \bar{a}_2)} \hat{V} D^{-1}(\cdot, \cdot + \bar{a}_1) \hat{V} D^{-1}(\cdot + \bar{a}_1, \cdot + \bar{a}_1 + \bar{a}_2), \qquad \bar{a}_1, \bar{a}_2 \in \text{supp } \hat{V}. \end{aligned} \tag{3.13}$$

$$\vdots$$

$$\begin{aligned} F(\bar{a}_1, \bar{a}_2, \cdots \bar{a}_s) = & \sum_{\text{perm } (\bar{a}_1, \bar{a}_2 \cdots \bar{a}_s)} \hat{V} D^{-1}(\cdot, \cdot + \bar{a}_1) \hat{V} D^{-1}(\cdot + \bar{a}_1, \cdot + \bar{a}_1 + \bar{a}_2) \\ & \cdots \hat{V} D^{-1}(\cdot + \sum_{\ell=1}^{s-1} \bar{a}_\ell, \cdot + \sum_{\ell=1}^{s} \bar{a}_\ell), \quad \bar{a}_1, \bar{a}_2, \cdots \bar{a}_s \in \text{supp } \hat{V}. \end{aligned} \tag{3.14}$$

Lemma 3.2. *Assume $\bar{m} = (m, n) \in \Pi' \cap S'$ and increase K to $K > c^2 k^2 + c\|V\|_\infty + 1$ ($\mathbb{N} \ni c > 8$), so $|m\alpha + n\beta| > K > c^2 k^2 + c\|V\|_\infty + 1$, for all $(\alpha, \beta) \in \mathbb{Z}^2 \backslash \{0\}$, $|\alpha|$, $|\beta| \leq ck$. Then*

$$|F(\bar{a}_1, \bar{a}_2, \cdots \bar{a}_s)| \leq \mathcal{O}(1) \prod_{\ell=1}^{s} \frac{|\hat{V}(a_\ell, b_\ell)|}{|ma_\ell + nb_\ell + \frac{a_\ell^2 + b_\ell^2}{2} - \frac{\lambda}{2}|}, \tag{3.15}$$

where $\bar{a}_1, \bar{a}_2, \cdots \bar{a}_s \in$ supp $\hat{V}$, $1 \leq s \leq c$ and $\lambda = E - m^2 - n^2$ as in (3.7).

Proof. When $s = 1$, (3.15) follows from the definition (3.12). When $s = 2$

$$\begin{aligned} 4F(\bar{a}_1, \bar{a}_2) = & \frac{\hat{V}(a_1, b_1) \hat{V}(a_2, b_2)}{(ma_1 + nb_1 + \frac{a_1^2 + b_1^2}{2} - \frac{\lambda}{2})(m(a_1 + a_2) + n(b_1 + b_2) + \frac{(a_1 + a_2)^2 + (b_1 + b_2)^2}{2} - \frac{\lambda}{2})} \\ & + \frac{\hat{V}(a_2, b_2) \hat{V}(a_1, b_1)}{(ma_2 + nb_2 + \frac{a_2^2 + b_2^2}{2} - \frac{\lambda}{2})(m(a_1 + a_2) + n(b_1 + b_2) + \frac{(a_1 + a_2)^2 + (b_1 + b_2)^2}{2} - \frac{\lambda}{2})}. \end{aligned} \tag{3.16}$$

To simplify notation, let

$$A_\ell = ma_\ell + nb_\ell + \frac{a_\ell^2 + b_\ell^2}{2} - \frac{\lambda}{2}, \tag{3.17}$$

and more generally

$$A_{\ell_1\cdots\ell_s} = m\sum_{\ell=1}^{s} a_\ell + n\sum_{\ell=1}^{s} b_\ell + \frac{(\sum_{\ell=1}^{s} a_\ell)^2 + (\sum_{\ell=1}^{s} b_\ell)^2}{2} - \frac{\lambda}{2}. \tag{3.18}$$

So

$$\begin{aligned}(3.16) &= \hat{V}(a_1,b_1)\hat{V}(a_2,b_2)[\frac{1}{A_1A_{12}} + \frac{1}{A_2A_{12}}] \\ &= \hat{V}(a_1,b_1)\hat{V}(a_2,b_2)[\frac{1}{A_1A_2} + \frac{\mathcal{O}(1)}{A_1A_2A_{12}}].\end{aligned} \tag{3.19}$$

Taking the absolute value, we obtain (3.15) for $s = 2$.

We now make an induction on s. Assume

$$F(\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_s) = \prod_{\ell=1}^{s} \hat{V}(a_\ell, b_\ell)\cdot[\frac{1}{A_1A_2\cdots A_s} + \frac{\mathcal{O}(1)}{A_1A_2\cdots A_sA_{1\cdots s}}] \tag{3.20}$$

holds at s. To arrive at $s+1$, we write

$$F(\bar{a}_1, \bar{a}_2, \cdots \bar{a}_{s+1}) = \sum_\sigma F(\sigma)\frac{\hat{V}(\sigma^c)}{A_{1\cdots s+1}}, \tag{3.21}$$

where $\sigma \subset \{\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_{s+1}\}$, $|\sigma| = s$, σ^c is the complement, which only has one element. Using (3.20) for $F(\sigma)$, we obtain

$$\begin{aligned}F(\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_{s+1}) &= \prod_{\ell=1}^{s+1} \hat{V}(a_\ell, b_\ell)\cdot \\ &[\frac{A_1 + A_2 + \cdots + A_{s+1}}{A_1A_2\cdots A_sA_{s+1}A_{1\cdots s+1}} + \sum_\sigma \frac{\mathcal{O}(1)}{(\prod_{\ell_s\in\sigma} A_{\ell_s})A_\sigma A_{1\cdots s+1}}].\end{aligned} \tag{3.22}$$

$$A_1 + A_2 + \cdots + A_{s+1} = A_{1\cdots s+1} + \mathcal{O}(1) \tag{3.23}$$

and since

$$\frac{1}{A_\sigma A_{1\cdots s+1}} = [\frac{1}{A_\sigma} - \frac{1}{A_{1\cdots s+1}}]\cdot\frac{\mathcal{O}(1)}{A_{\sigma^c}}, \tag{3.24}$$

(3.22) gives

$$F(\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_{s+1}) = \prod_{\ell=1}^{s+1} \hat{V}(a_\ell, b_\ell)\cdot[\frac{1}{A_1A_2\cdots A_{s+1}} + \frac{\mathcal{O}(1)}{A_1A_2\cdots A_{s+1}A_{1\cdots s+1}}]. \tag{3.25}$$

Increasing K to $K > c^2k^2 + c\|V\|_\infty + 1$ in view of the $\mathcal{O}(1)$ in (3.23) and taking the absolute value, we obtain (3.15). []

Proof of Proposition 3.1. Assume $|p \cap S| \geq 2$, otherwise set $\mathcal{M}_{ij} = 0$ ($i \neq j, i, j \in \{p \cap S\}$) in the argument below. Let

$$\begin{aligned}&i \in p\cap S,\, \Lambda_0^c = p\backslash\{\Lambda_0 \cup \{p\cap S\}\}, \\ &M^c = P_cH_pP_c,\, M_0^c = H_{\Lambda_0^c}, \\ &\Gamma = M^c - (M_0 \oplus M_0^c),\end{aligned} \tag{3.26}$$

where Λ_0 and M_0 as defined in (3.3, 3.4).

Using (3.2, 3.5, 3.26) and the resolvent equation, we have

$$\begin{aligned}&\mathcal{M}_{ii} - M'_{ii} = \hat{V}(E-M_0)^{-1}\Gamma(E-M^c)^{-1}\hat{V}\\ &=\sum_{i',i''}[\hat{V}D^{-1}]^4(i,i')[\hat{V}(E-M_0)^{-1}\Gamma(E-M^c)^{-1}\hat{V}](i',i'')[D^{-1}\hat{V}]^4(i'',i),\end{aligned}\tag{3.27}$$

where we used the fact that (3.3) implies

$$\text{dist }(i, \text{supp }\Gamma) > 3k. \tag{3.28}$$

Using (3.15) for $s=4$ and

$$\begin{aligned}&\|(E-M_0)^{-1}\| = \mathcal{O}(1/K),\\ &\|(E-M^c)^{-1}\| = \mathcal{O}(1),\\ &\|\Gamma\| = \mathcal{O}(1),\end{aligned}$$

we obtain

$$|\mathcal{M}_{ii} - M'_{ii}| \leq \mathcal{O}(1/K)\cdot \Big[\frac{(2k+1)^8}{4!}\Big]^2 \cdot \sup{}_{(a_\ell,b_\ell)\in \text{supp }\hat{V}} \prod_{\ell=1}^{8} \frac{1}{|ma_\ell + nb_\ell + \frac{a_\ell^2+b_\ell^2}{2} - \frac{\lambda}{2}|}, \tag{3.29}$$

where $(m,n)=i$. Similarly,

$$\mathcal{M}_{ij} = \sum_{i'}[\hat{V}D^{-1}]^8(i,i')[\hat{V}(E-M^c)^{-1}\hat{V}](i',j), \quad i\neq j,\, i,j\in\{p\cap S\}, \tag{3.30}$$

where we used $|i-j|_\infty > ck > 8k$. So

$$|\mathcal{M}_{ij}| \leq \frac{(2k+1)^{16}}{8!}\min{}_{(m,n)=i,j}\cdot \sup{}_{(a_\ell,b_\ell)\in \text{supp }\hat{V}} \prod_{\ell=1}^{8} \frac{1}{|ma_\ell + nb_\ell + \frac{a_\ell^2+b_\ell^2}{2} - \frac{\lambda}{2}|} \tag{3.31}$$

using (3.15) and $\mathcal{M}_{ij} = \mathcal{M}_{ji}$. (3.29, 3.31) imply (3.7, 3.8). []

The scalar Hamiltonian.

Let

$$\begin{aligned}M_{ii} &= M'_{ii} - E + |i|^2\\ &= M'_{ii} - \lambda\\ &= [\hat{V}(E-M_0)^{-1}\hat{V}](i,i), \quad i\in p\cap S\subset \mathbb{Z}^2,\end{aligned}\tag{3.32}$$

from (3.3-3.5). M_{ii} is the scalar Hamiltonian that we will study in detail in section 4. Here it suffices to note that for fixed E and $|i|$, M_{ii} is only a function of the angle θ: $M_{ii} = M_{ii}(\theta)$. Moreover for $i\in p\cap S'\subset\mathbb{R}^2$, defining Λ_0 as in (3.3), $\Lambda_0\subset\mathbb{Z}_i^2 = i+\mathbb{Z}^2$, M_0 defined in (3.4) extends to a matrix on $\ell^2(\Lambda_0)$.

For i such that $\Lambda_0 \subset p \in \Pi'$, $(E-M_0)^{-1}$ is well defined for $E \in [R-1/2, R+1/2]$ with

$$\|(E-M_0)^{-1}\|_{\ell^2(\Lambda_0\backslash\{i\})} \leq \mathcal{O}(1/K). \tag{3.33}$$

This is because for any two points (m', n'), $(m'', n'') \in \Lambda_0$, assuming $(m', n') \neq (m'', n'')$, one writes $m'' = m' + \alpha$, $n'' = n' + \beta$ with $\mathbb{Z}^2 \ni (\alpha, \beta) \neq 0$. Assuming

$$\begin{aligned} {m'}^2 + {n'}^2 &= R', \\ {m''}^2 + {n''}^2 &= R'', \end{aligned} \tag{3.34}$$

one obtains

$$|m'\alpha + n'\beta| = \frac{1}{2}|R' - R'' - (\alpha^2+\beta^2)| > K > c^2k^2 + c\|V\|_\infty + 1, \tag{3.35}$$

since $p \in \Pi'$and we used the larger K from Proposition 3.1.

From (3.3), sup $(|\alpha|, |\beta|) \leq 8k+2$, so $(\alpha^2+\beta^2)/2 \leq (8k+2)^2$. Using this in (3.35) we have $|R'-R''| > 2(K-(8k+2)^2) > 2((c^2-65)k^2 + c\|V\|_\infty + 1)$. Since $i \in \Lambda_0$ satisfies $||i|^2 - R| \leq \|V\|_\infty + 1$ and $\mathbb{N} \ni c > 8$, this proves (3.33). We now view M_{ii} as a function of θ, defined on appropriate arcs of the circle $S_i' = \{(x,y)|\,|x^2+y^2 = |i|^2\} \subset \mathbb{R}^2$.

4. Polynomial approximation and generic V

In this section, we investigate the scalar Hamiltonian

$$M_{ii} = [\hat{V}(E-M_0)^{-1}\hat{V}](i,i) \tag{4.1}$$

as defined in (3.32, 3.3-3.5). From (3.33), $|M_{ii}| \leq \mathcal{O}(1/K)$, where $K > c^2k^2 + c\|V\|_\infty + 1$ and $\mathbb{N} \ni c > 8$. Since $E \in [R-1/2, R+1/2]$, for the purpose of this section, we only need to consider i such that $i = (m,n) = (\sqrt{R}\cos\theta, \sqrt{R}\sin\theta)$. Let $\tilde{S} = \{(m,n) \in \mathbb{R}^2 | m^2+n^2 = R\}$ and D the diagonal part of $E - M_0$ as before:

$$D_{\ell\ell} = E - |\ell|^2 = R + \lambda - |\ell|^2, \quad \lambda \in [-1/2, 1/2]. \tag{4.2}$$

Writing ∂ for $\partial_\theta D$ and using the resolvent equation, we have

$$\frac{\partial M_{ii}}{\partial\theta} = [\hat{V}(E-M_0)^{-1}(\partial)(E-M_0)^{-1}\hat{V}](i,i) \tag{4.3}$$

$$= (\hat{V}D^{-1})\partial(D^{-1}\hat{V}) \tag{4.4}$$

$$+ [(\hat{V}D^{-1})\partial(D^{-1}\hat{V})^2 + (\hat{V}D^{-1})^2\partial(D^{-1}\hat{V})] \tag{4.5}$$

$$+ [(\hat{V}D^{-1})^2\partial(D^{-1}\hat{V})^2 + (\hat{V}D^{-1})\partial(D^{-1}\hat{V})^3 + (\hat{V}D^{-1})^3\partial(D^{-1}\hat{V})] \tag{4.6}$$

$$\begin{aligned} &+ [(\hat{V}D^{-1})\partial((E-M_0)^{-1}\hat{V})(D^{-1}\hat{V})^3 + (\hat{V}D^{-1})^3(\hat{V}(E-M_0)^{-1})\partial(D^{-1}\hat{V}) \\ &\quad + (\hat{V}D^{-1})^2\partial(D^{-1}\hat{V})^3 + (\hat{V}D^{-1})^3\partial(D^{-1}\hat{V})^2] \end{aligned} \tag{4.7}$$

$$\begin{aligned} &+ [(\hat{V}D^{-1})^2\partial((E-M_0)^{-1}\hat{V})(D^{-1}\hat{V})^3 + (\hat{V}D^{-1})^3(\hat{V}(E-M_0)^{-1})\partial(D^{-1}\hat{V})^2 \\ &\quad + (\hat{V}D^{-1})^3\partial(D^{-1}\hat{V})^3] \end{aligned} \tag{4.8}$$

$$+ [(\hat{V}D^{-1})^3\partial((E-M_0)^{-1}\hat{V})(D^{-1}\hat{V})^3 + (\hat{V}D^{-1})^3(\hat{V}(E-M_0)^{-1})\partial(D^{-1}\hat{V})^3] \tag{4.9}$$

$$+ [(\hat{V}D^{-1})^3(\hat{V}(E-M_0)^{-1})\partial((E-M_0)^{-1}\hat{V})(D^{-1}\hat{V})^3], \tag{4.10}$$

where it is understood that (4.4-4.10) pertain to the (i,i) entry. It follows immediately from Lemma 3.2 and $\|\partial_\theta D\| = \mathcal{O}(\sqrt{R})$:

Lemma 4.1.

$$\frac{1}{\sqrt{R}}|[(4.6)+\cdots+(4.10)]| \leq \mathcal{O}(1)\sum_{(a_\ell,b_\ell)\in\text{supp }\hat{V}}\frac{1}{\prod_{\ell=1}^{4}|ma_\ell+nb_\ell+\frac{a_\ell^2+b_\ell^2}{2}-\frac{\lambda}{2}|}, \quad (4.11)$$

where $(m,n)=i$.

The rest of this section is devoted to estimate the main terms (4.4, 4.5). Before that we first estimate (4.1), which gives an appoximation to λ.

Lemma 4.2.

$$|M_{ii}| \leq \mathcal{O}(1)\Big[\sum_{(a_\ell,b_\ell)\in\text{supp }\hat{V}}\frac{1}{|ma_\ell+nb_\ell|}\Big]^2, \quad (4.12)$$

where $(m,n)=i$ *and* $|ma_\ell+nb_\ell|>K>c^2k^2+c\|V\|_\infty+1$ ($\mathbb{N}\ni c>8$).

Proof. Using the resolvent equation, we have

$$M_{ii}=\hat{V}D^{-1}\hat{V}+\hat{V}D^{-1}\hat{V}D^{-1}\hat{V}+\hat{V}D^{-1}\hat{V}(E-M_0)^{-1}\hat{V}D^{-1}\hat{V}, \quad (4.13)$$

where the right side only refers to the (i,i) entry.

For any (a,b), $(a',b')\in\mathbb{Z}^2\backslash\{0\}$, we say $(a,b)\sim(a',b')$ if $(a,b)=s(a',b')$ or $(a',b')=s(a,b)$, $s\in\mathbb{Z}\backslash\{0\}$. We call this equivalent class $\mathcal{C}_{a,b}$ if (a,b) is such that $a\geq 0$, $a+b\geq 0$ and $|(a,b)|\leq|(a',b')|$ for all (a',b') such that $(a,b)\sim(a',b')$. We define the geometric support of $\hat{V}$ to be

$$\text{gsupp }\hat{V}\stackrel{\text{def}}{=}\{(a,b)\in\mathbb{Z}^2\backslash\{0\}|\exists s\geq 1 \text{ such that } (sa,sb)\in \text{ supp }\hat{V}\}. \quad (4.14)$$

Assume $(a,b)\in\text{gsupp }\hat{V}$, we define

$$\lambda_{a,b}=\sum_{(a'b')\in\mathcal{C}_{a,b}}\hat{V}(a',b')D^{-1}(i+(a',b'),i+(a',b'))\hat{V}(-a',-b') \quad (4.15)$$

$$=\sum_{\substack{s\geq 1\\(sa,sb)\in\text{supp }\hat{V}}}\hat{V}(sa,sb)D^{-1}(i+(sa,sb),i+(sa,sb))\hat{V}(-sa,-sb) \quad (4.16)$$

$$+\hat{V}(-sa,-sb)D^{-1}(i-(sa,sb),i-(sa,sb))\hat{V}(sa,sb) \quad (4.17)$$

Using the above, we have

$$\hat{V}D^{-1}\hat{V}=\sum_{(a,b)\in\text{gsupp }\hat{V}}\lambda_{a,b}, \quad (4.18)$$

where

$$\lambda_{a,b}=\sum_{s\geq 1}|\hat{V}(sa,sb)|^2\Big[-\frac{1}{2sam+2sbn+s^2a^2+s^2b^2-\lambda}+\frac{1}{2sam+2sbn-(s^2a^2+s^2b^2-\lambda)}\Big]$$

$$=\sum_{s\geq 1}\frac{|\hat{V}(sa,sb)|^2\cdot(s^2a^2+s^2b^2-\lambda)}{2s^2}\cdot\frac{1}{(ma+nb)^2-(\frac{s^2a^2+s^2b^2-\lambda}{2s})^2}, \quad (4.19)$$

$|a|$, $|b| \le k$, $(a,b) \ne (0,0)$ and $\lambda \in [-1/2, 1/2]$. So

$$\hat{V} D^{-1} \hat{V} = \mathcal{O}(1) \sum_{(a_\ell, b_\ell) \in \text{gsupp } \hat{V}} \frac{1}{(ma_\ell + nb_\ell)^2}, \tag{4.20}$$

if $|ma_\ell + nb_\ell| > K > c^2k^2 + c\|V\|_\infty + 1$ ($\mathbb{N} \ni c > 8$).

The second and third terms in the right side of (4.13) are bounded above by

$$\mathcal{O}(1)\Big[\sum_{(a_\ell, b_\ell) \in \text{supp } \hat{V}} \frac{1}{|ma_\ell + nb_\ell|}\Big]^2 \text{ and } \mathcal{O}(1/K)\Big[\sum_{(a_\ell, b_\ell) \in \text{supp } \hat{V}} \frac{1}{|ma_\ell + nb_\ell|}\Big]^2. \tag{4.21}$$

(4.20, 4.21) imply (4.12). []

Since (4.4) $= \frac{\partial}{\partial \theta}$(4.18), we take the derivative of (4.19) and have

$$\begin{aligned} \frac{1}{\sqrt{R}} \frac{\partial}{\partial \theta} \lambda_{a,b} = &- \sum_{1 \le s \le k} \frac{|\hat{V}(sa, sb)|^2 (s^2a^2 + s^2b^2 - \lambda)}{s^2} \\ &(a \sin\theta - b\cos\theta) \cdot \frac{(ma+nb)}{[(ma+nb)^2 - (\frac{s^2a^2+s^2b^2-\lambda}{2s})^2]^2}, \end{aligned} \tag{4.22}$$

$|a|$, $|b| \le k$, $(a,b) \ne (0,0)$.

For a fixed $(a,b) \in \text{gsupp } \hat{V}$, (4.22) have a sign. More precisely, the vectors (a,b) and $(-b,a)$ divide $\mathbb{R}^2$ into four quadrants. If (m,n) is in the first and third, $\frac{1}{\sqrt{R}} \frac{\partial}{\partial\theta} \lambda_{a,b} > 0$, otherwise $\frac{1}{\sqrt{R}} \frac{\partial}{\partial\theta} \lambda_{a,b} < 0$. But the quadrants vary according to (a,b) leading to cancellations in the sum:

$$\frac{1}{\sqrt{R}} \sum_{(a,b) \in \text{gsupp } \hat{V}} \frac{\partial}{\partial \theta} \lambda_{a,b} = \frac{(4.4)}{\sqrt{R}}. \tag{4.23}$$

The following separation property of gsupp $\hat{V}$ plays an essential role in determining zeroes of (4.23).

Lemma 4.3. *Let $(m,n) \in \tilde{S}$, the circle centered at $(0,0)$ of radius $\sqrt{R}$ in $\mathbb{R}^2$. If there exists $(a,b) \in$ gsupp $\hat{V}$, $\mathbb{N} \ni |a|$, $|b| \le k$, such that*

$$|ma + nb| < \epsilon\sqrt{R}, \quad \epsilon > 0, \tag{4.24}$$

then for all

$$(a', b') \in \textit{gsupp } \hat{V} \backslash \{(a,b)\} \tag{4.25}$$

$$|ma' + nb'| > [\mathcal{O}(1/k) - \epsilon]\sqrt{R}. \tag{4.26}$$

Proof.

$$ma + nb = (m,n) \cdot (a,b) = \sqrt{R} \cdot \sqrt{a^2+b^2} \cdot \cos\theta = \sqrt{R} \cdot \sqrt{a^2+b^2} \cdot \sin\phi, \tag{4.27}$$

where θ is the angle between (m,n) and (a,b), $\phi = \pi/2 - \theta$.

$$\begin{aligned} ma' + nb' =& \sqrt{R} \cdot \sqrt{a'^2 + b'^2} \cdot \cos\theta' = \sqrt{R} \cdot \sqrt{a'^2+b'^2} \cdot \sin\phi' \\ =& \sqrt{R} \cdot \sqrt{a'^2+b'^2} \cdot \sin(\phi' - \phi + \phi), \end{aligned} \tag{4.28}$$

where θ' is the angle between (m,n) and (a',b'), $\phi' = \pi/2 - \theta'$. Since $\min|\phi' \pm \phi| = \mathcal{O}(1/k)$ for $(a'b')$ satisfying (4.25), using (4.24) in (4.26, 4.27), we obtain (4.26). []

Using Lemma 4.3 in (4.12), we obtain that $\lambda \in [-\mathcal{O}(1/K^2), \mathcal{O}(1/K^2)]$ in (4.22).

Lemma 4.4. *Let $(m, n) \in \tilde{S}$, the circle centered at $(0, 0)$ of radius $\sqrt{R}$ in $\mathbb{R}^2$. If there exists $(a, b) \in$ gsupp $\hat{V}$ such that*

$$|ma + nb| < \epsilon\sqrt{R}, \quad (0 < \epsilon < 1/k^3), \tag{4.29}$$

then

$$\begin{aligned} \frac{|(4.4) + (4.5)|}{\sqrt{R}} &> \mathcal{O}(1)\frac{1}{|ma + nb|^3} \\ &> \mathcal{O}(\frac{1}{\epsilon^3})\frac{1}{R^{3/2}}, \end{aligned} \tag{4.30}$$

if

$$\sum_{1\leq s\leq k} |\hat{V}(sa, sb)|^2(a^2 + b^2) - \sum_{1\leq|s|,|s'|\leq k} \frac{\hat{V}(sa, sb)\hat{V}(s'a, s'b)\hat{V}((s - s')a, (s - s')b))}{4s^2s'} \neq 0. \tag{4.31}$$

Proof. We first assume $a \geq 0$ and $b \geq 0$. Write $m = \sqrt{R}\cos\theta$ and $n = \sqrt{R}\sin\theta$. We distinguish in (4.5) the terms only involve $\hat{V}(sa, sb)$ with $(a, b) \in$ gsupp $\hat{V}$ satisfying (4.29), $1 \leq |s| \leq k$ and call the sum $\mu_{a,b}$. We have

$$\begin{aligned} \mu_{a,b} = &\sum_{1\leq|s|,|s'|\leq k} \frac{\hat{V}(sa, sb)\hat{V}(s'a, s'b)\hat{V}((s - s')a, (s - s')b))}{4s^2s'} \\ &(a\sin\theta - b\cos\theta) \cdot \frac{1}{(ma + nb)^3} + \mathcal{O}(\frac{1}{(ma + nb)^5}). \end{aligned} \tag{4.32}$$

There are three cases: $a > 0$, $b > 0$; $a = 0$, $b > 0$ and $a > 0$, $b = 0$.

(i) $a > 0$, $b > 0$, (4.29) implies $\cos\theta\sin\theta \leq 0$. Otherwise $|ma + nb| > |m| + |n| > \sqrt{R}$. So $|a\sin\theta - b\cos\theta| \geq 1$ and

$$\frac{1}{\sqrt{R}}|\frac{\partial}{\partial\theta}\lambda_{a,b} + \mu_{a,b}| > \mathcal{O}(1) \cdot \frac{1}{|ma + nb|^3}$$

from (4.22, 4.32), where we used (4.31, 4.12).

(ii) $a = 0$, $b > 0$, (4.29) reduces to

$$|nb| < \epsilon\sqrt{R}.$$

So

$$|\sin\theta| < \mathcal{O}(\epsilon), \quad |\cos\theta| > 1 - \mathcal{O}(\epsilon). \tag{4.33}$$

Using (4.29, 4.33) in (4.22, 4.32), we obtain

$$\frac{1}{\sqrt{R}}|\frac{\partial}{\partial\theta}\lambda_{0,b} + \mu_{0,b}| > \mathcal{O}(1) \cdot \frac{1}{|ma + nb|^3} \tag{4.34}$$

(ii) $a = 0$, $b > 0$, similarly,

$$\frac{1}{\sqrt{R}}|\frac{\partial}{\partial\theta}\lambda_{a,0} + \mu_{a,0}| > \mathcal{O}(1) \cdot \frac{1}{|ma + nb|^3} \tag{4.35}$$

Clearly, same estimates hold for $a \geq 0$ and $b \leq 0$. So we have

$$\begin{aligned}
\frac{1}{\sqrt{R}}|\frac{\partial}{\partial\theta}\lambda_{a,b}+\mu_{a,b}| &> \mathcal{O}(1)\frac{1}{|ma+nb|^3} \\
&> \mathcal{O}(\frac{1}{\epsilon^3})\cdot\frac{1}{R^{3/2}} > \mathcal{O}(\frac{1}{\epsilon^3})\cdot\sum_{\substack{(a'b')\neq(a,b)\\(a'b')\in\text{gsupp }\hat{V}}}\frac{1}{|ma'+nb'|^3} \\
&> \mathcal{O}(\frac{1}{\epsilon^3})\frac{1}{\sqrt{R}}\sum_{\substack{(a'b')\neq(a,b)\\(a'b')\in\text{gsupp }\hat{V}}}|\frac{\partial}{\partial\theta}\lambda_{a',b'}|,
\end{aligned} \tag{4.36}$$

where we used (4.29, 4.32-4.35, 4.22) and Lemma 4.3.

So

$$\begin{aligned}
\frac{|(4.4)+\mu_{a,b}|}{\delta^2\sqrt{R}} &> \frac{1}{\sqrt{R}}[|\frac{\partial}{\partial\theta}\lambda_{a,b}+\mu_{a,b}| - \sum_{\substack{(a'b')\neq(a,b)\\(a'b')\in\text{gsupp }\hat{V}}}|\frac{\partial}{\partial\theta}\lambda_{a',b'}|] \\
&> (1-\mathcal{O}(\epsilon^3k^2))\frac{1}{\sqrt{R}}|\frac{\partial}{\partial\theta}\lambda_{a,b}+\mu_{a,b}| \\
&> \mathcal{O}(1)\frac{1}{|ma+nb|^3} \\
&> \mathcal{O}(\frac{1}{\epsilon^3})\frac{1}{R^{3/2}}.
\end{aligned} \tag{4.37}$$

Since each term in (4.5) aside from $\mu_{a,b}$ is third order in D^{-1} involving at least one $(a',b') \in$ gsupp $\hat{V}$, $(a',b') \neq (a,b)$, (4.24, 4.26) imply

$$|\frac{(4.5)-\mu_{a,b}}{\sqrt{R}}| < \mathcal{O}(k^2)\frac{1}{|ma+nb|^2|ma'+nb'|} < \mathcal{O}(\epsilon k^3)\frac{1}{|ma+nb|^3}.$$

Combining with (4.37), this proves the lemma for $0 < \epsilon < 1/k^3$. []

Combining Lemme 4.1 and 4.4, we have

Proposition 4.5. *Let $(m,n) \in \tilde{S} \cap \Pi'$. If there exists $(a,b) \neq (0,0)$, $(a,b) \in gsupp\ \hat{V}$ such that*

$$|ma+nb| < \epsilon\sqrt{R}, \quad (0 < \epsilon < 1/k^3), \tag{4.38}$$

then

$$\frac{1}{\sqrt{R}}|\frac{\partial M_{ii}}{\partial\theta}| > \frac{\mathcal{O}(1)}{|ma+nb|^3} > \frac{\mathcal{O}(1)}{R^{3/2}}, \quad (|ma+nb| > K > c^2k^2 + c\|V\|_\infty + 1, \mathbb{N} \ni c > 8), \tag{4.39}$$

provided (4.31) holds.

Proof. This follows immediately from (4.30, 4.11, 4.3). []

Polynomial approximation.

It follows from Proposition 4.5 that in order to control the zeroes of $\frac{\partial M_{ii}}{\partial \theta}$, we only need to restrict to (m,n) such that

$$|ma+nb| \geq \epsilon\sqrt{R} \tag{4.40}$$

for all $(a,b) \in \text{gsupp } \hat{V}$. More precisely, in view of Lemma 4.1, we want to exclude θ satisfying (4.40) such that

$$\frac{1}{\sqrt{R}}\Big|\sum_{(a,b)\in \text{gsupp } \hat{V}} \frac{\partial}{\partial\theta}\lambda_{a,b} + (4.5)\Big| = \Big|\frac{(4.4)+(4.5)}{\sqrt{R}}\Big| \leq \frac{\mathcal{O}(1)}{R^2}. \tag{4.41}$$

Let

$$\Lambda \stackrel{\text{def}}{=} \frac{1}{\sqrt{R}}\Big(\sum_{(a,b)\in \text{gsupp } \hat{V}} \frac{\partial}{\partial\theta}\lambda_{a,b} + (4.5)\Big), \tag{4.42}$$

$$v_{a,b} \stackrel{\text{def}}{=} \sum_{1\leq s\leq k} |\hat{V}(sa,sb)|^2(a^2+b^2), \quad (a,b)\in \text{gsupp } \hat{V}, \tag{4.43}$$

$$g_{a,b;c,d} \stackrel{\text{def}}{=} \sum_{\substack{-k\leq s,s'\leq k\\ s,s'\neq 0}} \frac{\hat{V}(sa,sb)\hat{V}(s'c-sa,s'd-sb)\hat{V}(s'c,s'd)}{4s^2s'}, \quad (a,b),(c,d)\in \text{gsupp } \hat{V}. \tag{4.44}$$

Assume (m,n) such that (4.40) hold for all $(a,b)\in \text{gsupp } \hat{V}$ and $\lambda = \mathcal{O}(1/R)$, then

$$\begin{aligned}\Lambda =& \frac{1}{R^{3/2}}\Big(\sum_{(a,b)\in \text{gsupp } \hat{V}} v_{a,b}\frac{a\sin\theta - b\cos\theta}{(a\cos\theta+b\sin\theta)^3}\\ &+\sum_{(a,b),(c,d)\in \text{gsupp } \hat{V}} g_{a,b;c,d}\frac{a\sin\theta-b\cos\theta}{(a\cos\theta+b\sin\theta)^2(c\cos\theta+d\sin\theta)}\Big)\\ &+\mathcal{O}(\frac{1}{R^2})+\mathcal{O}(\frac{1}{R^{5/2}})\end{aligned} \tag{4.45}$$

$$\stackrel{\text{def}}{=} \frac{1}{R^{3/2}}\Lambda_1 + \mathcal{O}(\frac{1}{R^2})+\mathcal{O}(\frac{1}{R^{5/2}}), \tag{4.46}$$

where $v_{a,b}$ and $g_{a,b;c,d}$ as in (4.43).

Let $\nu = |\text{gsupp } \hat{V}| \leq \mathcal{O}(k^2)$, (4.40) define 2ν arcs Γ' of the circle $\tilde{S}$. Let $(a,b)_\perp$ be the ray perpendicular to $(a,b)\in \text{gsupp } \hat{V} : (a,b)_\perp \cdot (a,b) = 0$. Then for all Γ', $\Gamma' \cap (a,b)_\perp = \emptyset$, for all $(a,b)\in \text{gsupp } \hat{V}$, and Λ_1 is well defined on Γ'.

Let

$$\begin{aligned} x &= \tan\theta, \quad \text{when } |\tan\theta| \leq 1,\\ x &= \coth\theta, \quad \text{when } |\coth\theta| < 1.\end{aligned} \tag{4.47}$$

Rewrite Λ_1 in terms of x and call the resulting function f. We have

$$f = (1+x^2)\Big(\sum_{(a,b)\in \mathrm{gsupp}\ \hat{V}} v_{a,b}\frac{ax-b}{(a+bx)^3} + \sum_{(a,b),(c,d)\in \mathrm{gsupp}\ \hat{V}} g_{a,b;c,d}\frac{ax-b}{(a+bx)^2(c+dx)}\Big),$$
$$|\tan\theta| \le 1, x=\tan\theta, |x|\le 1, \tag{4.48}$$

$$f = (1+x^2)\Big(\sum_{(a,b)\in \mathrm{gsupp}\ \hat{V}} v_{a,b}\frac{a-bx}{(ax+b)^3} + \sum_{(a,b),(c,d)\in \mathrm{gsupp}\ \hat{V}} g_{a,b;c,d}\frac{a-bx}{(ax+b)^2(cx+d)}\Big),$$
$$|\coth\theta| \le 1, x=\coth\theta, |x|< 1. \tag{4.49}$$

Both f are rational functions and can be written as

$$f = \frac{P_1}{P_2}, \tag{4.50}$$

where P_1 and P_2 are polynomials in x of degrees at most $3(\nu^2+\nu) < 4\nu^2$ and

$$0 < |P_2| < \mathcal{O}(1) \tag{4.51}$$

on arcs Γ', defined above (4.47). Moreover P_1 is a polynomial whose coefficients *only* depends on $\hat{V}$ and supp $\hat{V}$ in view of (4.43, 4.48, 4.49). It is of the form

$$P_1 = A_p x^p + A_{p-1}x^{p-1} + \cdots + A_0, \quad A_p \ne 0, 0 < p < 4\nu^2, \tag{4.52}$$

and

$$A_j = A_j(\hat{V}, \mathrm{supp}\ \hat{V}). \tag{4.53}$$

From (4.51), the set

$$I \overset{\mathrm{def}}{=} \{x \,||f(x)| < \frac{1}{\sqrt{R}}\} \subseteq I_1 \overset{\mathrm{def}}{=} \{x\,||P_1(x)| < \frac{\mathcal{O}(1)}{\sqrt{R}}\}. \tag{4.54}$$

To bound the measure of I_1, we use the resultant. From (4.52),

$$P_1' = pA_p x^{p-1} + (p-1)A_{p-1}x^{p-2} + \cdots + A_1. \tag{4.55}$$

By definition,

$$\text{Resultant } (P_1, P_1') = \det \begin{pmatrix} A_p & A_{p-1} & A_{p-2} & \cdots & A_1 & A_0 & \cdots & \cdots \\ 0 & A_p & A_{p-1} & \cdots & \cdots & A_1 & A_0 & \cdots \\ \vdots & & \vdots & & A_p & A_{p-1} & \cdots & A_0 \\ pA_p & (p-1)A_{p-1} & (p-2)A_{p-2} & \cdots & A_1 & 0 & \cdots & \cdots \\ 0 & \cdots & 0 & \cdots & \cdots & pA_p & \cdots & A_1 \end{pmatrix}, \tag{4.56}$$

and let $D(\hat{V})$ denote the above resultant. (4.57)

If $D(\hat{V}) \ne 0$, P_1 and P_1' have no common roots. Let $\Gamma \subset \tilde{S}$ be the largest set such that on Γ, (4.40) hold for all $(a,b) \in$ gsupp $\hat{V}$. Since R is fixed, we also use Γ to denote the corresponding set of angles $\theta \in [0, 2\pi)$.

Lemma 4.6. *Assume V is such that $D(\hat{V}) \neq 0$ for both P_1 defined from (4.48-4.50), and $\lambda = \mathcal{O}(1/R)$, then*

$$mes\ \{\theta \in \Gamma || \Lambda(\theta)| \leq \frac{\gamma}{R^2}\} \leq \frac{\gamma C_V}{\sqrt{R}}, \tag{4.58}$$

where Λ is as defined in (4.42). Moreover the set in (4.58) has at most $\mathcal{O}(k^4)$ connected components.

Proof. P_1 is of degree at most $4\nu^2$, with $\nu = |\text{gsupp } \hat{V}| = \mathcal{O}(k^2)$. So P_1 has at most $4\nu^2$ zeroes. Since $D(\hat{V}) \neq 0$,

$$\min \{|P_1'(x)| P_1(x) = 0\} > \frac{1}{C_V} > 0. \tag{4.59}$$

So

$$\text{mes } \{x | P_1(x)| \leq \frac{\gamma}{\sqrt{R}}\} \leq \frac{\gamma C_V}{\sqrt{R}}, \tag{4.60}$$

(4.59, 4.48-4.50, 4.45) and the fact that

$$d\theta = \pm \frac{1}{1+x^2} dx$$

imply (4.58). []

5. Proof of the Theorem

Assume V is a generic trigonometric polynomial of degree k satisfying the genericity conditions (i, ii) in sect. 1, so that Lemmas 4.4 and 4.6 are available. Let $\tilde{S}$ be the circle over $\mathbb{R}^2$ of radius $\sqrt{R}$, $R \in \mathbb{N}$ as before. Take $c = 9$ in Lemma 2.2 and define the geometric singular set

$$\Theta_g \overset{\text{def}}{=} \{\theta \in [0, 2\pi) || \alpha \cos\theta + \beta \sin\theta| \leq \frac{K}{\sqrt{R}} \text{ for some } (\alpha, \beta) \in [-9k, 9k]^2 \backslash \{0\}, \tag{5.1}$$

where

$$K > c^2k^2 + c\|V\|_\infty + 1 = 81k^2 + 9\|V\|_\infty + 1. \tag{5.2}$$

Θ_g has at most $\mathcal{O}(k^2)$ connected components and

$$\text{mes } \Theta_g = \frac{\mathcal{O}(1)}{\sqrt{R}} \text{ on } [0, 2\pi). \tag{5.3}$$

We also use Θ_g to denote the corresponding arcs of $\tilde{S}$, As before, let

$$\Gamma = \{(m, n) \in \tilde{S} || ma + nb| \geq \epsilon\sqrt{R}, \quad \forall (a, b) \in \text{gsupp } \hat{V}\}, \tag{5.4}$$

where $0 < \epsilon < 1/k^3$.

Assume $\lambda = \mathcal{O}(1/R)$ and let $\Theta_a \subset \Gamma$ be the algebraic singular set defined in (4.58) with $\gamma > k^{10}/\epsilon^4 > k^{22}$ in view of (4.11) and (4.40),

$$\text{mes } \Theta_a \leq \frac{\mathcal{O}(1)}{\sqrt{R}} \text{ on } [0, 2\pi), \tag{5.5}$$

where $\mathcal{O}(1) = \gamma C_V$. Define

$$\Theta = \Theta_g \cup \Theta_a \tag{5.6}$$

and let Θ also denote the corresponding set on $\tilde{S}$. Θ has at most $\mathcal{O}(k^4)$ connected components,

$$\text{mes } \Theta = \mathcal{O}(1) \text{ on } \tilde{S}. \tag{5.7}$$

Let $p \in \Pi$, Π as defined in Lemma 2.1 and $\bar{S} = \tilde{S} \cap \mathbb{Z}^2$. Assume

$$p \cap \Theta = \emptyset, \quad p \cap \bar{S} \neq \emptyset. \tag{5.8}$$

Define

$$\mathcal{M} \stackrel{\text{def}}{=} \mathcal{M}_p \tag{5.9}$$

as in (3.2), M' as in (3.3-3.6) and M_{ii} as in (4.1) first for $i \in p \cap \bar{S}$, then for $i \in \tilde{S} \backslash \Theta_g$.

Lemma 5.1. *Assume*

$$\lambda' + M_{ii}(\lambda') = 0, \tag{5.10}$$

where $\lambda' = E - |i|^2 = E - R$. Then on Γ defined in (5.4),

$$\lambda' = \mathcal{O}(\frac{1}{R}); \tag{5.11}$$

and on each connected component of

$$S'' \stackrel{\text{def}}{=} \tilde{S} \backslash \Theta \tag{5.12}$$

either

$$\frac{1}{\sqrt{R}} \frac{d\lambda'}{d\theta} \geq \mathcal{O}(1) \sum_{(a_\ell, b_\ell) \in \text{gsupp } \hat{V}} \frac{1}{\prod_{\ell=1}^4 |ma_\ell + nb_\ell|}, \tag{5.13}$$

$$\text{or } \frac{1}{\sqrt{R}} \frac{d\lambda'}{d\theta} \leq -\mathcal{O}(1) \sum_{(a_\ell, b_\ell) \in \text{gsupp } \hat{V}} \frac{1}{\prod_{\ell=1}^4 |ma_\ell + nb_\ell|}, \tag{5.14}$$

where $i = (m, n) = \sqrt{R}(\cos \theta, \sin \theta)$.

Proof. (5.11) follows from Lemma 4.2. Using (4.30) or (4.58) in (4.4, 4.5), Lemma 4.1 in (4.6-4.10), we obtain

$$\frac{1}{\sqrt{R}} | \frac{\partial M_{ii}}{\partial \theta} \| \geq \mathcal{O}(1) \sum_{(a_\ell, b_\ell) \in \text{gsupp } \hat{V}} \frac{1}{\prod_{\ell=1}^4 |ma_\ell + nb_\ell|}, \tag{5.15}$$

$\mathbb{R}^2 \ni i = (m, n)$ on $\tilde{S} \backslash \Theta$. Here we also used (5.11), when $\theta \in \Gamma$. Moreover it is sign definite on each connected component of $\tilde{S} \backslash \Theta = S''$. From (5.10)

$$-\frac{d\lambda'}{d\theta} = \frac{\partial M_{ii}}{\partial \theta} + \frac{\partial M_{ii}}{\partial \lambda'} \cdot \frac{d\lambda'}{d\theta}. \tag{5.16}$$

So

$$\frac{d\lambda'}{d\theta} = (-1 + \mathcal{O}(1/K^2)) \cdot \frac{\partial M_{ii}}{\partial \theta}, \tag{5.17}$$

where we used (3.33). Using (5.15), we obtain the lemma. []

Proposition 5.2. *The set $S'' = \tilde{S} \backslash \Theta$ has at most $\mathcal{O}(k^4)$ connected components. Let $\Gamma'' \subset S''$ be a connected component. Assume $p,\, p' \in \Pi$ and $p,\, p' \cap \bar{S} \neq \emptyset$ be such that $p,\, p' \cap \tilde{S} \subset \Gamma''$. Let $i \in \{p, p' \cap \bar{S}\}$, $\lambda_i = E - |i|^2 = E - R \in [-1/2, 1/2]$ be such that*

$$0 \in \sigma(\mathcal{M}(\lambda_i)), \tag{5.18}$$

where $\mathcal{M} = \mathcal{M}_p$ or $\mathcal{M}_{p'}$. Let $\lambda_i' = E' - |i|^2 = E' - R \in [-1/2, 1/2]$ be such that

$$0 = \lambda_i' + M_{ii}(\lambda_i'). \tag{5.19}$$

Then

$$|\lambda_i - \lambda_i'| = \mathcal{O}(1) \sum_{(a_\ell, b_\ell) \in \text{gsupp } \hat{V}} \frac{1}{\prod_{\ell=1}^{8} |ma_\ell + nb_\ell|}, \quad (m,n) = i, \tag{5.20}$$

and

$$|\lambda_i - \lambda_{j'}| > \sup_{(m,n)=i,j'} \sum_{(a_\ell, b_\ell) \in \text{gsupp } \hat{V}} \frac{\mathcal{O}(1)}{\prod_{\ell=1}^{4} |ma_\ell + nb_\ell|}, \quad i, j' \in \{p, p' \cap \bar{S}\}, i \neq j'. \tag{5.21}$$

Proof. The number of connected components follow from the definition of Θ in (5.1, 4.58, 5.6). Assume $\mathcal{M} = \mathcal{M}_p$ has rank ≥ 2. Write the right side of (3.29) as $\mathcal{O}_i$. We have

$$\begin{aligned} F(E) &= \det(\mathcal{M}(E)) \\ &= \prod_j (E - |j|^2 + M_{jj}(E) + \mathcal{O}_j) + \mathcal{O}(\sum \prod \mathcal{M}_{ij}), \end{aligned} \tag{5.22}$$

where the second product contains at least two off diagonal elements. So

$$\begin{aligned} F'(E) = &\sum_j \prod_{j' \neq j} (E - |j'|^2 + M_{jj}(E) + \mathcal{O}_{j'})(1 + \frac{\partial M_{jj}}{\partial E}(E) + \mathcal{O}_j) \\ &+ \mathcal{O}_{ij}, \end{aligned} \tag{5.23}$$

where we used $\mathcal{O}_{ij}$ to denote the right side of (3.31) and analyticity in E to reach (5.23).

Let $E = |i|^2 + \lambda_i'$ and write $F(\lambda_i')$, $F'(\lambda_i')$ for $F(|i|^2 + \lambda_i')$, $F'(|i|^2 + \lambda_i')$ respectively. Since $\frac{\partial M_{jj}}{\partial E} = \mathcal{O}(1/K^2)$, (5.23) gives

$$|F'(\lambda_i')| \geq (1 - \mathcal{O}(1/K^2)) \begin{cases} |\lambda_i' - \lambda_{j'}'| - \max_{ij} \mathcal{O}_{ij}, & \text{if } \exists j' \in p,\, j' \neq i,\, |j'|^2 = R, \\ \mathcal{O}(1), & \text{otherwise,} \end{cases} \tag{5.24}$$

where we also used (4.12). Using (5.13) or (5.14), we have for any $\tilde{j} \in \Gamma''$, $|\tilde{j} - i| \geq 1$ and

$$\begin{aligned} |\lambda_i' - \lambda_{\tilde{j}}'| = |\int_{\tilde{j}}^{i} (\frac{d\lambda'}{d\theta}) d\theta| &\geq \max_{(m,n)=i,\tilde{j}} \sum_{(a_\ell, b_\ell) \in \text{gsupp } \hat{V}} \frac{1}{\prod_{\ell=1}^{4} |ma_\ell + nb_\ell|} \\ &\gg \max_{i\tilde{j}} \mathcal{O}_{i\tilde{j}}. \end{aligned} \tag{5.25}$$

So

$$|F'(\lambda_i')| \geq (1-\mathcal{O}(1/K^2))\begin{cases} |\lambda_i'-\lambda_{j'}'|, & \text{if } \exists j'\in p,\, j'\neq i,\, |j'|^2=R, \\ \mathcal{O}(1), & \text{otherwise.}\end{cases} \tag{5.26}$$

From (5.22),

$$|F(\lambda_i')| \leq \mathcal{O}_i\begin{cases} |\lambda_i'-\lambda_{j'}'|, & \text{if } \exists j'\in p,\, j'\neq i,\, |j'|^2=R, \\ \mathcal{O}(1), & \text{otherwise.}\end{cases} \tag{5.27}$$

Let $0<a\ll 1$.

$$F(\lambda_i'\pm a) = F(\lambda_i') \pm aF'(\lambda_i') + \mathcal{O}(a^2) = F'(\lambda_i')(\frac{F(\lambda_i')}{F'(\lambda_i')} \pm a) + \mathcal{O}(a^2). \tag{5.28}$$

Since

$$|\frac{F(\lambda_i')}{F'(\lambda_i')}| \leq \mathcal{O}_i \stackrel{\text{def}}{=} \mathcal{O}(1) \sum_{(a_\ell,b_\ell)\in \text{gsupp } \hat{V}} \frac{1}{\prod_{\ell=1}^{8} |ma_\ell+nb_\ell|} \tag{5.29}$$

from (3.29), for $a>10\mathcal{O}_i$, $F(\lambda'+a)$ and $F(\lambda'-a)$ have opposite signs. Since F is analytic, this implies $F(\lambda_i)=0$ for some

$$\lambda_i \in \lambda_i' + (-11\mathcal{O}_i, 11\mathcal{O}_i), \tag{5.30}$$

which proves (5.20). Since similar statements hold for $\lambda_{j'}$, we obtain

$$|\lambda_i-\lambda_{j'}| > \frac{1}{2}|\lambda_i'-\lambda_{j'}'|, \tag{5.31}$$

implying (5.21) by using (5.25).

Clearly simpler arguments apply when $\mathcal{M}$ is a scalar as $F'(\lambda_i') = (1-\mathcal{O}(1/K^2)) > 1/2$ and $F'(\lambda_i') = \mathcal{O}_i$. Combining the two cases, we obtain the proposition. []

Proof of the Theorem.

$$\sigma(H) = \sigma(\hat{H}) \subseteq \cup_{R\in\mathbb{Z}}[R-1/2, R+1/2]. \tag{5.32}$$

In the following lines, we go back to the convention of writing H for $\hat{H}$. Since $R\geq -\|V\|_\infty$ and for $-\|V\|_\infty < R \leq 0$, (1,2, 1.3) are obvious, we only need to be concerned with $R\in\mathbb{N}$. From Proposition 5.2, given $E\in[R-1/2, R+1/2]$, $R\in\mathbb{N}$, there exist at most $\mathcal{O}(k^4)$ $p\in\Pi$, such that

$$\text{dist }(E, \sigma(H_p)) \leq o(\frac{1}{R^2}). \tag{5.33}$$

First recall

$$S' = \{(x,y)\in\mathbb{R}^2 |\, |x^2+y^2-R| \leq \|V\|_\infty + 1\}, \quad S = S'\cap\mathbb{Z}^2,$$
$$\tilde{S} = \{(x,y)\in\mathbb{R}^2 |\, x^2+y^2=R\}, \quad \bar{S} = \tilde{S}\cap\mathbb{Z}^2$$

and Θ as defined in (5.1, 4.58, 5.6). This is because

- if $p \cap \Theta = \emptyset$ and $p \cap \bar{S} = \emptyset$, then

$$\text{dist } (\sigma(H_p), [R - 1/2, R + 1/2]) \geq \mathcal{O}(1),$$

  from Proposition 3.1, Lemma 4.2 and analyticity of $\mathcal{M}_p$ in E,

- $S'' = \tilde{S} \backslash \Theta$ has at most $\mathcal{O}(k^4)$ connected components Γ''. On each Γ'', (5.21) hold,
- Θ has at most $\mathcal{O}(k^4)$ connected components and mes $\Theta = \mathcal{O}(1)$ on $\tilde{S}$,
- for all $p \in \Pi$, $|p \cap \tilde{S}| = \mathcal{O}(R^{1/6})|$, $|p \cap S| \leq \kappa$.

Assume $p \cap S \neq \emptyset$. Since

$$\text{dist } (\{p \cap S\}, \partial p) = \mathcal{O}(R^{1/6}) \tag{5.34}$$

by construction (Lemma 2.1), this implies

$$\text{dist } (\sigma(H_p), \sigma(H)) = \mathcal{O}(e^{-R^{1/6}}). \tag{5.35}$$

This is because if $\hat{\phi}$ is an eigenfunction of H_p with eigenvalue E, then $(H - E)\hat{\phi} = \mathcal{O}(e^{-R^{1/6}})$, which implies (5.35).

In fact more generally, for all Λ such that either $\Lambda \subset p$ or $\Lambda \supseteq p$:

$$\text{dist } (\sigma(H_p), \sigma(H_\Lambda)) = \mathcal{O}(e^{-\min\,(d_1, d_2)}). \tag{5.36}$$

where

$$d_1 = \text{dist } (\{p \cap S\}, \partial \Lambda\}, \tag{5.37}$$
$$d_2 = \text{dist } (\{p \cap S\}, \partial p\}. \tag{5.38}$$

Let $E \in \sigma(H)$, since each H_p has at most κ eigenvalues in $[R - 1/2, R + 1/2]$, (5.33, 5.35) give that $\sigma(H)$ is of multiplicity at most $\mathcal{O}(k^4)$.

To prove localization of the Fourier series $\hat{\phi}$ of the eigenfunction ϕ, we proceed as follows. Let $p \in \Pi$ be such that dist $(E, \sigma(H_p)) \leq o(\frac{1}{R^2})$. Let $\mathcal{S}$ be this set of singular p. From the argument above, there are only $\mathcal{O}(k^4)$ such p. Let

$$\mathcal{R} = \{(m, n) \in S \cap \mathcal{S}\}. \tag{5.39}$$

Then $|\mathcal{R}| = \mathcal{O}(k^4)$, since $|p \cap S| \leq \kappa$. (Note that $|\mathcal{R}| \geq 1$ from (5.35). So the following construction is not empty.)

Since $\hat{\phi} \in \ell^2$, we may assume $\|\hat{\phi}\|_\infty \leq 1$ by normalization: $\|\hat{\phi}\|_2 = 1$. So

$$|\hat{\phi}(j)| \leq 1 \text{ for } j \in \mathcal{R}. \tag{5.40}$$

To prove decay of $\hat{\phi}(j)$ for $j \notin \mathcal{R}$, we let $i_1 \in \mathcal{R}$ be such that

$$|i_1 - j| = \min{}_{i \in \mathcal{R}} |i - j|. \tag{5.41}$$

(If there are two sites which are minimal, choose one and name it i_1.) Let Λ be a square of size $\mathcal{O}(|j-i_1|)$ such that $i_1\in\Lambda$, $j\in\Lambda$ and

$$\text{dist }(j,\partial\Lambda)=2|j-i_1|. \tag{5.42}$$

Let

$$\tilde{\Lambda}=\Lambda\backslash\mathcal{R}. \tag{5.43}$$

(i) If $|j-i_1|\le R^{1/7}$, then

$$\|(H_{\tilde{\Lambda}}-E)^{-1}\|\le\mathcal{O}(1), \tag{5.44}$$

since $\tilde{\Lambda}\cap S=\emptyset$.

(ii) Otherwise

$$\|(H_{\tilde{\Lambda}}-E)^{-1}\|\le\mathcal{O}(R^2) \tag{5.45}$$

from (5.21, 5.36).

Define

$$\mathcal{V}=H-(H_{\tilde{\Lambda}}\oplus H_{\mathbb{Z}^2\backslash\tilde{\Lambda}}). \tag{5.46}$$

Since

$$(H-E)\hat{\phi}=0, \tag{5.47}$$

we have

$$\Pi_{\tilde{\Lambda}}\hat{\phi}=\Pi_{\tilde{\Lambda}}(H_{\tilde{\Lambda}}-E)^{-1}\mathcal{V}\hat{\phi}. \tag{5.48}$$

(i)

$$|\hat{\phi}(j)|\le C\sum_{j_\ell\in\mathcal{R}\cap\Lambda}e^{-|j-j_\ell|} \tag{5.49}$$

follows from Neumann series expansion about the diagonal.

(ii) Let $\mathcal{R}'=\tilde{\Lambda}\cap S$. For $i'\in\mathcal{R}'$, let Λ' be the square centered at i' of size $L'=(\log R)^2$. There are two possibilities: dist $(\{\Lambda'\cap S\},\,\partial\Lambda')=\mathcal{O}((\log R)^2)$ or dist $(\{\Lambda'\cap S\},\,\partial\Lambda')<\mathcal{O}((\log R)^2)$. In the latter case, let $L''=100L'$ and Λ'' be the square centered at i' of size L''. By construction

$$\text{dist }(\{\Lambda''\cap S\},\partial\Lambda'')=\mathcal{O}((\log R)^2), \tag{5.50}$$

this is because from Lemma 2.1, for a given integer in S there is at most 1 other integer in S which is at distance $\asymp\mathcal{O}((\log R)^2)$ apart. Rename Λ'' as Λ'.

We have from (5.36, 5.21)

$$\text{dist }(E,\sigma(H_{\Lambda'}))\ge\mathcal{O}(\frac{1}{R^2}) \tag{5.51}$$

and moreover

$$|(H_{\Lambda'}-E)^{-1}(x,y)|\le e^{-|x-y|} \tag{5.52}$$

for $|x-y|\ge L'/10$ by using Neumann series, (5.51) and the fact that $|x_1-x_2|\le L'/100$ for all $x_1,\,x_2\in\{\Lambda'\cap S\}$. Clearly (5.51, 5.52) hold for all Λ' of size $\mathcal{O}((\log R)^2)$, $\Lambda'\subset\tilde{\Lambda}$, $\Lambda'\cap S=\emptyset$.

Expanding $(H_{\tilde\Lambda}-E)^{-1}$ repeatedly in $(H_{\Lambda'}-E)^{-1}$ using the resolvent equation:

$$(H_{\tilde\Lambda}-E)^{-1}=(H_{\Lambda'}-E)^{-1}\tilde\Gamma(H_{\tilde\Lambda}-E)^{-1},$$

where $\tilde\Gamma\overset{\text{def}}{=}H_{\tilde\Lambda}-(H_{\Lambda'}\oplus H_{\tilde\Lambda\backslash\Lambda'})$, (5.51, 5.52, 5.45) give

$$|\hat\phi(j)|\le C\sum_{j_\ell\in\mathcal{R}\cap\Lambda}e^{-|j-j_\ell|}.$$

Combining cases (i,ii), we obtain (1.2, 1.3) and hence the Theorem. []

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