Complete nonmeasurability in regular families

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Abstract. We show that for a σ-ideal $\mathcal{I}$ with a Borel base of subsets of an uncountable Polish space, if $\mathcal{A}$ is (in several senses) a "regular" family of subsets from $\mathcal{I}$ then there is a subfamily of $\mathcal{A}$ whose union is completely nonmeasurable i.e. its intersection with every Borel set not in $\mathcal{I}$ does not belong to the smallest σ-algebra containing all Borel sets and $\mathcal{I}$. Our results generalize results from [**3**] and [**4**].

1. Notation and Terminology

Throughout this paper, X, Y will denote uncountable Polish spaces and $\mathcal{B}(X)$ the Borel σ-algebra of X. We say that the ideal $\mathcal{I}$ on X has *Borel base* if every element $A \in \mathcal{I}$ is contained in a Borel set in $\mathcal{I}$. (It is assumed that an ideal is always proper.) The ideal consisting of all countable subsets of X will be denoted by $[X]^{\leq\omega}$ and the ideal of all meager subsets of X will be denoted by $\mathbb{K}$. Let μ be a continous probability measure on X. The ideal consisting of all μ-null sets will be denoted by $\mathbb{L}_\mu$. By the following well known result, $\mathbb{L}_\mu$ can be identified with the σ-ideal of Lebesgue null sets.

Theorem 1.1 ([**6**], Theorem 3.4.23)**.** *If μ is a continous probability on $\mathcal{B}(X)$, then there is a Borel isomorphism $h : X \to [0,1]$ such that for every Borel subset B of $[0,1]$, $\lambda(B) = \mu(h^{-1}(B))$, where λ is a Lebesgue measure.*

Definition 1.1. *We say that $(Z, \mathcal{I})$ is Polish ideal space if Z is Polish uncountable space and $\mathcal{I}$ is a σ-ideal on Z having Borel base and containing all singletons. In this case, we set*

$$\mathcal{B}_+(Z) = \mathcal{B}(Z) \setminus \mathcal{I}.$$

A subset of Z not in $\mathcal{I}$ will be called a $\mathcal{I}$-positive set; sets in $\mathcal{I}$ will also be called $\mathcal{I}$-null. Also, the σ-algebra generated by $\mathcal{B}(Z) \cup \mathcal{I}$ will be denoted by $\overline{\mathcal{B}}(Z)$, called the $\mathcal{I}$-completion of $\mathcal{B}(Z)$.

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It is easy to check that $A \in \overline{\mathcal{B}}(Z)$ if and only if there is an $I \in \mathcal{I}$ such that $A \triangle I$ (the symetric difference) is Borel.

Example 1.1. *Let μ be a continous probability measure on X. Then $(X, [X]^{\leq\omega})$, $(X, \mathbb{K})$, $(X, \mathbb{L}_\mu)$ are Polish ideal spaces.*

Definition 1.2. *A Polish ideal group is 3-tuple $(G, \mathcal{I}, +)$ where $(G, \mathcal{I})$ is Polish ideal space and $(G, +)$ is an abelian topological group with respect to the Polish topology of G.*

Definition 1.3. *Let $(X, \mathcal{I})$ be a Polish ideal space and $A \subseteq X$. We say that A is $\mathcal{I}$–nonmeasurable, if $A \notin \overline{\mathcal{B}}(X)$. Further, we say that A is completely $\mathcal{I}$–nonmeasurable if*

$$\forall B \in \mathcal{B}_+(X) \;\; A \cap B \neq \emptyset \wedge A^c \cap B \neq \emptyset.$$

Clearly every completely $\mathcal{I}$–nonmeasurable set is $\mathcal{I}$–nonmeasurable. In the literature, completely $[X]^{\leq\omega}$–nonmeasurable sets are called Bernstein sets. Also, note that A is completely $\mathbb{L}_\mu$–nonmeasurable if and only if the inner measure of A is zero and the outer measure one.
For any set E, $|E|$ will denote the cardinality of E.
Let $(X, \mathcal{I})$ be a Polish ideal space and $\mathcal{F} \subseteq \mathcal{I}$. We set

$$\begin{aligned}
add(\mathcal{I}) &= \min\{|\mathcal{A}| : \; \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup \mathcal{A} \notin \mathcal{I}\}\\
cov(\mathcal{I}) &= \min\{|\mathcal{A}| : \; \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup \mathcal{A} = X\}\\
cov(\mathcal{F}) &= \min\{|\mathcal{A}| : \; \mathcal{A} \subseteq \mathcal{F} \wedge \bigcup \mathcal{A} = X\}\\
cov_h(\mathcal{I}) &= \min\{|\mathcal{A}| : \; \mathcal{A} \subseteq \mathcal{I} \wedge \exists B \in \mathcal{B}_+(X) B \subseteq \bigcup \mathcal{A}\}\\
cov_h(\mathcal{F}) &= \min\{|\mathcal{A}| : \; \mathcal{A} \subseteq \mathcal{F} \wedge \exists B \in \mathcal{B}_+(X) B \subseteq \bigcup \mathcal{A}\}
\end{aligned}$$

An ideal $\mathcal{I}$ is c.c.c. if every family of pairwise disjoint non-empty $\mathcal{I}$-positive Borel sets is countable. Now let $(X, \mathcal{I})$ be a Polish ideal space with $\mathcal{I}$ c.c.c. and $A \subseteq X$. Let $\mathcal{A}$ be a maximal family of pairwise disjoint $\mathcal{I}$-positive Borel sets contained in A^c. Set $B = (\bigcup \mathcal{A})^c$. Then B is Borel, $A \subseteq B$ and for every Borel set $C \supseteq A$, $B \setminus C \in \mathcal{I}$. Any such set B is called a *Borel envelope* of A and will be denoted by $[A]_\mathcal{I}$. Note that a Borel envelope of A is unique modulo $\mathcal{I}$ and it is minimal (modulo $\mathcal{I}$) Borel set containing A.
It follows that $\overline{\mathcal{B}}(X)$ is Marczewski complete (see [**6**], p.114). Therefore, it is closed under Souslin operation (see [**6**], Theorem 3.5.22). It follows that if $\mathcal{I}$ is also c.c.c., $\overline{\mathcal{B}}(X)$ contains all analytic sets.
For any set $F \subseteq X \times Y$ and $x \in X$, $y \in Y$ let

$$F_x = \{y \in Y : \; (x, y) \in F\}$$

and

$$F^y = \{x \in X : \; (x, y) \in F\}.$$

Further, for any $T \subseteq Y$, we set

$$F^{-1}(T) = \{x \in X : F_x \cap T \neq \emptyset\}.$$

A multifunction $F : X \to Y$ is called *$\mathcal{A}$–measurable* if for every open set U in Y, $F^{-1}(U) \in \mathcal{A}$, where $\mathcal{A}$ is a σ-algebra on X.
Let π be a partition of X and $A \subseteq X$. The smallest π-invariant subset of X containing A is called the *saturation* of A and is denoted by A^*. Thus,

$$A^* = \bigcup\{E \in \pi : E \cap A \neq \emptyset\}.$$

We call π *Borel measurable* if the saturation of every open set is Borel; it is *strongly Borel measurable* if the saturation of every closed set is Borel measurable. Since X is second countable, every strongly Borel measurable partition is Borel measurable.
The rest of our notations and terminology are standard. For other notation and terminology in Descriptive Set Theory we follow [**6**].

2. Main results

The following results are the main results of the paper.

Theorem 2.1. *Let $(X,\mathcal{I})$ be a Polish ideal space such that every set in $\mathcal{B}_+(X)$ contains a $\mathcal{I}$-positive closed set. Suppose $\mathcal{A}$ is a strongly Borel measurable partition of X into $\mathcal{I}$-null closed sets. Then there is a subfamily $\mathcal{A}_0 \subseteq \mathcal{A}$ such that $\bigcup \mathcal{A}_0$ is completely $\mathcal{I}$–nonmeasurable.*

Theorem 2.2. *Let $(X,\mathcal{I})$ be a Polish ideal space. Suppose $f : X \to Y$ is a $\overline{\mathcal{B}}(X)$-measurable map such that for every $y \in Y$, $f^{-1}(y) \in \mathcal{I}$. Then there is a $T \subseteq Y$ such that $f^{-1}(T)$ is completely $\mathcal{I}$–nonmeasurable.*

Theorem 2.3. *Let $(X,\mathcal{I})$ be a Polish ideal space with $\mathcal{I}$ c.c.c. Let $F : X \to Y$ be a $\overline{\mathcal{B}}(X)$-measurable multifunction such that for every $x \in X$, $F(x)$ is finite. Then there exists a $T \subseteq Y$ such that $F^{-1}(T)$ is completely $\mathcal{I}$–nonmeasurable.*

Theorem 2.4. *Let $(X,\mathcal{I})$ be a Polish ideal space with $\mathcal{I}$ c.c.c. Suppose F is an analytic subset of $X \times Y$ satisfying the following conditions:*

(1) $(\forall y \in Y)(F^y \in \mathcal{I})$;
(2) $X \setminus \pi_X(F) \in \mathcal{I}$, *where $\pi_X : X \times Y \to X$ is the projection map;*
(3) $(\forall x \in X)(|F_x| < \omega)$.

Then there exists a $T \subseteq Y$ such that $F^{-1}(T)$ is completely $\mathcal{I}$–nonmeasurable.

These results generalize results from [**3**] and [**4**]. In the next section, we present the proofs of our theorems.

3. Proofs of the main results

One of the key ideas of this paper is the following theorem (see [**4**]). For reader's convenience we will give the proof of it.

Theorem 3.1. *Let $(X,\mathcal{I})$ be a Polish ideal space. Assume that a family $\mathcal{A} \subseteq \mathcal{I}$ satisfies the following conditions:*

(1) $X \setminus \bigcup \mathcal{A} \in \mathcal{I}$,
(2) $Z = \{x \in X : \bigcup\{A \in \mathcal{A} : x \in A\} \notin \mathcal{I}\} \in \mathcal{I}$,
(3) $cov_h(\mathcal{F}) = 2^\omega$, *where* $\mathcal{F} = \{\bigcup\{A \in \mathcal{A} : x \in A\} : x \in X \setminus Z\}$.

Then there exists a subfamily $\mathcal{A}_0 \subseteq \mathcal{A}$ *such that* $\bigcup \mathcal{A}_0$ *is completely* $\mathcal{I}$*–nonmeasurable.*

Proof. First of all, we can assume that $Z = \emptyset$ in the second assumption. Now, let us enumerate the family of all positive Borel sets with respect to the ideal $\mathcal{I}$ i.e. $\mathcal{B}_+(X) = \{B_\alpha : \alpha < 2^\omega\}$. By transfinite induction we will construct a sequence

$$\langle (d_\xi, A_\xi) \in B_\xi \times \mathcal{A} : \xi < 2^\omega \rangle$$

satisfying the following conditions

(1) $A_\xi \cap B_\xi \neq \emptyset$,
(2) $d_\xi \notin \bigcup_{\alpha < 2^\omega} A_\alpha$.

Assume that we have constructed a sequence $\langle (d_\xi, A_\xi) \in B_\xi \times \mathcal{A} : \xi < \alpha \rangle$. Since $\bigcup_{\xi<\alpha}\{A \in \mathscr{A} : d_\xi \in A\}$ does not cover any positive Borel set, we are able to find $a_\alpha \in B_\alpha \setminus \bigcup_{\xi<\alpha}\{A \in \mathcal{A} : d_\xi \in A\}$. Let A_α be any element of $\mathcal{A}$ such that $a_\alpha \in A_\alpha$ and find $d_\alpha \in B_\alpha \setminus \bigcup_{\xi \leq \alpha} A_\xi$. It finishes α step of our construction.
Now, let us define $\mathcal{A}_0 = \{A_\xi : \xi \in 2^\omega\}$. For every positive Borel set we have that $\bigcup \mathcal{A}_0 \cap B \neq \emptyset$ and $\{d_\xi : \xi \in 2^\omega\} \cap B \neq \emptyset$. Moreover, $\{d_\xi : \xi \in 2^\omega\} \cap \bigcup \mathcal{A}_0 = \emptyset$. It shows that $\bigcup \mathcal{A}_0$ is completely $\mathcal{I}$–nonmeasurable. □

Remark 3.1. *We can replace the last assumption in Theorem 3.1 by the set theoretic assumption* $cov_h(\mathcal{I}) = 2^\omega$.

As a corollary we have:

Corollary 3.1 (ZFC+CH)**.** *Let* $(X, \mathcal{I})$ *be a Polish ideal space. Let* $\mathcal{A} \subseteq \mathcal{I}$ *be a point-countable family i.e.* $\forall x \in X\ |\{A \in \mathcal{A} : x \in A\}| \leq \omega$ *and* $\bigcup \mathcal{A} = X$. *Then there exists a subfamily* $\mathcal{A}_0 \subseteq \mathcal{A}$ *such that* $\bigcup \mathcal{A}_0$ *is completely* $\mathcal{I}$*–nonmeasurable.*

It is also known that above corollary is independent from ZFC theory (see [**5**]).

Proof of Theorem 2.1. By Theorem 3.1, it is sufficient to prove that $cov_h(\mathcal{A}) = 2^\omega$. Towards proving this, take any $B \in \mathcal{B}_+(X)$. Let$F \subseteq B$ be a $\mathcal{I}$-positive closed set. Let

$$\pi = \{E \cap F : E \in \mathcal{F}\}.$$

Note that π is uncountable and strongly Borel measurable partition of F into closed sets. Since every strongly Borel measurable partition is Borel measurable, it is Borel measurable. Hence, it admits a Borel cross-selection S (see [**6**], Theorem 5.4.3, see [**1**]). Clearly S is uncountable and, therefore of cardinality 2^ω. This implies that $|\pi| = 2^\omega$. □

As a corollary we get the following result for Polish groups:

Corollary 3.2. *Let $(G, \mathcal{I}, +)$ be a compact Polish ideal group. Suppose $\mathcal{I}$ is closed under translations. Assume that each set from $\mathcal{B}_+(G)$ contains a $\mathcal{I}$-positive closed set. Let $H < G$ be a perfect subgroup and $H \in \mathcal{I}$. Then there exists a $T \subseteq G$ such that $T + H$ is completely $\mathcal{I}$–nonmeasurable in G.*

PROOF. This follows from Theorem 2.1 by taking $\mathcal{A}$ to be the set of all left cosets of H. □

To prove Theorem 2.2, we need the following result from [**3**].

Theorem 3.2 (Brzuchowski, Cichoń, Grzegorek, Ryll-Nardzewski)**.** *Let $(X, \mathcal{I})$ be a Polish ideal space and $\mathcal{A} \subseteq \mathcal{I}$ a point-finite cover of X. Then there is a subfamily $\mathcal{A}_0 \subseteq \mathcal{A}$ whose union is not in $\overline{\mathcal{B}}(X)$.*

PROOF OF THEOREM 2.2. Fix a countable base $\{U_n\}$ for the topology of Y. For each n, let $I_n \in \mathcal{I}$ such that $f^{-1}(U_n) \triangle I_n$ is Borel. Let $X' = X \setminus \bigcup_n I_n$. Then $f : X' \to Y$ is Borel. Thus, without any loss of generality, we assume that f is Borel measurable.
Now, let $B \in \mathcal{B}_+(X)$. Set

$$A = \pi_Y((B \times Y) \cap graph(f)).$$

Then A, being analytic, is either countable or of cardinality 2^ω. If A were countable, B is covered by countable subfamily of $\mathcal{I}$, a contradiction, Thus, $cov_h\{f^{-1}(y) : y \in Y\} = 2^\omega$. Our result now follows from Theorem 3.1. □

Theorem 3.3. *Let $(X, \mathcal{I})$ be a Polish ideal space. Let $I \in \mathcal{I}$ and $f : X \setminus I \to Y$ a Borel map such that for every $y \in Y$, $f^{-1}(y)$ is $\mathcal{I}$-null. Then there is a $T \subseteq Y$ such that $f^{-1}(T)$ is completely $\mathcal{I}$–nonmeasurable set.*

PROOF. Let $B \supseteq I$ be a Borel $\mathcal{I}$-null set. Now apply Theorem 2.2 to $f\lceil(X \setminus B)$. □

The next theorem is a technical result which helps us to prove stronger theorems in case $\mathcal{I}$ is c.c.c.

Theorem 3.4. *Let $(X, \mathcal{I})$ be a Polish ideal space with $\mathcal{I}$ c.c.c. Assume that we have a family $\mathcal{F} \subseteq \mathcal{I}$ satisfying the following conditions:*
(1) *$\mathcal{F}$ is point-finite;*
(2) $(\forall B \in \mathcal{B}_+(X))(B \subseteq [\bigcup \mathcal{F}]_{\mathcal{I}} \to |\{F \in \mathcal{F} : F \cap B \neq \emptyset\}| = 2^\omega)$.
Then there exists a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ such that $\bigcup \mathcal{F}'$ is completely $\mathcal{I}$–nonmeasurable in $[\bigcup \mathcal{F}]_{\mathcal{I}}$.

PROOF.

Step 1. *There exists a subfamily $\mathcal{F}_0 \subseteq \mathcal{F}$ having the following properties*
(1) $[\bigcup \mathcal{F}_0]_{\mathcal{I}} = [\bigcup \mathcal{F}]_{\mathcal{I}}$,
(2) $(\forall B \in \mathcal{B}_+(X))(B \subseteq \bigcup \mathcal{F}_0 \to \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{F}_0 \wedge B \subseteq \bigcup \mathcal{A}\} = 2^\omega)$.

Proof. Let us recall that for a set $D \subseteq X$ a symbol $]D[_{\mathcal{I}}$ denotes a maximal Borel set (mod $\mathcal{I}$) contained in D. We will construct a sequence $(\mathcal{A}_n)$ satistying the following conditions

(1) $|\mathcal{A}_n| < 2^\omega$,
(2) $\mathcal{A}_n \subseteq \mathcal{F} \setminus \bigcup_{i<n} \mathcal{A}_i$,
(3) $]\bigcup \mathcal{A}_n[_{\mathcal{I}}$ is maximal element in the family $\{]\bigcup \mathcal{A}[_{\mathcal{I}}\colon |\mathcal{A}| < 2^\omega \wedge \mathcal{A} \subseteq \mathcal{F} \setminus \bigcup_{i<n} \mathcal{A}_i\}$.

Notice that the existance of the maximal element in the family $\{]\bigcup \mathcal{A}[_{\mathcal{I}}\colon |\mathcal{A}| < 2^\omega \wedge \mathcal{A} \subseteq \mathcal{F} \setminus \bigcup_{i<n} \mathcal{A}_i\}$ is implied by the c.c.c property of the ideal $\mathcal{I}$.
We finish the construction if $\{]\bigcup \mathcal{A}[_{\mathcal{I}}\colon |\mathcal{A}| < 2^\omega \wedge \mathcal{A} \subseteq \mathcal{F} \setminus \bigcup_{i<n} \mathcal{A}_i\} = \{\emptyset\}$. Our construction has to end up after finitely many steps. Notice that $]\bigcup \mathcal{A}_{n+1}[_{\mathcal{I}} \subseteq]\bigcup \mathcal{A}_n[_{\mathcal{I}}$ and $]\bigcup \mathcal{A}_n[_{\mathcal{I}} \neq \emptyset$. So, assuming that there is infinitely many $\mathcal{A}_n$'s we find a point $x \in X$ which belongs to infinitely many $\bigcup \mathcal{A}_n$'s. Then x belongs to infinitely many members of $\mathcal{F}$, what gives a contradiction with point-finiteness of the family $\mathcal{F}$. So, our construction ends up after k steps ($k < \omega$).
Now, put $\mathcal{F}_0 = \mathcal{F} \setminus \bigcup\{\mathcal{A}_n : n \leq k\}$. It is a desired family. □

Step 2. *There exists a subfamily $\mathcal{F}' \subseteq \mathcal{F}_0$ such that $\bigcup \mathcal{F}'$ is completely $\mathcal{I}$-nonmeasurable in $[\bigcup \mathcal{F}_0]_{\mathcal{I}}$.*

Proof. Let us enumerate two families of positive Borel sets. Namely,

$$\mathcal{B}^0 = \{B^0_\alpha : \alpha < 2^\omega\} = \left\{B \in \mathcal{B}_+(X) : B \subseteq \left[\bigcup \mathcal{F}_0\right]_{\mathcal{I}} \setminus \left]\bigcup \mathcal{F}_0\right[_{\mathcal{I}}\right\},$$

$$\mathcal{B}^1 = \{B^1_\alpha : \alpha < 2^\omega\} = \left\{B \in \mathcal{B}_+(X) : B \subseteq \left]\bigcup \mathcal{F}_0\right[_{\mathcal{I}}\right\}.$$

By transfinite induction we construct a sequence

$$((F^0_\xi, F^1_\xi, d_\xi) \in \mathcal{F}_0 \times \mathcal{F}_0 \times B^1_\xi : \ \ \xi < 2^\omega)$$

satisfying the following conditions

(1) $F^0_\xi \cap B^0_\xi \neq \emptyset, \quad F^1_\xi \cap B^1_\xi \neq \emptyset$,
(2) $d_\xi \notin \bigcup_{\xi<2^\omega}(F^0_\xi \cup F^1_\xi)$.

Assume that we have constructed a sequence $((F^0_\xi, F^1_\xi, d_\xi) \in \mathcal{F}_0 \times \mathcal{F}_0 \times B^1_\xi : \ \ \xi < \alpha)$ Since $|\{F \in \mathcal{F}_0 : \ d_\xi \in F \text{ for some } \xi < \alpha\}| < 2^\omega$, we are able to find F^0_α, F^1_α such that $F^0_\alpha, F^1_\alpha \notin \{F \in \mathcal{F}_0 : \ d_\xi \in F \text{ for some } \xi < \alpha\}$ and $F^0_\alpha \cap B^0_\alpha \neq \emptyset, F^1_\alpha \cap B^1_\alpha \neq \emptyset$. What is more $\bigcup\{F^0_\xi, F^1_\xi : \xi \leq \alpha\}$ does not cover B^1_α. So, we can pick $d_\alpha \in B^1_\alpha \setminus \bigcup\{F^0_\xi, F^1_\xi : \xi \leq \alpha\}$. It finishes α step of our construction.
Now, let us define $\mathcal{F}' = \{F^0_\xi, F^1_\xi : \ \ \xi \in 2^\omega\}$. We have that $\bigcup \mathcal{F}'$ has not empty intersection with any positive Borel set contained in $[\bigcup \mathcal{F}_0]_{\mathcal{I}}$ and $\{d_\xi : \xi \in 2^\omega\}$ has not empty intersection with every positive Borel set contained in $]\bigcup \mathcal{F}_0[_{\mathcal{I}}$. Moreover, $\{d_\xi : \ \ \xi \in 2^\omega\} \cap \bigcup \mathcal{F}' = \emptyset$ It implies that $\bigcup \mathcal{F}'$ does not contain any positive Borel set. It shows that $\bigcup \mathcal{F}'$ is completely $\mathcal{I}$–nonmeasurable in $[\bigcup \mathcal{F}_0]_{\mathcal{I}}$. □

Since $[\bigcup \mathcal{F}]_{\mathcal{I}} = [\bigcup \mathcal{F}_0]_{\mathcal{I}}$, it finishes the proof. □

Remark 3.2. *Assuming that $cov(\mathcal{I}) > \omega_1$ we can prove the same theorem for wider class of families. Namely, it is enough to assume that a family $\mathcal{F} \subseteq \mathcal{I}$ is point-countable, i.e. $(\forall x \in X)(|\{F \in \mathcal{F} : x \in f\}| \leq \omega$. Since $cov(\mathcal{I}) > \omega_1$, there is a point which belongs to ω_1 many Borel sets with the same envelope.*

Proof of Theorem 2.3. By an argument contained in the proof of Theorem 2.2, without loss of generality, we can assume that $F^{-1}(U)$ is Borel for every open set U in Y. Fix any $B \in \mathcal{B}_+(X)$. By Kuratowski–Ryll-Nardzewski selection theorem (see [**6**], Theorem 5.2.1, see [**2**]), $F\lceil B$ admits a Borel selection s. The range of s, being uncountable, is of cardinality 2^ω. This implies that the condition (2) of Theorem 3.4 is satisfied by $\mathcal{F} = \{F^{-1}(y) : y \in Y\}$. Since each $F(x)$ is finite, $\mathcal{F}$ is point-finite. The result now follows from Theorem 3.4. □

Proof of Theorem 2.4. Without loss of generality, we can assume that $\pi_X(F) = X$. Since I is c.c.c., every analytic set in X is in $\overline{\mathcal{B}}(X)$ (see Section 1). It follows that F is the graph of $\overline{\mathcal{B}}(X)$-measurable, finite set valued multifunction. The result follows from Theorem 2.3. □

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