SPECTRAL ASYMPTOTICS FOR ROBIN PROBLEMS WITH A DISCONTINUOUS COEFFICIENT

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Abstract. The spectral behavior of the difference between the resolvents of two realizations of a second-order strongly elliptic symmetric differential operator A defined by different Robin conditions $\chi u = b_1\gamma_0 u$ and $\chi u = b_2\gamma_0 u$, can in the case where all coefficients are C^∞ be determined by use of a general result by the author in 1984 on singular Green operators. We here treat the problem for nonsmooth b_i, showing that if b_1 and b_2 are in L_∞, the s-numbers s_j satisfy $s_j j^{3/(n-1)} \le C$ for all j. This improves a recent result for $A = -\Delta$ by Behrndt et al., that $\sum_j s_j^p < \infty$ for $p > (n-1)/3$, under a hypothesis of boundedness of b_i^{-1}. Moreover, we show that if b_1 and b_2 are in C^ε for some $\varepsilon > 0$, with jumps at a smooth hypersurface, then $s_j j^{3/(n-1)} \to c$ for $j \to \infty$, with a constant defined from the principal symbol of A and $b_2 - b_1$.

We also show that the usual principal spectral asymptotic estimate for pseudodifferential operators of negative order on a closed manifold extends to products of pseudodifferential operators of negative order interspersed with piecewise continuous functions.

Introduction.

Consider a second-order strongly elliptic symmetric operator

$$A = -\sum_{j,k=1}^{n} \partial_j(a_{jk}\partial_k u) + a_0 u \tag{0.1}$$

on a bounded smooth domain $\Omega \subset \mathbb{R}^n$, and denote by A_γ, A_ν, resp. $\widetilde{A}$, the realizations in $L_2(\Omega)$ defined by the Dirichlet condition $\gamma_0 u = 0$, the Neumann condition $\nu_A u = 0$, resp. a Robin condition $\nu_A u - bu = 0$ with b real. (Here $\gamma_0 u = u|_{\partial\Omega}$, and ν_A is the conormal derivative $\nu_A u = \sum_{j,k=1}^n n_j \gamma_0(a_{jk}\partial_k u)$, with $\vec{n} = (n_1, \dots, n_n)$ denoting the interior normal to $\partial\Omega$.) It is a classical result of Birman [B62], shown also for exterior domains, that the difference between the resolvents of the Robin realization and the Dirichlet realization is compact and has the spectral behavior, for large negative λ,

$$s_j\big((\widetilde{A} - \lambda)^{-1} - (A_\gamma - \lambda)^{-1}\big) j^{2/(n-1)} \le C \text{ for all } j; \tag{0.2}$$

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here $s_j(T)$ denotes the j-th eigenvalue of $(T^*T)^{\frac12}$ (the j-th s-number or singular value of T), counted with multiplicities. This was shown assuming merely that $b\in L_\infty(\partial\Omega)$. For the situation where all coefficients are C^∞, the estimate was later improved to an asymptotic estimate

$$s_j\big((\widetilde A-\lambda)^{-1}-(A_\gamma-\lambda)^{-1}\big)j^{2/(n-1)}\to c \text{ for } j\to\infty; \tag{0.3}$$

this follows from Grubb [G74], Sect. 8 (with generalizations to higher-order operators), and Birman and Solomiak [BS80] (including exterior domains). The paper [G84] gave tools to extend (0.3) to nonselfadjoint situations (also for exterior domains by a cutoff technique), by showing that for any singular Green operator G on Ω of order $-t<0$ and class 0,

$$s_j(G)j^{t/(n-1)}\to c \text{ for } j\to\infty; \tag{0.4}$$

here G belongs to the calculus of pseudodifferential boundary operators, introduced by Boutet de Monvel [B71] and further developed in [G84], [G96]. In fact, the resolvent differences considered above are singular Green operators of order -2 and class 0, when all coefficients are smooth.

Considering another resolvent difference, J. Behrndt, M. Langer, I. Lobanov, V. Lotoreichik and I. Popov showed in a recent paper [BLLLP10], on the basis of a theory of quasi-boundary triples by J. Behrndt and M. Langer [BL07], that when $A=-\Delta$ and b is a real function in $L_\infty(\partial\Omega)$ with $b^{-1}\in L_\infty(\partial\Omega)$, the difference between the resolvent of $\widetilde A$ and the resolvent of the Neumann realization A_ν satisfies an estimate with 2 replaced by 3, for λ in the intersection of resolvent sets $\varrho(\widetilde A)\cap\varrho(A_\nu)$:

$$(\widetilde A-\lambda)^{-1}-(A_\nu-\lambda)^{-1}\in\mathcal C_p \text{ for } p>3/(n-1); \tag{0.5}$$

here $\mathcal C_p$ denotes the space of compact operators T with singular value sequences $(s_j(T))_{j\in\mathbb N}\in\ell_p$; the Schatten class of order p. (Besides real b, also cases with a fixed sign on $\operatorname{Im} b$ were treated.)

In the smooth case this follows for arbitrary $b\in C^\infty(\partial\Omega)$ from (0.4) with a more precise estimate:

$$s_j\big((\widetilde A-\lambda)^{-1}-(A_\nu-\lambda)^{-1}\big)j^{3/(n-1)}\to c \text{ for } j\to\infty; \tag{0.6}$$

as noted also in [G10], Cor. 8.4 and Ex. 8.5.

The result of [BLLLP10] is more general by treating nonsmooth b, but has an assumption of boundedness of b^{-1} that excludes many C^∞-functions. The authors have informed us of a forthcoming work removing that assumption.

We shall give a proof in this paper without the hypothesis of boundedness of b^{-1}, that an upper bound

$$s_j\big((\widetilde A-\lambda)^{-1}-(A_\nu-\lambda)^{-1}\big)j^{3/(n-1)}\le C \text{ for all } j, \tag{0.7}$$

holds for any complex $b\in L_\infty(\partial\Omega)$ (this implies (0.5)).

Moreover, we shall show that when b has a little smoothness, e.g. is in a Hölder space C^ε for some $\varepsilon>0$, then the singular values satisfy the asymptotic estimate (0.6), where

c is a constant determined from b and the principal symbol of A. Finally, we show that such asymptotic estimates hold even when b has jumps at smooth hypersurfaces of $\partial\Omega$.

For the results leading to (0.7), the method is, as in [BLLLP10], an application of functional analysis, building on a theory of extensions (here Grubb [G68]) together with a general knowledge of elliptic boundary value problems. The extension of (0.6) to the nonsmooth situations draws on methods and results for pseudodifferential boundary operators in [G84] and a result on operators with restricted kernels by Laptev [L77, L81]. As an auxiliary result of independent interest we show that a product of classical pseudodifferential operators of negative order on a closed manifold, interspersed with piecewise continuous functions having jumps at a smooth hypersurface, has a principal spectral asymptotics estimate as in the smooth case.

We consider a slightly more general operator A than in (0.1) including first-order terms, assuming that it is associated with a symmetric sesquilinear form that is coercive on $H^1(\Omega)$.

There exist sophisticated methods for piecewise smooth boundary conditions, see e.g. Peetre [P61, P63], Shamir [S63], Eskin [E81], Rempel and Schulze [RS83], Harutyunyan and Schulze [HS08], giving microlocal treatments, but they are not needed for the present results. Let us also mention that we do not here address the question of nonsmooth domains, as e.g. in Gesztesy and Mitrea [GM09, GM10] and [AGW10], and their references.

To keep the paper short, some introductory material found in other sources will not be repeated here.

The main details of the extension theory [G68]–[G74] have been recalled and explained in several recent papers [BGW09], [G08], [G10]; resulting Krein-type resolvent formulas are shown in [BMW09].

Sobolev spaces are recalled in numerous places. The basic facts we shall need on these and other function spaces such as Besov and Bessel-potential spaces, are recalled e.g. in [AGW10], Sect. 2.

The calculus of pseudodifferential boundary operators is explained in Boutet de Monvel [B71] and in [G84], [G96], [G09].

1. The Robin realization.

Let Ω be a bounded smooth subset of $\mathbb{R}^n$ with boundary $\partial\Omega = \Sigma$, and let

$$a(u,v) = \sum_{j,k=1}^{n} (a_{jk}\partial_k u, \partial_j v) + \sum_{j=1}^{n}((a_j\partial_j u, v) + (a'_j u, \partial_j v)) + (a_0 u, v), \tag{1.1}$$

be a sesquilinear form with coeficients in $C^\infty(\overline{\Omega})$ such that the associated second-order operator

$$Au = -\sum_{j,k=1}^{n} \partial_j(a_{jk}\partial_k u) + \sum_{j=1}^{n}(a_j\partial_j u - \partial_j(a'_j u)) + a_0 u, \tag{1.2}$$

is formally selfadjoint and strongly elliptic. We assume moreover that $a(u,u)$ is real for $u \in H^1(\Omega)$ and (with $c > 0$, $k \geq 0$)

$$a(u,u) \geq c\|u\|_1^2 - k\|u\|_0^2, \text{ for } u \in H^1(\Omega). \tag{1.3}$$

This holds if the matrix $(a_{jk})_{j,k=1}^n$ is real, symmetric and positive definite, $a_j' = \overline{a_j}$, and a_0 is real, at each $x \in \overline{\Omega}$.

Let $b \in L_\infty(\Sigma)$, and define the sesquilinear form a_b by

$$a_b(u,v) = a(u,v) + (b\gamma_0 u, \gamma_0 v)_{L_2(\Sigma)}. \tag{1.4}$$

Since $\|\gamma_0 u\|^2_{L_2(\Sigma)} \le c'\|u\|^2_{\frac34} \le \varepsilon\|u\|_1^2 + C(\varepsilon)\|u\|_0^2$ for any ε, we infer from (1.3) that

$$\operatorname{Re} a_b(u,u) \ge c_1\|u\|_1^2 - k_1\|u\|_0^2, \text{ for } u \in H^1(\Omega), \tag{1.5}$$

where $c_1 < c$ is close to c and $k_1 \ge k$ is a large constant.

The sesquilinear form a_b on $V = H^1(\Omega)$ in $H = L_2(\Omega)$ defines a realization $\widetilde{A}$ of A by Lions' version of the Lax-Milgram lemma (as recalled e.g. in [G09], Ch. 12), with domain

$$D(\widetilde{A}) = \{u \in H^1(\Omega) \cap D(A_{\max}) \mid (Au, v) = a_b(u,v) \text{ for all } v \in H^1(\Omega)\}.$$

The operator $\widetilde{A}$ is closed, densely defined with spectrum in a sectorial region in $\{\operatorname{Re}\lambda \ge -k_1\}$, and its adjoint $\widetilde{A}^*$ is the analogous operator defined from

$$a_b^*(u,v) = \overline{a(v,u)} + (\bar{b}\gamma_0 u, \gamma_0 v)_{L_2(\Sigma)}. \tag{1.6}$$

In particular, when b is real, $\widetilde{A}$ is selfadjoint.

It will be useful to observe:

Lemma 1.1. *For any small $\theta > 0$ there is an $\alpha \ge 0$ such that the spectrum of $\widetilde{A}$ is contained in the region*

$$M_{\theta,\alpha,k_1} = \{z \in \mathbb{C} \mid |\operatorname{Im} z| \le \theta(\operatorname{Re} z + \alpha),\ \operatorname{Re} z \ge -k_1\}. \tag{1.7}$$

Proof. Let $K = \|\operatorname{Im} b\|_{L_\infty(\Sigma)}$. From the inequalities for $a_b(u,u)$ we see that for $u \in H^1(\Omega)$,

$$\begin{aligned}|\operatorname{Im} a_b(u,u)| &= |\operatorname{Im}(b\gamma_0 u, \gamma_0 u)| \le K(\varepsilon\|u\|_1^2 + C(\varepsilon)\|u\|_0^2)\\ &\le K\varepsilon c_1^{-1}(\operatorname{Re} a_b(u,u) + k_1\|u\|_0^2) + KC(\varepsilon)\|u\|_0^2\\ &= K\varepsilon c_1^{-1}\operatorname{Re} a_b(u,u) + (K\varepsilon c_1^{-1}k_1 + KC(\varepsilon))\|u\|_0^2.\end{aligned}$$

This (together with (1.5)) shows that for $u \ne 0$, $a_b(u,u)/\|u\|_0^2$ has its values in M_{θ,α,k_1}, where $\theta = K\varepsilon c_1^{-1}$ can be taken arbitrarily small, $\alpha = K\varepsilon c_1^{-1}k_1 + KC(\varepsilon)$. The numerical ranges of $\widetilde{A}$ and $\widetilde{A}^*$ are contained in this set, which then also contains the spectra. (More details for this kind of argument can be found in [G09], Sect. 12.4.) ☐

We denote $\sum_j n_j\gamma_0\partial_j u = \gamma_1 u$. The Neumann-type boundary operator

$$\chi u = \sum_{j,k=1}^n n_j\gamma_0 a_{jk}\partial_k u + \sum_{j=1}^n n_j\gamma_0 a_j' u, \tag{1.8}$$

enters in the "halfways Green's formula"

$$(Au,v)-a(u,v)=(\chi u,\gamma_0 v)_{L_2(\Sigma)}, \tag{1.9}$$

for smooth u and v. It is known e.g. from [LM68] that γ_1 and χ extend to continuous mappings from $H^1(\Omega)\cap D(A_{\max})$ to $H^{-\frac12}(\Sigma)$, such that for $u\in H^1(\Omega)\cap D(A_{\max})$, $v\in H^1(\Omega)$, (1.9) holds with the scalar product over Σ replaced by the sesquilinear duality between $H^{-\frac12}(\Sigma)$ and $H^{\frac12}(\Sigma)$. Then

$$(Au,v)-a_b(u,v)=(\chi u,\gamma_0 v)_{H^{-\frac12}(\Sigma),H^{\frac12}(\Sigma)}-(b\gamma_0 u,\gamma_0 u)_{L_2(\Sigma)},$$

and hence

$$D(\widetilde A)=\{u\in H^1(\Omega)\cap D(A_{\max})\mid \chi u=b\gamma_0 u \text{ in } H^{-\frac12}(\Sigma)\}.$$

Note that $\chi u=\nu_A u+s\gamma_0 u$ for a smooth function s, so the more traditional Robin conditions $\nu_A u-b'\gamma_0 u=0$ can be written in the form $\chi u-b\gamma_0 u=0$ by taking $b=b'+s$. (We also have that $\nu_A u=s_0\gamma_1 u+\mathcal{A}_1\gamma_0 u$ with an invertible smooth function s_0 and a first-order tangential differential operator $\mathcal{A}_1$.)

For $b=0$, the condition is $\chi u=0$, defining what we shall call the Neumann realization A_χ; it is selfadjoint with $D(A_\chi)\subset H^2(\Omega)$. It is well-known that when b is smooth, then $D(\widetilde A)\subset H^2(\Sigma)$.

Lemma 1.2. *When $b\in L_\infty(\Sigma)$, the domain of $\widetilde A$ satisfies*

$$D(\widetilde A)\subset H^{\frac32}(\Omega)\cap D(A_{\max}).$$

Proof. When $u\in D(\widetilde A)$, then $u\in H^1(\Omega)$ implies $\gamma_0 u\in H^{\frac12}(\Sigma)\subset L_2(\Sigma)$. Multiplication by b is continuous on $L_2(\Sigma)$, so $b\gamma_0 u\in L_2(\Sigma)$. Then also $\chi u=b\gamma_0 u$ is in $L_2(\Sigma)$. By the ellipticity of the Neumann problem, $Au\in L_2(\Omega)$ with $\chi u\in L_2(\Sigma)$ imply $u\in H^{\frac32}(\Omega)$. □

When b has more smoothness or piecewise smoothness, we can get more regularity: It is known that when b is in the Bessel potential space $H^r_p(\Sigma)$ with $r>(n-1)/p$, $p\ge 2$, then multiplication by b is continuous in $H^s(\Sigma)$ for $|s|\le r$ (cf. e.g. Johnsen [J95]). In relation to Hölder spaces C^r and Besov spaces $B^r_{p.q}$ there are inclusions

$$C^{r+2\varepsilon}(\Sigma)\hookrightarrow B^{r+\varepsilon}_{\infty,2}(\Sigma)\hookrightarrow B^{\alpha+\varepsilon}_{p,2}(\Sigma)\hookrightarrow H^r_p(\Sigma),\quad \varepsilon>0, \tag{1.10}$$

so also functions in these spaces preserve $H^s(\Omega)$ for $|s|\le r$. (A summary of the relevant facts on function spaces is given e.g. in [AGW10], Sect. 2.)

When $X(\Sigma)$ is a function space over Σ, we say that b is piecewise in X, when the $(n-1)$-dimensional manifold Σ is a union $\Sigma_1\cup\dots\cup\Sigma_J$ of smooth subsets Σ_j with disjount interiors (such that the interfaces are smooth $(n-2)$-dimensional manifolds), and b equals a function $b_j\in X(\Sigma)$ on each of the interiors.

It is well-known that multiplication by 1_{Σ_j} is continuous on $H^s(\Sigma)$ for all $|s|<\frac12$.

Proposition 1.3.

1° *Let* $b\in H^r_p(\Sigma)$ *with* $r>(n-1)/p$, $p\ge 2$ *(it holds if* b *is in one of the spaces in* (1.10)*). Then* $D(\widetilde A)\subset H^{\frac32+r}(\Omega)$ *if* $r<\frac12$, $D(\widetilde A)\subset H^2(\Omega)$ *if* $r\ge\frac12$.

2° *Let* b *be piecewise in* $H^r_p(\Sigma)$ *with* $r>(n-1)/p$, $p\ge 2$. *Then* $D(\widetilde A)\subset H^{\frac32+r}(\Omega)$ *if* $r<\frac12$, $D(\widetilde A)\subset H^{2-\varepsilon}(\Omega)$ *for any* $\varepsilon>0$ *if* $r\ge\frac12$.

Proof. As already noted, $u\in H^1(\Omega)$ implies $\gamma_0u\in H^{\frac12}(\Sigma)$. In the case 1°, multiplication by b preserves $H^s(\Sigma)$ for $|s|\le r$, so $b\gamma_0u\in H^{\min\{r,\frac12\}}(\Sigma)$. Then also $\chi u=b\gamma_0u$ is in $H^{\min\{r,\frac12\}}(\Sigma)$, and now $Au\in L_2(\Omega)$ with $\chi u\in H^{\min\{r,\frac12\}}(\Sigma)$ imply $u\in H^{\frac32+r}(\Omega)$ if $r<\frac12$, $u\in H^2(\Omega)$ if $r\ge\frac12$, by the ellipticity of the Neumann problem.

In the case 2°, since $b=\sum_{j=1}^J b_j1_{\Sigma_j}$, multiplication by b maps $H^r(\Sigma)$ into itself if $r<\frac12$, and into $H^{\frac12-\varepsilon}$, any $\varepsilon>0$, if $r\ge\frac12$. Completing the proof as under 1°, we find that $u\in H^{\frac32+r}(\Omega)$ if $r<\frac12$, $u\in H^{2-\varepsilon}(\Omega)$ if $r\ge\frac12$. □

Let us regard $\widetilde A$ from the point of view of the general extension theory of [G68], as recalled in [BGW09], [G08], [G10].

We take the Dirichlet realization A_γ as the reference operator, assumed to have a positive lower bound. (Seen from the point of view of [G68], [BL07] uses instead the Neumann realization A_χ as the reference operator.) The operator $\widetilde A$ corresponds, by the general theory, to a closed densely defined operator $T\colon V\to W$, where V and W are closed subsets of $Z=\ker A_{\max}$ and $D(T)$ is dense in V; and this in turn is carried over by use of the homeomorphism $\gamma_0:Z\xrightarrow{\sim}H^{-\frac12}(\Sigma)$, to a closed operator $L:X\to Y^*$, with domain $D(L)$ dense in X, where X and Y are closed subspaces of $H^{-\frac12}(\Sigma)$. Here $X=\gamma_0V$, $Y=\gamma_0W$ and $D(L)=\gamma_0D(T)=\gamma_0D(\widetilde A)$.

Proposition 1.4. *The operator* $L:X\to Y$ *corresponding to* $\widetilde A$ *by* [G68] *has* $X=Y=H^{-\frac12}(\Sigma)$*, and acts like* $b-P^0_{\gamma,\chi}$ *with a domain contained in* $H^1(\Sigma)$*. When* b *is real,* L *is selfadjoint as an unbounded operator from* $H^{-\frac12}(\Sigma)$ *to* $H^{\frac12}(\Sigma)$.

Proof. Besides the description in [BGW09], we shall use the observations on operators defined by sesquilinear forms worked out in [G70] (and partly recalled in [G09], Ch. 13.2, see in particular Th. 13.19). Since the domain of $a_b(u,v)$ equals $H^1(\Omega)$, T is defined from a sesquilinear form $t(z,w)$ with domain $H^1(\Omega)\cap Z$ dense in Z, and hence $V=W=Z$. It follows that $X=Y=H^{-\frac12}(\Sigma)$, and L is densely defined and closed as an operator from $H^{-\frac12}(\Sigma)$ to $H^{\frac12}(\Sigma)$. The adjoint L^* is of the same type and corresponds to $\widetilde A^*$. When b is real, $\widetilde A$ is selfadjoint as noted above; then L is selfadjoint.

In the interpretation of the extension theory, $\widetilde A$ represents the boundary condition

$$\gamma_0u\in D(L),\quad \Gamma u=L\gamma_0u;$$

where $\Gamma u=\chi u-P^0_{\gamma,\chi}\gamma_0u$, so $L\gamma_0u=\chi u-P^0_{\gamma,\chi}\gamma_0u$ when $u\in D(\widetilde A)$. ($P^\lambda_{\gamma,\chi}$ is the operator mapping Dirichlet boundary values to Neumann boundary values for solutions of $(A-\lambda)u=0$; more on this below.) Since the functions in $D(\widetilde A)$ also satisfy $\chi u=b\gamma_0u$, we see that L acts like

$$L\varphi=(b-P^0_{\gamma,\chi})\varphi.$$

By Lemma 1.2, $D(\widetilde{A}) \subset H^{\frac{3}{2}}(\Sigma)$, so $D(L) = \gamma_0 D(\widetilde{A}) \subset H^1(\Sigma)$. $\square$

When we replace A by $A - \lambda$, where λ is in the resolvent set $\varrho(A_\gamma)$ of A_γ, we get for the corresponding operator L^λ:

$$L^\lambda \text{ acts like } b - P^\lambda_{\gamma,\chi}, \text{ with } D(L^\lambda) = D(L) \subset H^1(\Sigma).$$

For $\lambda \in \varrho(A_\gamma) \cap \varrho(\widetilde{A})$, there holds a Krein resolvent formula (shown in [BGW09], Th. 3.4):

$$(\widetilde{A} - \lambda)^{-1} = (A_\gamma - \lambda)^{-1} + K^\lambda_\gamma (L^\lambda)^{-1} (K^{\bar{\lambda}}_\gamma)^*. \tag{1.11}$$

Here K^λ_γ is the Poisson operator for the Dirichlet problem, i.e. the solution operator $K^\lambda_\gamma : \varphi \mapsto u$ for the problem

$$(A - \lambda)u = 0 \text{ on } \Omega, \quad \gamma_0 u = \varphi \text{ on } \Sigma;$$

it maps $H^{s-\frac{1}{2}}(\Sigma) \to H^s(\Omega)$ continuously for all s, and the adjoint maps e.g. $L_2(\Omega)$ to $H^{\frac{1}{2}}(\Sigma)$.

We can use this to show a spectral estimate for $(\widetilde{A} - \lambda)^{-1} - (A_\chi - \lambda)^{-1}$, going via differences with the Dirichlet resolvent. The argumentation is not the same as that of [BLLLP10], which uses a Krein formula based on the Poisson operator for the Neumann problem, and needs to assume essentially that b has a bounded inverse.

The spectrum of A_γ is contained in a positive halfline $[c_0, \infty[$, and the spectrum of A_χ is contained in a larger halfline $]-k, \infty[$, cf. (1.3). For $\lambda \in \mathbb{C} \setminus]-k, \infty[$, the Dirichlet-to-Neumann operator $P^\lambda_{\gamma,\chi} = \chi K^\lambda_\gamma$ is a homeomorphism from $H^s(\Sigma)$ to $H^{s-1}(\Sigma)$ for all $s \in \mathbb{R}$, with inverse $P^\lambda_{\chi,\gamma}$, the Neumann-to-Dirichlet operator. Then we can write

$$L^\lambda \varphi = (b - P^\lambda_{\gamma,\chi})\varphi = (b P^\lambda_{\chi,\gamma} - 1) P^\lambda_{\gamma,\chi} \varphi, \text{ for } \varphi \in D(L). \tag{1.12}$$

Since $P^\lambda_{\chi,\gamma}$ is of order -1, it is compact in $L_2(\Sigma)$. Then $bP^\lambda_{\chi,\gamma} - 1$ is a Fredholm operator in $L_2(\Sigma)$, as noted also in [BLLLP10]. If λ is such that: (1) L^λ is invertible (from $D(L)$ to $H^{\frac{1}{2}}(\Sigma)$), (2) $bP^\lambda_{\chi,\gamma} - 1$ is invertible in $L_2(\Sigma)$, then the two inverses must coincide on $H^{\frac{1}{2}}(\Sigma)$.

For $bP^\lambda_{\chi,\gamma} - 1$, we get invertibility as follows: We have as a simple application of the principles in [G96] (cf. Th. 2.5.6, (A.25–26)) that

$$\|P^\lambda_{\gamma,\chi}\varphi\|_{H^{s,\mu}(\Sigma)} \simeq \|\varphi\|_{H^{s+1,\mu}(\Sigma)}, \quad \|\varphi\|_{H^{s-1,\mu}(\Sigma)} \simeq \|P^\lambda_{\chi,\gamma}\varphi\|_{H^{s,\mu}(\Sigma)},$$

uniformly in $\mu = |\lambda|^{\frac{1}{2}}$ for $\lambda \to \infty$ on rays in $\mathbb{C} \setminus \mathbb{R}_+$; this holds since $P^\lambda_{\gamma,\chi}$ is parameter-elliptic of order 1 and regularity $+\infty$ on the rays in $\mathbb{C} \setminus \mathbb{R}_+$. In particular, one has on such a ray $\{\lambda = \mu^2 e^{i\eta}\}$ with $\eta \in]0, 2\pi[$, for $s \in [0,1]$ and $\mu \geq 1$,

$$\|P^\lambda_{\chi,\gamma}\varphi\|_{H^s(\Sigma)} + \langle\mu\rangle^s \|P^\lambda_{\chi,\gamma}\varphi\|_{L_2(\Sigma)} \leq c \min\{\|\varphi\|_{H^{s-1}(\Sigma)}, \langle\mu\rangle^{s-1} \|\varphi\|_{L_2(\Sigma)}\},$$

so the norm of $P^\lambda_{\chi,\gamma}$ in $L_2(\Sigma)$ is $O(\langle\mu\rangle^{-1})$ on the ray. Take μ_0 so large that $\|bP^\lambda_{\chi,\gamma}\|_{\mathcal{L}(L_2(\Sigma)}\le \delta<1$ for $\mu\ge\mu_0$, then $bP^\lambda_{\chi,\gamma}-1$ is invertible as an operator in $L_2(\Sigma)$ for $\mu\ge\mu_0$, with a bounded inverse $(bP^\lambda_{\chi,\gamma}-1)^{-1}$:

$$(bP^\lambda_{\chi,\gamma}-1)^{-1}=-1-\sum_{k=1}^\infty (bP^\lambda_{\chi,\gamma})^k,\text{ converging in }\mathcal{L}(L_2(\Sigma)). \tag{1.13}$$

Then $b-P^\lambda_{\gamma,\chi}$ has an inverse

$$(b-P^\lambda_{\gamma,\chi})^{-1}=P^\lambda_{\chi,\gamma}(bP^\lambda_{\chi,\gamma}-1)^{-1}. \tag{1.14}$$

For L^λ we know from the extension theory that L^λ is bijective from $D(L)$ to $H^{\frac12}(\Sigma)$ if and only if $\lambda\in\varrho(\widetilde A)$. It follows from Lemma 1.1 by a simple geometric consideration that for each ray $\{\lambda=\mu^2e^{i\eta}\}$ with $\eta\in\,]0,2\pi[\,$, there is a μ_1 such that such that $\lambda\in\varrho(\widetilde A)$ for $\mu\ge\mu_1$.

For $\mu\ge\max\{\mu_0,\mu_1\}$, both (1) and (2) are satisfied, so then

$$(L^\lambda)^{-1}=(b-P^\lambda_{\gamma,\chi})^{-1}=P^\lambda_{\chi,\gamma}(bP^\lambda_{\chi,\gamma}-1)^{-1}\text{ on }H^{\frac12}(\Sigma). \tag{1.15}$$

We note in particular that

$$D(L^\lambda)=\{\varphi\in H^1(\Sigma)\mid (b-P^\lambda_{\gamma,\chi})\varphi\in H^{\frac12}(\Sigma)\}, \tag{1.16}$$

for such λ. Now $D(L)=D(L^\lambda)$, and $P^0_{\gamma,\chi}-P^\lambda_{\gamma,\chi}$ is bounded from $H^{-\frac12}(\Sigma)$ to $H^{\frac12}(\Sigma)$ (cf. [BGW09], Rem. 3.2), so we conclude that

$$D(L)=\{\varphi\in H^1(\Sigma)\mid (b-P^0_{\gamma,\chi})\varphi\in H^{\frac12}(\Sigma)\}. \tag{1.17}$$

It follows moreover that (1.16) holds for *all* $\lambda\in\varrho(A_\gamma)$.

This shows the main part of:

Theorem 1.5. *The domain of L satisfies* (1.17)*, and it is also described by* (1.16) *for any $\lambda\in\varrho(A_\gamma)$.*

On each ray in $\mathbb{C}\setminus\mathbb{R}_+$, λ is in $\varrho(\widetilde A)$ and (1.15) *holds for $|\lambda|$ sufficiently large. For such λ,*

$$(\widetilde A-\lambda)^{-1}-(A_\gamma-\lambda)^{-1}=K^\lambda_\gamma P_{\chi,\gamma}(bP^\lambda_{\chi,\gamma}-1)^{-1}K^{\bar\lambda\,*}_\gamma. \tag{1.18}$$

Proof. The statements before formula (1.18) were accounted for above, and the formula follows by insertion of (1.15) in (1.11). □

2. Spectral estimates.

Spectral estimates for resolvent differences will now be studied. A classical reference for the basic concepts is the book of Gohberg and Krein [GK69]; some particularly relevant facts were collected in [G84], supplied with additional results. We shall include a short summary here:

For $p > 0$, the space $\mathcal{C}_p$ is the Schatten class of compact linear operators T (in a Hilbert space H) with singular value sequences $(s_j(T))_{j\in\mathbb{N}} \in \ell_p$, and $\mathfrak{S}_p$ denotes the quasi-normed space of compact operators T with $s_j(T) = O(j^{-1/p})$; here $\mathfrak{S}_p \subset \mathcal{C}_{p+\varepsilon}$ for all $\varepsilon > 0$.

The rules shown by Ky Fan [F51]

$$s_{j+k-1}(T + T') \le s_j(T) + s_k(T'), \quad s_{j+k-1}(TT') \le s_j(T)s_k(T'), \tag{2.1}$$

imply that $\mathcal{C}_p$ and $\mathfrak{S}_p$ are vector spaces, and that a product rule holds:

$$\mathfrak{S}_p \cdot \mathfrak{S}_q \subset \mathfrak{S}_{1/(1/p+1/q)}, \quad \mathcal{C}_p \cdot \mathcal{C}_q \subset \mathcal{C}_{1/(1/p+1/q)}. \tag{2.2}$$

Moreover, the rule

$$s_j(ATB) \le \|A\| s_j(T) \|B\| \tag{2.3}$$

implies that $\mathfrak{S}_p$ and $\mathcal{C}_p$ are preserved under compositions with bounded operators. We mention two perturbation results:

Lemma 2.1.

1° If $s_j(T)j^{1/p} \to C_0$ and $s_j(T')j^{1/p} \to 0$ for $j \to \infty$, then $s_j(T + T')j^{1/p} \to C_0$ for $j \to \infty$.

2° If $T = T_M + T'_M$ for each $M \in \mathbb{N}$, where $s_j(T_M)j^{1/p} \to C_M$ for $j \to \infty$ and $s_j(T'_M)j^{1/p} \le \varepsilon_M$ for $j \in \mathbb{N}$, with $C_M \to C_0$ and $\varepsilon_M \to 0$ for $M \to \infty$, then $s_j(T)j^{1/p} \to C_0$ for $j \to \infty$.

The statement in 1° is the Weyl-Ky Fan theorem (cf. e.g. [GK69] Th. II 2.3), and 2° is a refinement shown in [G84], Lemma 4.2.2°.

We also recall that when Ξ is a compact n-dimensional smooth manifold (possibly with boundary) and T is a bounded linear operator from $L_2(\Xi)$ to $H^t(\Xi)$ for some $t > 0$, then $T \in \mathfrak{S}_{n/t}$ as an operator in $L_2(\Xi)$, with

$$s_j(T)j^{t/n} \le C\|T\|_{\mathcal{L}(L_2,H^t)}, \tag{2.4}$$

C depending only on Ξ and t. See [G84], Lemma 4.4ff. for references.

The Poisson operator K^λ_γ is continuous from $H^{s-\frac12}(\Sigma)$ to $H^s(\Omega)$ for all $s \in \mathbb{R}$, and its adjoint $K^{\lambda *}_\gamma$ is a trace operator of class 0 and order -1 in the pseudodifferential boundary operator calculus, hence is continuous from $H^s(\Omega)$ to $H^{s+\frac12}(\Sigma)$ for $s > -\frac12$. Then the composition $K^{\lambda *}_\gamma K^\lambda_\gamma$ is continuous from $L^2(\Sigma)$ to $H^1(\Sigma)$, so in view of (2.4), $K^{\lambda *}_\gamma K^\lambda_\gamma \in \mathfrak{S}_{n-1}$ and hence $K^\lambda_\gamma \in \mathfrak{S}_{(n-1)/(1/2)}$, as operators in $L_2(\Sigma)$. The singular numbers of $K^{\lambda *}_\gamma$ have the same behavior. Moreover, since $P^\lambda_{\chi,\gamma}$ is a pseudodifferential operator of order -1 on Σ, it lies in $\mathfrak{S}_{n-1}$ when considered as an operator in $L_2(\Sigma)$. .

Theorem 2.2. *Let $b \in L_\infty(\Sigma)$. For any $\lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\chi)$,*

$$(\widetilde{A} - \lambda)^{-1} - (A_\chi - \lambda)^{-1} \in \mathfrak{S}_{(n-1)/3}. \tag{2.5}$$

Proof. First assume that λ lies so far out on a ray in $\mathbb{C} \setminus \mathbb{R}_+$ that the statements in Theorem 1.5 are valid.

Applying (1.18) to our $\widetilde{A}$ and also to the case $b = 0$ (the Neumann realization), we find by subtraction:

$$\begin{aligned}(\widetilde{A} - \lambda)^{-1} - (A_\chi - \lambda)^{-1} &= (\widetilde{A} - \lambda)^{-1} - (A_\gamma - \lambda)^{-1} - ((A_\chi - \lambda)^{-1} - (A_\gamma - \lambda)^{-1}) \\ &= K_\gamma^\lambda P_{\chi,\gamma}^\lambda [(bP_{\chi,\gamma}^\lambda - 1)^{-1} + 1] K_\gamma^{\bar\lambda *} \\ &= K_\gamma^\lambda P_{\chi,\gamma}^\lambda (bP_{\chi,\gamma}^\lambda - 1)^{-1} b P_{\chi,\gamma}^\lambda K_\gamma^{\bar\lambda *}.\end{aligned} \tag{2.6}$$

The last expression is composed of the operator K_γ^λ in $\mathfrak{S}_{(n-1)/(1/2)}$, the adjoint of $K_\gamma^{\bar\lambda}$ with the same property, two factors $P_{\chi,\gamma}^\lambda$ in $\mathfrak{S}_{n-1}$ and the bounded operators $(bP_{\chi,\gamma}^\lambda - 1)^{-1}$ and b, so it belongs to $\mathfrak{S}_{(n-1)/3}$, by (2.2).

Now let λ' be an arbitrary number in $\varrho(\widetilde{A}) \cap \varrho(A_\chi)$. We use the following refined resolvent identity as in [BLLLP10]:

$$\begin{aligned}&(S - \lambda')^{-1} - (T - \lambda')^{-1} \\ &\quad = (1 + (\lambda' - \lambda)(T - \lambda')^{-1})((S - \lambda)^{-1} - (T - \lambda)^{-1})(1 + (\lambda' - \lambda)(S - \lambda')^{-1}),\end{aligned} \tag{2.7}$$

valid for $\lambda, \lambda' \in \varrho(T) \cap \varrho(S)$. Applying it to $S = \widetilde{A}$ and $T = A_\chi$ for λ as above and $\lambda' \in \varrho(\widetilde{A}) \cap \varrho(A_\chi)$, we find that $(\widetilde{A} - \lambda')^{-1} - (A_\chi - \lambda')^{-1}$ is a composition of an operator in $\mathfrak{S}_{(n-1)/3}$ with two bounded operators, hence lies in $\mathfrak{S}_{(n-1)/3}$, as was to be shown. □

The authors of [BLLLP10] have informed us that they can obtain the result of that paper without assuming that $b^{-1} \in L_\infty(\Sigma)$; details of proof will be included in a forthcoming paper.

There is an obvious corollary:

Corollary 2.3. *Let $b_1, b_2 \in L_\infty(\Sigma)$, and denote the corresponding realizations of Robin conditions $\chi u = b_1 \gamma_0 u$ resp. $\chi u = b_2 \gamma_0 u$ by $\widetilde{A}_1$ resp. $\widetilde{A}_2$. For any $\lambda \in \varrho(\widetilde{A}_1) \cap \varrho(\widetilde{A}_2)$,*

$$(\widetilde{A}_1 - \lambda)^{-1} - (\widetilde{A}_2 - \lambda)^{-1} \in \mathfrak{S}_{(n-1)/3}. \tag{2.8}$$

Proof. Write $(\widetilde{A}_1 - \lambda)^{-1} - (\widetilde{A}_2 - \lambda)^{-1}$ as the difference between $(\widetilde{A}_1 - \lambda)^{-1} - (A_\chi - \lambda)^{-1}$ and $(\widetilde{A}_2 - \lambda)^{-1} - (A_\gamma - \lambda)^{-1}$, then the result follows from Theorem 2.2 since $\mathfrak{S}_p$ is a vector space. □

Formula (1.18) also allows us to show a *spectral asymptotics* estimate for $(\widetilde{A} - \lambda)^{-1} - (A_\gamma - \lambda)^{-1}$ that was obtained in the smooth case for selfadjoint realizations and negative λ in Grubb [G74], Sect. 8, and Birman and Solomiak [BS80]. In the former paper it is shown, also for $2m$-order problems, that the operator is, on the complement of its nullspace, *isometric* to an elliptic pseudodifferential operator on Σ of order $-2m$ (which has the asserted spectral asymptotics); in the latter paper exterior domains are included.

Theorem 2.4. *Let $b \in L_\infty(\Sigma)$. For any $\lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\gamma)$,*

$$s_j((\widetilde{A}-\lambda)^{-1} - (A_\gamma - \lambda)^{-1}) j^{2/(n-1)} \to C_0 \text{ for } j \to \infty, \tag{2.9}$$

where C_0 is the same constant as in the case $b = 0$ (where $\widetilde{A} = A_\chi$).

Proof. For large λ on rays in $\mathbb{C} \setminus \mathbb{R}_+$ as in Theorem 1.5 we write formula (1.13) as

$$(bP^\lambda_{\chi,\gamma} - 1)^{-1} = -1 - bP^\lambda_{\chi,\gamma} S, \text{ where } S = \sum_{k=0}^\infty (bP^\lambda_{\chi,\gamma})^k \in \mathcal{L}(L_2(\Sigma)). \tag{2.10}$$

Then we have from (1.18):

$$\begin{aligned}(\widetilde{A}-\lambda)^{-1} - (A_\gamma - \lambda)^{-1} &= K^\lambda_\gamma P_{\chi,\gamma}(-1 - bP^\lambda_{\chi,\gamma} S) K^{\bar\lambda *}_\gamma \\ &= -K^\lambda_\gamma P_{\chi,\gamma} K^{\bar\lambda *}_\gamma - K^\lambda_\gamma P_{\chi,\gamma} b P^\lambda_{\chi,\gamma} S K^{\bar\lambda *}_\gamma .\end{aligned} \tag{2.11}$$

The first term equals $(A_\chi - \lambda)^{-1} - (A_\gamma - \lambda)^{-1}$ and is known to satisfy a spectral asymptotics estimate (2.9). The second term is in $\mathfrak{S}_{(n-1)/3}$, in view of the mapping properties of its factors, as in the proof of Theorem 2.2. By Lemma 2.1.1°, it follows that the sum of the two terms has the asymptotic behavior (2.9).

General $\lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\gamma)$ are included by use of the resolvent identity (2.7), which gives the operator as a sum of a term with the behavior (2.9) and terms in $\mathfrak{S}_{(n-1)/(2+t)}$ with $t > 0$, using that $(A_\gamma - \lambda)^{-1} \in \mathfrak{S}_{n/2}$ and $(\widetilde{A} - \lambda)^{-1} \in \mathfrak{S}_{n/(3/2)}$. Then Lemma 2.1.1° applies to show (2.9) for the sum. □

Spectral asymptotics estimates for the resolvent difference (2.5) are harder to get at, since b here enters in the principal part of the operator. However, with a little smoothness of b we can obtain the spectral estimate by reduction to a case that allows an approximation procedure.

We consider the resolvent difference of two Robin problems from the start, since the asymptotic property is not in general additive.

Theorem 2.5. *Assume that $b_1, b_2 \in H^r_p(\Sigma)$ with $r > (n-1)/p$, $p \geq 2$; this holds if the b_i are in one of the spaces in* (1.10)*, where r can be taken arbitrarily small positive. Define $\widetilde{A}_i$ as in Corollary* 2.3*. Then for $\lambda \in \varrho(\widetilde{A}_1) \cap \varrho(\widetilde{A}_2)$,*

$$s_j((\widetilde{A}_1 - \lambda)^{-1} - (\widetilde{A}_2 - \lambda)^{-1}) j^{3/(n-1)} \to C(g^0)^{3/(n-1)} \text{ for } j \to \infty, \tag{2.12}$$

where $C(g^0)$ is a constant defined from $b_2 - b_1$ and the principal symbols of K^λ_γ and $P^\lambda_{\chi,\gamma}$, described in detail in (2.18)–(2.19) *below.*

Proof. First let λ be large on a ray in $\mathbb{C} \setminus \mathbb{R}_+$ such that Theorem 1.5 applies to $\widetilde{A}_1$ and $\widetilde{A}_2$. Using (2.10) in the form

$$(b_i P^\lambda_{\chi,\gamma} - 1)^{-1} = -1 - b_i P^\lambda_{\chi,\gamma} - (b_i P^\lambda_{\chi,\gamma})^2 S_i$$

we have that

$$(b_1 P^\lambda_{\chi,\gamma} - 1)^{-1} - (b_2 P^\lambda_{\chi,\gamma} - 1)^{-1} = (b_2 - b_1) P^\lambda_{\chi,\gamma} - (b_1 P^\lambda_{\chi,\gamma})^2 S_1 - (b_2 P^\lambda_{\chi,\gamma})^2 S_2.$$

Then we get using (2.6):

(2.13)

$$
\begin{aligned}
(\widetilde A_1-\lambda)^{-1}-(\widetilde A_2-\lambda)^{-1}&=(\widetilde A_1-\lambda)^{-1}-(A_\chi-\lambda)^{-1}-((\widetilde A_2-\lambda)^{-1}-(A_\chi-\lambda)^{-1})\\
&=K^\lambda_\gamma P^\lambda_{\chi,\gamma}[(b_1P^\lambda_{\chi,\gamma}-1)^{-1}+1]K^{\bar\lambda *}_\gamma-K^\lambda_\gamma P^\lambda_{\chi,\gamma}[(b_2P^\lambda_{\chi,\gamma}-1)^{-1}+1]K^{\bar\lambda *}_\gamma\\
&=K^\lambda_\gamma P^\lambda_{\chi,\gamma}(b_2-b_1)P^\lambda_{\chi,\gamma}K^{\bar\lambda *}_\gamma-K^\lambda_\gamma P^\lambda_{\chi,\gamma}(b_1P^\lambda_{\chi,\gamma})^2S_1K^{\bar\lambda *}_\gamma+K^\lambda_\gamma P^\lambda_{\chi,\gamma}(b_2P^\lambda_{\chi,\gamma})^2S_2K^{\bar\lambda *}_\gamma\\
&=G+F_1+F_2.
\end{aligned}
$$

In the terms F_i we use for one of the factors $b_iP^\lambda_{\chi,\gamma}$ that b_i preserves $H^s(\Sigma)$ for $|s|\le r$ (see the text before Proposition 1.3), so that $b_iP^\lambda_{\chi,\gamma}$ maps $L_2(\Sigma)$ continuously into $H^{r'}(\Sigma)$, $r'=\min\{r,1\}$. So this factor is in $\mathfrak S_{(n-1)/r'}$, together with the usual two factors in $\mathfrak S_{(n-1)/(1/2)}$ and two factors in $\mathfrak S_{n-1}$, whereby the full composed operator F_i is in $\mathfrak S_{(n-1)/(3+r')}$. It will not influence the spectral asymptotics.

In the term G, let us denote $b_2-b_1=b$. We write b for each $M\in\mathbb N$ as a sum

$$b=b_M+b'_M, \tag{2.14}$$

where $b_M\in C^\infty(\Sigma)$ and $\sup_{x'\in\Sigma}|b'_M(x')|\le 1/M$; this is possible since b is continuous on the smooth compact manifold Σ. Accordingly, we write $G=G_M+G'_M$ with

$$G_M=-K^\lambda_\gamma P^\lambda_{\chi,\gamma}b_MP^\lambda_{\chi,\gamma}K^{\bar\lambda *}_\gamma,\quad G'_M=-K^\lambda_\gamma P^\lambda_{\chi,\gamma}b'_MP^\lambda_{\chi,\gamma}K^{\bar\lambda *}_\gamma.$$

Here G'_M is a composition of fixed operators with the usual $\mathfrak S_p$-properties and a factor b'_M whose norm in $\mathcal L(L_2(\Sigma))$ is $\le 1/M$; this implies that

$$\sup_j s_j(G'_M)j^{3/(n-1)}\le C/M,\ \text{all } M, \tag{2.15}$$

for a suitable constant C, in view of (2.3).

The term G_M is treated by a more serious application of the tools in [G84]. Since $b_M\in C^\infty$, G_M is a genuine singular Green operator of order -3 and class 0, with polyhomogeneous symbol. The principal symbol g^0_M is the symbol of the boundary symbol operator (in local coordinates)

$$g^0_M(x',\xi',D_n)=k^0(x',\xi',D_n)p^0(x',\xi')b_M(x')p^0(x',\xi')k^0(x',\xi',D_n)^*$$

(where we have omitted some indexations and used that λ does not enter in the principal symbols). It follows from [G84], Th. 4.10, that

$$s_j(G_M)j^{3/(n-1)}\to C(g^0_M)^{3/(n-1)}\text{ for } j\to\infty, \tag{2.16}$$

where

(2.17)

$$C(g^0_M)=\tfrac{1}{(n-1)(2\pi)^{n-1}}\int_\Sigma\int_{|\xi'|=1}\operatorname{tr}\big[\big(g^0_M(x',\xi',D_n)^*g^0_M(x',\xi',D_n)\big)^{(n-1)/6}\big]\,d\omega(\xi')dx'.$$

(See [G84] for further explanation.) Notice here that $b_M(x')$ and its conjugate enter as pointwise multiplication factors in g^0_M and in $(g^0_M)^*$. When $M\to\infty$, $b_M(x')\to b(x')$ uniformly in x', so

$$(2.18)\quad C(g^0_M)\to C(g^0)=: \frac{1}{(n-1)(2\pi)^{n-1}}\int_\Sigma\int_{|\xi'|=1}\operatorname{tr}\left[\left(k^0(p^0)^*\,\bar b\,(p^0)^*(k^0)^*k^0p^0\,b\,p^0(k^0)^*\right)^{(n-1)/6}\right]d\omega(\xi')dx',$$
$$\text{with } b=b_2-b_1.$$

Now we first apply Lemma 2.1.2° to the decompositions $G=G_M+G'_M$; this shows that G has the spectral behavior in (2.12). When F_1 and F_2 are added to G, we can use Lemma 2.1.1° to conclude that also $G+F_1+F_2$ has the spectral behavior in (2.12).

Finally, general $\lambda\in\varrho(\widetilde A)\cap\varrho(A_\chi)$ are included by use of the resolvent formula (2.7) as in the preceding proof. □

Remark 2.6. Formula (2.18) can be considerably simplified, when we observe that $k^0(x',\xi',D_n)\colon\mathbb C\to L_2(\mathbb R_+)$ maps $v\in\mathbb C$ to $\tilde k^0(x',x_n,\xi')v$, where $\tilde k^0(x',x_n,\xi')\in\mathcal S(\overline{\mathbb R}_+)$ is the symbol-kernel. In the case $A=-\Delta$ it equals $e^{-|\xi'|x_n}$, and it has a similar structure for general A (cf. e.g. [GS01], Sect. 2.d). The operator $k^0(x',\xi',D_n)^*\colon L_2(\mathbb R_+)\to\mathbb C$ maps $u(x_n)$ to $(u,\tilde k^0)_{L_2(\mathbb R_+)}$. Thus $k^0(x',\xi',D_n)^*k^0(x',\xi',D_n)$ is the multiplication by $\|\tilde k^0\|^2_{L_2(\mathbb R_+)}$, and $k^0(x',\xi',D_n)k^0(x',\xi',D_n)^*$ is the rank 1 operator mapping u to $(u,\tilde k^0)\tilde k^0$. The latter operator has the sole eigenvector $\tilde k^0_1=\tilde k^0/\|\tilde k^0\|$ with a positive eigenvalue $\|\tilde k^0\|^2$ (besides eigenvectors in the nullspace), so its trace equals the eigenvalue. The other factors p^0, $(p^0)^*=\bar p^0$, b and $\bar b$ are multiplication operators. Thus $k^0(p^0)^*\,\bar b\,(p^0)^*(k^0)^*k^0p^0\,b\,p^0(k^0)^*$ is the rank 1 operator in $L_2(\mathbb R_+)$:

$$u\mapsto\|\tilde k^0\|^4|p^0|^4|b|^2(u,\tilde k^0_1)\tilde k^0_1;$$

the trace equals the eigenvalue, and the trace of a power equals the power of the eigenvalue. Therefore the formula for the constant $C(g^0)$ reduces to

$$(2.19)\quad C(g^0)=\frac{1}{(n-1)(2\pi)^{n-1}}\int_\Sigma\int_{|\xi'|=1}(\|\tilde k^0\|^2_{L_2(\mathbb R_+)}|p^0|^2|b|)^{(n-1)/3}\,d\omega(\xi')dx',\quad b=b_2-b_1.$$

3. Coefficients with jumps.

It possible to extend the result of Theorem 2.5 to cases where b has jump discontinuities, by use of special results for pseudodifferential operators (from now on abbreviated to ψdo's). In showing this, we also supply the general knowledge on spectral asymptotics for ψdo's multiplied with nonsmooth functions.

Let Ξ be a compact n'-dimensionl C^∞-manifold without boundary, and assume that it is divided by a smooth $(n'-1)$-dimensional hypersurface into two subsets Ξ_+ and Ξ_- (n'-dimensional C^∞-manifolds with boundary) such that $\Xi=\Xi_+\cup\Xi_-$, $\Xi_+^\circ\cap\Xi_-^\circ=\emptyset$, $\partial\Xi_+=\partial\Xi_-$. (Since the sets need not be connected, this covers the situation of J smooth subsets described before Proposition 1.3.) We denote by $r^\pm$ the restrictions from Ξ to $\Xi_\pm$, and by $e^\pm$ the extension-by-zero operators from functions on $\Xi_\pm$ to functions on Ξ:

$$e^\pm u=\begin{cases}u\text{ on }\Xi_\pm\\0\text{ on }\Xi_\mp.\end{cases}$$

Multiplication by the characteristic function 1_{Ξ_+} for Ξ_+ can also be written e^+r^+; similarly $1_{\Xi_-}=e^-r^-$.

It is well-known (as recalled e.g. in [G84], Lemma 4.5) that when P is an $N\times N$-matrix formed classical ψdo on Ξ of negative order $-t$, then it satisfies the spectral asymptotics formula

$$s_j(P)j^{t/n'}\to C(p^0)^{t/n'} \text{ for } j\to\infty, \tag{3.1}$$

where

$$C(p^0)=\tfrac{1}{n'(2\pi)^{n'}}\int_\Xi\int_{|\xi|=1}\operatorname{tr}\big[\big(p^0(x,\xi)^*p^0(x,\xi)\big)^{n'/2t}\big]\,d\omega(\xi)dx. \tag{3.2}$$

Let us also recall the result of Laptev [L77, L81]:

Proposition 3.1. *Let P be a classical pseudodifferential operator on Ξ of negative order $-t$. Then $1_{\Xi_+}P1_{\Xi_-}\in\mathfrak{S}_{(n'-1)/t}$.*

(Expressed in local coordinates, this means that the operator whose kernel is the restriction of the kernel of P to the second or fourth quadrant, picks up the boundary dimension in its spectral behavior. For ψdo's having the transmission property at $\partial\Xi_+$, this is confirmed by the results of [G84].)

The rules in the following are valid also for $N\times N$-matrix formed operators P and factors b, and would then need a trace indication tr in the integrals; we leave this aspect out here for simplicity.

Theorem 3.2. *Let P be a classical pseudodifferential operator of negative order $-t$, such that $(Pu,u)\ge 0$ for $u\in L_2(\Xi)$. Then $P_{(+)}=1_{\Xi_+}P1_{\Xi_+}$ satisfies the spectral asymptotics formula*

$$s_j(P_{(+)})j^{t/n'}\to c(P_{(+)})^{t/n'} \textit{ for } j\to\infty, \tag{3.3}$$

where

$$\begin{aligned} c(P_{(+)})&=\tfrac{1}{n'(2\pi)^{n'}}\int_{\Xi_+}\int_{|\xi|=1}\big(p^0(x,\xi)^*p^0(x,\xi)\big)^{n'/2t}\,d\omega(\xi)dx\\ &=\tfrac{1}{n'(2\pi)^{n'}}\int_{\Xi_+}\int_{|\xi|=1}p^0(x,\xi)^{n'/t}\,d\omega(\xi)dx.\end{aligned} \tag{3.4}$$

Proof. The principal symbol p^0 is ≥ 0; which explains the second identity in (3.4). Introduce two C^∞ cutoff functions ζ_1 and ζ_2 taking values in $[0,1]$ such that $\zeta_1=1$ on Ξ_+ and vanishes outside a neighborhood of Ξ_+, and $\zeta_2=0$ on Ξ_- and is 1 outside a neighborhood of Ξ_-. We shall then compare $P_{(+)}$ with the operators

$$P_1=\zeta_1P\zeta_1 \text{ and } P_2=\zeta_2P\zeta_2.$$

When $u\in L_2(\Xi)$, denote $e^\pm r^\pm u=u_\pm$. We have for P_1:

$$\begin{aligned}(P_1u,u)&=(P_1u_+,u_+)+(P_1u_+,u_-)+(P_1u_-,u_+)+(P_1u_-,u_-)\\ &=(P_{(+)}u,u)+(Ru,u)+(P\zeta_1u_-,\zeta_1u_-),\end{aligned}$$

where $R = 1_{\Xi_-} P_1 I_{\Xi_+} + 1_{\Xi_+} P_1 I_{\Xi_-}$. Since P_1 is a classical ψdo of order $-t$ on Ξ, it has the spectral behavior in (3.1)–(3.2) with the limit $C(p_1^0)^{t/n'}$; here

$$C(p_1^0) = \tfrac{1}{n'(2\pi)^{n'}} \int_{\operatorname{supp}\zeta_1} \int_{|\xi|=1} (\zeta_1 p^0(x,\xi)\zeta_1)^{n'/t}\, d\omega(\xi) dx.$$

Moreover, R is of the type considered in Proposition 3.1, hence lies in $\mathfrak{S}_{(n'-1)/t}$. Then by Lemma 2.1.1°, $P_1 - R$ likewise has the spectral behavior in (3.1)–(3.2) with the limit $C(p_1^0)^{t/n'}$. Now observe that since P is nonnegative, $(P\zeta_1 u_-, \zeta_1 u_-) \geq 0$ for all $u \in L_2(\Xi)$. Thus we have:

$$(P_{(+)}u, u) \leq ((P_1 - R)u, u), \text{ for all } u \in L_2(\Xi). \tag{3.5}$$

Both operators $P_{(+)}$ and $P_1 - R$ are selfadjoint nonnegative, so the s-numbers are the same as the eigenvalues, and the minimum-maximum principle implies in view of (3.5) that

$$s_j(P_{(+)}) \leq s_j(P_1 - R), \text{ for all } j. \tag{3.6}$$

It then follows from the limit property of the $s_j(P_1 - R)$ that

$$\limsup_{j\to\infty} s_j(P_{(+)}) j^{t/n'} \leq C(p_1^0)^{t/n'}. \tag{3.7}$$

There is a similar proof that

$$\liminf_{j\to\infty} s_j(P_{(+)}) j^{t/n'} \geq C(p_2^0)^{t/n'}. \tag{3.8}$$

Since $C(p_1^0)$ and $C(p_2^0)$ come arbitrarily close to $c(P_{(+)})$ when the support of ζ_1 shrinks towards Ξ_+ and the support of $1 - \zeta_2$ shrinks towards Ξ_-, we conclude that (3.3) with (3.4) holds. □

This leads to a result on compositions of ψdo's with discontinuous factors, which seems to have an interest in itself:

Theorem 3.3. *Let P be an operator composed of l classical pseudodifferential operators $P_1, \dots, P_l$ of negative orders $-t_1, \dots, -t_l$ and $l+1$ functions $b_1, \dots, b_{l+1}$ that are piecewise continuous on Ξ with possible jumps at $\partial\Xi_+$ (so the b_k extend to continous funcxtions on Ξ_+ and on Ξ_-);*

$$P = b_1 P_1 \dots b_l P_l b_{l+1}. \tag{3.9}$$

Let $t = t_1 + \dots + t_l$. Then P has the spectral behavior:

$$s_j(P) j^{t/n'} \to c(P)^{t/n'} \text{ for } j \to \infty, \tag{3.9}$$

where

$$\begin{aligned} c(P) &= \tfrac{1}{n'(2\pi)^{n'}} \int_\Xi \int_{|\xi|=1} \big(\bar b_{l+1}(x) p_l^0(x,\xi)^* \dots p_1^0(x,\xi)^* \bar b_1(x) \cdot \\ &\qquad \cdot b_1(x) p_1^0(x,\xi) \dots p_l^0(x,\xi) b_{l+1}(x)\big)^{n'/2t}\, d\omega(\xi) dx \\ &= \tfrac{1}{n'(2\pi)^{n'}} \int_\Xi \int_{|\xi|=1} |b_1 \dots b_{l+1} p_1^0 \dots p_l^0|^{n'/t}\, d\omega(\xi) dx. \end{aligned} \tag{3.10}$$

Proof. We can write

$$
\begin{aligned}
P^*P &= \bar b_{l+1}P_l^* \dots P_1^*\bar b_1 b_1 P_1 \dots P_l b_l \\
&= 1_{\Xi_+}\bar b_{l+1}P_l^* \dots P_1^*\bar b_1 b_1 P_1 \dots P_l b_l 1_{\Xi_+} + 1_{\Xi_-}\bar b_{l+1}P_l^* \dots P_1^*\bar b_1 b_1 P_1 \dots P_l b_l 1_{\Xi_-} + R,
\end{aligned}
$$

where R is a sum of terms of order $-t$, each containing at least one factor of the type in Proposition 3.1. Thus $R \in \mathfrak{S}_{n'/(t+\delta)}$ with a $\delta > 0$. For the term $1_{\Xi_+}P^*P1_{\Xi_+}$, we proceed as in Theorem 2.5. We can assume that b_k is extended from Ξ_+ to a continuous function b_k on Ξ. Each b_k is approximated by a uniformly convergent sequence b_{kM} of C^∞-functions on Ξ. For each M,

$$
P_M^*P_M = \bar b_{l+1,M}P_l^* \dots P_1^*\bar b_{1M} b_{1M} P_1 \dots b_{lM} P_l b_{l+1,M}
$$

is a classical nonnegative ψdo of order $-t$, so Theorem 3.2 applies to the operator with 1_{Ξ_+} before and after, and gives the corresponding spectral asymptotics formula. Since $P_M^*P_M - P^*P$ can be written as a sum of terms where each has a factor $b_{kM} - b_k$ or $\bar b_{kM} - \bar b_k$, we have for $M \to \infty$ that

$$
\sup_j s_j(1_{\Xi_+}P_M^*P_M 1_{\Xi_+} - 1_{\Xi_+}P^*P1_{\Xi_+})j^{t/n'} \to 0.
$$

Then Lemma 2.1.2° implies a spectral asymptotics formula for $1_{\Xi_+}P^*P1_{\Xi_+}$, with the constant as in (3.10) but integrated over Ξ_+. — There is a similar result for $1_{\Xi_-}P^*P1_{\Xi_-}$, relative to Ξ_-.

Now since $L_2(\Xi)$ identifies with the orthogonal sum of $L_2(\Xi_+)$ and $L_2(\Xi_-)$, the spectra are simply superposed when the operators are added together. The statement $\lambda_j(T)j^{t/n'} \to c(T)^{t/n'}$ for $j \to \infty$ is equivalent with $N'(a;T)a^{n'/t} \to c(T)$ for $a \to \infty$, where $N'(a;T)$ counts the number of eigenvalues in $[1/a, \infty[$; superposition of the spectra means addition of the counting functions. (More on counting functions e.g. in [G96], Sect. A.6.) Thus $1_{\Xi_+}P^*P1_{\Xi_+} + 1_{\Xi_-}P^*P1_{\Xi_-}$ has a spectral asymptotics behavior where the constant is obtained by adding the integrals for $1_{\Xi_+}P^*P1_{\Xi_+}$ and $1_{\Xi_-}P^*P1_{\Xi_-}$, so it is as described in (3.9)–(3.10). By Lemma 2.1.1°, the behavior keeps this form when we add R to the operator. □

A similar theorem holds for matrix formed operators P_k and factors b_k, with $c(P)$ defined by the first expression in (3.10); here of course it cannot be reduced to the second expression unless all the factors commute.

A special case of the situation in Theorem 3.3 is the case of bP, where P is a classical ψdo and b is a piecewise continuous functios. We need a case with interspersed factors b_k in our application below.

We can now show:

Theorem 3.4. *The conclusion of Theorem* 2.5 *holds also when* b_1 *and* b_2 *are piecewise in* $H_p^r(\Sigma)$ *for some* $r > 0$, $b_2 - b_1$ *having jumps at a smooth hypersurface.*

Proof. We use again the decomposition in (2.13)

$$
\begin{aligned}
&(\widetilde A_1 - \lambda)^{-1} - (\widetilde A_2 - \lambda)^{-1} \\
&\quad = K_\gamma^\lambda P_{\chi,\gamma}^\lambda (b_2 - b_1) P_{\chi,\gamma}^\lambda K_\gamma^{\bar\lambda *} - K_\gamma^\lambda P_{\chi,\gamma}^\lambda (b_1 P_{\chi,\gamma}^\lambda)^2 S_1 K_\gamma^{\bar\lambda *} + K_\gamma^\lambda P_{\chi,\gamma}^\lambda (b_2 P_{\chi,\gamma}^\lambda)^2 S_2 K_\gamma^{\bar\lambda *} \\
&\quad = G + F_1 + F_2,
\end{aligned}
$$

and F_1 and F_2 are handled as after (2.13), using that $b_i P^\lambda_{\chi,\gamma}$ maps $L_2(\Sigma)$ into $H^{r'}$, $r' = \min\{r, \frac12 - \varepsilon\}$. Then they are in $\mathfrak{S}_{(n-1)/(3+r')}$. We denote again $b_2 - b_1 = b$.

For G we proceed as follows: Let λ be large negative, so that Theorem 1.5 holds; since $\lambda \in \mathbb{R}$, $K^{\bar\lambda}_\gamma = K^\lambda_\gamma$, and $P^\lambda_{\chi,\gamma}$ is selfadjoint. The j-th eigenvalue of G^*G satisfies

$$\lambda_j(G^*G) = \lambda_j(K^\lambda_\gamma P^\lambda_{\chi,\gamma}\bar b P^\lambda_{\chi,\gamma} K^{\lambda^*}_\gamma K^\lambda_\gamma P^\lambda_{\chi,\gamma} b P^\lambda_{\chi,\gamma} K^{\lambda^*}_\gamma).$$

Here $K^{\lambda^*}_\gamma K^\lambda_\gamma$ equals a selfadjoint ψdo P_1 of order -1; it is nonnegative on $L_2(\Sigma)$ and injective, since K^λ_γ is injective:

$$(P_1\varphi,\varphi) = (K^{\lambda^*}_\gamma K^\lambda_\gamma \varphi,\varphi) = \|K^\lambda_\gamma\varphi\|^2_{\frac12} \geq c\|\varphi\|^2_{-\frac12},$$

hence elliptic. It follows from Seeley [S67] that P_1 has a squareroot $P_2 = P_1^{\frac12}$ which is a classical elliptic ψdo of order $-\frac12$. Then we find using

$$\lambda_j(TT') = \lambda_j(T'T), \tag{3.13}$$

that

$$\begin{aligned}\lambda_j(G^*G) &= \lambda_j(K^\lambda_\gamma P^\lambda_{\chi,\gamma}\bar b P^\lambda_{\chi,\gamma} P_2 P_2 P^\lambda_{\chi,\gamma} b P^\lambda_{\chi,\gamma} K^{\lambda^*}_\gamma)\\ &= \lambda_j(P_2 P^\lambda_{\chi,\gamma} b P^\lambda_{\chi,\gamma} K^{\lambda^*}_\gamma K^\lambda_\gamma P^\lambda_{\chi,\gamma}\bar b P^\lambda_{\chi,\gamma} P_2)\\ &= \lambda_j(P_2 P^\lambda_{\chi,\gamma} b P^\lambda_{\chi,\gamma} P_1 P^\lambda_{\chi,\gamma}\bar b P^\lambda_{\chi,\gamma} P_2).\end{aligned}$$

The operator $Q = P_2 P^\lambda_{\chi,\gamma} b P^\lambda_{\chi,\gamma} P_1 P^\lambda_{\chi,\gamma}\bar b P^\lambda_{\chi,\gamma} P_2$ is an operator to which Theorem 3.4 applies, and it gives a spectral asymptotics formula with the constant defined as in (3.10), with $n' = n-1$. In view of Remark 2.6, it can be rewritten as (2.19).

The proof is now completed in the same way as in the proof of Theorem 2.5. □

The results can be extended to exterior domains by the method of [G10a].

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