# Tight embedding of subspaces of $L_p$ in $\ell_p^n$ for even $p$ *

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**Abstract**

Using a recent result of Batson, Spielman and Srivastava, We obtain a tight estimate on the dimension of $\ell_p^n$, $p$ an even integer, needed to almost isometrically contain all $k$-dimensional subspaces of $L_p$.

In a recent paper [BSS] Batson, Spielman and Srivastava introduced a new method for sparsification of graphs which already proved to have functional analytical applications. Here we bring one more such application. Improving over a result of [BLM] (or see [JS] for a survey on this and related results), we show that for even $p$ and for $k$ of order $n^{2/p}$ any $k$ dimensional subspace of $L_p$ nicely embeds into $\ell_p^n$. This removes a log factor from the previously known estimate. The result in Theorem 2 is actually sharper than stated here and gives the best possible result in several respects, in particular in the dependence of $k$ on $n$.

The theorem of [BSS] we shall use is not specifically stated in [BSS], but is stated as Theorem 1.6 in Srivastava's thesis [Sr]:

**Theorem 1** *[BSS] Suppose* $0 < \varepsilon < 1$ *and* $A = \sum_{i=1}^m v_i v_i^T$ *are given, with* $v_i$ *column vectors in* $\mathbb{R}^k$. *Then there are nonnegative weights* $\{s_i\}_{i=1}^m$, *at most* $\lceil k/\varepsilon^2 \rceil$ *of which are nonzero, such that, putting* $\tilde{A} = \sum_{i=1}^m s_i v_i v_i^T$,

$$(1-\varepsilon)^{-2} x^T A x \le x^T \tilde{A} x \le (1+\varepsilon)^2 x^T A x$$

*for all* $x \in \mathbb{R}^k$.

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**Corollary 1** *Let $X$ be a $k$-dimensional subspace of $\ell_2^m$ and let $0 < \varepsilon < 1$. Then there is a set $\sigma \subset \{1, 2, \ldots, m\}$ of cardinality $n \le C\varepsilon^{-2}k$ ($C$ an absolute constant) and positive weights $\{s_i\}_{i\in\sigma}$ such that*

$$\|x\|_2 \le (\sum_{i\in\sigma} s_i x^2(i))^{1/2} \le (1+\varepsilon)\|x\|_2$$

*for all $x = (x(1), x(2), \ldots, x(m)) \in X$.*

**Proof:** Let $u_1, u_2, \ldots, u_k$ be an orthonormal basis for $X$; $u_j = (u_{1,j}, u_{2,j}, \ldots, u_{m,j})$, $j = 1, \ldots, k$. Put $v_i^T = (u_{i,1}, u_{i,2}, \ldots, u_{i,k})$, $i = 1, \ldots, m$. Then $\sum_{i=1}^m v_i v_i^T = I_k$, the $k \times k$ identity matrix. Let $s_i$ be the weights given by Theorem 1 (and $\sigma$ their support). Then, for all $x = \sum_{i=1}^k a_i u_i \in X$,

$$(1-\varepsilon)^{-2}\|x\|_2 = a^T \sum_{i=1}^m v_i v_i^T a \le a^T \sum_{i=1}^m s_i v_i v_i^T a \le (1+\varepsilon)^2 \|x\|_2.$$

Finally, notice that, for each $i = 1, \ldots, m$, $a^T v_i v_i^T a = x(i)^2$, the square of the $i$-th coordinate of $x$. Thus,

$$a^T \sum_{i=1}^m s_i v_i v_i^T a = \sum_{i=1}^m s_i x(i)^2.$$

■

We first prove a simpler version of the main result.

**Proposition 1** *Let $X$ is a $k$ dimensional subspace of $L_p$ for some even $p$ and let $0 < \varepsilon < 1$. Then $X$ $(1+\varepsilon)$-embeds in $\ell_p^n$ for $n = O((\varepsilon p)^{-2} k^{p/2})$.*

**Proof:** Assume as we may that $X$ is a $k$ dimensional subspace of $\ell_p^m$ for some finite $m$. Consider the set of all vectors which are coordinatewise products of $p/2$ vectors from $X$; i.e, of the form

$$(x_1(1)x_2(1)\ldots x_{p/2}(1), x_1(2)x_2(2)\ldots x_{p/2}(2), \ldots, x_1(m)x_2(m)\ldots x_{p/2}(m))$$

where $x_j = (x_j(1), x_j(2), \ldots, x_j(m))$, $j = 1, 2, \ldots, p/2$, are elements of $X$. We shall denote the vector above as $x_1 \cdot x_2 \cdot \cdots \cdot x_{p/2}$. The span of this set in $\mathbb{R}^m$, which we denote by $X^{p/2}$, is clearly a linear space of dimension at most $k^{p/2}$. Consequently, by Corollary 1, there is a set $\sigma \subset \{1, 2, \ldots, m\}$ of cardinality at most $C(\varepsilon p)^{-2}k^{p/2}$ and weights $\{s_i\}_{i\in\sigma}$ such that

$$\|y\|_2 \leq (\sum_{i\in\sigma} s_i y^2(i))^{1/2} \leq (1 + \frac{\varepsilon p}{4})\|y\|_2$$

for all $y \in X^{p/2}$. Restricting to $y$-s of the form

$$y = (x(1)^{p/2}, x(2)^{p/2}, \ldots, x(m)^{p/2})$$

with $x = (x(1), x(2), \ldots, x(m)) \in X$, we get

$$\|x\|_p^{p/2} \leq (\sum_{i\in\sigma} s_i x^p(i))^{1/2} \leq (1 + \frac{\varepsilon p}{4})\|x\|_p^{p/2}.$$

Raising these inequalities to the power $2/p$ gives the result. ∎

We now state and prove the main result.

**Theorem 2** *Let $X$ be a $k$ dimensional subspace of $L_p$ for some even $p \leq k$ and let $0 < \varepsilon < 1$. Then $X$ $(1+\varepsilon)$-embeds in $\ell_p^n$ for $n = O(\varepsilon^{-2}(\frac{10k}{p})^{p/2})$. Equivalently, for some universal $c > 0$, for any $n$ and any $k \leq c\varepsilon^{4/p}pn^{2/p}$, any $k$-dimensional subspace of $L_p$ $(1+\varepsilon)$-embeds in $\ell_p^n$.*

**Proof:** The only change from the previous proof is a better estimate of the dimension of the auxiliary subspace involved. An examination of the proof above shows that it is enough to apply Corollary 1 to any subspace containing all the vectors $x^{p/2} = x \cdot x \cdot \cdots \cdot x$ ($p/2$ times), $x = (x(1), \ldots, x(m)) \in X$. The smallest such subspace is the space of degree $p/2$ homogeneous polynomials in elements of $X$. Its basis is the set of monomials of the form $u_{j_1}^{p_1} \cdot u_{j_2}^{p_2} \cdot \cdots \cdot u_{j_l}^{p_l}$ with $p_1 + \cdots + p_l = p/2$, where $u_1, \ldots, u_k$ is a basis for $X$. The dimension of this space, which is the number of such monomials, is $\binom{k+p/2-1}{p/2} \leq (\frac{10k}{p})^{p/2}$. ∎

Next we remark on the estimate $k \leq c\varepsilon^{4/p}pn^{2/p}$.
This estimate improves (unfortunately, only for even $p$) over the known estimates (the best of which is in [BLM]) by removing a $\log n$ factor that was presented in the best estimate till now. Also, the $p$ in front of the $n^{2/p}$ is a nice feature. It is known that the dependence of $k$ on $p$ and $n$ in this estimate is best possible even if one restrict to subspaces of $L_p$ isometric to $\ell_2^k$ (see [BDGJN]). Actually the result above indicates that $\ell_2^k$ are the "worse" subspaces.

As for the dependence on $\varepsilon$, the published proofs gives at best quadratic dependence while here we get a quadratic dependence for $p = 4$ and better ones as $p$ grows. For the special case of $X = \ell_2^k$ and $p = 4$ a better result is known: One can embed $\ell_2^k$ isometrically into $\ell_4^{4n^2}$ ([Ko]). But for $p = 6, 8, \ldots$ we get better result here even for this special case than what was previously known (The best I knew was a linear dependence on $\varepsilon$ - this is unpublished. Here we get $\varepsilon^{2/3}$ for $p = 6$ and better for larger $p$-s.) It is not clear that this is the best possible dependence on $\varepsilon$, but note that for some combinations of $\varepsilon$ and $p$ and in particular for every $\varepsilon$ and large enough $p$ ($p > c \log 1/\varepsilon$) the dependence on $\varepsilon$ becomes a constant and, up to universal constants, we get the best possible result with respect to all parameters.

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