Kernels for products of L-functions

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Abstract

The Rankin-Cohen bracket of two Eisenstein series provides a kernel yielding products of the periods of Hecke eigenforms at critical values. Extending this idea leads to a new type of Eisenstein series built with a double sum. We develop the properties of these series and their non-holomorphic analogs and show their connection to values of L-functions outside the critical strip.

1 Introduction

In 1952, Rankin [21] introduced the fruitful idea of expressing the product of two critical values of the L-function of a weight k Hecke eigenform f for $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ in terms of the Petersson scalar product of f and a product of Eisenstein series:

$$\langle E_{k_1}E_{k_2}, f\rangle = (-1)^{k_1/2}2^{3-k}\frac{k_1k_2}{B_{k_1}B_{k_2}}L^*(f,1)L^*(f,k_2) \tag{1.1}$$

for $k = k_1+k_2$, the Bernoulli numbers B_j and the completed, entire L-function of f,

$$L^*(f,s) := \frac{\Gamma(s)}{(2\pi)^s}\sum_{m=1}^{\infty}\frac{a_f(m)}{m^s} = \int_0^{\infty} f(iy)y^{s-1}\,dy.$$

Zagier [25, p. 149] extended (1.1) to get

$$\langle [E_{k_1},E_{k_2}]_n, f\rangle = (-1)^{k_1/2}(2\pi i)^n 2^{3-k}\binom{k-2}{n}\frac{k_1k_2}{B_{k_1}B_{k_2}}L^*(f,n+1)L^*(f,n+k_2) \tag{1.2}$$

where $k = k_1+k_2+2n$ and $[g_1,g_2]_n$ stands for the *Rankin-Cohen bracket of index* n

$$[g_1,g_2]_n := \sum_{r=0}^{n}(-1)^r\binom{k_1+n-1}{n-r}\binom{k_2+n-1}{r}g_1^{(r)}g_2^{(n-r)}. \tag{1.3}$$

The *periods* of f in the critical strip are the numbers

$$L^*(f,1), L^*(f,2), \dots, L^*(f,k-1). \tag{1.4}$$

Zagier in [25, §5] and Kohnen-Zagier in [12] proved important results of the Eichler-Shimura-Manin theory on the algebraicity of these critical values using (1.2). We describe this in more depth in §8.1.

On the face of it, the techniques of [25], employing (1.2), apply only to critical values; an extension to non-critical values, $L^*(f,j)$ for integers $j \leqslant 0$ or $j \geqslant k$, would seem to require Rankin-Cohen brackets of negative index n or holomorphic Eisenstein series of negative weight, neither of which are defined. Analyzing the structure of the Rankin-Cohen bracket of two Eisenstein series in §5 reveals a natural construction which we call a *double Eisenstein series*[1]:

$$\sum_{\substack{\gamma,\,\delta\in\Gamma_\infty\backslash\Gamma\\ \gamma\delta^{-1}\neq\Gamma_\infty}} \left(c_{\gamma\delta^{-1}}\right)^{-l} j(\gamma,z)^{-k_1} j(\delta,z)^{-k_2} \tag{1.5}$$

where, for $\gamma\in\Gamma$, we write

$$\gamma=\begin{pmatrix} a_\gamma & b_\gamma\\ c_\gamma & d_\gamma\end{pmatrix}, \quad j(\gamma,z):=c_\gamma z+d_\gamma.$$

By comparison, the usual holomorphic Eisenstein series is

$$E_k(z):=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma,z)^{-k}. \tag{1.6}$$

The double Eisenstein series (1.5) converges to a weight k_1+k_2 cuspform when $l>2-k_1, 2-k_2$. For positive integers l it behaves as a Rankin-Cohen bracket of negative index, see Proposition 2.4. This allows us to further generalize (1.1), (1.2) and in §8 we begin to extend the Eichler-Shimura-Manin theory outside the critical strip by characterizing the field containing the ratios of non-critical values in terms of double Eisenstein series and their Fourier coefficients.

An extension of Zagier's kernel formula (1.2) in the non-holomorphic direction is given in §9.3. There we show that the holomorphic double Eisenstein series have non-holomorphic counterparts:

$$\sum_{\substack{\gamma,\,\delta\in\Gamma_\infty\backslash\Gamma\\ \gamma\delta^{-1}\neq\Gamma_\infty}} |c_{\gamma\delta^{-1}}|^{-s-s'}\mathrm{Im}(\gamma z)^s\mathrm{Im}(\delta z)^{s'}.$$

These weight 0 functions possess analytic continuations and functional equations resembling those for the classical non-holomorphic Eisenstein series. As kernels, they produce products of L-functions for *Maass cusp forms*, see Theorem 2.8. The main motivation for this construction was its potential use in the rapidly developing study of periods of Maass cusp forms [16, 1, 18, 19].

2 Statement of main results

2.1 Preliminaries

All our notation is as in [3]. Throughout, Γ is the modular group $\mathrm{SL}(2,\mathbb{Z})$ acting on the upper half plane $\mathbb{H}$. Let $S_k(\Gamma)$ be the $\mathbb{C}$-vector space of holomorphic, weight k cusp forms for Γ and $M_k(\Gamma)$ the space of modular forms. These spaces are acted on by the Hecke operators T_m, see (3.6). Let $\mathcal{B}_k$ be the unique basis of S_k consisting of Hecke eigenforms, normalized to have first Fourier coefficient 1. We assume throughout this paper that $f\in\mathcal{B}_k$. Since $\langle T_m f, f\rangle=\langle f, T_m f\rangle$ it follows that all the Fourier coefficients of f are real and hence $\overline{L^*(f,s)}=L^*(f,\overline{s})$. Also, recall the functional equation

$$L^*(f,k-s)=(-1)^{k/2}L^*(f,s). \tag{2.1}$$

We summarize some standard properties of the non-holomorphic Eisenstein series, see for example [7, Chapters 3, 6].

Definition 2.1. *For $z=x+iy\in\mathbb{H}$ and $s=\sigma+it\in\mathbb{C}$ with $\sigma>1$, the weight zero,* non-holomorphic Eisenstein series *is*

$$E(z,s):=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\mathrm{Im}(\gamma z)^s=\frac{y^s}{2}\sum_{\substack{c,d\in\mathbb{Z}\\(c,d)=1}}|cz+d|^{-2s}. \tag{2.2}$$

[1] In the context of multiple zeta functions, the authors in [4] give a different definition of 'double Eisenstein series'.

This function satisfies the bound

$$E(z,s) = y^s + O(y^{1-\sigma}) \quad \text{as} \;\; y \to \infty \tag{2.3}$$

for an implied constant depending on s. Let $\theta(s) := \pi^{-s}\Gamma(s)\zeta(2s)$. Then $E(z,s)$ has a Fourier expansion [7, Theorem 3.4] which we may write in the form

$$E(z,s) = y^s + \frac{\theta(1-s)}{\theta(s)} y^{1-s} + \sum_{m \neq 0} \phi(m,s)|m|^{-1/2} W_s(mz) \tag{2.4}$$

where $W_s(mz) = 2(|m|y)^{1/2} K_{s-1/2}(2\pi|m|y)e^{2\pi i m x}$ is the Whittaker function for $z = x+iy \in \mathbb{H}$ and $\theta(s)\phi(m,s) = \sigma_{2s-1}(|m|)|m|^{1/2-s}$. As usual, $\sigma_s(m) := \sum_{d|m} d^s$ is the divisor function.

We will also need the weight $k \in 2\mathbb{Z}$, non-holomorphic Eisenstein series. Generalizing (2.2), set

$$E_k(z,s) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma z)^s \left(\frac{j(\gamma,z)}{|j(\gamma,z)|} \right)^{-k}. \tag{2.5}$$

Then (2.5) converges to an analytic function of $s \in \mathbb{C}$, and a smooth function of $z \in \mathbb{H}$, for $\mathrm{Re}(s) > 1$. Also $y^{-k/2}E_k(z,s)$ has weight k in z. Define the completed non-holomorphic Eisenstein series as

$$E_k^*(z,s) := \theta_k(s)E_k(z,s) \quad \text{for} \quad \theta_k(s) := \pi^{-s}\Gamma(s+|k|/2)\zeta(2s).$$

With (2.4), we see that $E(z,s)$ has a meromorphic continuation to all $s \in \mathbb{C}$. The same is true of $E_k(z,s)$, see [3, §2.1] for example. We have the functional equations

$$\theta(s/2) = \theta((1-s)/2) \tag{2.6}$$
$$E_k^*(z,s) = E_k^*(z,1-s). \tag{2.7}$$

2.2 Holomorphic double Eisenstein series

Define the subgroup

$$\Gamma_\infty^* := \{ \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) | \, n \in \mathbb{Z} \}. \tag{2.8}$$

Then Γ_∞, the subgroup of Γ fixing ∞, is $\Gamma_\infty^* \cup -\Gamma_\infty^*$. For $\gamma \in \Gamma_\infty \backslash \Gamma$ the quantities c_γ, d_γ and $j(\gamma,z)$ are only defined up to sign (though even powers are well defined). For $\gamma \in \Gamma_\infty^* \backslash \Gamma$ there is no ambiguity in the signs of c_γ, d_γ and $j(\gamma,z)$.

Definition 2.2. *Let $z \in \mathbb{H}$ and $w \in \mathbb{C}$. For integers k_1, $k_2 \geqslant 3$ we define the* double Eisenstein series

$$\boldsymbol{E}_{k_1,k_2}(z,w) := \zeta(w+k_1)\zeta(w+k_2) \sum_{\substack{\gamma,\,\delta \in \Gamma_\infty^* \backslash \Gamma \\ c_{\gamma\delta^{-1}} > 0}} \left(c_{\gamma\delta^{-1}}\right)^{-w} j(\gamma,z)^{-k_1} j(\delta,z)^{-k_2}. \tag{2.9}$$

As we see in Proposition 4.1, this series is well-defined and for $\mathrm{Re}(w)$ large enough converges to a holomorphic function of z that is a weight $k = k_1+k_2$ cusp form. It vanishes identically when k_1, k_2 have different parity. Also, it is important to note that $c_{\gamma\delta^{-1}} = \left| \begin{smallmatrix} c_\gamma & d_\gamma \\ c_\delta & d_\delta \end{smallmatrix} \right|$.

To get the most general kernel, we set

$$\boldsymbol{E}_{s,k-s}(z,w) := \sum_{\substack{a,b,c,d \in \mathbb{Z} \\ ad-bc>0}} (ad-bc)^{-w} \left(\frac{az+b}{cz+d} \right)^{-s} (cz+d)^{-k}. \tag{2.10}$$

For $\omega \in \mathbb{C}/(-\infty, 0]$ and $s \in \mathbb{C}$ define $\omega^s := e^{s \log \omega}$ using the principle branch of $\log$. Note that $ad - bc > 0$ implies $(az+b)/(cz+d) \in \mathbb{H}$ for $z \in \mathbb{H}$ and so $((az+b)/(cz+d))^s$ in (2.10) is well-defined. We will see in §6.1 that $\boldsymbol{E}_{s,k-s}(z,w)$, defined in (2.10), converges absolutely and uniformly on compact sets for which $\mathrm{Re}(w) \geqslant 0$ and $2 < \mathrm{Re}(s) < k-2$ and agrees with $\boldsymbol{E}_{k_1,k_2}(z,w)$ for $s = k_1$.

Define the *completed double Eisenstein series* as

$$\boldsymbol{E}^*_{s,k-s}(z,w) := \frac{e^{si\pi/2}\Gamma(s)\Gamma(k-s)\Gamma(w+k-1)}{2^{w+2}\pi^{w+k}\Gamma(k-1)}\boldsymbol{E}_{s,k-s}(z,w). \tag{2.11}$$

Theorem 2.3. *Let $k \in \mathbb{Z}_{\geqslant 6}$. The series $\boldsymbol{E}^*_{s,k-s}(z,w)$ has an analytic continuation to all $s, w \in \mathbb{C}$ and as a function of z is always in $S_k(\Gamma)$. We have*

$$\langle\, \boldsymbol{E}^*_{s,k-s}(\cdot,w), f \,\rangle = L^*(f,s)L^*(f,1-w) \tag{2.12}$$

for any f in $\mathcal{B}_k$. We also have the two functional equations:

$$\boldsymbol{E}^*_{s,k-s}(z,w) \quad = \quad (-1)^{k/2}\boldsymbol{E}^*_{k-s,s}(z,w), \tag{2.13}$$

$$\boldsymbol{E}^*_{s,k-s}(z,w) \quad = \quad (-1)^{k/2}\boldsymbol{E}^*_{s,k-s}(z,2-k-w). \tag{2.14}$$

The next result shows how $\boldsymbol{E}^*_{s,k-s}$ is a generalization of the Rankin-Cohen bracket $[E_{k_1}, E_{k_2}]_n$.

Proposition 2.4. *For $n \in \mathbb{Z}_{\geqslant 1}$ and even $k_1, k_2 \geqslant 4$,*

$$n![E_{k_1}, E_{k_2}]_n = \frac{2(-1)^{k_1/2}\pi^k\Gamma(k-1)}{(2\pi i)^n\zeta(k_1)\zeta(k_2)\Gamma(k_1)\Gamma(k_2)\Gamma(k-n-1)}\boldsymbol{E}^*_{k_1+n,k_2+n}(z,-n).$$

Another way to understand these double Eisenstein series is through their connections to non-holomorphic Eisenstein series. Any smooth function, transforming with weight k and with polynomial growth as $y \to \infty$ may be projected into S_k with respect to the Petersson scalar product. See [3, §3.2] and the contained references. Denote this holomorphic projection by π_{hol}.

Proposition 2.5. *Let $k = k_1 + k_2 \geqslant 6$ for even $k_1, k_2 \geqslant 0$. Then for all $s, w \in \mathbb{C}$*

$$\boldsymbol{E}^*_{s,k-s}(z,1-w) = \pi_{hol}\Big[(-1)^{k_2/2}y^{-k/2}E^*_{k_1}(z,u)E^*_{k_2}(z,v)/(2\pi^{k/2})\Big]$$

where

$$u = (s+w-k+1)/2, \quad v = (-s+w+1)/2.$$

2.3 Non-critical values of L-functions.

Let $m \in \mathbb{Z}$ satisfy $m \leqslant 0$ or $m \geqslant k$. We have, according to [13, §3.4] and the references therein,

$$L^*(f,m) \in \mathscr{P}[1/\pi]$$

where $\mathscr{P}$ is the ring of periods: complex numbers that may be expressed as an integral of an algebraic function over an algebraic domain. In contrast to the periods (1.4), we do not have much more precise information about the algebraic properties of the values $L^*(f,m)$. A special case of a theorem by Koblitz [11] shows, for example, that

$$L^*(f,m) \quad \notin \quad \mathbb{Z}\cdot L^*(f,1) + \mathbb{Z}\cdot L^*(f,2) + \cdots + \mathbb{Z}\cdot L^*(f,k-1).$$

However, we can prove the following theorem, giving the link between Fourier coefficients of double Eisenstein series and values of L-functions inside and outside the critical strip.

Theorem 2.6. *Let K_{all} be the field obtained by adjoining to $\mathbb{Q}$ the Fourier coefficients of $E^*_{n,k-n}(z,0)$ and $E^*_{k-2,2}(z,n)$ for all even n. Then for each $f \in \mathcal{B}_k$ there exist $\omega_+(f)$, $\omega_-(f) \in \mathbb{R}$ such that*

$$L^*(f,s)/\omega_+(f), \quad L^*(f,w)/\omega_-(f) \in K_{all}K_f$$

for all s even, w odd.

Therefore, we would like to know more about the Fourier coefficients of the double Eisenstein series in Theorem 2.6. See the further discussion in §8.2 with the aim of fully characterizing the field K_{all}.

In §7 we also prove results analogous to Theorem 2.6 for non-critical values of the (completed) L-function of f twisted by $e^{2\pi imp/q}$ for $p/q \in \mathbb{Q}$:

$$L^*(f,s;p/q) := \frac{\Gamma(s)}{(2\pi)^s}\sum_{m=1}^{\infty}\frac{a_f(m)e^{2\pi imp/q}}{m^s} = \int_0^\infty f(iy+p/q)y^{s-1}\,dy. \tag{2.15}$$

As a side result in §7, we uncover a connection between twisted and non-twisted L-functions:

$$\frac{(2\pi)^{w+k-1}}{\Gamma(w+k-1)}L^*(f,s)L^*(f,w+k-1) = \sum_{v=1}^{\infty}\frac{1}{v^{w+k-s}}\sum_{u=0}^{v-1}\zeta_{u,v}(w+s)L^*(f,s;u/v) \tag{2.16}$$

when $f \in \mathcal{B}_k$, $\mathrm{Re}(w) \geqslant 0$ and $2 < \mathrm{Re}(s) < k-2$. The term $\zeta_{u,v}$ is the Dirichlet series

$$\zeta_{u,v}(s) := \sum_{n=1}^{\infty}\frac{\lceil\frac{n-u}{v}\rceil}{n^s} \qquad (\mathrm{Re}(s) > 2). \tag{2.17}$$

Note that (2.16) is similar in structure to a formula of Manin [18, Theorem 2.2], expressing L-functions as a natural sum over rationals.

2.4 Non-holomorphic double Eisenstein series

Definition 2.7. *For $z \in \mathbb{H}$, w, s, $s' \in \mathbb{C}$, we define the* non-holomorphic double Eisenstein series *as*

$$\mathcal{E}(z,w;s,s') := \sum_{\substack{\gamma,\,\delta\in\Gamma_\infty\backslash\Gamma \\ \gamma\delta^{-1}\neq\Gamma_\infty}}\frac{\mathrm{Im}(\gamma z)^s\mathrm{Im}(\delta z)^{s'}}{|c_{\gamma\delta^{-1}}|^w}. \tag{2.18}$$

A simple comparison with (2.2) shows it is absolutely and uniformly convergent for $\mathrm{Re}(s)$, $\mathrm{Re}(s') > 1$ and $\mathrm{Re}(w) > 0$. (This domain of convergence is improved in Proposition 4.2.) The most symmetric form of (2.18) is when $w = s + s'$. Define

$$\mathcal{E}^*(z;s,s') := 4\pi^{-s-s'}\Gamma(s)\Gamma(s')\zeta(3s+s')\zeta(s+3s')\mathcal{E}(z,s+s';s,s') + 2\theta(s)\theta(s')E(z,s+s'). \tag{2.19}$$

Theorem 2.8. *The completed double Eisenstein series $\mathcal{E}^*(z;s,s')$ has a meromorphic continuation to all $s, s' \in \mathbb{C}$ and satisfies the functional equations*

$$\mathcal{E}^*(z;s,s') = \mathcal{E}^*(z;s',s), \tag{2.20}$$
$$\mathcal{E}^*(z;s,s') = \mathcal{E}^*(z;1-s,1-s'). \tag{2.21}$$

For any even Maass Hecke eigenform u_j,

$$\langle\, \mathcal{E}^*(z;s,s'), u_j\,\rangle = L^*(u_j, s+s'-1/2)L^*(u_j, s'-s+1/2).$$

3 Further background results and notation

We need to introduce two more families of modular forms. First, recall our notation Γ_∞^* from (2.8). Suppose $h\left(\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right)\gamma\right) = h(\gamma)$ for all $n \in \mathbb{Z}$ and $\gamma \in \Gamma$. Then, ignoring convergence for now,

$$\sum_{\gamma \in \Gamma_\infty^* \backslash \Gamma} h(\gamma) = \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} h\left(\left(\begin{smallmatrix}* & *\\ c & d\end{smallmatrix}\right)\right).$$

Also

$$\sum_{\gamma \in \Gamma_\infty \backslash \Gamma} h(\gamma) = \frac{1}{2} \sum_{\gamma \in \Gamma_\infty^* \backslash \Gamma} h(\gamma)$$

when $h(\gamma) = h(-\gamma)$ for all $\gamma \in \Gamma$.

Definition 3.1. *For $z \in \mathbb{H}$, $k \geqslant 4$ in $2\mathbb{Z}$ and $m \in \mathbb{Z}_{\geqslant 0}$ the holomorphic* Poincaré series *is*

$$P_k(z;m) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \frac{e^{2\pi i m \gamma z}}{j(\gamma,z)^k} \quad = \quad \frac{1}{2} \sum_{\gamma \in \Gamma_\infty^* \backslash \Gamma} \frac{e^{2\pi i m \gamma z}}{j(\gamma,z)^k}. \tag{3.1}$$

For $m \geqslant 1$ the series $P_k(z;m)$ span $S_k(\Gamma)$. The Eisenstein series $E_k(z) = P_k(z;0)$ is not a cusp form but is in the space $M_k(\Gamma)$.

Definition 3.2. *The* generalized Cohen kernel *is given by*

$$\mathcal{C}_k(z,s;p/q) := \frac{1}{2} \sum_{\gamma \in \Gamma} (\gamma z + p/q)^{-s} j(\gamma,z)^{-k} \tag{3.2}$$

for $p/q \in \mathbb{Q}$ and $s \in \mathbb{C}$ with $1 < \mathrm{Re}(s) < k-1$.

In [3, Section 5] we studied $\mathcal{C}_k(z,s;p/q)$ (the factor $1/2$ is included to keep the notation consistent with [3] where $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$). We showed that, for each $s \in \mathbb{C}$ with $1 < \mathrm{Re}(s) < k-1$, $\mathcal{C}_k(z,s;p/q)$ converges to an element of $S_k(\Gamma)$, with a meromorphic continuation to all $s \in \mathbb{C}$. From [3, Prop. 5.4] we have

$$\langle \mathcal{C}_k(\cdot,s;p/q), f \rangle = 2^{2-k} \pi e^{-si\pi/2} \frac{\Gamma(k-1)}{\Gamma(s)\Gamma(k-s)} L^*(f,k-s;p/q). \tag{3.3}$$

For simplicity we write $\mathcal{C}_k(z,s)$ for $\mathcal{C}_k(z,s;0)$. The twisted L-functions satisfy

$$\overline{L^*(f,s;p/q)} = L^*(f,\overline{s};-p/q). \tag{3.4}$$

and

$$q^s L^*(f,s;p/q) = (-1)^{k/2} q^{k-s} L^*(f,k-s;-p'/q) \tag{3.5}$$

for $pp' \equiv 1 \mod q$, as in [14, App. A.3].

Define $\mathcal{M}_n := \left\{ \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \;\middle|\; a,b,c,d \in \mathbb{Z}, ad-bc = n \right\}$. Thus $\mathcal{M}_1 = \Gamma$. For $k \in \mathbb{Z}$ and $g : \mathbb{H} \to \mathbb{C}$ set

$$(g|_k\gamma)(z) := g(\gamma z) j(\gamma,z)^{-k}$$

for all $\gamma \in \Gamma$. The weight k Hecke operator T_n acts on $g \in M_k$ by

$$T_n g := n^{k-1} \sum_{\gamma \in \Gamma \backslash \mathcal{M}_n} (g|_k\gamma) = n^{k-1} \sum_{\substack{ad=n \\ a,d>0}} d^{-k} \sum_{0 \leqslant b < d} g\left(\frac{az+b}{d}\right). \tag{3.6}$$

4 Convergence of double Eisenstein series

To first determine where the Eisenstein series $\boldsymbol{E}_{k_1,k_2}(z,w)$, defined in (2.9), is absolutely convergent we consider

$$\sum_{\substack{\gamma,\,\delta\in\Gamma^*_\infty\backslash\Gamma \\ c_{\gamma\delta^{-1}}>0}} \left| \left(c_{\gamma\delta^{-1}}\right)^{-w} j(\gamma,z)^{-k_1} j(\delta,z)^{-k_2} \right|. \tag{4.1}$$

Set $u=\mathrm{Re}(w)$. Recalling that $\mathrm{Im}(\gamma z)=y|j(\gamma,z)|^{-2}$ we deduce that (4.1) equals

$$y^{-(k_1+k_2)/2} \sum_{\substack{\gamma,\,\delta\in\Gamma^*_\infty\backslash\Gamma \\ c_{\gamma\delta^{-1}}>0}} \left|c_{\gamma\delta^{-1}}\right|^{-u} \mathrm{Im}(\gamma z)^{k_1/2}\mathrm{Im}(\delta z)^{k_2/2}$$
$$= 2y^{-(k_1+k_2)/2} \sum_{\substack{\gamma,\,\delta\in\Gamma_\infty\backslash\Gamma \\ \gamma\delta^{-1}\not\in\Gamma_\infty}} \left|c_{\gamma\delta^{-1}}\right|^{-u} \mathrm{Im}(\gamma z)^{k_1/2}\mathrm{Im}(\delta z)^{k_2/2}. \tag{4.2}$$

For $u=\mathrm{Re}(w)\geqslant 0$, we have $|c_{\gamma\delta^{-1}}|^{-u}\leqslant 1$ when $\gamma\delta^{-1}\not\in\Gamma_\infty$. Hence (4.2) is bounded by

$$2y^{-(k_1+k_2)/2} \sum_{\substack{\gamma,\,\delta\in\Gamma_\infty\backslash\Gamma \\ \gamma\delta^{-1}\not\in\Gamma_\infty}} \mathrm{Im}(\gamma z)^{k_1/2}\mathrm{Im}(\delta z)^{k_2/2}$$
$$= 2y^{-(k_1+k_2)/2}\Big(E(z,k_1/2)E(z,k_2/2)-E(z,k_1/2+k_2/2)\Big) \tag{4.3}$$

on noting that $\mathrm{Im}(\gamma z)=\mathrm{Im}(\delta z)$ for $\gamma\delta^{-1}\in\Gamma_\infty$. Recalling (2.2), these non-holomorphic Eisenstein series are absolutely convergent for k_1, $k_2>2$.

We next examine the case $\mathrm{Re}(w)<0$. Set $\varepsilon(\gamma,z):=j(\gamma,z)/|j(\gamma,z)|=e^{i\arg(j(\gamma,z))}$. It is easy to verify that, for all γ, $\delta\in\Gamma^*_\infty\backslash\Gamma$ and $z\in\mathbb{H}$,

$$\begin{aligned}
c_{\gamma\delta^{-1}} &= c_\gamma j(\delta,z)-c_\delta j(\gamma,z)\\
&= \left(\frac{j(\gamma,z)-\overline{j(\gamma,z)}}{2iy}\right) j(\delta,z) - \left(\frac{j(\delta,z)-\overline{j(\delta,z)}}{2iy}\right) j(\gamma,z)\\
&= \left(\varepsilon(\delta,z)^{-2}-\varepsilon(\gamma,z)^{-2}\right) j(\gamma,z)j(\delta,z)/(2iy).
\end{aligned}$$

Therefore

$$\begin{aligned}
|c_{\gamma\delta^{-1}}| &= \left|\frac{\varepsilon(\gamma,z)}{\varepsilon(\delta,z)}-\frac{\varepsilon(\delta,z)}{\varepsilon(\gamma,z)}\right| \mathrm{Im}(\gamma z)^{-1/2}\mathrm{Im}(\delta z)^{-1/2}/2\\
&= \left|\mathrm{Im}\left(\frac{\varepsilon(\gamma,z)}{\varepsilon(\delta,z)}\right)\right| \mathrm{Im}(\gamma z)^{-1/2}\mathrm{Im}(\delta z)^{-1/2}\\
&\leqslant \mathrm{Im}(\gamma z)^{-1/2}\mathrm{Im}(\delta z)^{-1/2}.
\end{aligned} \tag{4.4}$$

Hence, with $u=\mathrm{Re}(w)<0$, using (4.4) in (4.2) we have that (4.1) is bounded by

$$2y^{-(k_1+k_2)/2}\left(E\left(z,\frac{u+k_1}{2}\right)E\left(z,\frac{u+k_2}{2}\right)-E\left(z,u+\frac{k_1+k_2}{2}\right)\right). \tag{4.5}$$

Therefore the series is absolutely convergent for $u+k_1$, $u+k_2>2$. We have shown that (2.9) is absolutely convergent for $k_1,k_2\geqslant 3$ in $\mathbb{Z}$ and $\mathrm{Re}(w)>\max(2-k_1,2-k_2)$. This convergence is uniform for z in compact sets of $\mathbb{H}$ and w in compact sets in $\mathbb{C}$ satisfying the above constraint.

We next verify that $\boldsymbol{E}_{k_1,k_2}|_{k_1+k_2}\tau = \boldsymbol{E}_{k_1,k_2}$ for all $\tau \in \Gamma$:

$$\begin{aligned}
\frac{\boldsymbol{E}_{k_1,k_2}(\tau z, w)}{j(\tau, z)^{k_1+k_2}} &= \sum_{\substack{\gamma,\,\delta\in\Gamma^*_\infty\backslash\Gamma \\ c_{\gamma\delta^{-1}}>0}} \left(c_{\gamma\delta^{-1}}\right)^{-w} j(\gamma\tau, z)^{-k_1} j(\delta\tau, z)^{-k_2} \\
&= \sum_{\substack{\gamma,\,\delta\in\Gamma^*_\infty\backslash\Gamma \\ c_{(\gamma\tau)(\delta\tau)^{-1}}>0}} \left(c_{(\gamma\tau)(\delta\tau)^{-1}}\right)^{-w} j(\gamma\tau, z)^{-k_1} j(\delta\tau, z)^{-k_2} \\
&= \boldsymbol{E}_{k_1,k_2}(z, w).
\end{aligned}$$

We finally show that $\boldsymbol{E}_{k_1,k_2}$ is a cusp form. By (2.3),

$$y^{-\frac{k_1+k_2}{2}}\Big(E(z, k_1/2)E(z, k_2/2) - E(z, k_1/2 + k_2/2)\Big) = O\Big(y^{1-k_1} + y^{1-k_2} + y^{2-k_1-k_2}\Big)$$

and approaches 0 as $y \to \infty$. Thus, by (4.3), the function $\boldsymbol{E}_{k_1,k_2}(z, w)$ vanishes at the cusp ∞ and is therefore a cusp form if $\mathrm{Re}(w) \geqslant 0$. The argument for $\mathrm{Re}(w) < 0$ is similar. Assembling these results, we have shown the following:

Proposition 4.1. *Let k_1, $k_2 \geqslant 3$ be in $\mathbb{Z}$ and $z \in \mathbb{H}$, $w \in \mathbb{C}$. The series $\boldsymbol{E}_{k_1,k_2}(z, w)$ is absolutely and uniformly convergent for z and w in compact sets with $\mathrm{Re}(w) > \max(2 - k_1, 2 - k_2)$. For each such w we have $\boldsymbol{E}_{k_1,k_2}(z, w) \in S_{k_1+k_2}(\Gamma)$ as a function of z.*

The same techniques prove the next result, for the non-holomorphic double Eisenstein series.

Proposition 4.2. *Let $z \in \mathbb{H}$, $s, s', w \in \mathbb{C}$ with $\sigma = \mathrm{Re}(s)$ and $\sigma' = \mathrm{Re}(s')$. The series $\mathcal{E}(z, w; s, s')$, defined in* (2.18) *is absolutely and uniformly convergent for z, w, s and s' in compact sets satisfying*

$$\sigma, \sigma' > 1 \quad \text{and} \quad \mathrm{Re}(w) > 2\max(1 - \sigma, 1 - \sigma').$$

Unlike $\boldsymbol{E}_{k_1,k_2}(z, w)$, the series $\mathcal{E}(z, w; s, s')$ will have polynomial growth as $y \to \infty$.

5 Applying the Rankin-Cohen bracket to Poincaré series

The main objective of this section is to show how double Eisenstein series arise naturally when the Rankin-Cohen bracket is applied to the usual Eisenstein series E_k. Proposition 2.4 will be a consequence of this. In fact, since there is no difficulty in extending these methods, we compute the Rankin-Cohen bracket of two arbitrary Poincaré series

$$[P_{k_1}(z; m_1), P_{k_2}(z; m_2)]_n$$

for $m_1, m_2 \geqslant 0$. The result may be expressed in terms of the *double Poincaré series*, defined below. In this way, the action of the Rankin-Cohen brackets on spaces of modular forms can be completely described. See also Corollary 5.5 at the end of this section.

Definition 5.1. *Let $z \in \mathbb{H}$, k_1, $k_2 \geqslant 3$ in $\mathbb{Z}$ and m_1, $m_2 \in \mathbb{Z}_{\geqslant 0}$. For $w \in \mathbb{C}$ with $\mathrm{Re}(w) > \max(2 - k_1, 2 - k_2)$, we define the* double Poincaré series

$$\boldsymbol{P}_{k_1,k_2}(z, w; m_1, m_2) := \zeta(w + k_1)\zeta(w + k_2) \sum_{\substack{\gamma,\,\delta\in\Gamma^*_\infty\backslash\Gamma \\ c_{\gamma\delta^{-1}}>0}} \left(c_{\gamma\delta^{-1}}\right)^{-w} \frac{e^{2\pi i(m_1\gamma z + m_2\delta z)}}{j(\gamma, z)^{k_1} j(\delta, z)^{k_2}}. \tag{5.1}$$

The series (5.1) will vanish identically unless k_1 and k_2 have the same parity. Clearly we have $\boldsymbol{E}_{k_1,k_2}(z,w) = \boldsymbol{P}_{k_1,k_2}(z,w;0,0)$. Since $|e^{2\pi i(m_1\gamma z+m_2\delta z)}| \leqslant 1$, it is a simple matter to verify that the work in §4 proves that $\boldsymbol{P}_{k_1,k_2}(z,w;m_1,m_2)$ converges absolutely and uniformly on compacta to a cusp form in $S_{k_1+k_2}(\Gamma)$.

For $l \in \mathbb{Z}_{\geqslant 0}$ it is convenient to set

$$Q_k(z,l;m) := \begin{cases} P_k(z;m) & \text{if } l = 0, \\ \frac{1}{2}\sum_{\gamma\in\Gamma^*_\infty\backslash\Gamma} \frac{e^{2\pi i m\gamma z}(c_\gamma)^l}{j(\gamma,z)^{k+l}} & \text{if } l \geqslant 1. \end{cases} \tag{5.2}$$

As in the proof of Proposition 4.1, this is an absolutely convergent series for k even and at least 4. The next result may be verified by induction.

Lemma 5.2. *For every $j \in \mathbb{Z}_{\geqslant 0}$, have the formulas*

$$\begin{aligned} \frac{d^j}{dz^j}E_k(z) &= (-1)^j\frac{(k+j-1)!}{(k-1)!}Q_k(z,j;0), \\ \frac{d^j}{dz^j}P_k(z;m) &= \sum_{l=0}^{j}(-1)^{l+j}(2\pi i m)^l\frac{j!}{l!}\binom{k+j-1}{k+l-1}Q_{k+2l}(z,j-l;m) \qquad (m>0). \end{aligned}$$

Set

$$A_{k_1,k_2}(l,u)_n := \frac{(k_1+n-1)!(k_2+n-1)!}{l!u!(n-l-u)!(k_1+l-1)!(k_2+u-1)!}.$$

Proposition 5.3. *For $m_1, m_2 \in \mathbb{Z}_{\geqslant 1}$ we have*

$$\begin{aligned} &[P_{k_1}(z;m_1), P_{k_2}(z;m_2)]_n \\ &\quad = \sum_{\substack{l,u\geqslant 0\\ l+u\leqslant n}} A_{k_1,k_2}(l,u)_n\frac{(-2\pi i m_1)^l(2\pi i m_2)^u}{2\zeta(k_1+2l)\zeta(k_2+2u)}\boldsymbol{P}_{k_1+n+l-u,k_2+n-l+u}(z,-n+l+u;m_1,m_2) \\ &\qquad + P_{k_1+k_2+2n}(z;m_1+m_2)\sum_{\substack{l,u\geqslant 0\\ l+u=n}} A_{k_1,k_2}(l,u)_n(-2\pi i m_1)^l(2\pi i m_2)^u. \end{aligned}$$

Proof. We have

$$\begin{aligned} &[P_{k_1}(z;m_1), P_{k_2}(z;m_2)]_n \\ &\quad = \sum_{l=0}^{n}\sum_{u=0}^{n}(2\pi i m_1)^l(2\pi i m_2)^u\frac{(k_1+n-1)!(k_2+n-1)!}{l!u!(k_1+l-1)!(k_2+u-1)!} \\ &\qquad \times\sum_{r=l}^{n-u}(-1)^{n+l+u+r}\frac{Q_{k_1+2l}(z,r-l;m_1)Q_{k_2+2u}(z,n-r-u;m_2)}{(r-l)!(n-r-u)!}. \end{aligned} \tag{5.3}$$

The inner sum over r is

$$\begin{aligned} &\frac{(-1)^l}{4(n-l-u)!}\sum_{\gamma,\delta\in\Gamma^*_\infty\backslash\Gamma}\frac{e^{2\pi i(m_1\gamma z+m_2\delta z)}}{j(\gamma,z)^{k_1+2l}j(\delta,z)^{k_2+2u}} \\ &\qquad \times\sum_{r=l}^{n-u}\binom{n-l-u}{r-l}\left(\frac{c_\gamma}{j(\gamma,z)}\right)^{r-l}\left(\frac{-c_\delta}{j(\delta,z)}\right)^{n-r-u} \end{aligned} \tag{5.4}$$

and, employing the binomial theorem, (5.4) reduces to

$$\frac{(-1)^l}{4(n-l-u)!}\sum_{\gamma,\delta\in\Gamma^*_\infty\backslash\Gamma}\frac{e^{2\pi i(m_1\gamma z+m_2\delta z)}}{j(\gamma,z)^{k_1+n+l-u}j(\delta,z)^{k_2+n-l+u}}\big(c_\gamma j(\delta,z)-c_\delta j(\gamma,z)\big)^{n-l-u} \tag{5.5}$$

for $l+u<n$ and

$$\frac{(-1)^l}{4(n-l-u)!}\sum_{\gamma,\delta\in\Gamma^*_\infty\backslash\Gamma}\frac{e^{2\pi i(m_1\gamma z+m_2\delta z)}}{j(\gamma,z)^{k_1+n+l-u}j(\delta,z)^{k_2+n-l+u}} \tag{5.6}$$

for $l+u=n$. Noting that $c_\gamma j(\delta,z)-c_\delta j(\gamma,z)=\begin{vmatrix} c_\gamma & d_\gamma \\ c_\delta & d_\delta\end{vmatrix}=c_{\gamma\delta^{-1}}$ means that (5.5) becomes

$$\frac{(-1)^l}{(n-l-u)!2\zeta(k_1+2l)\zeta(k_2+2u)}\boldsymbol{P}_{k_1+n+l-u,k_2+n-l+u}(z,-n+l+u;m_1,m_2) \tag{5.7}$$

and (5.6) equals

$$\frac{(-1)^l}{(n-l-u)!}\left(\frac{\boldsymbol{P}_{k_1+n+l-u,k_2+n-l+u}(z,-n+l+u;m_1,m_2)}{2\zeta(k_1+2l)\zeta(k_2+2u)}+P_{k_1+k_2+2n}(z;m_1+m_2)\right). \tag{5.8}$$

Putting (5.7) and (5.8) into (5.3) finishes the proof. □

In fact, Proposition 5.3 is also valid for m_1 or m_2 equalling 0 provided we agree that $(-2\pi i m_1)^l=1$ in the ambiguous case where $m_1=l=0$ and similarly that $(2\pi i m_2)^u=1$ when $m_2=u=0$. With this notational convention the proof of the last proposition gives

Corollary 5.4. *For $m>0$ we have*

$$[E_{k_1}(z),P_{k_2}(z;m)]_n=\sum_{u=0}^{n}A_{k_1,k_2}(0,u)_n\frac{(2\pi i m)^u}{2\zeta(k_1)\zeta(k_2+2u)}\boldsymbol{P}_{k_1+n-u,k_2+n+u}(z,-n+u;0,m)$$
$$+P_{k_1+k_2+2n}(z;m)\cdot A_{k_1,k_2}(0,n)_n(2\pi i m)^n$$

and

$$[E_{k_1}(z),E_{k_2}(z)]_n=\frac{A_{k_1,k_2}(0,0)_n}{2\zeta(k_1)\zeta(k_2)}\boldsymbol{E}_{k_1+n,k_2+n}(z,-n)+E_{k_1+k_2}(z)\cdot\delta_{n,0}. \tag{5.9}$$

Proposition 2.4 follows directly from (5.9). A basic property of Rankin-Cohen brackets has also naturally emerged:

Corollary 5.5. *For $g_1\in M_{k_1}(\Gamma)$ and $g_2\in M_{k_2}(\Gamma)$ we have $[g_1,g_2]_n\in S_{k_1+k_2+2n}(\Gamma)$ for $n>0$.*

Proof. The space $M_{k_1}(\Gamma)$ is spanned by E_{k_1} and the Poincaré series $P_{k_1}(z;m)$ for $m\in\mathbb{Z}_{\geqslant 1}$. So we may write g_1, and similarly g_2, as a linear combination of Eisenstein and Poincaré series. Hence $[g_1,g_2]_n$ is a linear combination of the Rankin-Cohen brackets appearing in Proposition 5.3 and Corollary 5.4. By these results $[g_1,g_2]_n$ is a linear combination of double Poincaré and double Eisenstein series which are in $S_{k_1+k_2+2n}(\Gamma)$, we have already shown. □

It would be interesting to know if $\boldsymbol{P}_{k_1,k_2}(z,w;m_1,m_2)$ has a meromorphic continuation in w. As a corollary of work in the next section we establish the continuation of $\boldsymbol{P}_{k_1,k_2}(z,w;0,0)$ to all $w\in\mathbb{C}$.

6 Further results on double Eisenstein series

6.1 Analytic Continuation

We first establish an initial domain of convergence for $\boldsymbol{E}_{s,k-s}(z,w)$, defined in (2.10). The elementary identity $|\omega^s| = |\omega|^{\mathrm{Re}(s)} \cdot e^{-\arg(\omega)\mathrm{Im}(s)}$ implies that

$$|z^{-s}| \leqslant |z|^{-\sigma} \cdot e^{\pi t} \quad \text{for} \quad s = \sigma + it \in \mathbb{C}, \quad z \in \mathbb{H}.$$

Hence, for $\mathrm{Re}(w) \geqslant 0$ we have

$$\begin{aligned}
\sum_{\substack{a,b,c,d\in\mathbb{Z}\\ ad-bc>0}} \left|(ad-bc)^{-w}\left(\frac{az+b}{cz+d}\right)^{-s}(cz+d)^{-k}\right| &\ll \sum_{\substack{a,b,c,d\in\mathbb{Z}\\ ad-bc>0}} \left|\frac{az+b}{cz+d}\right|^{-\sigma}|cz+d|^{-k} \\
&\leqslant \sum_{\substack{a,b,c,d\in\mathbb{Z}\\ ad-bc>0}} |az+b|^{-\sigma}|cz+d|^{-(k-\sigma)}. \qquad (6.1)
\end{aligned}$$

By comparison with (2.2) this last series is absolutely convergent for $\sigma > 2$ and $k-\sigma > 2$. Therefore (2.10) converges absolutely and uniformly on compact sets for which $2 < \mathrm{Re}(s) < k-2$ and $\mathrm{Re}(w) \geqslant 0$. For these s, w we also have

$$\begin{aligned}
\boldsymbol{E}_{s,k-s}(z,w) &= \sum_{u,v=1}^{\infty} \sum_{\substack{a,b,c,d\in\mathbb{Z}\\ (a,b)=(c,d)=1\\ ad-bc>0}} (au\cdot dv - bu\cdot cv)^{-w}\left(\frac{au\cdot z+bu}{cv\cdot z+dv}\right)^{-s}(cv\cdot z+dv)^{-k} \\
&= \sum_{u,v=1}^{\infty} u^{-w-s}v^{-w-k+s} \sum_{\substack{a,b,c,d\in\mathbb{Z}\\ (a,b)=(c,d)=1\\ ad-bc>0}} (ad-bc)^{-w}\left(\frac{az+b}{cz+d}\right)^{-s}(cz+d)^{-k} \\
&= \zeta(w+s)\zeta(w+k-s) \sum_{\substack{\gamma,\,\delta\in\Gamma_\infty^*\backslash\Gamma\\ c_{\gamma\delta^{-1}}>0}} \left(c_{\gamma\delta^{-1}}\right)^{-w}\left(\frac{j(\gamma,z)}{j(\delta,z)}\right)^{-s} j(\delta,z)^{-k}. \qquad (6.2)
\end{aligned}$$

Comparing (6.2) with (4.1) and arguing as in §4, completes the proof of the following extension of Proposition 4.1.

Proposition 6.1. *Let $z \in \mathbb{H}$, $k \in \mathbb{Z}$ and let $s, w \in \mathbb{C}$ satisfy $2 < \mathrm{Re}(s) < k-2$ and $\mathrm{Re}(w) \geqslant 0$. The series $\boldsymbol{E}_{s,k-s}(z,w)$ is absolutely and uniformly convergent for s, w and z in compact sets satisfying the above constraints. For each such s, w we have $\boldsymbol{E}_{s,k-s}(z,w) \in S_k(\Gamma)$ as a function of z.*

Replace s by k_1 in (6.2) to see that (2.10) agrees with (2.9).

Proof of Theorem 2.3. Our next task is to prove the meromorphic continuation of $\boldsymbol{E}_{s,k-s}(z,w)$ in s and w. For s, w in the initial domain of convergence, we begin with

$$\begin{aligned}
\boldsymbol{E}_{s,k-s}(z,w) &= \sum_{\substack{a,b,c,d\in\mathbb{Z}\\ ad-bc>0}} |ad-bc|^{-w}\left(\frac{az+b}{cz+d}\right)^{-s}(cz+d)^{-k} \\
&= \sum_{n=1}^{\infty}\frac{1}{n^w} \sum_{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\mathcal{M}_n} \left(\frac{az+b}{cz+d}\right)^{-s}(cz+d)^{-k} \\
&= 2\sum_{n=1}^{\infty}\frac{T_n\mathcal{C}_k(z,s)}{n^{w+k-1}}, \qquad (6.3)
\end{aligned}$$

recalling (3.2). With Proposition 6.1, we see

$$\boldsymbol{E}_{s,k-s}(z,w) = \sum_{f\in\mathcal{B}_k} \frac{\langle\, \boldsymbol{E}_{s,k-s}(\cdot,w), f\,\rangle}{\langle\, f,f\,\rangle} f(z) = 2\sum_{n=1}^{\infty} \frac{1}{n^{w+k-1}} \sum_{f\in\mathcal{B}_k} \frac{\langle\, T_n\mathcal{C}_k(\cdot,s), f\,\rangle}{\langle\, f,f\,\rangle} f(z).$$

Then

$$\langle\, T_n\mathcal{C}_k(z,s), f\,\rangle = \langle\, \mathcal{C}_k(z,s), T_n f\,\rangle = a_f(n)\langle\, \mathcal{C}_k(z,s), f\,\rangle$$

and with (3.3) we obtain

$$\boldsymbol{E}_{s,k-s}(z,w) = 2^{w+2}\pi^{w+k}e^{-si\pi/2}\frac{\Gamma(k-1)}{\Gamma(s)\Gamma(k-s)\Gamma(w+k-1)} \times \sum_{f\in\mathcal{B}_k} L^*(f,k-s)L^*(f,w+k-1)\frac{f(z)}{\langle\, f,f\,\rangle}. \tag{6.4}$$

Define the completed double Eisenstein series $\boldsymbol{E}^*$ with (2.11). Then (6.4) becomes

$$\boldsymbol{E}^*_{s,k-s}(z,w) = \sum_{f\in\mathcal{B}_k} L^*(f,s)L^*(f,1-w)\frac{f(z)}{\langle\, f,f\,\rangle}. \tag{6.5}$$

We also now see from (6.5) that $\boldsymbol{E}^*_{s,k-s}(z,w)$ has a analytic continuation to all s, w in $\mathbb{C}$ and satisfies (2.12) and the two functional equations (2.13), (2.14). □

Combining this result, Theorem 2.3, with Proposition 2.4 gives a new proof of Zagier's formula (1.2). His original proof in [25, Prop. 6] employed Poincaré series.

Proof of Proposition 2.5. Let $F_{s,w}(z) = (-1)^{k_2/2}y^{-k/2}E^*_{k_1}(z,u)E^*_{k_2}(z,v)/(2\pi^{k/2})$ with $u = (s+w-k+1)/2$, $v = (-s+w+1)/2$ as before. Then $F_{s,w}(z)$ has weight k and polynomial growth as $y\to\infty$. It is proved in [3, Prop. 2.1] that

$$\langle\, F_{s,w}, f\,\rangle = L^*(f,s)L^*(f,w) \tag{6.6}$$

for all $f\in B_k$. Comparing (6.6) with (2.12) shows that $\boldsymbol{E}^*_{s,k-s}(\cdot,1-w) = \pi_{hol}(F_{s,w})$ as required. □

6.2 Twisted double Eisenstein series

Let $p/q\in\mathbb{Q}$ with $q>0$. In this section, we further extend our definition of double Eisenstein series to

$$\boldsymbol{E}_{s,k-s}(z,w;p/q) := \sum_{\substack{a,b,c,d\in\mathbb{Z}\\ ad-bc>0}} (ad-bc)^{-w}\left(\frac{az+b}{cz+d}+\frac{p}{q}\right)^{-s}(cz+d)^{-k} \tag{6.7}$$

and establish its basic required properties.

Writing

$$(ad-bc)^{-w}\left(\frac{az+b}{cz+d}+\frac{p}{q}\right)^{-s} = q^{w+s}\big((aq+cp)d-(bq+dp)c\big)^{-w}\left(\frac{(aq+cp)z+(bq+dp)}{cz+d}\right)^{-s}$$

we see that

$$\boldsymbol{E}_{s,k-s}(z,w;p/q) = q^{w+s}\sum_{\substack{a',b',c,d\in\mathbb{Z}\\ a'd-b'c>0}} (a'd-b'c)^{-w}\left(\frac{a'z+b'}{cz+d}\right)^{-s}(cz+d)^{-k}$$

with $a' \equiv cp \mod q$ and $b' \equiv dp \mod q$. Hence (6.7) is majorized by (6.1) and so $\boldsymbol{E}_{s,k-s}(z, w; p/q)$ converges absolutely and uniformly to an element of S_k (as a function of z) for $2 < \mathrm{Re}(s) < k-2$ and $\mathrm{Re}(w) > 0$.

The analog of (6.3) is

$$\boldsymbol{E}_{s,k-s}(z, w; p/q) = 2 \sum_{n=1}^{\infty} \frac{T_n \mathcal{C}_k(z, s; p/q)}{n^{w+k-1}}. \tag{6.8}$$

Hence, with (3.3),

$$\boldsymbol{E}_{s,k-s}(z, w; p/q) = 2^{w+2} \pi^{w+k} e^{-si\pi/2} \frac{\Gamma(k-1)}{\Gamma(s)\Gamma(k-s)\Gamma(w+k-1)} \times \sum_{f \in \mathcal{B}_k} L^*(f, k-s; p/q) L^*(f, w+k-1) \frac{f(z)}{\langle f, f \rangle}. \tag{6.9}$$

Define the completed double Eisenstein series $\boldsymbol{E}^*_{s,k-s}(z, w; p/q)$ with the same factor as (2.11) and we obtain

$$\langle \boldsymbol{E}^*_{s,k-s}(\cdot, w; p/q), f \rangle = L^*(f, k-s; p/q) L^*(f, w+k-1) \tag{6.10}$$

for any f in $\mathcal{B}_k$. Then (6.9) implies $\boldsymbol{E}^*_{s,k-s}(z, w; p/q)$ has an analytic continuation to all s, w in $\mathbb{C}$. It satisfies the two functional equations:

$$\begin{aligned} \boldsymbol{E}^*_{s,k-s}(z, 2-k-w; p/q) &= (-1)^{k/2} \boldsymbol{E}^*_{s,k-s}(z, w; p/q), \\ q^s \boldsymbol{E}^*_{k-s,s}(z, w; p/q) &= (-1)^{k/2} q^{k-s} \boldsymbol{E}^*_{s,k-s}(z, w; -p'/q) \end{aligned}$$

for $pp' \equiv 1 \mod q$ using (2.1) and (3.5), respectively.

7 The Hecke action

The expression (6.3), giving $\boldsymbol{E}_{s,k-s}$ in terms of $\mathcal{C}_k$ acted upon by the Hecke operators, can be studied further and yields an interesting relation between $\boldsymbol{E}_{s,k-s}(z, w)$ and the generalized Cohen kernel $\mathcal{C}_k(z, s; p/q)$.

For each $\rho \in \Gamma\backslash\mathcal{M}_n$ and $\gamma \in \Gamma$, there is a unique $\rho' \in \Gamma\backslash\mathcal{M}_n$ and $\gamma' \in \Gamma$ such that $\rho\gamma = \gamma'\rho'$ (see, e.g. [24, Prop. 3.36] or [6, §6.2]). Therefore

$$\begin{aligned} T_n \mathcal{C}_k(z, s; p/q) &= n^{k-1} \sum_{\rho \in \Gamma\backslash\mathcal{M}_n} \left(\sum_{\gamma \in \Gamma} \left(\gamma\rho z + \frac{p}{q} \right)^{-s} j(\gamma, \rho z)^{-k} \right) j(\rho, z)^{-k} \\ &= n^{k-1} \sum_{\rho \in \Gamma\backslash\mathcal{M}_n} \sum_{\gamma \in \Gamma} \left(\gamma\rho z + \frac{p}{q} \right)^{-s} j(\gamma\rho, z)^{-k} \\ &= n^{k-1} \sum_{\rho \in \Gamma\backslash\mathcal{M}_n} \sum_{\gamma \in \Gamma} \left(\rho\gamma z + \frac{p}{q} \right)^{-s} j(\rho\gamma, z)^{-k} \\ &= n^{k-1} \sum_{\substack{ad=n \\ a,d>0}} d^{-k} \sum_{0 \leqslant b < d} \sum_{\gamma \in \Gamma} \left(\frac{a}{d}\gamma z + \frac{b}{d} + \frac{p}{q} \right)^{-s} j(\gamma, z)^{-k}. \end{aligned}$$

Hence

$$T_n \mathcal{C}_k(z, s; p/q) = n^{k-s-1} \sum_{d|n} d^{2s-k} \sum_{0 \leqslant b < d} \mathcal{C}_k \left(z, s; \frac{d(bq+dp)}{qn} \right).$$

For $2 < \mathrm{Re}(s) < k-2$ and $\mathrm{Re}(w) \geqslant 0$ we find

$$\begin{aligned}
\frac{1}{2}\boldsymbol{E}_{s,k-s}(z,w) &= \sum_{n=1}^{\infty} \frac{T_n \mathcal{C}_k(z,s)}{n^{w+k-1}} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^{w+s}} \sum_{d|n} d^{2s-k} \sum_{0\leqslant b<d} \mathcal{C}_k\left(z,s;\frac{bd}{n}\right) \\
&= \sum_{d=1}^{\infty} d^{2s-k} \sum_{v=1}^{\infty} \frac{1}{(dv)^{w+s}} \sum_{0\leqslant b<d} \mathcal{C}_k\left(z,s;\frac{b}{v}\right) \\
&= \sum_{v=1}^{\infty} \frac{1}{v^{w+s}} \sum_{b=0}^{\infty} \mathcal{C}_k\left(z,s;\frac{b}{v}\right) \sum_{d>b} \frac{1}{d^{w-s+k}} \\
&= \sum_{v=1}^{\infty} \frac{1}{v^{w+s}} \sum_{b=0}^{v-1}\sum_{r=0}^{\infty} \mathcal{C}_k\left(z,s;\frac{b+rv}{v}\right) \sum_{d>b+rv} \frac{1}{d^{w-s+k}}.
\end{aligned}$$

We have $\mathcal{C}_k(z,s;1+p/q) = \mathcal{C}_k(z,s;p/q)$ and, recalling (2.17),

$$\zeta_{b,v}(s) = \sum_{r=0}^{\infty} \sum_{d>b+rv} \frac{1}{d^s}$$

since $0 \leqslant b < v$. Consequently, for $2 < \mathrm{Re}(s) < k-2$ and $\mathrm{Re}(w) \geqslant 0$

$$\boldsymbol{E}_{s,k-s}(z,w) = 2\sum_{v=1}^{\infty} \frac{1}{v^{w+s}} \sum_{u=0}^{v-1} \zeta_{u,v}(w-s+k)\mathcal{C}_k\left(z,s;\frac{u}{v}\right). \tag{7.1}$$

Taking the inner product of both sides of (7.1) with f and rearranging with (2.1) yields (2.16). Looking to simplify (7.1) leads to the natural question of whether there are further relations between the $\mathcal{C}_k(z,s;u/v)$ for rational u/v in the interval $[0,1)$. For example, it is a simple exercise with (3.3) and (3.5) to show that

$$q^{-s}\mathcal{C}_k(z,s;p/q) = e^{-si\pi}q^{-k+s}\mathcal{C}_k(z,k-s;-p'/q)$$

for $pp' \equiv 1 \mod q$. With $s = k/2$ at the center of the critical strip we get an even simpler relation:

$$\mathcal{C}_k(z,k/2;p/q) = (-1)^{k/2}\mathcal{C}_k(z,k/2;-p'/q). \tag{7.2}$$

A more interesting, but speculative, possibility would be to argue in the reverse direction in order to derive information about L-functions twisted by exponentials with *non-rational* exponents. Specifically, if we established, by other means, relations between the $\mathcal{C}_k(z,s;x)$ for $x \notin \mathbb{Q}$, then (7.1) and other results proven here might lead to relations for L-functions twisted by exponentials with non-rational exponents. That would be important because such L-functions play a prominent role in Kaszorowski and Perelli's programme of classifying the Selberg class (see e.g. [10]). Relations between these L-functions seem to be necessary for the extension of Kaszorowski and Perelli's classification to degree 2, to which L-functions of GL(2) cusp forms belong.

8 Periods of cusp forms

8.1 Values of L-functions inside the critical strip

We first review the proof of Manin's Periods Theorem. This exhibits a general principle of proving algebraicity we will be using in the next sections.

For all $s, w \in \mathbb{C}$ it is convenient to define $H_{s,w} \in S_k$ by the conditions

$$\langle H_{s,w}, f \rangle = L^*(f,s)L^*(f,w) \quad \text{for all} \quad f \in \mathcal{B}_k.$$

This defines $H_{s,w}$ uniquely, giving

$$H_{s,w} = \sum_{f \in \mathcal{B}_k} \frac{\langle H_{s,w}, f \rangle}{\langle f, f \rangle} f = \sum_{f \in \mathcal{B}_k} \frac{L^*(f,s)L^*(f,w)}{\langle f, f \rangle} f. \tag{8.1}$$

By (2.1) and (8.1) we obviously have $H_{s,w} = H_{w,s}$ and $H_{s,w} = (-1)^{k/2} H_{k-s,w} = (-1)^{k/2} H_{s,k-w}$. We need the following result.

Lemma 8.1. *For $g \in S_k$ with Fourier coefficients in the field K_g and $f \in \mathcal{B}_k$ with coefficients in K_f,*

$$\big\langle g, f \big\rangle / \big\langle f, f \big\rangle \in K_g K_f.$$

Proof. See Shimura's general result [23, Lemma 4]. It is also a simple extension of [3, Lemma 4.3]. □

Let $K_{critical}$ be the field obtained by adjoining to $\mathbb{Q}$ all the Fourier coefficients of

$$\Big\{ H_{s,k-1}, H_{k-2,w} \;\Big|\; 1 \leqslant s, w \leqslant k-1, s \text{ even }, w \text{ odd} \Big\}.$$

Thus, with $f \in \mathcal{B}_k$ and employing Lemma 8.1,

$$L^*(f,k-1)L^*(f,k-2) = \big\langle H_{k-1,k-2}, f \big\rangle = c_f \big\langle f, f \big\rangle \tag{8.2}$$

for $c_f \in K_{critical} K_f$ and the left side of (8.2) is nonzero because the Euler product for $L^*(f,s)$ converges for $\mathrm{Re}(s) > k/2 + 1/2$. Set

$$\omega_+(f) := \frac{c_f \big\langle f, f \big\rangle}{L^*(f,k-1)}, \quad \omega_-(f) := \frac{\big\langle f, f \big\rangle}{L^*(f,k-2)}. \tag{8.3}$$

Then $\omega_+(f)\omega_-(f) = \big\langle f, f \big\rangle$ and we have:

Lemma 8.2. *For each $f \in \mathcal{B}_k$*

$$L^*(f,s)/\omega_+(f), \quad L^*(f,w)/\omega_-(f) \in K_{critical} K_f$$

for all s, w with $1 \leqslant s, w \leqslant k-1$ and s even, w odd.

Proof. For such s and w,

$$\begin{aligned} \frac{L^*(f,s)}{\omega_+(f)} &= \frac{L^*(f,s)L^*(f,k-1)}{c_f \big\langle f, f \big\rangle} = \frac{\big\langle H_{s,k-1}, f \big\rangle}{c_f \big\langle f, f \big\rangle} = \frac{c'_f \big\langle f, f \big\rangle}{c_f \big\langle f, f \big\rangle} \in K_{critical} K_f \\ \frac{L^*(f,w)}{\omega_-(f)} &= \frac{L^*(f,w)L^*(f,k-2)}{c_f \big\langle f, f \big\rangle} = \frac{\big\langle H_{k-2,w}, f \big\rangle}{c_f \big\langle f, f \big\rangle} = \frac{c''_f \big\langle f, f \big\rangle}{c_f \big\langle f, f \big\rangle} \in K_{critical} K_f. \end{aligned}$$ □

To deduce Manin's Theorem from Lemma 8.2, we use Zagier's explicit expression for $H_{s,w}$. For $n \geqslant 0$, even $k_1, k_2 \geqslant 4$ and $k = k_1 + k_2 + 2n$, (1.2) implies

$$(-1)^{k_1/2} 2^{3-k} \frac{k_1 k_2}{B_{k_1} B_{k_2}} \binom{k-2}{n} H_{n+1,n+k_2} = \frac{[E_{k_1}, E_{k_2}]_n}{(2\pi i)^n}. \tag{8.4}$$

The Fourier coefficients of E_{k_1}, E_{k_2} are rational and hence the right side of (8.4) has rational coefficients. Then $H_{n+1,n+k_2}$ has Fourier coefficients in $\mathbb{Q}$ (and also for k_1, $k_2 = 2$ as described in [12, p. 214]). It follows that $K_{critical} = \mathbb{Q}$ and Lemma 8.2 becomes

Theorem 8.3. (Manin's Periods Theorem) *For each $f \in \mathcal{B}_k$ there exist $\omega_+(f), \omega_-(f) \in \mathbb{R}$ such that*

$$L^*(f,s)/\omega_+(f), \quad L^*(f,w)/\omega_-(f) \in K_f$$

for all s, w with $1 \leqslant s, w \leqslant k-1$ and s even, w odd.

8.2 L-values outside the critical strip

We will now apply the technique of the last section to incorporate values of the L-function outside the critical strip. Let K_{all} be the field obtained by adjoining to $\mathbb{Q}$ all the Fourier coefficients of

$$\{H_{s,k-1}, H_{k-2,w} \mid s \text{ even }, w \text{ odd }\}.$$

Then, as in the proof of Lemma 8.2,

$$L^*(f,s)/\omega_+(f), \quad L^*(f,w)/\omega_-(f) \in K_{all}K_f$$

for each $f \in \mathcal{B}_k$ and all s even, w odd. We may characterize the field K_{all} using double Eisenstein series. By Theorem 2.3, we have $H_{s,w}(z) = \boldsymbol{E}^*_{s,k-s}(z, 1-w)$ and so

$$\begin{aligned} H_{s,k-1}(z) &= (-1)^{k/2}H_{s,1}(z) = (-1)^{k/2}\boldsymbol{E}^*_{s,k-s}(z,0), \\ H_{k-2,w}(z) &= \boldsymbol{E}^*_{k-2,2}(z, 1-w). \end{aligned}$$

We have proved:

Theorem 8.4. *Let K_{all} be the field obtained by adjoining to $\mathbb{Q}$ the Fourier coefficients of $\boldsymbol{E}^*_{n,k-n}(z,0)$ and $\boldsymbol{E}^*_{k-2,2}(z,n)$ for all even n. Then for each $f \in \mathcal{B}_k$*

$$L^*(f,s)/\omega_+(f), \quad L^*(f,w)/\omega_-(f) \in K_{all}K_f$$

for all s even, w odd.

We indicate briefly how the Fourier coefficients required in Theorem 8.4 may be calculated using a slight extension of the methods in [3, §3]. We wish to find the l-th Fourier coefficient, $a_{s,w}(l)$, of $H_{s,w}(z) = \boldsymbol{E}^*_{s,k-s}(z, 1-w)$ for s even and w odd (and we assume $s, w \geqslant k/2 > 1$). With Proposition 2.5, this is $(-1)^{k_2/2}/(2\pi^{k/2})$ times the l-th Fourier coefficient of

$$\pi_{hol}\Big[y^{-k/2}E^*_{k_1}(z,u)E^*_{k_2}(z,v)\Big]$$

for $u = (s+w-k+1)/2$ and $v = (-s+w+1)/2$ both in $\mathbb{Z}$. Let

$$\begin{aligned} F(z) := {} & y^{-k/2}E^*_{k_1}(z,u)E^*_{k_2}(z,v) \\ & - \frac{\theta_{k_1}(u)\theta_{k_2}(1-v)}{\theta_k(s+1-k/2)}y^{-k/2}E^*_k(z, s+1-k/2) - \frac{\theta_{k_1}(u)\theta_{k_2}(v)}{\theta_k(w+1-k/2)}y^{-k/2}E^*_k(z, w+1-k/2). \end{aligned}$$

Then $\pi_{hol}\left(y^{-k/2}E^*_{k_1}(z,u)E^*_{k_2}(z,v)\right) = \pi_{hol}\left(F(z)\right)$ because $\pi_{hol}\left(y^{-k/2}E^*_k(z,s)\right) = 0$ for every s. We have constructed F so that $F(z) \ll y^{-\epsilon}$ as $y \to \infty$ and we may use [3, Lemma 3.3] to obtain

$$a_{s,w}(l) = \frac{(-1)^{k_2/2}(4\pi l)^{k-1}}{(2\pi^{k/2})(k-2)!}\int_0^\infty F_l(y)e^{-2\pi l y}y^{k-2}\,dy,$$

on writing $F(z) = \sum_{l \in \mathbb{Z}} e^{2\pi i l x}y^{-k/2}F_l(y)$. The functions $F_l(y)$ are sums involving the Fourier coefficients of $E^*_{k_1}(z,u)$ and $E^*_{k_2}(z,v)$ with $u, v \in \mathbb{Z}$. As shown in [3, Theorem 3.1] these coefficients are simply expressed in terms of divisor functions, Bernoulli numbers and a combinatorial part. For s, w in the critical strip, this calculation yields an explicit finite formula for $a_{s,w}(l)$ in [3, Theorem 1.3] (and another proof that $H_{s,w}$ in (8.4) has rational Fourier coefficients and that $K_{critical} = \mathbb{Q}$). For s, w outside the critical strip, we obtain infinite series representations for $a_{s,w}(l)$, but again involving nothing more complicated than divisor functions and Bernoulli numbers. Further details of this computation will appear in [20].

8.3 Twisted Periods

There is an analog of Manin's Periods Theorem for twisted L-functions. Let $p/q \in \mathbb{Q}$ and let u be an integer with $1 \leqslant u \leqslant k-1$. Manin shows in [17, (13)] (see also [15, Chapter 5]) that $i^u \int_0^{p/q} f(iy)y^{u-1}\,dy$ is an integral linear combination of periods $i^v \int_0^\infty f(iy)y^{v-1}\,dy$ for $v = 1, \dots, k-1$. With (2.15) this proves

$$i^u q^{k-2} L^*(f,u;p/q) \quad \in \quad \mathbb{Z}\cdot iL^*(f,1) + \mathbb{Z}\cdot i^2 L^*(f,2) + \cdots + \mathbb{Z}\cdot i^{k-1}L^*(f,k-1).$$

Therefore, Theorem 8.3 implies the next result.

Theorem 8.5. *For all $f \in \mathcal{B}_k$, $p/q \in \mathbb{Q}$ and integers u with $1 \leqslant u \leqslant k-1$,*

$$L^*(f,u;p/q) \in K_f(i)\omega_+(f) + K_f(i)\omega_-(f).$$

Going further, for all $s, w \in \mathbb{C}$, $p/q \in \mathbb{Q}$ we now define $H_{s,w}(z;p/q) \in S_k$ by the conditions

$$\langle\, H_{s,w}(\cdot;p/q), f\,\rangle = L^*(f,s;p/q)L^*(f,w) \quad \text{for all} \quad f \in \mathcal{B}_k.$$

Hence

$$H_{s,w}(z;p/q) = \sum_{f\in\mathcal{B}_k} \frac{L^*(f,s;p/q)L^*(f,w)}{\langle\, f,f\,\rangle} f(z).$$

Let K'_{all} be the field obtained by adjoining to $\mathbb{Q}$ all the Fourier coefficients of

$$\Big\{ H_{s,k-1}(z;p/q), H_{w,k-2}(z;p/q) \,\Big|\, s \text{ even}, w \text{ odd}, p/q \in \mathbb{Q} \Big\}.$$

Arguing as in the proof of Lemma 8.2 again, for each $f \in \mathcal{B}_k$ and all s even, w odd, $p/q \in \mathbb{Q}$

$$L^*(f,s;p/q)/\omega_+(f), \quad L^*(f,w;p/q)/\omega_-(f) \in K'_{all}K_f.$$

We may characterize the field K'_{all} using double Eisenstein series. By (6.10), we have

$$H_{s,w}(z;p/q) = \boldsymbol{E}^*_{k-s,s}(z, 1-w; p/q).$$

Theorem 8.6. *Let K'_{all} be the field obtained by adjoining to $\mathbb{Q}$ the Fourier coefficients of*

$$\boldsymbol{E}^*_{k-n,n}(z, 2-k; p/q), \quad \boldsymbol{E}^*_{k-n-1,n+1}(z, 3-k; p/q)$$

for all even n and all $p/q \in \mathbb{Q}$. Then for each $f \in \mathcal{B}_k$

$$L^*(f,s;p/q)/\omega_+(f), \quad L^*(f,w;p/q)/\omega_-(f) \quad \in \quad K'_{all}K_f$$

for all s even, w odd and $p/q \in \mathbb{Q}$.

9 The non-holomorphic case

9.1 Background results and notation

We will need a non-holomorphic analog of the Cohen kernel $\mathcal{C}_k(z,s)$:

Definition 9.1. *With $z \in \mathbb{H}$, s, $s' \in \mathbb{C}$ define the* non-holomorphic kernel $\mathcal{K}$ *as*

$$\mathcal{K}(z;s,s') := \frac{1}{2}\sum_{\gamma\in\Gamma} \frac{\operatorname{Im}(\gamma z)^{s+s'}}{|\gamma z|^{2s}}. \tag{9.1}$$

This series is absolutely convergent, uniformly on compacta, for $z \in \mathbb{H}$ and $\mathrm{Re}(s), \mathrm{Re}(s') > 1/2$, as shown in [3, §5.2]. It resembles the hyperbolic Eisenstein series, see [22].

For $\Gamma = \mathrm{SL}(2, \mathbb{Z})$, the discrete spectrum of the Laplace operator $\Delta = -4y^2 \partial_z \partial_{\overline{z}}$ is given by u_0, the constant eigenfunction, and u_j for $j \in \mathbb{Z}_{\geqslant 1}$ an orthogonal system of Maass cuspforms (see e.g. [7, Chapters 4,7]) with Fourier expansions

$$u_j(z) = \sum_{n \neq 0} |n|^{-1/2} \nu_j(n) W_{s_j}(nz)$$

where u_j has eigenvalue $s_j(1 - s_j)$ and by Weyl's law [7, (11.5)]

$$\#\{j : |\mathrm{Im}(s_j)| \leqslant T\} = T^2/12 + O(T \log T). \tag{9.2}$$

We may assume the u_j are Hecke eigenforms normalized to have $\nu_j(1) = 1$. Necessarily we have $\nu_j(n) \in \mathbb{R}$. Let ι be the antiholomorphic involution $(\iota u_j)(z) := u_j(-\overline{z})$. We may also assume each u_j is an eigenfunction of this operator, necessarily with eigenvalues ± 1. If $\iota u_j = u_j$ then $\nu_j(n) = \nu_j(-n)$ and u_j is called *even*. If $\iota u_j = -u_j$ then $\nu_j(n) = -\nu_j(-n)$ and u_j is *odd*.

The L-function associated to the Maass cusp form u_j is $L(u_j, s) = \sum_{n=1}^{\infty} \nu_j(n)/n^s$, convergent for $\mathrm{Re}(s) > 3/2$ since $\nu_j(n) \ll n^{1/2}$ by [7, (8.8)]. The completed L-function for an even form u_j is

$$L^*(u_j, s) := \pi^{-s} \Gamma\left(\frac{s + s_j - 1/2}{2}\right) \Gamma\left(\frac{s - s_j + 1/2}{2}\right) L(u_j, s) \tag{9.3}$$

and it satisfies

$$L^*(u_j, 1 - s) = L^*(u_j, s) = \overline{L^*(u_j, \overline{s})}. \tag{9.4}$$

See [2, p. 107] for (9.3), (9.4) and the analogous odd case.

To $E(z, w)$ (recall (2.4)) we associate the L-function

$$L(E(\cdot, w), s) := \sum_{m=1}^{\infty} \frac{\phi(m, w)}{m^s}$$

The well-known identity $\sum_{n=1}^{\infty} \sigma_x(n)/n^s = \zeta(s)\zeta(s - x)$ implies

$$L(E(\cdot, s), w) = \frac{2\pi^s}{\Gamma(s)} \frac{\zeta(w + s - 1/2)\zeta(w - s + 1/2)}{\zeta(2s)}. \tag{9.5}$$

9.2 The non-holomorphic kernel $\mathcal{K}$

Throughout this section we use $s = \sigma + it$, $s' = \sigma' + it'$. Recall $\mathcal{K}(z; s, s')$ defined in (9.1) for $\mathrm{Re}(s)$, $\mathrm{Re}(s') > 1/2$. Our goal is to prove the meromorphic continuation of $\mathcal{K}(z; s, s')$ in s and s' using its spectral decomposition. See [7, §7.4] for a similar decomposition and continuation of the automorphic Green function.

A routine verification (using [8, Lemma 9.2] for example) yields

$$\Delta \mathcal{K}(z; s, s') = (s + s')(1 - s - s')\mathcal{K}(z; s, s') + 4ss'\mathcal{K}(z; s + 1, s' + 1). \tag{9.6}$$

Put

$$\xi_{\mathbb{Z}}(z, s) := \sum_{m \in \mathbb{Z}} \frac{1}{|z + m|^{2s}}.$$

Then

$$\mathcal{K}(z; s, s') = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma z)^{s + s'} \xi_{\mathbb{Z}}(\gamma z, s). \tag{9.7}$$

Use the Poisson summation formula as in [7, §3.4] or [5, Th. 3.1.8] to see that

$$\xi_{\mathbb{Z}}(z,s) = \frac{\pi^{1/2}\Gamma(s-1/2)}{\Gamma(s)} y^{1-2s} + \frac{2\pi^s}{\Gamma(s)} y^{1/2-s} \sum_{m\neq 0} |m|^{s-1/2} K_{s-1/2}(2\pi|m|y) e^{2\pi i m x} \tag{9.8}$$

for $\mathrm{Re}(s) > 1/2$. Set

$$\xi^{\sharp}_{\mathbb{Z}}(z,s) := \sum_{m\neq 0} |m|^{s-1/2} K_{s-1/2}(2\pi|m|y) e^{2\pi i m x}. \tag{9.9}$$

Let $B_\rho := \{z \in \mathbb{C} : |z| \leqslant \rho\}$. Then with [9, Lemma 6.4]

$$\sqrt{y} K_{s-1/2}(2\pi y) \ll e^{-2\pi y}\left(y^{\rho+3} + y^{-\rho-3}\right)$$

for all $s \in B_\rho$ and $\rho, y > 0$ with the implied constant depending only on ρ. Hence

$$\xi^{\sharp}_{\mathbb{Z}}(z,s) \ll \sum_{m=1}^{\infty} e^{-2\pi m y}\left(m^{\rho+\sigma+2} y^{\rho+5/2} + m^{-\rho+\sigma-4} y^{-\rho-7/2}\right).$$

We also have [9, Lemma 6.2]

$$\sum_{m=1}^{\infty} m^{\rho} e^{-2m\pi y} \ll e^{-2\pi y}\left(1 + y^{-\rho-1}\right)$$

for all $y > 0$ with the implied constant depending only on $\rho \geqslant 0$. Therefore

$$\xi^{\sharp}_{\mathbb{Z}}(z,s) \ll e^{-2\pi y}\left(y^{\rho+5/2} + y^{-\rho-9/2}\right). \tag{9.10}$$

Consider the weight 0 series

$$\mathcal{K}^{\sharp}(z;s,s') := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma z)^{s'+1/2} \xi^{\sharp}_{\mathbb{Z}}(\gamma z, s). \tag{9.11}$$

With (9.10), we have

$$\mathcal{K}^{\sharp}(z;s,s') \ll \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \left(\mathrm{Im}(\gamma z)^{\sigma'+\rho+3} + \mathrm{Im}(\gamma z)^{\sigma'-\rho-4}\right) e^{-2\pi \mathrm{Im}(\gamma z)} \tag{9.12}$$

so that $\mathcal{K}^{\sharp}(z;s,s')$ is absolutely convergent for $\mathrm{Re}(s') > \rho + 5$.

Proposition 9.2. *Let $\rho > 0$ and $s, s' \in \mathbb{C}$ satisfy $s \in B_\rho$, $\mathrm{Re}(s) > 1/2$ and $\mathrm{Re}(s') > \rho + 5$. Then*

$$\mathcal{K}(z;s,s') = \frac{\pi^{1/2}\Gamma(s-1/2)}{\Gamma(s)} E(z, s'-s+1) + \frac{2\pi^s}{\Gamma(s)} \mathcal{K}^{\sharp}(z;s,s') \tag{9.13}$$

and, for an implied constant depending only on s, s',

$$\mathcal{K}^{\sharp}(z;s,s') \ll y^{5+\rho-\sigma'} \quad \text{as} \quad y \to \infty. \tag{9.14}$$

Proof. It is clear that (9.13) follows from (9.7), (9.8), (9.9) and (9.11) when s and s' are in the stated range. With (9.12) and employing (2.3) we deduce that as $y \to \infty$

$$\begin{aligned}
\mathcal{K}^{\sharp}(z;s,s') &\ll \left(y^{\sigma'+\rho+3} + y^{\sigma'-\rho-4}\right) e^{-2\pi y} + \sum_{\gamma\in\Gamma_\infty\backslash\Gamma, \gamma\neq\Gamma_\infty} \left(\mathrm{Im}(\gamma z)^{\sigma'+\rho+3} + \mathrm{Im}(\gamma z)^{\sigma'-\rho-4}\right) \\
&\ll e^{-\pi y} + y^{1-(\sigma'+\rho+3)} + y^{1-(\sigma'-\rho-4)} \\
&\ll y^{5+\rho-\sigma'}.
\end{aligned}$$

□

Clearly, for $\mathrm{Re}(s') > \rho+5$, (9.13) gives the meromorphic continuation of $\mathcal{K}(z;s,s')$ to all $s \in B_\rho$. For these s, s' it follows from (9.14) that $\mathcal{K}^\sharp$, as a function of z, is bounded. Also use (9.6) and (9.13) to show that

$$\Delta\mathcal{K}^\sharp(z;s,s') = (s+s')(1-s-s')\mathcal{K}^\sharp(z;s,s') + 4\pi s'\mathcal{K}^\sharp(z;s+1,s'+1)$$

and hence $\Delta\mathcal{K}^\sharp$ is also bounded. Therefore, with [7, Theorems 4.7, 7.3], $\mathcal{K}^\sharp$ has the spectral decomposition

$$\mathcal{K}^\sharp(z;s,s') = \sum_{j=0}^{\infty} \frac{\langle\, \mathcal{K}^\sharp(\cdot;s,s'), u_j \,\rangle}{\langle\, u_j, u_j \,\rangle} u_j(z) + \frac{1}{4\pi i}\int_{(1/2)} \langle\, \mathcal{K}^\sharp(\cdot;s,s'), E(\cdot,r)\,\rangle E(z,r)\,dr \tag{9.15}$$

where the integral is from $1/2-i\infty$ to $1/2+i\infty$ and the convergence of (9.15) is pointwise absolute in z and uniform on compacta.

Lemma 9.3. *For $s \in B_\rho$ and $\mathrm{Re}(s') > \rho+5$ we have*

$$\langle\, \mathcal{K}^\sharp(\cdot;s,s'), u_j \,\rangle = \frac{\pi^{1/2-s}}{4\Gamma(s')} L^*(u_j, s'-s+1/2)\Gamma\left(\frac{s'+s+s_j-1}{2}\right)\Gamma\left(\frac{s'+s-s_j}{2}\right)$$

when u_j is an even Maass cuspform. If u_j is odd or constant then the inner product is zero.

Proof. Unfolding,

$$\begin{aligned}
\langle\, \mathcal{K}^\sharp(\cdot;s,s'), u_j \,\rangle &= \int_{\Gamma\backslash\mathbb{H}} \mathcal{K}^\sharp(z;s,s')\overline{u_j(z)}\,d\mu(z) \\
&= \int_0^\infty \int_0^1 \left(\sum_{m\neq 0} y^{s'+1/2}|m|^{s-1/2}K_{s-1/2}(2\pi|m|y)e^{2\pi i m x}\right)\overline{u_j(z)}\,\frac{dxdy}{y^2} \\
&= 2\sum_{m\neq 0} \nu_j(m)|m|^{s-1/2}\int_0^\infty y^{s'}K_{s-1/2}(2\pi|m|y)K_{\overline{s_j}-1/2}(2\pi|m|y)\frac{dy}{y}.
\end{aligned}$$

Evaluating the integral [7, p. 205] yields

$$\langle\, \mathcal{K}^\sharp(\cdot;s,s'), u_j \,\rangle = \frac{L(u_j, s'-s+1/2)}{4\pi^{s'}\Gamma(s')}\prod \Gamma\left(\frac{s' \pm (s-1/2) \pm (\overline{s_j}-1/2)}{2}\right).$$

Using (9.3) and that $\overline{s_j} = 1-s_j$ finishes the proof. □

In the same way, when $\mathrm{Re}(r) = 1/2$,

$$\langle\, \mathcal{K}^\sharp(\cdot;s,s'), E(\cdot,r)\,\rangle = \frac{L(\overline{E(\cdot,r)}, s'-s+1/2)}{4\pi^{s'}\Gamma(s')}\prod \Gamma\left(\frac{s' \pm (s-1/2) \pm (\overline{r}-1/2)}{2}\right).$$

Further, $\overline{E(z,r)} = E(z,\overline{r}) = E(z,1-r)$ and with (9.5) we have shown the following.

Lemma 9.4. *For $s \in B_\rho$ and $\mathrm{Re}(s') > \rho+5$*

$$\begin{aligned}
\langle\, \mathcal{K}^\sharp(\cdot;s,s'), E(\cdot,r)\,\rangle &= \frac{\pi^{1/2-s}}{2\Gamma(s')\theta(1-r)} \\
&\quad \times \Gamma\left(\frac{s'+s-r}{2}\right)\Gamma\left(\frac{s'+s-1+r}{2}\right)\theta\left(\frac{s'-s+r}{2}\right)\theta\left(\frac{s'-s+1-r}{2}\right).
\end{aligned}$$

Recall that $\theta(s) := \pi^{-s}\Gamma(s)\zeta(2s)$ as in §2.1. Let

$$\begin{aligned}
\mathcal{K}_1(z;s,s') &:= \frac{\pi^{1/2}\Gamma(s-1/2)}{\Gamma(s)}E(z,s'-s+1)\\
\mathcal{K}_2(z;s,s') &:= \frac{\pi^{1/2}}{2\Gamma(s)\Gamma(s')}\sum_{\substack{j=1\\ u_j \text{ even}}}^{\infty} L^*(u_j,s'-s+1/2)\Gamma\left(\frac{s'+s+s_j-1}{2}\right)\Gamma\left(\frac{s'+s-s_j}{2}\right)\frac{u_j(z)}{\langle u_j,u_j\rangle}\\
\mathcal{K}_3(z;s,s') &:= \frac{\pi^{1/2}}{\Gamma(s)\Gamma(s')}\frac{1}{4\pi i}\int_{(1/2)}\Gamma\left(\frac{s'+s-r}{2}\right)\Gamma\left(\frac{s'+s-1+r}{2}\right)\\
&\qquad\times\theta\left(\frac{s'-s+r}{2}\right)\theta\left(\frac{s'-s+1-r}{2}\right)\frac{E(z,r)}{\theta(1-r)}\,dr.
\end{aligned}$$

Assembling Proposition 9.2, (9.15) and Lemmas 9.3, 9.4 we have proven the decomposition

$$\mathcal{K}(z;s,s') = \mathcal{K}_1(z;s,s') + \mathcal{K}_2(z;s,s') + \mathcal{K}_3(z;s,s') \tag{9.16}$$

for $s \in B_\rho$ and $\mathrm{Re}(s') > \rho + 5$.

Clearly $\mathcal{K}_1(z;s,s')$ is a meromorphic function of s and s' in all of $\mathbb{C}$. The same is true for $\mathcal{K}_2(z;s,s')$ since the factors $L(u_j,s'-s+1/2)\frac{u_j(z)}{\langle u_j,u_j\rangle}$ have at most polynomial growth as $\mathrm{Im}(s_j)\to\infty$ while the Γ factors have exponential decay by Stirling's formula. See (9.2) and [7, §§7,8] for the necessary bounds.

Theorem 9.5. *The non-holomorphic kernel $\mathcal{K}(z;s,s')$ has a meromorphic continuation to all $s,s'\in\mathbb{C}$.*

Proof. As we have discussed, $\mathcal{K}_1(z;s,s')$ and $\mathcal{K}_2(z;s,s')$ are meromorphic functions of $s,s'\in\mathbb{C}$. The poles of $\Gamma(w)$ are at $w=0,-1,-2,\dots$ and $\theta(w)$ has poles exactly at $w=0,1/2$ (with residues $-1/2$, $1/2$ respectively). Therefore, the integral in $\mathcal{K}_3(z;s,s')$ is certainly an analytic function of s,s' for $\sigma' > \sigma+1/2$ and $\sigma>1/2$ since the Γ and θ factors have exponential decay as $|r|\to\infty$. Next consider s fixed (with $\sigma>1/2$) and s' varying. Consider a point r_0 with $\mathrm{Re}(r_0)=1/2$. Let $B(r_0)$ be a small disc centered at r_0 and $B(1-r_0)$ an identical disc at $1-r_0$. By deforming the path of integration to a new path C to the left of $B(r_0)$ and to the right of $B(1-r_0)$, we may, by Cauchy's theorem, analytically continue $\mathcal{K}_3(z;s,s')$ to s' with $s'-s\in B(r_0)$. Let C_1 be a clockwise contour around the left side of $B(r_0)$ and C_2 be a counter-clockwise contour around the right side of $B(1-r_0)$ so that $C=(1/2)+C_1+C_2$. For $s'-s$ inside C_1 (and $1-(s'-s)$ inside C_2) we have

$$\pi^{-1/2}\Gamma(s)\Gamma(s')\cdot\mathcal{K}_3(z;s,s') = \frac{1}{4\pi i}\int_C * = \frac{1}{4\pi i}\int_{(1/2)} * + \frac{1}{4\pi i}\int_{C_1} * + \frac{1}{4\pi i}\int_{C_2} *$$

with $*$ denoting the integrand in the definition of $\mathcal{K}_3$. Then

$$\begin{aligned}
\frac{1}{4\pi i}\int_{C_1} &= \frac{-2\pi i}{4\pi i}\left(\operatorname*{Res}_{r=s'-s}\theta\left(\frac{s'-s+1-r}{2}\right)\right)\Gamma(s)\Gamma(s'-1/2)\frac{\theta(s'-s)}{\theta(1-s'+s)}E(z,s'-s)\\
&= \frac{1}{2}\Gamma(s)\Gamma(s'-1/2)\frac{\theta(s'-s)}{\theta(1-s'+s)}E(z,s'-s)\\
&= \frac{1}{2}\Gamma(s)\Gamma(s'-1/2)E(z,s-s'+1).
\end{aligned}$$

We get the same result for $\frac{1}{4\pi i}\int_{C_2}$ and it follows that for all s' with $\sigma-1/2<\mathrm{Re}(s')<\sigma+1/2$, the continuation of $\mathcal{K}_3(z;s,s')$ is given by

$$\pi^{-1/2}\Gamma(s)\Gamma(s')\cdot\mathcal{K}_3(z;s,s') = \Gamma(s)\Gamma(s'-1/2)E(z,s-s'+1) + \frac{1}{4\pi i}\int_{(1/2)} *. \tag{9.17}$$

Similarly, as s' crosses the line with real part $\sigma - 1/2$, the term $-\Gamma(s-1/2)\Gamma(s')E(z, s'-s+1)$ must be added to the right side of (9.17). Thus, for all s with $1/2 < \mathrm{Re}(s') < \sigma - 1/2$, the continuation of $\mathcal{K}(z; s, s')$ is

$$\mathcal{K}(z; s, s') = \frac{\pi^{1/2}\Gamma(s'-1/2)}{\Gamma(s')} E(z, s-s'+1) + \mathcal{K}_2(z; s, s') + \mathcal{K}_3(z; s, s'). \tag{9.18}$$

Clearly, with (9.17), (9.18) we have demonstrated the meromorphic continuation of $\mathcal{K}(z; s, s')$ to all $s, s' \in \mathbb{C}$ with $\mathrm{Re}(s), \mathrm{Re}(s') > 1/2$. The continuation to all $s, s' \in \mathbb{C}$ follows in the same way with further terms in the expression for $\mathcal{K}(z; s, s')$ appearing from the residues of the poles of $\Gamma\left(\frac{s'+s-r}{2}\right)\Gamma\left(\frac{s'+s-1+r}{2}\right)$ as $\mathrm{Re}(s'+s) \to -\infty$. □

Proposition 9.6. *We have the functional equation*

$$\mathcal{K}(z; s, s') = \mathcal{K}(z; s', s). \tag{9.19}$$

Proof. We may verify (9.19) by comparing (9.16) with (9.18) and using that $\mathcal{K}_2(z; s, s') = \mathcal{K}_2(z; s', s)$ by (9.4), and $\mathcal{K}_3(z; s, s') = \mathcal{K}_3(z; s', s)$ by (2.6). There is a second, easier proof: with $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, replace γ in (9.1) by $S\gamma$. □

Proposition 9.7. *For all $s, s' \in \mathbb{C}$ and any even Maass Hecke eigenform u_j,*

$$\langle \mathcal{K}(\cdot; s, s'), u_j \rangle = \frac{\pi^{1/2}}{2\Gamma(s)\Gamma(s')} \Gamma\left(\frac{s'+s+s_j-1}{2}\right)\Gamma\left(\frac{s'+s-s_j}{2}\right) \cdot L^*(u_j, s'-s+1/2).$$

Proof. Since each u_j is orthogonal to Eisenstein series we have by (9.16) (for $s \in B_\rho$ and $\mathrm{Re}(s') > \rho + 5$) that

$$\langle \mathcal{K}(\cdot; s, s'), u_j \rangle = \langle \mathcal{K}_2(\cdot; s, s'), u_j \rangle.$$

The result follows, extending to all $s, s' \in \mathbb{C}$ by analytic continuation. □

9.3 Non-holomorphic double Eisenstein series

A similar argument to the proof of (6.3) shows that, for $\mathrm{Re}(s), \mathrm{Re}(s') > 1$ and $\mathrm{Re}(w) \geqslant 0$,

$$\zeta(w+2s)\zeta(w+2s')\mathcal{E}(z, w; s, s') = \frac{1}{2}\sum_{n=1}^{\infty} \frac{T_n \mathcal{K}(z; s, s')}{n^{w-1/2}} \tag{9.20}$$

where, in this context [5, (3.12.3)], the appropriately normalized Hecke operator acts as

$$T_n \mathcal{K}(z) = \frac{1}{n^{1/2}} \sum_{\gamma \in \Gamma\backslash \mathcal{M}_n} \mathcal{K}(\gamma z).$$

For each Maass form we have $T_n u_j = \nu_j(n) u_j$ and for the Eisenstein series [5, Prop. 3.14.2] implies $T_n E(z, s) = n^{s-1/2}\sigma_{1-2s}(n) E(z, s)$. Therefore, as in (9.5),

$$\sum_{n=1}^{\infty} \frac{T_n E(z, s)}{n^{w-1/2}} = E(z, s) \sum_{n=1}^{\infty} \frac{\sigma_{1-2s}(n)}{n^{w-s}} = E(z, s)\zeta(w-s)\zeta(w+s-1).$$

Now choose any $\rho > 0$. For $s \in B_\rho$, $\mathrm{Re}(s) > 1$, $\mathrm{Re}(s') > \rho + 5$ and $\mathrm{Re}(w) \geqslant 0$ we may apply T_n to both sides of (9.16) and obtain

$$\zeta(w+2s)\zeta(w+2s')\mathcal{E}(z,w;s,s') = \frac{\pi^{1/2}\Gamma(s-1/2)}{2\Gamma(s)}\zeta(s'-s+w)\zeta(s-s'+w-1)E(z,s'-s+1)$$
$$+\frac{\pi^{1/2}}{4\Gamma(s)\Gamma(s')}\sum_{\substack{j=1\\ u_j \text{ even}}}^{\infty} L^*(u_j,s'-s+1/2)\Gamma\left(\frac{s'+s+s_j-1}{2}\right)\Gamma\left(\frac{s'+s-s_j}{2}\right)L(u_j,w-1/2)\frac{u_j(z)}{\langle u_j,u_j\rangle}$$
$$+\frac{\pi^{1/2}}{2\Gamma(s)\Gamma(s')}\frac{1}{4\pi i}\int_{(1/2)}\theta\left(\frac{s'-s+r}{2}\right)\theta\left(\frac{s'-s+1-r}{2}\right)\Gamma\left(\frac{s'+s-r}{2}\right)\Gamma\left(\frac{s'+s-1+r}{2}\right)$$
$$\times\zeta(w-r)\zeta(w-1+r)\frac{E(z,r)}{\theta(1-r)}\,dr. \quad (9.21)$$

Put

$$\Omega(s,s';r) := \theta\left(\frac{s'+s-r}{2}\right)\theta\left(\frac{s'+s-1+r}{2}\right)\theta\left(\frac{s'-s+r}{2}\right)\theta\left(\frac{s'-s+1-r}{2}\right)\Big/\theta(1-r).$$

Define the completed double Eisenstein series as in (2.19) and write

$$U(z;s,s') := \sum_{\substack{j=1\\ u_j \text{ even}}}^{\infty} L^*(u_j,s+s'-1/2)L^*(u_j,s'-s+1/2)\frac{u_j(z)}{\langle u_j,u_j\rangle}.$$

As in the last section, Ω and U have exponential decay as $|r|$ and $|\mathrm{Im}(s_j)| \to \infty$. Specializing (9.21) to $w = s+s'$, we have proved the next result.

Lemma 9.8. *For $s \in B_\rho$, $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(s') > \rho + 5$*

$$\mathcal{E}^*(z;s,s') = 2\theta(s)\theta(s')E(z;s+s') + 2\theta(1-s)\theta(s')E(z,s'-s+1)$$
$$+U(z;s,s') + \frac{1}{2\pi i}\int_{(1/2)}\Omega(s,s';r)E(z,r)\,dr. \quad (9.22)$$

From this we show the following.

Theorem 9.9. *The completed double Eisenstein series $\mathcal{E}^*(z;s,s')$ has a meromorphic continuation to all $s, s' \in \mathbb{C}$ and we have the functional equations*

$$\mathcal{E}^*(z;s,s') = \mathcal{E}^*(z;s',s), \quad (9.23)$$
$$\mathcal{E}^*(z;s,s') = \mathcal{E}^*(z;1-s,1-s'). \quad (9.24)$$

Proof. First note that (9.22) gives the meromorphic continuation of $\mathcal{E}^*(z;s,s')$ to all s, s' with $s \in B_\rho$ and $\mathrm{Re}(s') > \rho + 5$. As in the proof of Theorem 9.5, we see that the further continuation in s' is given by (9.22) along with residues that are picked up as the line of integration is crossed: for $s \in B_\rho$ fixed and $\mathrm{Re}(s') \to -\infty$ the continuation of $\mathcal{E}^*(z;s,s')$ is given by (9.22) plus each of the following

$$\begin{array}{rll} 2\theta(s)\theta(1-s')E(z,s-s'+1) & \text{when} & \mathrm{Re}(s') < \sigma + 1/2, \\ -2\theta(1-s)\theta(s')E(z,s'-s+1) & \text{when} & \mathrm{Re}(s') < \sigma - 1/2, \\ 2\theta(1-s)\theta(1-s')E(z,2-s-s') & \text{when} & \mathrm{Re}(s') < -\sigma + 1/2, \\ -2\theta(s)\theta(s')E(z;s+s') & \text{when} & \mathrm{Re}(s') < -\sigma - 1/2. \end{array}$$

We have therefore shown the meromorphic continuation of $\mathcal{E}^*(z;s,s')$ to all $s \in B_\rho$ and $s' \in \mathbb{C}$. Hence, for all s' with $\mathrm{Re}(s') < -\rho - 4$, say, we have

$$\mathcal{E}^*(z;s,s') = 2\theta(1-s)\theta(1-s')E(z;2-s-s') + 2\theta(s)\theta(1-s')E(z,s-s'+1) + U(z;s,s') + \frac{1}{2\pi i}\int_{(1/2)} \Omega(s,s';r)E(z,r)\,dr. \quad (9.25)$$

The functional equation (9.24) is a consequence of the easily checked symmetries $U(z;1-s,1-s') = U(z;s,s')$, $\Omega(1-s,1-s';r) = \Omega(s,s';r)$ and a comparison of (9.22) and (9.25). The equation (9.23) has a similar proof, or more simply follows from the definition (2.19). □

Proposition 9.10. *For any even Maass Hecke eigenform u_j (as in §9.1) and all $s, s' \in \mathbb{C}$*

$$\langle \mathcal{E}^*(\cdot;s,s'), u_j \rangle = L^*(u_j, s+s'-1/2)L^*(u_j, s'-s+1/2).$$

Proof. As in Proposition 9.7, only $U(z;s,s')$ in (9.22) will contribute to the inner product. □

With Theorem 9.9 and Proposition 9.10, we have proved Theorem 2.8.

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