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Oeljeklaus-Toma manifolds admitting no complex subvarieties

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To Professor Vasile Brînzănescu at his sixty-fifth birthday

Abstract

The Oeljeklaus-Toma (OT-) manifolds are complex manifolds constructed by Oeljeklaus and Toma from certain number fields, and generalizing the Inoue surfaces S_m. On each OT-manifold we construct a holomorphic line bundle with semipositive curvature form ω_0 and trivial Chern class. Using this form, we prove that the OT-manifolds admitting a locally conformally Kähler structure have no non-trivial complex subvarieties. The proof is based on the Strong Approximation theorem for number fields, which implies that any leaf of the null-foliation of ω_0 is Zariski dense.

Contents

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1 Introduction

1.1 OT-manifolds and their subvarieties

The Oeljeklaus-Toma (OT-) manifolds are an important class of compact complex manifolds not admitting a Kähler metric. They were discovered by Oeljeklaus and Toma in 2005 ([OT]). The construction of OT-manifolds uses the Dirichlet unit theorem from number theory (Subsection 1.2; see [PV] for additional details of this construction and many related questions). Starting from a degree 3 number field, one obtains a 2-dimensional OT-manifold known as Inoue surface S_m (see [I]).

For some number fields, the OT-manifolds are locally conformally Kähler. A locally conformally Kähler (LCK) structure on a complex manifold is a Kähler metric on its universal cover $\tilde{M}$, such that the deck transform maps act on $\tilde{M}$ by homotheties. The OT-manifolds serve an important function in the theory of LCK manifolds, providing a counterexample to a longstanding conjecture of I. Vaisman, who asked whether there exists a compact, non-Kähler LCK-manifold M with all odd Betti numbers even: $b_{2p+1}(M) \vdots 2$. The Oeljeklaus-Toma manifolds in dimension 3 are the only known examples of compact LCK-manifolds with even odd Betti numbers, $b_1 = b_5 = 2, b_2 = 0$.

An OT-manifold is LCK if (and only if) it is constructed from a number field K which has precisely 2 complex (non-real) embeddings, that is, two distinct homomorphisms $K \xrightarrow{\sigma,\overline{\sigma}} \mathbb{C}$.

Oeljeklaus and Toma proved that an OT-manifold has no global meromorphic functions. We give a generalization of this theorem, proving that an OT-manifold which is locally conformally Kähler has no non-trivial complex subvarieties.

The idea of the proof of this result is quite simple. We construct a holomorphic Hermitian line bundle, called **the weight bundle**, on any OT-manifold M. This bundle is topologically trivial, and has semipositive curvature form ω_0. The weight bundle also admits a flat connection, compatible with the holomorphic structure.

To learn about complex subvarieties of an OT-manifold, we study the zero-foliation Σ of ω_0, proving that all its leaves are Zariski dense in M. For an OT-manifold M constructed from a number field K admitting exactly $2t$ distinct complex (non-real) embeddings to $\mathbb{C}$, the leaves of Σ are t-dimensional. When $t = 1$, M is locally conformally Kähler, and Σ is one-dimensional. In this case, we prove that for any positive-dimensional complex subvariety $Z \subset M$, Z contains with each point $z \in Z$ a leaf Σ_z

passing through z. Since all leaves of Σ are Zariski dense, the same is true for Z.

The weight bundle L is quite useful for many other purposes. As it was done in [Ve3], one can take the α-th tensor power of L, denoted by L^α, for any real α; this power is well defined, because L is equipped with a natural C^∞-trivialization. The Gauduchon degree $\deg_g$ of L^α, taken with respect to any Gauduchon metric, satisfies $\frac{1}{\alpha}\deg_g L^\alpha = \deg_g L > 0$, hence M admits a line bundle with any prescribed Gauduchon degree. This implies, in particular, that the connected component of the Picard group $Pic(M)$ is non-compact. Also, this implies that any vector bundle on M has degree zero after tensoring with an appropriate power of L; this is useful for the study of Hermitian-Einstein bundles on M, providing useful tools for the classification of stable bundles, and, eventually, coherent sheaves on M.

A similar argument was used in [Ve3] to study holomorphic vector bundles and subvarieties on homogeneous elliptic fibrations, such as Calabi-Eckmann manifolds and quasi-regular Vaisman manifolds. We pose two questions, very much unsolved, but quite natural in the context presented by [Ve3] and the present paper. Notice that from their construction it is clear that OT-manifolds are affine flat, that is, equipped with a flat, affine, torsion-free connection.

Question 1.1: Are there any OT-manifolds with non-trivial closed complex subvarieties? Can we classify these subvarieties? Are they always completely geodesic with respect to the flat affine connection?

Question 1.2: Does there exist a stable holomorphic vector bundle of rank > 1 on any OT-manifold of dimension > 2? Do all holomorphic vector bundles admit a flat connection, compatible with the holomorphic structure?

Remark 1.3: It is well known that generic complex tori have no non-trivial complex subvarieties. In [Ve2], it was shown that all stable bundles on a generic complex torus of dimension > 2 have rank 1, and all holomorphic vector bundles admit flat connections.

1.2 Number theory and the construction of OT-manifolds

Let $[K:\mathbb{Q}]$ be a number field, that is, a finite extension of $\mathbb{Q}$, of degree n, with $\sigma_1, ..., \sigma_s$ the real embeddings of K into $\mathbb{C}$, and $\sigma_{s+1}, ..., \sigma_n$ the complex embeddings. Let $\sigma = (\sigma_1, \dots, \sigma_{s+t}) : K \to \mathbb{C}^{s+t}$ be the corresponding group homomorphism. Since the complex embeddings of K into $\mathbb{C}$ occur in pairs of complex conjugate embeddings, the number $n-s$ is even, $n-s=t$.

Let $\mathcal{O}_K$ be the ring of algebraic integers of K, $\mathcal{O}_K^*$ its multiplicative group of units and $\mathcal{O}_K^{*,+}$ the group of units which are positive in all the real embeddings of K.

Denote by $\mathbb{H}$ be the upper complex half-plane. Using the Dirichlet's unit theorem, Oeljeklaus and Toma proved that $\mathcal{O}_K \rtimes \mathcal{O}_K^{*,+}$ acts freely on $\mathbb{H}^s \times \mathbb{C}^t$ by

$$\begin{aligned} T_a(z_i) &= (z_i + \sigma_i(a)), \quad i = 1, \dots, s+t, \quad a \in \mathcal{O}_K, \\ R_u(z_i) &= (\sigma_i(u) z_i), \quad i = 1, \dots, s+t, \quad u \in \mathcal{O}_K^{*,+}. \end{aligned}$$

(see [OT], [PV]). Moreover, an *admissible* subgroup $U \subset \mathcal{O}_K^{*,+}$ can always be found such that the action of $\Gamma := \mathcal{O}_K \rtimes U$ is also properly discontinuous. For $t=1$, every U of finite index in $\mathcal{O}_K^{*,+}$ has this property.

Definition 1.4: The manifold $M_K := (\mathbb{H}^s \times \mathbb{C}^t)/\Gamma$ is called an **Oeljeklaus-Toma manifold**. It is a compact complex manifold of dimension $s+t$.

For $s=t=1$, M_K reduces to an Inoue surface S_m (where m is a matrix in $\mathrm{SL}(3,\mathbb{Z})$), see [I]. The corresponding number field K is $\mathbb{Q}[T]/P(t)$, where $P_m(t)$ is the characteristic polynomial of the matrix m. It is shown in [OT] that the manifolds M_K are never Kähler, but that for $t=1$, M_K is a locally conformally Kähler (LCK) manifold (see [DO] and the more recent survey [OV] for definitions and results in LCK geometry). We briefly explain the construction of this LCK metric.

Clearly, the function $\psi(z) = \prod_{i=1}^s (\operatorname{im} z_i) + |z_{s+1}|^2$ is plurisubharmonic on $\mathbb{H}^s \times \mathbb{C}$. It defines the Kähler form $\Omega := \partial\overline{\partial}\,\psi$ on $\mathbb{H}^s \times \mathbb{C}$. The group Γ acts on $(\mathbb{H}^s \times \mathbb{C}, \Omega)$ by homotheties:

$$\begin{aligned} T_a^*\Omega &= \Omega, \\ R_u^*\Omega &= |\sigma_{s+1}(u)|^2\Omega. \end{aligned}$$

Let now $\chi : \Gamma \to \mathbb{R}^{>0}$ be the character $\chi(\gamma) = \frac{\gamma^*\Omega}{\Omega}$. We call **automorphic** any p-form $\eta \in \Lambda^p(\mathbb{H}^s \times \mathbb{C})$ which satisfies $\gamma^*\eta = \chi(\gamma)\eta$. For any automorphic function φ on $\mathbb{H}^s \times \mathbb{C}$, the quotient $\frac{\Omega}{\varphi}$ is Γ-invariant and hence projects to an LCK metric ω on M_K. This form satisfies the equation $d\omega = \theta \wedge \omega$, for the closed 1-form θ (called **the Lee form**) which is the projection on M_K of $\tilde\theta = -d\log\varphi$:

$$d\omega = -\frac{d\varphi}{\varphi^2} \wedge \tilde\omega = -d(\log\varphi) \wedge \omega.$$

It is easily seen that the function $\varphi = \prod_{i=1}^{s}(\operatorname{im} z_i)^{-1}$ is automorphic, and hence it produces a LCK metric on M_K as described above. This LCK metric generalizes the one constructed by Tricerri on S_m, [Tr].

The main result of this paper shows that, just as Inoue surfaces S_m have no complex curves, OT-manifolds have no complex subvarieties:

Theorem 1.5: Let $[K : \mathbb{Q}]$ be a number field of degree $n = s + 2$, with s real embeddings and 2 complex embeddings, and M_K the corresponding OT-manifold. Then M_K has no non-trivial complex subvarieties.

Proof: See Theorem 3.1. ■

2 The weight bundle of an OT-manifold

Definition 2.1: Let $[K : \mathbb{Q}]$ be a number field of degree $n = s + 2t$, with s real embeddings and $2t$ complex embeddings, and $M_K = \mathbb{H}^s \times \mathbb{C}^t/\Gamma$ the associated OT-manifold. Denote by $z_1, ..., z_s$ the standard complex coordinates on $\mathbb{H}^s$, and let $\tilde\theta := d\log\prod_{i=1}^{s}(\operatorname{im} z_i)$. It is easy to see that the form $\tilde\theta$ is Γ-invariant. Therefore it is obtained as a lift of a form θ, called **the Lee form** of the OT-manifold. When $t = 1$, this is the Lee form constructed above.

Let M_K be an OT-manifold, and θ its Lee form. Consider a trivial Hermitian line bundle L with connection $\nabla := \nabla_0 + \sqrt{-1}\,\theta^c$, where $\theta^c := I(\theta)$, and ∇_0 is the trivial connection on L. Clearly, ∇ is Hermitian, and $\nabla^{0,1} = \overline\partial + \theta^{0,1}$, where $\theta^{0,1}$ is the (0,1)-part of θ.

Claim 2.2: In these assumptions, the curvature ω_0 of ∇ is $-\sqrt{-1}\,d\theta^c$. Moreover, this form is of type (1,1).

Proof: A simple computation shows that in the standard coordinates $z_1, ...z_s, z_{s+1}, ...z_{s+t}$, ω_0 can be written as follows:

$$\omega_0 = \sqrt{-1}\,\partial\overline\partial\log\varphi = \sqrt{-1}\,\sum_{i=1}^{s}\frac{dz_i\wedge d\overline z_i}{|\operatorname{im} z_i|^2},$$

■

Definition 2.3: Let M_K be an OT-manifold, and L the holomorphic Hermitian bundle defined above. Then L is called **the weight bundle** of M_K.

We restate Claim 2.2 as

Theorem 2.4: Let M_K be an OT-manifold, and L its weight bundle with the holomorphic Hermitian structure and the Chern connection ∇ defined above. Consider the form $\omega_0 := \sqrt{-1}\,\nabla^2$. Then ω_0 is a semi-positive form, which can be written in the standard coordinates $z_1, ...z_s, z_{s+1}, ..., z_{s+t}$ as follows:

$$\omega_0 = \sqrt{-1}\,\partial\overline{\partial}\log\varphi = \sqrt{-1}\,\sum_{i=1}^{s}\frac{\partial z_i \wedge \overline{\partial} z_i}{|\operatorname{im} z_i|^2}$$

■

Remark 2.5: The Vaisman manifolds are, by definition, LCK manifolds (M, I, g) satisfying the aditional condition $\nabla^g\theta = 0$, where ∇^g is the Levi-Civita connection of an LCK metric g. For all Vaisman manifolds, the 2-form $\omega_0 = d\theta^c$ is semi-positive, being zero only on the direction of $\theta^\sharp - I\theta^\sharp$. This is a general fact, proven in [Ve1], independent of the particular form of θ. OT-manifolds are far from being Vaisman (they never admit any Vaisman metric), but the particular expression of their Lee form gives ω_0 the same property as for Vaisman manifold. This is what inspired our construction.

Remark 2.6: An object of interest in conformal geometry and, in particular, LCK geometry is the **weight bundle**. It is the real line bundle $L \longrightarrow M$ associated to the representation $\mathrm{GL}(2n, \mathbb{R}) \ni A \mapsto |\det A|^{\frac{1}{n}}$ (see [OV]). Then L can be complexified and endowed with the Chern connection $\nabla_0 + \sqrt{-1}\,\theta^c$ (where ∇_0 is the trivial connection). It can be verified that $\omega_0 = \sqrt{-1}\nabla^2$, and hence ω_0 can be seen as the curvature form of this Chern connection. When $t = 1$ and M is an LCK-manifold, this construction gives the weight bundle defined above.

Remark 2.7: For any OT-manifold M, in addition to the Chern connection $\nabla_0 + \sqrt{-1}\,\theta^c$, the weight bundle L also admits the connection $\nabla_0 + \theta$, which is flat because $d\theta = 0$. It is clear that the $(0,1)$-part of ∇ coincides with the $(0,1)$-part of this flat connection.

The following claim is obvious from the explicit form of ω_0 (Theorem 2.4).

Claim 2.8: In the assumptions of Theorem 2.4, let $\tilde{\Sigma}$ be the holomorphic foliation on the covering $\tilde{M}_K = \mathbb{H}^s \times \mathbb{C}$ generated by the vector fields

$\frac{\partial}{\partial z_{s+1}}, ..., \frac{\partial}{\partial z_{s+t}}$. Then:

(i) The foliation $\tilde{\Sigma}$ is Γ-invariant, hence it is obtained as the pullback of a holomorphic foliation Σ on $M_K = \tilde{M}_K/\Gamma$.

(ii) The foliation Σ is the null-space of the form ω_0 constructed above.

■

Claim 2.9: Let $[K : \mathbb{Q}]$ be a number field of degree $n = s + 2$, with s real embeddings and 2 complex embeddings, M_K the corresponding LCK OT-manifold, and $\Sigma \subset TM$ the holomorphic foliation defined in Claim 2.8. Consider a complex closed subvariety $Z \subset M_K$. Then Σ is tangent to Z at any point of Z:

$$\forall z \in Z, \quad \Sigma\Big|_z \subset T_z Z. \tag{2.1}$$

Proof: The form ω_0 has $(n-1)$ positive eigenvalues, where $n = \dim_{\mathbb{C}} M$, and its zero eigenspace at z is $\Sigma\Big|_z$. Unless (2.1) holds at $z \in Z$, the restriction $\omega_0\Big|_Z$ has $m = \dim Z$ positive eigenvalues at z. Then $\int_Z \omega_0^m > 0$. This is impossible, because ω_0 is exact. ■

Corollary 2.10: In asssumption of Claim 2.9, let Σ_z be a leaf of Σ passing through $z \in Z$. Then $\Sigma_z \subset Z$.

■

3 Complex subvarieties in LCK OT-manifold

Using Corollary 2.10, we can easily prove the main result of this paper.

Theorem 3.1: Let $[K : \mathbb{Q}]$ be a number field of degree $n = s+2$, with s real embeddings and 2 complex embeddings, and let M_K be the corresponding OT-manifold. Then M_K has no non-trivial complex subvarieties.

Proof: Theorem 3.1 follows from Corollary 2.10 and the following more general proposition.

Proposition 3.2: Let $[K : \mathbb{Q}]$ be a number field of degree $n = s+2t$, $t > 0$, with s real embeddings and $2t$ complex embeddings, and let $M_K = \mathbb{H}^s \times \mathbb{C}^t/\Gamma$ be the associated (non-Kähler) OT-manifold. Let $\Sigma \subset TM_K$ be the foliation defined in Claim 2.8. Consider a leaf of Σ, and let Z be its closure. Then

(i) The preimage $\pi^{-1}(Z)$ of Z to $\tilde{M}_K = \mathbb{H}^s \times \mathbb{C}^t$ contains the set

$$Z_{\alpha_1,...,\alpha_s} := \{(z_1, ..., z_s, z_{s+1}, \dots, z_{s+t}) \mid \operatorname{im} z_i = \alpha_i\}$$

for some positive numbers $\alpha_1, \dots, \alpha_s \in \mathbb{R}^s$.

(ii) Any complex subvariety of M_K containing Z must coincide with M_K.

Proof: The implication (i) $\Rightarrow$ (ii) is clear, because any complex manifold containing $Z_{\alpha_1,...,\alpha_s}$ must have the same dimension as M_K. The proof of (i) is a bit more elaborate.

Let $\mathcal{O}$ be the ring of integers in K. By construction, the group $\Gamma = \pi_1(M_K)$ is an cross-product of the additive group $\mathcal{O}^+$ of $\mathcal{O}$ with a subgroup of the multiplicative group $\mathcal{O}^*$. Let $\tilde{\Sigma}$ be the pullback of the foliation Σ to $\tilde{M}_K = \mathbb{H}^s \times \mathbb{C}^t$. A leaf of $\tilde{\Sigma}$ is given as

$$T_{t_1,...,t_s} := \{(z_1, ..., z_s, z_{s+1}, ..., z_{s+t}) \mid z_i = t_i\}$$

for some $(t_1, ..., t_s) \in \mathbb{H}^s$. Let $\tilde{Z}$: $\pi^{-1}(Z)$ be the preimage of the corresponding closure of a leaf of Σ. Clearly, $\tilde{Z}$ is the closure of $\Gamma(T_{t_1,...,t_s})$. Therefore, to prove Proposition 3.2 (i) it is sufficient to show that the closure of $\Gamma(T_{t_1,...,t_s})$ contains $Z_{\alpha_1,...,\alpha_s}$. In fact, even the smaller group $\mathcal{O}^+ \subset \Gamma$ will suffice, as seen from the following lemma, which proves Proposition 3.2.

Lemma 3.3: Let $[K : \mathbb{Q}]$ be a number field of degree $n = s + 2t$, $t > 0$ with s real embeddings and $2t$ complex embeddings, and $\tilde{M}_K := \mathbb{H}^s \times \mathbb{C}^t$, equipped with the action of $\mathcal{O}^+$ as in Subsection 1.2. Consider the subset

$$T_{t_1,...,t_s} := \{(z_1, ..., z_s, z_{s+1}, \dots, z_{s+t}) \mid z_i = t_i\}$$

in $\tilde{M}_K$. Then the closure of $\mathcal{O}^+(T_{t_1,...,t_s})$ coincides with

$$Z_{\alpha_1,...,\alpha_s} := \{(z_1, ..., z_s, z_{s+1}, \dots, z_{s+t}) \mid \operatorname{im} z_i = \alpha_i, \}$$

with $\alpha_i := \operatorname{im} t_i$.

Proof: Equivalently, we may state that the closure of an orbit of the standard action of $\mathcal{O}^+$ in $\mathbb{H}^s$ is the set $\{(z_1, \dots, z_s, z_{s+1}, \dots, z_{s+t}) \mid \operatorname{im} z_i = \alpha_i\}$. This in turn is equivalent to the following

Lemma 3.4: Let $[K : \mathbb{Q}]$ be a number field of degree $n = s + 2t$, $t > 0$ with s real embeddings $\sigma_1, \dots, \sigma_s$ and $2t$ complex embeddings. Consider the additive group $\mathcal{O}^+$ of the corresponding ring of integers. Let $\sigma: \mathcal{O}^+ \longrightarrow \mathbb{R}^s$ map ξ to $\sigma_1(\xi), \dots, \sigma_s(\xi)$. Then the image of $\mathcal{O}^+$ is dense in $\mathbb{R}^s$.

Proof:[1] Let K be a number field, $\mathcal{O}_K$ its ring of integers, $\mathfrak{P}$ the set of all prime ideals of $\mathcal{O}_K$, V the product of all archimedean completions of K, and V_1 the product of some, but not all, archimedean completions. Denote by $\mathcal{O}_\nu$ the completion of $\mathcal{O}_K$ at $\nu \in \mathfrak{P}$, and let K_ν be the corresponding local field. Consider the adele space $\mathfrak{A}$, obtained as a subset of the product $V \times \prod_{\nu\in\mathfrak{P}} K_\nu$, where all components, except finitely many, belong to $\mathcal{O}_\nu$, and let $\mathfrak{A}_1$ be the image of projection of $\mathfrak{A}$ to $V_1 \times \prod_{\nu\in\mathfrak{P}} K_\nu$. Denote by $\tau: K \longrightarrow \mathfrak{A}_1$ the natural homomorphism, which is tautological componentwise.

From the Strong Approximation theorem (see [K] or [NT, Theorem 20.4.4][2]) it follows that the image $\tau(K)$ of K is dense in $\mathfrak{A}_1$. Let

$$\mathcal{O}_{\mathfrak{A}_1} := \mathfrak{A}_1 \cap \left(V_1 \times \prod_{\nu\in\mathfrak{P}} \mathcal{O}_\nu \right)$$

be the set of points of $\mathfrak{A}$, corresponding to the integer adeles. Clearly, $\mathcal{O}_{\mathfrak{A}_1}$ is open in $\mathfrak{A}_1$. Therefore, the intersection $\tau(K) \cap \mathcal{O}_{\mathfrak{A}_1}$ is dense in $\mathcal{O}_{\mathfrak{A}_1}$. On the other hand, $\tau(K) \cap \mathcal{O}_{\mathfrak{A}_1}$ consists of those elements of the number field which are integer at all non-archimedean places. This gives $\tau(K) \cap \mathcal{O}_{\mathfrak{A}_1} = \tau(\mathcal{O}_K)$. Therefore, the image of $\mathcal{O}_K$ to V_1 is dense.

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Remark 3.5: The above argument actually proves that the image of $\mathcal{O}_K$ in the product V_1 of all archimedean completions of K except one is dense in V_1.

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