Notes on algebras and vector spaces of functions

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Abstract

These informal notes are concerned with spaces of functions in various situations, including continuous functions on topological spaces, holomorphic functions of one or more complex variables, and so on.

Contents

Part I
Elements of functional analysis

1 Norms and seminorms

Let V be a vector space over the real numbers $\mathbf{R}$ or complex numbers $\mathbf{C}$. A nonnegative real-valued function $N(v)$ on V is said to be a *seminorm* on V if

$$N(t\,v) = |t|\,N(v) \tag{1.1}$$

for every $v \in V$ and $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, and

$$N(v+w) \le N(v) + N(w) \tag{1.2}$$

for every $v, w \in V$. Here $|t|$ denotes the absolute value of t when t is a real number, and the usual modulus of t when t is a complex number. A seminorm $N(v)$ on V is said to be a *norm* if $N(v) > 0$ for every $v \in V$. Of course, the absolute value defines a norm on $\mathbf{R}$, and the modulus defines a norm on $\mathbf{C}$.

As a basic class of examples, let E be a nonempty set, and let V be the vector space of real or complex-valued functions on E, with respect to pointwise addition and scalar multiplication. If $x \in E$ and $f \in V$, then

$$N_x(f) = |f(x)| \tag{1.3}$$

defines a seminorm on V. Let $\ell^\infty(E)$ be the linear subspace of V consisting of bounded functions on E, which may be denoted $\ell^\infty(E, \mathbf{R})$ or $\ell^\infty(E, \mathbf{C})$ to indicate whether the functions are real or complex-valued. It is easy to see that

$$\|f\|_\infty = \sup_{x \in E} |f(x)| \tag{1.4}$$

defines a norm on $\ell^\infty(E)$.

2 Norms and metrics

Let V be a vector space over the real or complex numbers, and let $\|v\|$ be a norm on V. It is easy to see that

$$d(v, w) = \|v - w\| \tag{2.1}$$

defines a metric on V, using the corresponding properties of a norm. More precisely, $d(v, w)$ is a nonnegative real-valued function defined for $v, w \in V$ which is equal to 0 if and only if $v = w$, $d(v, w)$ is symmetric in v and w, and

$$d(v, z) \le d(v, w) + d(w, z) \tag{2.2}$$

for every $v, w, z \in V$. Thus open and closed subsets of V, convergence of sequences, and so on may be defined as in the context of metric spaces.

Moreover, one can check that the topology on V determined by the metric associated to the norm is compatible with the algebraic structure corresponding to the vector space operations. This means that addition of vectors is continuous as a mapping from the Cartesian product of V with itself into V, and that scalar multiplication is continuous as a mapping from the Cartesian product of $\mathbf{R}$ or $\mathbf{C}$ with V into V. This can also be described in terms of the convergence of a sum of two convergent sequences in V, and the convergence of a product of a convergent sequence in $\mathbf{R}$ or $\mathbf{C}$ with a convergent sequence in V.

3 Seminorms and topologies

Let V be a real or complex vector space, and let $\mathcal{N}$ be a collection of seminorms on V. A set $U \subseteq V$ is said to be open with respect to $\mathcal{N}$ if for each $u \in U$ there are finitely many seminorms $N_1, \ldots, N_l \in \mathcal{N}$ and positive real numbers $r_1, \ldots, r_l$ such that

$$\{v \in V : N_j(u - v) < r_j,\ j = 1, \ldots, l\} \subseteq U. \tag{3.1}$$

It is easy to see that this defines a topology on V. If $u \in V$, $N \in \mathcal{N}$, and $r > 0$, then one can check that the corresponding ball

$$\{v \in V : N(u - v) < r\} \tag{3.2}$$

is an open set in V, using the triangle inequality. By construction, the collection of these open balls is a subbase for the topology on V associated to $\mathcal{N}$.

Let us say that $\mathcal{N}$ is *nice* if for every $v \in V$ with $v \neq 0$ there is an $N \in \mathcal{N}$ such that $N(v) > 0$. This is equivalent to the condition that $\{0\}$ be a closed set in V with respect to the topology associated to $\mathcal{N}$, which is to say that $V \backslash \{0\}$ is an open set in this topology. If $\mathcal{N}$ is nice, then the topology on V associated to $\mathcal{N}$ is Hausdorff. If $\|v\|$ is a norm on V, then the collection of seminorms on V consisting only of $\|v\|$ is nice, and the corresponding topology on V is the same as the one determined by the metric associated to $\|v\|$, as in the previous section.

If $\mathcal{N}$ is any collection of seminorms on V, then addition of vectors defines a continuous mapping from $V \times V$ into V, and scalar multiplication defines a continuous mapping from $\mathbf{R} \times V$ or $\mathbf{C} \times V$, as appropriate, into V. Thus V is a *topological vector space*, at least when $\mathcal{N}$ is nice, since it is customary to ask that $\{0\}$ be a closed set in a topological vector space. In particular, a vector space with a norm is a topological vector space, with respect to the topology determined by the metric associated to the norm, as in the previous section. If V is the space of real or complex-valued functions on a nonempty set E, and if $\mathcal{N}$ is the collection of seminorms of the form $N_x(f) = |f(x)|$, $x \in E$, as in Section 1, then $\mathcal{N}$ is a nice collection of seminorms on V. In this case, V can be identified with a Cartesian product of copies of $\mathbf{R}$ or $\mathbf{C}$, indexed by E, and the topology on V associated to $\mathcal{N}$ is the same as the product topology.

4 Convergent sequences

Remember that a sequence of elements $\{x_j\}_{j=1}^\infty$ of a topological space X is said to converge to an element x of X if for every open set U in X with $x \in U$ there is an $L \geq 1$ such that

$$x_j \in U \tag{4.1}$$

for each $j \geq L$. If the topology on X is determined by a metric $d(x, y)$, then this is equivalent to the condition that

$$\lim_{j \to \infty} d(x_j, x) = 0. \tag{4.2}$$

Similarly, if V is a real or complex vector space with a norm $\|\cdot\|$, and if $\{v_j\}_{j=1}^\infty$ is a sequence of elements of V, then $\{v_j\}_{j=1}^\infty$ converges to another element v of V when

$$\lim_{j \to \infty} \|v_j - v\| = 0. \tag{4.3}$$

If instead the topology on V is determined by a collection $\mathcal{N}$ of seminorms on V, then $\{v_j\}_{j=1}^\infty$ converges to v when

$$\lim_{j \to \infty} N(v_j - v) = 0 \tag{4.4}$$

for every $N \in \mathcal{N}$. In these last two cases, $\{v_j\}_{j=1}^\infty$ converges to v if and only if $\{v_j - v\}_{j=1}^\infty$ converges to 0.

A topological space X has a countable local base for the topology at $x \in X$ if there is a sequence $U_1(x), U_2(x), \ldots$ of open subsets of X such that $x \in U_l(x)$ for each l, and for each open set $U \subseteq X$ with $x \in U$ there is an $l \geq 1$ such that $U_l(x) \subseteq U$. In this case, one can also ask that $U_{l+1}(x) \subseteq U_l(x)$ for each l, by replacing $U_l(x)$ with the intersection of $U_1(x), \ldots, U_l(x)$ if necessary. Under this condition, if x is in the closure of a set $E \subseteq X$, then there is a sequence of elements of E that converges to x. Otherwise, one may have to use nets or filters instead of sequences. Of course, the limit of a convergent sequence of elements of a set $E \subseteq X$ is in the closure of E in any topological space X.

If X has a countable local base for the topology at each point, then the closed subsets of X can be characterized in terms of convergent sequences, as in the previous paragraph. Equivalently, the topology on X is determined by convergence of sequences. If the topology on X is defined by a metric, then X automatically satisfies this condition, with $U_l(x)$ equal to the open ball centered at x with radius $1/l$. In particular, this applies to a real or complex vector space V with a norm.

Suppose that the topology on V is given by a nice collection $\mathcal{N}$ of seminorms. If $\mathcal{N}$ consists of only finitely many seminorms $N_1, \ldots, N_l$, then

$$\|v\| = \max_{1 \le j \le l} N_j(v) \tag{4.5}$$

is a norm on V, and the topology on V associated to $\mathcal{N}$ is the same as the one associated to $\|v\|$. If $\mathcal{N}$ consists of an infinite sequence $N_1, N_2, \ldots$ of seminorms and $v \in V$, then

$$U_l(v) = \{w \in V : N_1(v - w), \ldots, N_l(v - w) \le 1/l\} \tag{4.6}$$

is a countable local base for the topology of V at v. Conversely, suppose that $U_1, U_2, \ldots$ is a sequence of open subsets of V such that $0 \in U_l$ for each l, and for each open set U in V with $0 \in U$ there is an $l \ge 1$ such that $U_l \subseteq U$. By the definition of the topology on V associated to $\mathcal{N}$, for each $l \ge 1$ there are finitely many seminorms $N_{l,1}, \ldots, N_{l,n_l} \in \mathcal{N}$ and positive real numbers $r_{l,1}, \ldots, r_{l,n_l}$ such that

$$\{v \in V : N_{l,j}(v) < r_{l,j}, \ j = 1, \ldots, n_l\} \subseteq U_l. \tag{4.7}$$

If $\mathcal{N}'$ is the collection of seminorms of the form $N_{l,j}$, $1 \le j \le n_l$, $l \ge 1$, then $\mathcal{N}'$ is a subset of $\mathcal{N}$ with only finitely or countably many elements. One can also check that the topology on V determined by $\mathcal{N}'$ is the same as the topology on V determined by $\mathcal{N}$.

5 Metrizability

Let X be a set, and let $\rho(x, y)$ be a nonnegative real-valued function defined for $x, y \in X$. We say that $\rho(x, y)$ is a *semimetric* on X if it satisfies the same conditions as a metric, except that $\rho(x, y)$ may be equal to 0 even when $x \ne y$. Thus $\rho(x, y)$ is a semimetric if $\rho(x, x) = 0$ for each $x \in X$,

$$\rho(x, y) = \rho(y, x) \tag{5.1}$$

for every $x, y \in X$, and

$$\rho(x, z) \le \rho(x, y) + \rho(y, z) \tag{5.2}$$

for every $x, y, z \in X$. If N is a seminorm on a real or complex vector space V, then

$$\rho(v, w) = N(v - w) \tag{5.3}$$

defines a semimetric on V.

If $\rho(x, y)$ is a semimetric on a set X and t is a positive real number, then

$$\rho_t(x, y) = \min(\rho(x, y), t) \tag{5.4}$$

is also a semimetric on X. The main point is that $\rho_t(x, y)$ also satisfies the triangle inequality, since $\rho(x, y)$ does. If $\rho(x, y)$ is a metric on X, then $\rho_t(x, y)$ is too, and they determine the same topology on X.

Let V be a real or complex vector space, and let $\mathcal{N}$ be a nice collection of seminorms on V. If $\mathcal{N}$ consists of only finitely many seminorms, then their maximum is a norm on V which determines the same topology on V as $\mathcal{N}$, as in the preceding section. If $\mathcal{N}$ consists of an infinite sequence of seminorms $N_1, N_2, \ldots$, then

$$d(v, w) = \max_{l \ge 1} \min(N_l(v - w), 1/l) \tag{5.5}$$

defines a metric on V that determines the same topology on V as $\mathcal{N}$. More precisely, if $v = w$, then $N_l(v - w) = 0$ for each l, and so $d(v, w) = 0$. If $v \neq w$, then $N_j(v - w) > 0$ for some j, because $\mathcal{N}$ is nice, and

$$\min(N_l(v - w), 1/l) \le 1/l < N_j(v - w) \tag{5.6}$$

for all but finitely many l, so that the maximum in the definition of $d(v, w)$ always exists. This also shows that $d(v, w) > 0$ when $v \neq w$, and $d(v, w)$ is obviously symmetric in v and w. It is not difficult to check that $d(v, w)$ satisfies the triangle inequality, using the fact that

$$\min(N_l(v - w), 1/l) \tag{5.7}$$

satisfies the triangle inequality for each l, as in the previous paragraphs. If r is a positive real number, then $d(v, w) < r$ if and only if $N_l(v - w) < r$ when $l \le 1/r$, and one can use this to show that $d(v, w)$ determines the same topology on V as $\mathcal{N}$.

Suppose now that $\mathcal{N}$ is a nice collection of seminorms on V, and that there is a countable local base for the topology on V associated to $\mathcal{N}$ at 0. This implies that there is a subset $\mathcal{N}'$ of $\mathcal{N}$ with only finitely or countably many elements that determines the same topology on V, as in the preceding section. It follows that there is a metric on V that determines the same topology on V, as in the previous paragraph. Note that this metric is invariant under translations on V, since it depends only on $v - w$.

6 Comparing topologies

Let V be a real or complex vector space, and let $\mathcal{N}$, $\mathcal{N}'$ be collections of seminorms on V. Suppose that every open set in V with respect to $\mathcal{N}'$ is also an open set with respect to $\mathcal{N}$. If $N' \in \mathcal{N}'$, then it follows that the open unit ball with respect to N' is an open set with respect to $\mathcal{N}$. This implies that there are finitely many seminorms $N_1, \ldots, N_l \in \mathcal{N}$ and positive real numbers $r_1, \ldots, r_l$ such that

$$(6.1) \qquad \{v \in V : N_j(v) < r_j,\ j = 1, \ldots, l\} \subseteq \{v \in V : N'(v) < 1\},$$

since 0 is an element of the open unit ball corresponding to N'. Equivalently,

$$(6.2) \qquad N'(v) < 1 \quad \text{when} \quad \max_{1 \le j \le l} r_j^{-1}\, N_j(v) < 1,$$

and so

$$(6.3) \qquad N'(v) \le \max_{1 \le j \le l} r_j^{-1}\, N_j(v)$$

for every $v \in V$. This implies in turn that

$$(6.4) \qquad N'(v) \le C \max_{1 \le j \le l} N_j(v)$$

for every $v \in V$, where C is the maximum of $r_1^{-1}, \ldots, r_l^{-1}$. Conversely, if for every $N' \in \mathcal{N}'$ there are finitely many seminorms $N_1, \ldots, N_l \in \mathcal{N}$ such that (6.4) holds for some $C \ge 0$, then every open set with respect to $\mathcal{N}'$ is also open with respect to $\mathcal{N}$. Of course, one can interchange the roles of $\mathcal{N}$ and $\mathcal{N}'$, so that they determine the same topology on V if and only if $\mathcal{N}$ and $\mathcal{N}'$ both satisfy this condition relative to the other.

Let us apply this to the case where $\mathcal{N}'$ consists of a single norm $\|v\|$. If every open set in V with respect to this norm is also an open set with respect to $\mathcal{N}$, then there are finitely many seminorms $N_1, \ldots, N_l \in \mathcal{N}$ such that

$$(6.5) \qquad \|v\| \le C \max_{1 \le j \le l} N_j(v)$$

for some $C > 0$ and every $v \in V$. In particular,

$$(6.6) \qquad \|v\|' = \max_{1 \le j \le l} N_j(v)$$

is also a norm on V in this case. Similarly, if every open set in V with respect to $\mathcal{N}$ is also an open set with respect to $\|v\|$, then for each $N \in \mathcal{N}$ there is a $C(N) \ge 0$ such that

$$(6.7) \qquad N(v) \le C(N)\, \|v\|$$

for every $v \in V$. If the topologies on V associated to $\mathcal{N}$ and $\|v\|$ are the same, then $\|v\|'$ also determines the same topology on V.

As a basic class of examples, let V be the vector space of real or complex-valued functions on a nonempty set E, and let $\mathcal{N}$ be the collection of seminorms on V of the form $N_x(f) = |f(x)|$, $x \in E$. If there is a norm $\|v\|$ on V such that the open unit ball in V with respect to $\|v\|$ is an open set with respect to $\mathcal{N}$, then it follows that the maximum of finitely many elements of $\mathcal{N}$ is a norm on V, as in the previous paragraphs. This implies that E has only finitely many elements. Conversely, if E has only finitely many elements, then the maximum of $N_x(f)$, $x \in E$, is a norm on V that determines the same topology. Note that the topology on V is metrizable if and only if E has only finitely or countably many elements, as in the preceding section.

Now let E be the set $\mathbf{Z}_+$ of positive integers, and let V be the vector space of real or complex-valued functions on $\mathbf{Z}_+$ that are rapidly decreasing in the sense that $f(j)$ is bounded by a constant multiple of j^{-k} for each nonnegative integer k. Put

$$N_k(f) = \sup_{j \ge 1} j^k \, |f(j)| \tag{6.8}$$

for each $k \ge 0$, which is a norm on V that reduces to the ℓ^∞ norm when $k = 0$ and is monotone increasing in k. It is easy to see that the topology on V associated to this collection of norms is not determined by finitely many of these norms. Hence the topology on V associated to this collection of norms is not determined by any single norm at all. However, this topology is metrizable, as in the preceding section.

7 Continuous linear functionals

Let V be a real or complex vector space with a nice collection of seminorms $\mathcal{N}$. As usual, a linear functional on V is a linear mapping from V into the real or complex numbers, as appropriate. Let V^* be the space of linear functionals on V that are continuous with respect to the topology on V determined by $\mathcal{N}$. This may be described as the topological dual of V, to distinguish it from the algebraic dual of all linear functionals on V. These dual spaces are also vector spaces over the real or complex numbers, as appropriate, using pointwise addition and scalar multiplication of functions.

If $\lambda \in V^*$, then the set of $v \in V$ such that $|\lambda(v)| < 1$ is open, because λ is continuous. Of course, 0 is an element of this set, because $\lambda(0) = 0$. It follows that there are finitely many seminorms $N_1, \ldots, N_l \in \mathcal{N}$ and positive real numbers $r_1, \ldots, r_l$ such that

$$\{v \in V : N_j(v) < r_j, \ j = 1, \ldots, l\} \subseteq \{v \in V : |\lambda(v)| < 1\}. \tag{7.1}$$

As in the previous section, this implies that

$$|\lambda(v)| \le \max_{1 \le j \le l} r_j^{-1} \, N_j(v) \tag{7.2}$$

for every $v \in V$. In particular, if C is the maximum of $r_1^{-1}, \ldots, r_l^{-1}$, then

$$|\lambda(v)| \le C \max_{1 \le j \le l} N_j(v) \tag{7.3}$$

for every $v \in V$.

Conversely, suppose that λ is a linear functional on V for which there are finitely many seminorms $N_1, \ldots, N_l \in \mathcal{N}$ and a nonnegative real number C such that (7.3) holds. In this case,

$$|\lambda(v) - \lambda(w)| = |\lambda(v - w)| \le C \max_{1 \le j \le l} N_j(v - w) \tag{7.4}$$

for every $v, w \in V$, because λ is linear. It is easy to see that λ is continuous on V with respect to the topology associated to $\mathcal{N}$ under these conditions. More precisely, for each $v \in V$ and $\epsilon > 0$, we have that

$$|\lambda(v) - \lambda(w)| < \epsilon \tag{7.5}$$

for every $w \in V$ such that $N_j(v - w) < C^{-1}\,\epsilon$ for $j = 1, \ldots, l$. Remember that open balls defined in terms of seminorms in $\mathcal{N}$ are automatically open sets with respect to $\mathcal{N}$, as in Section 3.

If the topology on V is determined by a single norm $\|v\|$, then the previous discussion can be simplified. If λ is a continuous linear functional on V, then $|\lambda(v)| < 1$ on an open ball around 0 in V. As before, this implies that there is a nonnegative real number C such that

$$|\lambda(v)| \le C\,\|v\| \tag{7.6}$$

for every $v \in V$. Conversely, if λ is a linear functional on V that satisfies (7.6) for some $C \ge 0$, then

$$|\lambda(v) - \lambda(w)| = |\lambda(v - w)| \le C\,\|v - w\| \tag{7.7}$$

for every $v, w \in V$, because of linearity. This clearly implies that λ is continuous with respect to the metric $d(v, w) = \|v - w\|$ associated to V, as in Section 2.

8 $\mathbf{R}^n$ and $\mathbf{C}^n$

Let n be a positive integer, and let $\mathbf{R}^n$, $\mathbf{C}^n$ be the space of n-tuples of real and complex numbers, respectively. As usual, these are vector spaces with respect to coordinatewise addition and scalar multiplication. Put

$$\|v\|_\infty = \max_{1 \le j \le n} |v_j| \tag{8.1}$$

for each $v = (v_1, \ldots, v_n) \in \mathbf{R}^n$ or $\mathbf{C}^n$. It is easy to see that this defines a norm on $\mathbf{R}^n$, $\mathbf{C}^n$, for which the corresponding topology is the standard topology. The latter is the same as the product topology on $\mathbf{R}^n$, $\mathbf{C}^n$ as the Cartesian product of n copies of $\mathbf{R}$, $\mathbf{C}$, with their standard topologies.

Another simple norm on $\mathbf{R}^n$, $\mathbf{C}^n$ is given by

$$\|v\|_1 = \sum_{j=1}^n |v_j|. \tag{8.2}$$

Note that

$$\|v\|_\infty \le \|v\|_1 \tag{8.3}$$

for every $v \in \mathbf{R}^n$ or $\mathbf{C}^n$. Similarly,

$$\|v\|_1 \le n\,\|v\|_\infty \tag{8.4}$$

for each $v \in \mathbf{R}^n$, $\mathbf{C}^n$. It follows that $\|v\|_1$ also determines the standard topology on $\mathbf{R}^n$, $\mathbf{C}^n$.

If $a_1, \ldots, a_n$ are real or complex numbers, then

$$\lambda(v) = \sum_{j=1}^n a_j\, v_j \tag{8.5}$$

defines a linear functional on $\mathbf{R}^n$ or $\mathbf{C}^n$, as appropriate. It is easy to see that λ is continuous with respect to the standard topology on $\mathbf{R}^n$ or $\mathbf{C}^n$. Of course, every linear functional on $\mathbf{R}^n$, $\mathbf{C}^n$ is of this form. More precisely, if λ is any linear functional on $\mathbf{R}^n$ or $\mathbf{C}^n$, then λ can be expressed as in (8.5), with

$$a_j = \lambda(e_j) \tag{8.6}$$

for each j, where $e_1, \ldots, e_n$ are the standard basis vectors in $\mathbf{R}^n$, $\mathbf{C}^n$. These are defined by taking the lth component of e_j equal to 1 when $j = l$ and 0 otherwise, so that

$$v = \sum_{j=1}^n v_j\, e_j \tag{8.7}$$

for each $v \in \mathbf{R}^n$ or $\mathbf{C}^n$.

If N is any seminorm on $\mathbf{R}^n$ or $\mathbf{C}^n$, then

$$N(v) = N\Big(\sum_{j=1}^n v_j\, e_j\Big) \le \sum_{j=1}^n N(e_j)\, |v_j|. \tag{8.8}$$

This implies that

$$N(v) \le \Big(\sum_{j=1}^n N(e_j)\Big)\, \|v\|_\infty \tag{8.9}$$

and

$$N(v) \le \Big(\max_{1 \le j \le n} N(e_j)\Big)\, \|v\|_1 \tag{8.10}$$

for every $v \in \mathbf{R}^n$ or $\mathbf{C}^n$. Thus N is automatically bounded by constant multiples of the basic norms $\|v\|_\infty$, $\|v\|_1$.

Using the triangle inequality, we get that

$$N(v) - N(w) \le N(v - w) \tag{8.11}$$

and

$$N(w) - N(v) \le N(v - w) \tag{8.12}$$

for every $v, w \in \mathbf{R}^n$ or $\mathbf{C}^n$, as appropriate. It follows that

$$|N(v) - N(w)| \le N(v - w) \tag{8.13}$$

for every v, w. Combining this with the estimates in the previous paragraph, we get that N is continuous as a real-valued function on $\mathbf{R}^n$ or $\mathbf{C}^n$.

Suppose now that N is a norm on $\mathbf{R}^n$ or $\mathbf{C}^n$. The set of $v \in \mathbf{R}^n$ or $\mathbf{C}^n$ with $\|v\|_\infty = 1$ is closed and bounded, and hence compact, with respect to the standard topology. Because N is continuous, it attains its minimum on this set, which is therefore positive. Hence there is a positive real number c such that

$$N(v) \ge c \tag{8.14}$$

when $\|v\|_\infty = 1$, which implies that

$$N(v) \ge c\,\|v\|_\infty \tag{8.15}$$

for every $v \in \mathbf{R}^n$ or $\mathbf{C}^n$, as appropriate, by homogeneity. We already know from (8.9) that $N(v)$ is bounded from above by a constant multiple of $\|v\|_\infty$, and we may now conclude that the topology on $\mathbf{R}^n$ or $\mathbf{C}^n$ determined by N is the same as the standard topology.

Let $\mathcal{N}$ be any nice collection of seminorms on $\mathbf{R}^n$ or $\mathbf{C}^n$, and let us check that the topology on $\mathbf{R}^n$ or $\mathbf{C}^n$ associated to $\mathcal{N}$ is the same as the standard topology. Let N_1 be an element of $\mathcal{N}$ that is not identically zero. If N_1 is a norm, then we stop, and otherwise we choose $N_2 \in \mathcal{N}$ such that $N_2(v) > 0$ for some $v \in \mathbf{R}^n$ or $\mathbf{C}^n$ with $v \ne 0$ and $N_1(v) = 0$. Note that the set of $v \in \mathbf{R}^n$ or $\mathbf{C}^n$ such that $N_1(v) = 0$ is a proper linear subspace of $\mathbf{R}^n$ or $\mathbf{C}^n$. If this linear subspace contains a nonzero element, then the set of $v \in \mathbf{R}^n$ or $\mathbf{C}^n$ such that $N_1(v) = N_2(v) = 0$ is a proper linear subspace of it. By repeating the process, we get finitely many seminorms $N_1, \ldots, N_l \in \mathcal{N}$ with $l \le n$ whose maximum defines a norm on $\mathbf{R}^n$ or $\mathbf{C}^n$, as appropriate. The topology on $\mathbf{R}^n$ or $\mathbf{C}^n$ associated to this norm is the same as the standard topology, as before. It follows that the topology on $\mathbf{R}^n$ or $\mathbf{C}^n$ associated to $\mathcal{N}$ is the same as the standard topology, since every seminorm on $\mathbf{R}^n$, $\mathbf{C}^n$ is bounded by a constant multiple of the usual norms $\|v\|_\infty$,$\|v\|_1$.

9 Weak topologies

Let V be a real or complex vector space. If λ is any linear functional on V, then

$$N_\lambda(v) = |\lambda(v)| \tag{9.1}$$

defines a seminorm on V. Let Λ be a collection of linear functionals on V, and let $\mathcal{N}(\Lambda)$ be the corresponding collection of seminorms N_λ, $\lambda \in \Lambda$. If Λ is nice in the sense that for each $v \in V$ with $v \ne 0$ there is a $\lambda \in \Lambda$ such that $\lambda(v) \ne 0$, then $\mathcal{N}(\Lambda)$ is a nice collection of seminorms on V. This leads to a topology on V, as in Section 3, which is the weak topology associated to Λ.

Under these conditions, each element of Λ is a continuous linear functional on V with respect to the weak topology associated to Λ. This implies that any finite linear combination of elements of Λ is also continuous with respect to this topology. Conversely, if λ is a continuous linear functional on V with respect to the weak topology associated to Λ, then there are finitely many elements

$\lambda_1, \dots, \lambda_n$ of Λ and a nonnegative real number C such that

$$|\lambda(v)| \le C \max_{1 \le j \le n} |\lambda_j(v)| \tag{9.2}$$

for every $v \in V$. In particular, $\lambda(v) = 0$ when $\lambda_j(v) = 0$ for $j = 1, \dots, n$, and an elementary argument in linear algebra shows that λ can be expressed as a linear combination of the λ_j's. One may wish to reduce first to the case where the λ_j's are linearly independent, by discarding any that are linear combinations of the rest.

Let E be a nonempty set, and let V be the vector space of real or complex-valued functions on E. Note that $\lambda_x(f) = f(x)$ is linear functional on V for each $x \in E$. This defines a nice collection of linear functionals on V, for which the corresponding collection of seminorms has been mentioned previously. It follows from the discussion in the previous paragraph that a linear functional λ on V is continuous with respect to the topology associated to this collection of seminorms if and only if it is a finite linear combination of λ_x's, $x \in E$.

Let V be any real or complex vector space, and let $\mathcal{N}$ be a nice collection of seminorms on V. This leads to the corresponding dual space V^* of continuous linear functionals on V. If $v \in V$ and $v \neq 0$, then there is a $\lambda \in V^*$ such that $\lambda(v) \neq 0$. This follows from the Hahn–Banach theorem, as in the next section. Thus V^* is itself a nice collection of linear functionals on V, which determines a weak topology on V as before, also known as the weak topology associated to $\mathcal{N}$. Note that every open set in V with respect to this weak topology is also an open set with respect to the topology associated to $\mathcal{N}$, because the elements of V^* are continuous with respect to the topology associated to $\mathcal{N}$. Every element of V^* is automatically continuous with respect to the weak topology on V, and conversely every continuous linear functional on V with respect to the weak topology is continuous with respect to the topology associated to $\mathcal{N}$. Hence V^* is also the space of continuous linear functionals on V with respect to the weak topology, which follows from the earlier discussion for the weak topology associated to any collection of linear functionals on V as well.

10 The Hahn–Banach theorem

Let V be a real or complex vector space, and let N be a seminorm on V. Also let λ be a linear functional on a linear subspace W of V such that

$$|\lambda(v)| \le C\, N(v) \tag{10.1}$$

for some $C \ge 0$ and every $v \in W$. The Hahn–Banach theorem states that there is an extension of λ to a linear functional on V that satisfies (10.1) for every $v \in V$, with the same constant C. We shall not go through the proof here, but we would like to mention some aspects of it, and some important consequences.

Sometimes the Hahn–Banach theorem is stated only in the case where N is a norm on V. This does not really matter, because essentially the same proof works for seminorms. Alternatively, if N is a seminorm on V, then

$$Z = \{v \in V : N(v) = 0\} \tag{10.2}$$

is a linear subspace of V. One can begin by extending λ to the linear span of W and Z by setting

$$\lambda(w + z) = \lambda(w) \tag{10.3}$$

for every $w \in W$ and $z \in Z$, which makes sense because $\lambda(v) = 0$ when v is in $W \cap Z$, by (10.1). One can then reduce to the case of norms by passing to the quotient of V by Z.

To prove the Hahn–Banach theorem in the real case, one first shows that λ can be extended to the linear span of W and any element of V, while maintaining (10.1). If W has finite codimension in V, then one can apply this repeatedly to extend λ to V. If N is a norm on V and V has a countable dense set, then one can apply this repeatedly to extend λ to a dense linear subspace of V, and then extend λ to all of V using continuity. Otherwise, the extension of λ to V is obtained using the axiom of choice, through Zorn's lemma or the Hausdorff maximality principle. The complex case can be reduced to the real case, by treating the real part of λ as a linear functional on W as a real vector space, and then complexifying the extension to V afterwards.

As an application, let $\mathcal{N}$ be a nice collection of seminorms on V, let $u \in V$ with $u \neq 0$ be given, and choose $N \in \mathcal{N}$ such that $N(u) > 0$. We can define λ on the one-dimensional subspace W of V spanned by u by

$$\lambda(t\,u) = t\,N(u) \tag{10.4}$$

for each $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate. This satisfies (10.1) with $C = 1$, and the Hahn–Banach theorem implies that there is an extension of λ to V that also satisfies (10.1) with $C = 1$. In particular, this extension is a continuous linear functional on V with respect to the topology associated to $\mathcal{N}$ such that $\lambda(u) \neq 0$, as in the previous section. One can also use the Hahn–Banach theorem to show that a closed linear subspace of V with respect to the topology associated to $\mathcal{N}$ is also closed with respect to the weak topology.

11 Dual norms

Let V be a real or complex vector space with a norm $\|v\|$. Remember that a linear functional λ on V is continuous with respect to the topology associated to $\|v\|$ if and only if there is a nonnegative real number C such that

$$|\lambda(v)| \le C\,\|v\| \tag{11.1}$$

for every $v \in V$. In this case, the dual norm $\|\lambda\|_*$ of λ is defined by

$$\|\lambda\|_* = \sup\{|\lambda(v)| : v \in V,\ \|v\| \le 1\}. \tag{11.2}$$

This is the same as the smallest value of C for which the previous inequality holds. It is not difficult to check that $\|\lambda\|_*$ defines a norm on the dual space V^* of continuous linear functionals on V.

If $v \in V$ and $v \neq 0$, then there is a $\lambda \in V^*$ such that $\|\lambda\|_* = 1$ and

$$\lambda(v) = \|v\|. \tag{11.3}$$

This uses the Hahn–Banach theorem, as in the previous section. More precisely, the argument in the previous section shows that $\|\lambda\|_* \le 1$, and equality holds because of the value of $\lambda(v)$.

Suppose that $V = \mathbf{R}^n$ or $\mathbf{C}^n$ for some positive integer n. If $a \in V$, then

$$\lambda_a(v) = \sum_{j=1}^n a_j\, v_j \tag{11.4}$$

defines a linear functional on V, and every linear functional on V is of this form. Note that

$$|\lambda_a(v)| \le \Big(\sum_{j=1}^n |a_j|\Big) \max_{1 \le j \le n} |v_j| = \|a\|_1\, \|v\|_\infty \tag{11.5}$$

for every $a, v \in V$, where $\|a\|_1$, $\|v\|_\infty$ are as in Section 8. This shows that the dual norm of λ_a on V with respect to $\|v\|_\infty$ is less than or equal to $\|a\|_1$. If one chooses $v \in V$ such that $\|v\|_\infty = 1$ and $a_j\, v_j = |a_j|$ for each j, then one gets that

$$|\lambda_a(v)| = \|a\|_1, \tag{11.6}$$

and hence the dual norm of λ_a with respect to $\|v\|_\infty$ is equal to $\|a\|_1$. Similarly,

$$|\lambda_a(v)| \le \|a\|_\infty\, \|v\|_1 \tag{11.7}$$

for every $a, v \in V$. This shows that the dual norm of λ_a on V with respect to $\|v\|_1$ is less than or equal to $\|a\|_\infty$, and one can check that the dual norm is equal to $\|a\|_\infty$ using standard basis vectors for v to get equality in the previous inequality.

12 Topological vector spaces

A *topological vector space* is basically a vector space with a topology that is compatible with the vector space operations. More precisely, let V be a vector space over the real or complex numbers, and suppose that V is also equipped with a topological structure. In order for V to be a topological vector space, the vector space operations of addition and scalar multiplication ought to be continuous. Addition of vectors corresponds to a mapping from the Cartesian product $V \times V$ of V with itself into V, and continuity of addition means that this mapping should be continuous, where $V \times V$ is equipped with the product topology associated to the given topology on V. Similarly, scalar multiplication corresponds to a mapping from $\mathbf{R} \times V$ or $\mathbf{C} \times V$ into V, depending on whether

V is a real or complex vector space. Continuity of scalar multiplication means that this mapping is continuous when $\mathbf{R} \times V$ or $\mathbf{C} \times V$ is equipped with the product topology associated to the standard topology on $\mathbf{R}$ or $\mathbf{C}$ and the given topology on V. It is customary to ask that topological vector spaces also satisfy a separation condition, which will be mentioned in a moment.

Note that continuity of addition implies that the translation mapping

$$\tau_a(v) = a + v \tag{12.1}$$

is continuous as a mapping from V into itself for every $a \in V$. This implies that τ_a is actually a homeomorphism from V onto itself for each $a \in V$, since τ_a is a one-to-one mapping from V onto itself whose inverse is τ_{-a}, which is also continuous for the same reason. In the same way, the dilation mapping

$$\delta_t(v) = t \cdot v \tag{12.2}$$

is a continuous mapping on V for every $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, because of continuity of scalar multiplication. If $t \neq 0$, then δ_t is a one-to-one mapping from V onto itself, with inverse equal to $\delta_{1/t}$, and hence a homeomorphism.

The additional separation condition for V to be a topological vector space is that the set $\{0\}$ consisting of the additive identity element 0 in V be a closed set in V. This implies that every subset of V with exactly one element is closed, because of the continuity of the translation mappings. One can also use continuity of addition at 0 to show that V is Hausdorff under these conditions.

It is easy to see that $\mathbf{R}^n$ and $\mathbf{C}^n$ are topological vector spaces with respect to their standard topologies. If a real or complex vector space V is equipped with a norm N, then V is a topological vector space with respect to the topology determined by the metric determined by N as in Section 2. If instead V is equipped with a nice collection $\mathcal{N}$ of seminorms, then V is a topological vector space with respect to the topology defined in Section 3. In particular, the requirement that $\mathcal{N}$ be nice corresponds exactly to the separation condition discussed in the previous paragraph.

In linear algebra, one is often interested in linear mappings between vector spaces. Similarly, in topology, one is often interested in continuous mappings between topological spaces. In the context of topological vector spaces, one is often interested in continuous linear mappings between topological vector spaces. This includes continuous linear functionals from a topological vector space into the real or complex numbers, as appropriate. Thus the topological dual V^* of a topological vector space V may be defined as the space of continuous linear functionals on V, as in Section 7.

Let V and W be topological vector spaces, both real or both complex. If ϕ is a one-to-one linear mapping from V onto W, then the inverse mapping ϕ^{-1} is a one-to-one linear mapping from W onto V as well. If $\phi : V \to W$ and $\phi^{-1} : W \to V$ are also continuous, so that ϕ is a homeomorphism from V onto W, then ϕ is said to be an *isomorphism* between V and W as topological vector spaces. It can be shown that a finite-dimensional real or complex topological vector space of dimension n is isomorphic to $\mathbf{R}^n$ or $\mathbf{C}^n$, as appropriate, with its standard topology.

A topological vector space V is said to be *locally convex* if there is a local base for the topology of V at 0 consisting of convex open subsets of V. If the topology on V is determined by a nice collection of seminorms, then it is easy to see that V is locally convex. Conversely, if V is locally convex, then one can show that the topology on V may be described by a nice collection of seminorms.

In any topological space, a necessary condition for the existence of a metric that describes the same topology is that there be a countable local base for the topology at each point. If a topological vector space V has a counatble local base for the topology at 0, then it has a countable local base for the topology at every point, because the topology is invariant under translations. In this case, it can be shown that there is a metric on V that describes the same topology and which is invariant under translations. If V has a countable local base for the topology at 0 and the topology on V is determined by a nice collection of seminorms, then only finitely or countably many seminorms are necessary to describe the topology, as in Section 4, and one can get a translation-invariant metric as in Section 5.

The definition of a Cauchy sequence can be extended to topological vector spaces, as follows. A sequence $\{v_j\}_{j=1}^\infty$ of elements of a topological vector space V is said to be a *Cauchy sequence* if for every open set U in V with $0 \in U$ there is a positive integer L such that

$$v_j - v_l \in U \tag{12.3}$$

for every $j, l \geq L$. If $d(v, w)$ is a metric on V that determines the given topology on V, and if $d(v, w)$ is invariant under translations on V in the sense that

$$d(v - z, w - z) = d(v, w) \tag{12.4}$$

for every $v, w, z \in V$, then it is easy to see that the usual definition of a Cauchy sequence in V with respect to $d(v, w)$ is equivalent to the preceding condition using the topological vector space structure.

Remember that a metric space X is said to be *complete* if every Cauchy sequence of elements of X converges to another element of X. Similarly, let us say that a topological vector space V is *sequentially complete* if every Cauchy sequence of elements of V as in the previous paragraph converges to an element of V. If there is a countable local base for the topology of V at 0, then this is equivalent to completeness of V with respect to any translation-invariant metric that determines the same topology on V. Otherwise, one can also consider Cauchy conditions for nets or filters on V.

13 Summable functions

Let E be a nonempty set, and let $f(x)$ be a real or complex valued function on E. We say that f is *summable* on E if the sums

$$\sum_{x \in A} |f(x)| \tag{13.1}$$

over finite subsets A of E are uniformly bounded. Of course, this holds trivially when E has only finitely many elements, since we can take $A = E$. If E is the set $\mathbf{Z}_+$ of positive integers, then this is equivalent to saying that $\sum_{j=1}^{\infty} |f(j)|$ converges, which means that $\sum_{j=1}^{\infty} f(j)$ converges absolutely.

We would like to define the sum

$$\sum_{x \in E} f(x) \tag{13.2}$$

when f is a summable function on E. Again this is trivial when E has only finitely many elements. If $E = \mathbf{Z}_+$, then the sum may be considered as a convergent infinite series, since it converges absolutely. If E is a countably infinite set, then one can reduce to the case where $E = \mathbf{Z}_+$ using an enumeration of E. Different enumerations lead to the same value of the sum, because the sum of an absolutely convergent series is invariant under rearrangements. If f is a summable function on any infinite set E, then one can check that the set of $x \in E$ such that $|f(x)| \geq \epsilon$ has only finitely many elements for each $\epsilon > 0$. This implies that the set of $x \in E$ such that $f(x) \neq 0$ has only finitely or countably many elements, so that the definition of the sum can be reduced to the previous case.

Alternatively, if f is a nonnegative real-valued summable function on E, then one can define the sum over E to be the supremum of the subsums (13.1) over all finite subsets of E. If f is any summable function on E, then f can be expressed as a linear combination of nonnegative real-valued summable functions, so that the definition of the sum can be reduced to that case. It is easy to see that this approach is compatible with the one in the previous paragraph. The space of summable functions on E is denoted $\ell^1(E)$, or more precisely $\ell^1(E, \mathbf{R})$ or $\ell^1(E, \mathbf{C})$ to indicate whether real or complex-valued functions on E are being used. One can check that this is a vector space with respect to pointwise addition and scalar multiplication, and that the sum over E defines a linear functional on $\ell^1(E)$.

If $f \in \ell^1(E)$, then put

$$\|f\|_1 = \sum_{x \in E} |f(x)|. \tag{13.3}$$

One can check that this is a norm on $\ell^1(E)$, and that

$$\left| \sum_{x \in E} f(x) \right| \leq \|f\|_1. \tag{13.4}$$

Let us say that a function f on E has *finite support* if $f(x) = 0$ for all but finitely many $x \in E$. It is not difficult to show that these functions are dense in ℓ^1, by considering finite sets $A \subseteq E$ for which $\sum_{x \in A} |f(x)|$ approximates $\|f\|_1$. Of course, $\sum_{x \in E} f(x)$ reduces to a finite sum when f has finite support on E. This gives another way to look at the sum of an arbitrary summable function on E. Namely, it is the unique continuous linear functional on $\ell^1(E)$ that is equal to the ordinary finite sum on the dense linear subspace of functions with finite support.

14 $c_0(E)$

Let E be a nonempty set, and let us say that a real or complex-valued function $f(x)$ on E *vanishes at infinity* if for each $\epsilon > 0$ the set of $x \in E$ such that $|f(x)| \geq \epsilon$ has only finitely many elements. The space of these functions is denoted $c_0(E)$, or $c_0(E, \mathbf{R})$, $c_0(E, \mathbf{C})$ to indicate whether real or complex-valued functions are being used. It is easy to see that these functions are bounded, and that they form a closed linear subspace of $\ell^\infty(E)$ with respect to the ℓ^∞ norm. Moreover, functions on E with finite support are dense in $c_0(E)$, and $c_0(E)$ is the closure of the linear subspace of functions on E with finite support in $\ell^\infty(E)$.

If f is a bounded function on E and g is a summable function on E, then $f\,g$ is a summable function on E, and

$$\|f\,g\|_1 \leq \|f\|_\infty \, \|g\|_1. \tag{14.1}$$

In particular,

$$\lambda_g(f) = \sum_{x \in E} f(x)\, g(x) \tag{14.2}$$

is well-defined and satisfies

$$|\lambda_g(f)| \leq \|f\|_\infty \, \|g\|_1. \tag{14.3}$$

This shows that λ_g defines a continuous linear functional on $\ell^\infty(E)$, whose dual norm with respect to the ℓ^∞ norm is less than or equal to $\|g\|_1$. The dual norm of λ_g with respect to the ℓ^∞ norm is actually equal to $\|g\|_1$, because one can choose $f \in \ell^\infty(E)$ so that $\|f\|_\infty = 1$ and $f(x)\, g(x) = |g(x)|$ for each $x \in E$.

We can also restrict λ_g to $c_0(E)$, to get a continuous linear functional on $c_0(E)$ whose dual norm is less than or equal to $\|g\|_1$. The dual norm of λ_g on $c_0(E)$ with respect to the ℓ^∞ norm is still equal to $\|g\|_1$, but we have to do a bit more to show that. The problem is that the function f mentioned at the end of the previous paragraph may not vanish at infinity on E. To fix that, we can choose for each nonempty finite set $A \subseteq E$ a function $f_A(x)$ such that $f_A(x) = 0$ when $x \in E \backslash A$, $f_A(x)\, g(x) = |g(x)|$ when $x \in A$, and $\|f_A\|_\infty = 1$. Thus f_A has finite support on E, and hence vanishes at infinity. By construction,

$$\lambda_g(f_A) = \sum_{x \in A} |g(x)|. \tag{14.4}$$

This shows that the dual norm of λ_g on $c_0(E)$ is greater than or equal to $\sum_{x \in A} |g(x)|$ for every nonempty finite set $A \subseteq E$, and it follows that the dual norm is equal to $\|g\|_1$, by taking the supremum over A.

Suppose now that λ is any continuous linear functional on $c_0(E)$. If $x \in E$, then let $\delta_x(y)$ be the function on E equal to 1 when $x = y$ and to 0 otherwise. Thus $\delta_x \in c_0(E)$, and we can put

$$g(x) = \lambda(\delta_x) \tag{14.5}$$

for each $x \in E$. If f is a function on E with finite support, then f can be expressed as a linear combination of δ's, and we get that

$$\lambda(f) = \sum_{x \in E} f(x)\, g(x), \tag{14.6}$$

by linearity. Using functions like f_A in the previous paragraph, we get that

$$\sum_{x \in A} |g(x)| \le \|\lambda\|_* \tag{14.7}$$

for every nonempty finite set $A \subseteq E$, where $\|\lambda\|_*$ is the dual norm of λ on $c_0(E)$. Hence $g \in \ell^1(E)$, and $\|g\|_1 \le \|\lambda\|_*$. We have already seen that $\lambda(f) = \lambda_g(f)$ when f has finite support on E, and it follows that this holds for every $f \in c_0(E)$, since functions with finite support are dense in $c_0(E)$, and both λ and λ_g are continuous on $c_0(E)$.

15 The dual of ℓ^1

Let E be a nonempty set, and suppose that f is a summable function on E, and that g is a bounded function on E. As in the previous section, $f\, g$ is a summable function on E, and

$$\|f\, g\|_1 \le \|f\|_1\, \|g\|_\infty. \tag{15.1}$$

Hence

$$\lambda_g(f) = \sum_{x \in E} f(x)\, g(x) \tag{15.2}$$

is well-defined and satisfies

$$|\lambda_g(f)| \le \|f\|_1\, \|g\|_\infty. \tag{15.3}$$

Thus λ_g defines a continuous linear functional on $\ell^1(E)$, with dual norm less than or equal to $\|g\|_\infty$. One can check that the dual norm of λ_g is actually equal to $\|g\|_\infty$, using functions f on E that are equal to 1 at one point and 0 elsewhere.

Conversely, suppose that λ is a bounded linear functional on $\ell^1(E)$. Let δ_x be as in the previous section, and put $g(x) = \lambda(\delta_x)$ for each $x \in E$, as before. Thus

$$|g(x)| = |\lambda(\delta_x)| \le \|\lambda\|_*\, \|\delta_x\|_1 = \|\lambda\|_* \tag{15.4}$$

for each $x \in E$, where $\|\lambda\|_*$ is the dual norm of λ on $\ell^1(E)$. This shows that $g \in \ell^\infty(E)$, and that $\|g\|_\infty \le \|\lambda\|_*$. In particular, λ_g is a continuous linear functional on $\ell^1(E)$, as in the preceding paragraph. If f has finite support on E, then f can be expressed as a linear combination of δ's, and hence $\lambda(f) = \lambda_g(f)$. It follows that this holds for every $f \in \ell^1(E)$, because functions with finite support are dense in $\ell^1(E)$, and λ, λ_g are continuous on ℓ^1.

Suppose now that E is an infinite set, and let $c(E)$ be the space of real or complex-valued functions $f(x)$ on E that have a limit at infinity. This means that there is a real or complex number a, as appropriate, such that for each $\epsilon > 0$,

$$|f(x) - a| < \epsilon \tag{15.5}$$

for all but finitely many $x \in E$. Equivalently, $f \in c(E)$ if there is an $a \in \mathbf{R}$ or $\mathbf{C}$ such that $f(x) - a \in c_0(E)$. As usual, this space may also be denoted $c(E, \mathbf{R})$ or $c(E, \mathbf{C})$, to indicate whether real or complex-valued functions are being used. It is easy to see that these functions are bounded, and that they form a closed linear subspace of $\ell^\infty(E)$ with respect to the ℓ^∞ norm.

If f, a are as in the previous paragraph, then put

$$\lim_{\substack{x \to \infty \\ x \in E}} f(x) = a. \tag{15.6}$$

It is easy to see that this limit is unique when it exists, and that

$$\Big|\lim_{\substack{x \to \infty \\ x \in E}} f(x)\Big| \le \|f\|_\infty. \tag{15.7}$$

Thus the limit defines a continuous linear functional on $c(E)$. The Hahn–Banach theorem implies that there is a continuous linear functional L on $\ell^\infty(E)$ with dual norm equal to 1 such that $L(f)$ is equal to this limit when $f \in c(E)$. However, one can also check that there is no $g \in \ell^1(E)$ such that $\lambda_g(f) = L(f)$ for every $f \in c(E)$, where λ_g is as in the previous section.

16 Filters

A nonempty collection $\mathcal{F}$ of nonempty subsets of a set X is said to be a *filter* if

$$A \cap B \in \mathcal{F} \text{ for every } A, B \in \mathcal{F}, \tag{16.1}$$

and

$$E \in \mathcal{F} \text{ for every } E \subseteq X \text{ for which there is an } A \in \mathcal{F} \text{ such that } A \subseteq E. \tag{16.2}$$

Suppose that X is a topological space, and that p is an element of X. A filter $\mathcal{F}$ on X is said to *converge* to p if

$$U \in \mathcal{F} \tag{16.3}$$

for every open set U in X with $p \in U$. If X is Hausdorff, then the limit of a convergent filter on X is unique.

A filter $\mathcal{F}'$ on a set X is said to be a *refinement* of another filter $\mathcal{F}$ on X if $\mathcal{F} \subseteq \mathcal{F}'$, as collections of subsets of X. Suppose that X is a topological space, and let p be an element of X. Remember that $\overline{A}$ denotes the closure in X of a subset A of X. If $\mathcal{F}$ is a filter on X and

$$p \in \bigcap_{A \in \mathcal{F}} \overline{A}, \tag{16.4}$$

then there is a refinement of $\mathcal{F}$ that converges to p. To see this, let $\mathcal{F}'$ be the collection of subsets E of X such that $A \cap U \subseteq E$ for some $A \in \mathcal{F}$ and open set $U \subseteq X$ with $p \in U$. By hypothesis, $p \in \overline{A}$, and hence $A \cap U \neq \emptyset$ under these conditions. One can check that the intersection of two elements of $\mathcal{F}'$ is also an element of $\mathcal{F}$, because of the corresponding properties of $\mathcal{F}$ and open neighborhoods of p. We also have that $E \in \mathcal{F}'$ for every $E \subseteq X$ for which there is a $B \in \mathcal{F}'$ such that $B \subseteq E$, by construction. Thus $\mathcal{F}'$ is a filter on X, which is clearly a refinement of $\mathcal{F}$, since we can take $U = X$. If U is any open set in X that contains p, then $U \in \mathcal{F}'$, since $A \cap U \subseteq U$ for every $A \in \mathcal{F}$. This shows that $\mathcal{F}'$ is a filter which is a refinement of $\mathcal{F}$ that converges to p, as desired.

Conversely, suppose that $\mathcal{F}'$ is a refinement of $\mathcal{F}$ that converges to p. If U is an open set in X that contains p, then $U \in \mathcal{F}'$, and so $A \cap U \in \mathcal{F}'$ for every $A \in \mathcal{F}'$. Hence $A \cap U \neq \emptyset$, and this holds in particular for every $A \in \mathcal{F}$, because $\mathcal{F}'$ is a refinement of $\mathcal{F}$. It follows that $p \in \overline{A}$, since this works for every open neighborhood U of p in X. Thus $p \in \overline{A}$ for every $A \in \mathcal{F}$, as before.

Now let V be a real or complex topological vector space. A filter $\mathcal{F}$ on V satisfies the *Cauchy condition* if for every open set U in V with $0 \in U$ there is an $A \in \mathcal{F}$ such that

$$A - A \subseteq U, \tag{16.5}$$

where

$$A - A = \{v - w : v, w \in A\}. \tag{16.6}$$

It is easy to see that convergent filters on V satisfy the Cauchy condition, using the continuity of

$$(v, w) \mapsto v - w \tag{16.7}$$

as a mapping from $V \times V$ into V. One can say that V is complete if every Cauchy filter on V converges, as in Section 12.

17 Compactness

Remember that a topological space X is *compact* if for every collection $\{U_\alpha\}_{\alpha \in A}$ of open subsets of X such that

$$X = \bigcup_{\alpha \in A} U_\alpha, \tag{17.1}$$

there are finitely many indices $\alpha_1, \ldots, \alpha_n \in A$ such that

$$X = U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}. \tag{17.2}$$

A collection $\{E_i\}_{i \in I}$ of closed subsets of X is said to have the *finite intersection property* if

$$E_{i_1} \cap \cdots \cap E_{i_n} \neq \emptyset \tag{17.3}$$

for every collection $i_1, \ldots, i_n$ of finitely many indices in I. If X is compact and $\{E_i\}_{i \in I}$ is a collection of closed subsets of X with the finite intersection property, then

$$\bigcap_{i \in I} E_i \neq \emptyset. \tag{17.4}$$

Otherwise, if $\bigcap_{i \in I} E_i = \emptyset$, then $U_i = X \backslash E_i$ would be an open covering of X with no finite subcovering.

Conversely, if X is not compact, then there is an open covering $\{U_\alpha\}_{\alpha \in A}$ of X for which there is no finite subcovering. If $E_\alpha = X \backslash U_\alpha$ for each $\alpha \in A$, then it is easy to see that $\{E_\alpha\}_{\alpha \in A}$ is a collection of closed subsets of X with the finite intersection property. However, the intersection of all of the E_α's is empty, because $\{U_\alpha\}_{\alpha \in A}$ is an open covering of X. This shows that X is compact when the intersection of any collection of closed subsets of X with the finite intersection property is nonempty.

Let $\mathcal{F}$ be a filter on X. As in the previous section, $\mathcal{F}$ has a refinement that converges to an element of X if and only if

$$\bigcap_{A \in \mathcal{F}} \overline{A} \neq \emptyset. \tag{17.5}$$

Note that $\{\overline{A} : A \in \mathcal{F}\}$ has the finite intersection property, because of the definition of a filter and the elementary fact that

$$\overline{A \cap B} \subseteq \overline{A} \cap \overline{B} \tag{17.6}$$

for every $A, B \subseteq X$. If X is compact, then it follows that every filter on X has a refinement that converges to an element of X.

Conversely, let $\{E_i\}_{i \in I}$ be a collection of closed subsets of X with the finite intersection property. Let $\mathcal{F}$ be the set of all $A \subseteq X$ such that

$$E_{i_1} \cap \cdots \cap E_{i_n} \subseteq A \tag{17.7}$$

for some finite collection of indices $i_1, \ldots, i_n \in I$. It is easy to see that $\mathcal{F}$ is a filter on X. If there is a refinement of $\mathcal{F}$ that converges to an element of X, then it follows that $\bigcap_{i \in I} E_i \neq \emptyset$. Thus X is compact when every filter on X has a refinement that converges to an element of X.

18 Ultrafilters

A maximal filter on a set X is said to be an *ultrafilter*. More precisely, a filter $\mathcal{F}$ on X is an ultrafilter if the only filter on X that is a refinement of $\mathcal{F}$ is itself. If $p \in X$ and $\mathcal{F}_p$ is the collection of subsets A of X such that $p \in A$, then it is easy to see that $\mathcal{F}_p$ is an ultrafilter on X. One can show that every filter has a refinement that is an ultrafilter, using the axiom of choice through Zorn's lemma or the Hausdorff maximality principle. If X is a compact topological space and $\mathcal{F}$ is an ultrafilter on X, then it follows that $\mathcal{F}$ converges to an element of X. More precisely, $\mathcal{F}$ has a refinement that converges, as in the previous section, and this refinement is the same as $\mathcal{F}$, since $\mathcal{F}$ is an ultrafilter. Conversely, if every ultrafilter on a topological space X converges, then X is compact. This is because every filter on X has a refinement which is an ultrafilter, and hence converges.

Suppose that $\mathcal{F}$ is a filter on a set X, and that B is a subset of X such that $A \cap B \neq \emptyset$ for every $A \in \mathcal{F}$. Let $\mathcal{F}_B$ be the collection of subsets E of X for which there is an $A \in \mathcal{F}$ such that

$$A \cap B \subseteq E. \tag{18.1}$$

It is easy to see that $\mathcal{F}_B$ is a filter on X that is a refinement of $\mathcal{F}$. If $\mathcal{F}$ is an ultrafilter on X, then it follows that $\mathcal{F}_B = \mathcal{F}$, and hence that $B \in \mathcal{F}$. Conversely, suppose that $\mathcal{F}$ is a filter on X such that $B \in \mathcal{F}$ for every $B \subseteq X$ such that $A \cap B \neq \emptyset$ for every $A \in \mathcal{F}$. If $\mathcal{F}'$ is a filter on X that is a refinement of $\mathcal{F}$, and if $B \in \mathcal{F}'$, then $A \cap B \in \mathcal{F}'$ for every $A \in \mathcal{F} \subseteq \mathcal{F}'$, and hence $A \cap B \neq \emptyset$ for every $A \in \mathcal{F}$. It follows that every $B \in \mathcal{F}'$ is also in $\mathcal{F}$, which means that $\mathcal{F}' = \mathcal{F}$. Thus $\mathcal{F}$ is an ultrafilter under these conditions.

Let $\mathcal{F}$ be an ultrafilter on a set X, and let B be a subset of X. If $A \cap B = \emptyset$ for some $A \in \mathcal{F}$, then $A \subseteq X \backslash B$, and hence $X \backslash B \in \mathcal{F}$. Otherwise, if $A \cap B \neq \emptyset$ for every $A \in \mathcal{F}$, then $B \in \mathcal{F}$, as in the previous paragraph. This shows that for every $B \subseteq X$, either

$$B \in \mathcal{F} \quad \text{or} \quad X \backslash B \in \mathcal{F} \tag{18.2}$$

when $\mathcal{F}$ is an ultrafilter. Conversely, if $\mathcal{F}$ is a filter on X with this property, then $\mathcal{F}$ is an ultrafilter. To see this, let $\mathcal{F}'$ be a filter on X that is a refinement of $\mathcal{F}$. If $B \in \mathcal{F}'$ and $X \backslash B \in \mathcal{F} \subseteq \mathcal{F}'$, then we get a contradiction, since $B \cap (X \backslash B) = \emptyset$. Thus each $B \in \mathcal{F}'$ is an element of $\mathcal{F}$, which implies that $\mathcal{F}' = \mathcal{F}$, as desired.

Let X, Y be sets, and let f be a mapping from X into Y. If $\mathcal{F}$ is a filter on X, then one can check that

$$f_*(\mathcal{F}) = \{A \subseteq Y : f^{-1}(A) \in \mathcal{F}\} \tag{18.3}$$

is a filter on Y. If $\mathcal{F}$ is an ultrafilter on X, then $f_*(\mathcal{F})$ is an ultrafilter on Y. To see this, let B be a subset of Y, and note that $f^{-1}(Y \backslash B) = X \backslash f^{-1}(B)$, so that $f^{-1}(B)$ or $f^{-1}(Y \backslash B)$ is an element of $\mathcal{F}$. Thus B or $Y \backslash B$ is an element of $f_*(\mathcal{F})$, which implies that $\mathcal{F}$ is an ultrafilter on Y, as in the previous paragraph.

19 Tychonoff's theorem

Let $\{X_i\}_{i \in I}$ be a collection of compact topological spaces, and let $X = \prod_{i \in I} X_i$ be their Cartesian product. A famous theorem of Tychonoff states that X is also compact with respect to the product topology. There is a well-known proof of this using ultrafilters, as follows. Let $\mathcal{F}$ be an ultrafilter on X, and let us show that $\mathcal{F}$ converges. Let p_i be the standard coordinate projection from X onto X_i for each $i \in I$. As before, $(p_i)_*(\mathcal{F})$ is an ultrafilter on X_i for each $i \in I$, and hence converges to an element x_i of X_i, by compactness. If $x \in X$ satisfies $p_i(x) = x_i$ for each i, then we would like to check that $\mathcal{F}$ converges to x. Let U be an open set in X such that $x \in U$. By the definition of the product topology, there are open sets $U_i \subseteq X_i$ for each $i \in I$ such that $x_i \in U_i$ for each i, $U_i = X_i$ for all but finitely many i, and

$$\prod_{i \in I} U_i \subseteq U. \tag{19.1}$$

Because $(p_i)_*(\mathcal{F})$ converges to x_i for each i, we get that $U_i \in (p_i)_*(\mathcal{F}_i)$ for each i, which means that $p_i^{-1}(U_i) \in \mathcal{F}$ for each i. Of course,

$$\prod_{i \in I} U_i = \bigcap_{i \in I} p_i^{-1}(U_i). \tag{19.2}$$

This is the same as the intersection of $p_i^{-1}(U_i)$ over finitely many $i \in I$, since $U_i = X_i$ and hence $p_i^{-1}(U_i) = X$ for all but finitely many i. It follows that the intersection is contained in $\mathcal{F}$, which implies that U is contained in $\mathcal{F}$, as desired.

20 The weak* topology

Let V be a real or complex topological vector space, and let V^* be the dual space of continuous linear functionals on V. If $v \in V$, then

$$L_v(\lambda) = \lambda(v) \tag{20.1}$$

defines a linear functional on V^*. This is automatically a nice collection of linear functionals on V^* in the sense of Section 9, since $\lambda = 0$ in V^* when $\lambda(v) = 0$ for each $v \in V$. The weak topology on V^* corresponding to this collection of linear functionals is known as the *weak* topology*.

Suppose now that V is equipped with a norm $\|v\|$ that determines the given topology on V, and let $\|\lambda\|_*$ be the corresponding dual norm on V^*. Observe that L_v is a continuous linear functional on V^* with respect to $\|\lambda\|_*$ for each $v \in V$. More precisely,

$$|L_v(\lambda)| \le \|\lambda\|_* \, \|v\| \tag{20.2}$$

for every $v \in V$ and $\lambda \in V^*$, by definition of $\|\lambda\|_*$. This shows that the dual norm of L_v as a continuous linear functional on V^* with respect to $\|\lambda\|_*$ is less than or equal to $\|v\|$ for each $v \in V$. The dual norm of L_v on V^* is actually equal to $\|v\|$, since for each $v \in V$ there is a $\lambda \in V^*$ such that $\|\lambda\|_* = 1$ and $\lambda(v) = \|v\|$, by the Hahn–Banach theorem.

Note that every open set in V^* with respect to the weak* topology is also an open set with respect to the dual norm. This follows from the fact that L_v is continuous on V^* with respect to the dual norm for each $v \in V$, as in the previous paragraph.

Consider the closed unit ball B^* in V^* with respect to the dual norm, which consists of all $\lambda \in V^*$ with $\|\lambda\|_* \le 1$. This is the same as the set of $\lambda \in V^*$ such that $|\lambda(v)| \le 1$ for every $v \in V$ with $\|v\| \le 1$. It follows easily from this description that B^* is a closed set in the weak* topology. The Banach–Alaoglu theorem states that B^* is a compact set with respect to the weak* topology. The basic idea is to show that B^* is homeomorphic to a closed subset of a product of closed intervals in the real case, or a product of closed disks in the complex case, and then use Tychonoff's theorem.

21 Filters on subsets

Let X be a nonempty set, and let E be a nonempty subset of X. If $\mathcal{F}_0$ is a filter on E, then there is a natural filter $\mathcal{F}_1$ on X associated to it, given by

$$\mathcal{F}_1 = \{B \subseteq X : B \cap E \in \mathcal{F}_0\}. \tag{21.1}$$

In particular, $\mathcal{F}_0 \subseteq \mathcal{F}_1$. Equivalently, if $f : E \to X$ is the inclusion mapping that sends every element of E to itself as an element of X, then $\mathcal{F}_1 = f_*(\mathcal{F}_0)$.

Conversely, if $\mathcal{F}_1$ is a filter on X such that $E \in \mathcal{F}_1$, then

$$\mathcal{F}_0 = \{A \subseteq E : A \in \mathcal{F}_1\} \tag{21.2}$$

is a filter on E. It is easy to see that this transformation between filters is the inverse of the one described in the previous paragraph. Thus we get a one-to-one correspondence between filters on E and filters on X that contain E as an element. Moreover, refinements of filters on E correspond exactly to refinements of filters on X that contain E as an element in this way. Of course, any refinement of a filter on X that contains E as an element also contains E as an element, and hence corresponds to a filter on E too.

Suppose now that X is a topological space. It is easy to check that a filter $\mathcal{F}_1$ on X that contains E as an element converges to a point $p \in E$ if and only if the corresponding filter $\mathcal{F}_0$ on E converges to p with respect to the induced topology on E. Using the remarks about refinements in the previous paragraph, it follows that $\mathcal{F}_1$ has a refinement on X that converges to an element of E if and only if $\mathcal{F}_0$ has a refinement on E that converges to an element of E with respect to the induced topology. Hence E is compact if and only if every filter $\mathcal{F}_1$ on X that contains E as an element has a refinement that converges to an element of E.

In the same way, ultrafilters on E correspond exactly to ultrafilters on X that contain E as an element, and E is compact if and only if every ultrafilter on X that contains E as an element converges to an element of E.

22 Bounded linear mappings

Let V and W be vector spaces, both real or both complex, and equipped with norms $\|v\|_V$, $\|w\|_W$, respectively. A linear mapping $T : V \to W$ is said to be *bounded* if there is a nonnegative real number A such that

$$\|T(v)\|_W \le A\,\|v\|_V \tag{22.1}$$

for every $v \in V$. Because of linearity, this implies that

$$\|T(v) - T(v')\|_W \le A\,\|v - v'\|_V \tag{22.2}$$

for every $v, v' \in V$, and hence that T is uniformly continuous with respect to the metrics on V and W associated to their norms. Conversely, if $T : V \to W$ is continuous at 0, then there is a $\delta > 0$ such that

$$\|T(v)\|_W < 1 \tag{22.3}$$

for every $v \in V$ with $\|v\|_V < \delta$. It is easy to see that this implies that T is bounded, with $A = 1/\delta$.

If T is a bounded linear mapping from V into W, then the *operator norm* $\|T\|_{op}$ of T is defined by

$$\|T\|_{op} = \sup\{\|T(v)\|_W : v \in V,\ \|v\|_V \le 1\}. \tag{22.4}$$

The boundedness of T says exactly that the supremum is finite, and is less than or equal to the nonnegative real number A mentioned in the previous paragraph. Equivalently, T satisfies the boundedness condition in the previous paragraph with $A = \|T\|_{op}$, and this is the smallest value of A with this property. Note that $\|T\|_{op} = 0$ if and only if $T = 0$.

Let T be a bounded linear mapping from V into W, and let a be a real or complex number, as appropriate. Of course, the product $a\,T$ of a and T is the linear mapping that sends $v \in V$ to $a\,T(v)$ in W. It is easy to see that $a\,T$ is also a bounded linear mapping, and that

$$\|a\,T\|_{op} = |a|\,\|T\|_{op}. \tag{22.5}$$

Similarly, if R is another bounded linear mapping from V into W, then the sum $R + T$ is defined as the linear mapping that sends $v \in V$ to $R(v) + T(v)$ in W. It is easy to see that $R + T$ is also a bounded linear mapping from V into W, and that

$$\|R + T\|_{op} \le \|R\|_{op} + \|T\|_{op}. \tag{22.6}$$

Let $\mathcal{BL}(V, W)$ be the space of bounded linear mappings from V into W. It follows from the remarks in the preceding paragraph that $\mathcal{BL}(V, W)$ is a real or complex vector space, as appropriate, with respect to pointwise addition and scalar multiplication of linear mappings, and that the operator norm defines a norm on this vector space. If W is the one-dimensional vector space of real or complex numbers, as appropriate, then $\mathcal{BL}(V, W)$ is the same as the dual space V^* of bounded linear functionals on V, and the operator norm is the same as the dual norm on V^*.

Suppose that W is complete as a metric space with respect to the metric associated to the norm, so that W is a *Banach space*. In this case, the space $\mathcal{BL}(V, W)$ of bounded linear mappings from V into W is also complete with respect to the operator norm, and thus a Banach space. To see this, let $\{T_j\}_{j=1}^\infty$ be a Cauchy sequence in $\mathcal{BL}(V, W)$. This means that for each $\epsilon > 0$ there is an $L(\epsilon) \geq 1$ such that

$$\|T_j - T_l\|_{op} \leq \epsilon \tag{22.7}$$

for every $j, l \geq L(\epsilon)$. Equivalently,

$$\|T_j(v) - T_l(v)\|_W \leq \epsilon \, \|v\|_V \tag{22.8}$$

for every $j, l \geq L(\epsilon)$ and $v \in V$, so that $\{T_j(v)\}_{j=1}^\infty$ is a Cauchy sequence in W for every $v \in V$. Because W is complete, it follows that $\{T_j(v)\}_{j=1}^\infty$ converges in W for every $v \in V$. Put

$$T(v) = \lim_{j \to \infty} T_j(v) \tag{22.9}$$

for every $v \in V$. It is easy to see that T is a linear mapping from V into W, because of the linearity of the T_j's.

Observe that

$$\|T_j(v) - T(v)\|_W \leq \epsilon \, \|v\| \tag{22.10}$$

for every $j \geq L(\epsilon)$ and $v \in V$, by taking the limit as $l \to \infty$ in (22.8). In particular,

$$\|T(v)\|_W \leq \|T_j(v)\|_W + \epsilon \, \|v\| \tag{22.11}$$

when $j \geq L(\epsilon)$. Applying this to $\epsilon = 1$ and $j = L(1)$, we get that

$$\|T(v)\|_W \leq \|T_{L(1)}(v)\|_W + \|v\| \leq \|T_{L(1)}\|_{op} \, \|v\| + \|v\|. \tag{22.12}$$

This implies that T is bounded, with $\|T\|_{op} \leq \|T_{L(1)}\|_{op} + 1$.

Using (22.10), we also get that

$$\|T_j - T\|_{op} \leq \epsilon \tag{22.13}$$

when $j \geq L(\epsilon)$. Thus $\{T_j\}_{j=1}^\infty$ converges to T with respect to the operator norm. This shows that every Cauchy sequence in $\mathcal{BL}(V, W)$ converges to an element of $\mathcal{BL}(V, W)$ when W is complete, as desired. In particular, we can apply this to $W = \mathbf{R}$ or $\mathbf{C}$, as appropriate, to get that the dual space V^* of bounded linear functionals on V is complete with respect to the dual norm, since the real and complex numbers are complete with respect to their standard metrics.

Suppose now that V_1, V_2, and V_3 are vector spaces, all real or all complex, and equipped with norms. Let T_1 be a bounded linear mapping from V_1 into V_2, and let T_2 be a bounded linear mapping from V_2 into V_3. As usual, the composition $T_2 \circ T_1$ is the linear mapping from V_1 into V_3 that sends $v \in V_1$ to $T_2(T_1(v))$. It is easy to see that $T_2 \circ T_1$ is also bounded, and that

$$\|T_2 \circ T_1\|_{op,13} \le \|T_1\|_{op,12} \, \|T_2\|_{op,23}. \tag{22.14}$$

Here the subscripts in the operator norms are included to indicate the vector spaces and norms being used.

23 Topological vector spaces, continued

Let V be a topological vector space over the real or complex numbers. If $v \in V$ and $A \subseteq V$, then put

$$v + A = \{v + a : a \in A\}. \tag{23.1}$$

If A is an open or closed set in V, then $v + A$ has the same property, because translations determine homeomorphisms on V. Similarly, if $A, B \subseteq V$, then put

$$A + B = \{a + b : a \in A, b \in B\}. \tag{23.2}$$

Equivalently,

$$A + B = \bigcup_{a \in A} (a + B) = \bigcup_{b \in B} (b + A), \tag{23.3}$$

which shows that $A + B$ is an open set in V as soon as either A or B is open, since it is the union of a collection of open sets.

If $A \subseteq V$ and t is a real or complex number, as appropriate, then we put

$$t\,A = \{t\,a : a \in A\}. \tag{23.4}$$

If $t \ne 0$ and A is an open or closed set in V, then $t\,A$ has the same property, because multiplication by t defines a homeomorphism on V. Of course, $t\,A = \{0\}$ when $t = 0$ and $A \ne \emptyset$. If $t = -1$, then $t\,A$ may be expressed simply as $-A$.

Suppose that U is an open set in V that contains 0. Continuity of addition at 0 in V implies that there are open sets $U_1, U_2 \subseteq V$ that contain 0 and satisfy

$$U_1 + U_2 \subseteq U. \tag{23.5}$$

If $v \in V$ and $v \ne 0$, then one can apply this to $U = V \backslash \{v\}$, to get that

$$U_1 \cap (v - U_2) = \emptyset. \tag{23.6}$$

Here $v - U_2 = v + (-U_2)$, which is an open set in V that contains v. This shows that V is Hausdorff, using also translation-invariance of the topology on V.

Let U be an open set in V that contains 0 again. Continuity of scalar multiplication at 0 implies that there is an open set $U_0 \subseteq V$ that contains 0 and a positive real number $\delta > 0$ such that

$$t\,U_0 \subseteq U \tag{23.7}$$

for every $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, with $|t| < \delta$. Consider

$$\widetilde{U}_0 = \bigcup_{|t|<\delta} t\, U_0, \tag{23.8}$$

where more precisely the union is taken over all real or complex numbers t such that $|t| < \delta$, as appropriate. Equivalently,

$$\widetilde{U}_0 = \bigcup_{0<|t|<\delta} t\, U_0, \tag{23.9}$$

since $0 \in U_0$, and hence $0 \in \widetilde{U}_0$. This shows that $\widetilde{U}_0$ is an open set in V, because it is a union of open sets.

A set $E \subseteq V$ is said to be *balanced* if

$$r\, E \subseteq E \tag{23.10}$$

for every $r \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, with $|r| \le 1$. Thus every nonempty balanced set contains 0 automatically. It is easy to see that the set $\widetilde{U}_0$ described in the previous paragraph is balanced by construction. This shows that for every open set $U \subseteq V$ with $0 \in U$ there is a nonempty balanced open set $\widetilde{U}_0 \subseteq U$. To put it another way, the nonempty balanced open sets in V form a local base for the topology of V at 0.

Let U be an open set in V that contains 0 again, and let $v \in V$ be given. Because $0\, v = 0 \in U$, continuity of scalar multiplication at v implies that there is a $\delta(v, U) > 0$ such that

$$t\, v \in U \tag{23.11}$$

for every $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, with $|t| < \delta(v, U)$.

Let A be any subset of V, and let $U \subseteq V$ be an open set that contains 0. If v is an element of the closure $\overline{A}$ of A in V, then

$$(v - U) \cap A \ne \emptyset, \tag{23.12}$$

since $v - U$ is an open set in V that contains v. Equivalently,

$$v \in A + U. \tag{23.13}$$

It follows that

$$\overline{A} \subseteq A + U. \tag{23.14}$$

24 Bounded sets

Let V be a topological vector space over the real or complex numbers. A set $E \subseteq V$ is said to be *bounded* if for every open set $U \subseteq V$ with $0 \in U$ there is a real or complex number t_1, as appropriate, such that

$$E \subseteq t_1\, U. \tag{24.1}$$

If U is balanced, then it follows that

$$E \subseteq t\,U \tag{24.2}$$

for every $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, such that $|t| \ge |t_1|$.

If U is any open set in V that contains 0, then there is a nonempty balanced open set $U' \subseteq U$, as in the previous section. In order to check that a set $E \subseteq V$ is bounded, it is therefore enough to consider nonempty balanced open sets in V, instead of arbitrary neighborhoods of 0. If U is an arbitrary neighborhood of 0 in V, then we also get that (24.2) holds for all real or complex numbers t, as appropriate, for which $|t|$ is sufficiently large. This follows by applying the stronger form of boundedness to a nonempty balanced open subset of U.

If $E \subseteq V$ has only finitely many elements, then it is easy to see that E is bounded, using the property (23.11) of neighborhoods of 0 in V. It is also easy to see that the union of finitely many bounded subsets of V is bounded, using the stronger form of boundedness described in the previous paragraphs. If the topology on V is determined by a norm $\|v\|$, then a set $E \subseteq V$ is bounded if and only if $\|v\|$ is bounded on E. Similarly, if the topology on V is determined by a nice collection of seminorms $\mathcal{N}$, then one can check that a set $E \subseteq V$ is bounded if and only if each seminorm $N \in \mathcal{N}$ is bounded on V. Of course, subsets of bounded sets are also bounded.

If U is a neighborhood of 0 in V, then

$$\bigcup_{n=1}^{\infty} n\,U = V, \tag{24.3}$$

because of (23.11). If $K \subseteq V$ is compact, then it follows that

$$K \subseteq n_1\,U \cup \cdots \cup n_l\,U \tag{24.4}$$

for some finite collection $n_1, \ldots, n_l$ of positive integers. If U is also balanced, then (24.4) implies that

$$K \subseteq n\,U, \tag{24.5}$$

where n is the maximum of $n_1, \ldots, n_l$. This shows that compact subsets of V are bounded, since it suffices to check boundedness with respect to nonempty balanced open subsets of V, as before.

Suppose that $E_1, E_2 \subseteq V$ are bounded, and let us check that $E_1 + E_2$ is bounded as well. Let U be a neighborhood of 0 in V, and let U_1, U_2 be neighborhoods of 0 such that $U_1 + U_2 \subseteq U$, as in the previous section. Thus

$$E_j \subseteq t\,U_j \tag{24.6}$$

when t is a real or complex number, as appropriate, for which $|t|$ is sufficiently large, and $j = 1, 2$. This implies that

$$E_1 + E_2 \subseteq t\,U_1 + t\,U_2 \subseteq t\,U \tag{24.7}$$

when $t \in \mathbf{R}$ or $\mathbf{C}$ is sufficiently large, as desired. In particular, it follows that translations of bounded sets are bounded, since sets with only one element are bounded.

Let us show now that the closure $\overline{E}$ of a bounded set $E \subseteq V$ is bounded. Let U be a neighborhood of 0 in V again, and let U_1, U_2 be neighborhoods of 0 such that $U_1 + U_2 \subseteq V$. Because E is bounded, there is a nonzero real or complex number t, as appropriate, such that

$$E \subseteq t\,U_1. \tag{24.8}$$

We also have that

$$\overline{E} \subseteq E + t\,U_2, \tag{24.9}$$

as in (23.14), since $t\,U_2$ is a neighborhood of 0 in V. Hence

$$\overline{E} \subseteq E + t\,U_2 \subseteq t\,U_1 + t\,U_2 \subseteq t\,U, \tag{24.10}$$

as desired.

If $E \subseteq V$ is bounded and r is a real or complex number, then it is easy to see that $r\,E$ is bounded too. More generally, suppose that W is another topological vector space over the real or complex numbers, depending on whether V is real or complex. If T is a continuous linear mapping from V into W and $E \subseteq V$ is bounded, then it is easy to see that $T(E)$ is bounded in W as well.

25 Uniform boundedness

Let M be a complete metric space, and let $\mathcal{E}$ be a nonempty collection of continuous nonnegative real-valued functions on M. Suppose that $\mathcal{E}$ is bounded pointwise on M, in the sense that

$$\mathcal{E}(x) = \{f(x) : x \in M\} \tag{25.1}$$

is a bounded set of real numbers for each $x \in M$. Put

$$\mathcal{E}_n = \{x \in M : f(x) \le n \text{ for every } f \in \mathcal{E}\} \tag{25.2}$$

for each positive integer n. Thus $\mathcal{E}_n$ is a closed set in M for every n, because the elements of $\mathcal{E}$ are supposed to be continuous functions on M, and

$$\bigcup_{n=1}^{\infty} \mathcal{E}_n = M, \tag{25.3}$$

by the hypothesis that $\mathcal{E}$ be bounded pointwise on M. The Baire category theorem implies that $\mathcal{E}_n$ has nonempty interior for some n, so that $\mathcal{E}$ is uniformly bounded on a nonempty open set in M.

Now let V be a real or complex vector space with a norm $\|v\|$, and let Λ be a nonempty collection of continuous linear functionals on V. Suppose that Λ is bounded pointwise on V, so that

$$\Lambda(v) = \{\lambda(v) : \lambda \in \Lambda\} \tag{25.4}$$

is a bounded set of real or complex numbers, as appropriate, for every $v \in V$. If V is also complete, then it follows from the argument in the previous paragraph that Λ is uniformly bounded on a nonempty open set in V. Using the linearity of the elements of V, one can show that Λ is actually bounded on the unit ball in V, which means that the dual norms of the elements of Λ are uniformly bounded. This is a version of the Banach–Steinhaus theorem.

Let V^* be the dual space of continuous linear functionals on V, as usual. Thus V^* is equipped with the dual norm $\|\lambda\|_*$, as in Section 11, and also the weak* topology, as in Section 20. It is easy to see that every bounded set in V^* with respect to the dual norm is also bounded with respect to the weak* topology, in the sense described in the previous section. Conversely, if V is complete, then every bounded set in V^* with respect to the weak* topology is also bounded with respect to the dual norm, by the principle of uniform boundedness described in the previous paragraph.

Similarly, we can consider V equipped with the weak topology associated to the collection of all continuous linear functionals on V with respect to the norm, as in Section 9. If $E \subseteq V$ is bounded with respect to the norm, then it is easy to see that E is also bounded with respect to the weak topology on V. Conversely, suppose that E is bounded with respect to the weak topology on V. This means that

$$E(\lambda) = \{\lambda(v) : v \in E\} \tag{25.5}$$

is a bounded set of real or complex numbers, as appropriate, for each $\lambda \in V^*$. As in Section 20,

$$L_v(\lambda) = \lambda(v) \tag{25.6}$$

defines a continuous linear functional on V^* with respect to the dual norm $\|\lambda\|_*$ for every $v \in V$. Let $V^{**} = (V^*)^*$ be the space of continuous linear functionals on V^* with respect to the dual norm on V^*. Thus V^{**} is also equipped with a weak* topology, as the dual of V^*. Consider

$$\mathcal{L} = \{L_v : v \in E\}, \tag{25.7}$$

as a subset of V^{**}. It is easy to see that $\mathcal{L}$ is bounded with respect to the weak* topology on V^{**}, because E is bounded with respect to the weak topology on V. We also know that V^* is complete with respect to the dual norm, as in Section 22. It follows from the discussion in the previous paragraph that $\mathcal{L}$ is bounded with respect to the dual norm on V^{**} associated to the dual norm on V^*. As in Section 20, the dual norm of L_v as a continuous linear functional on V^* is equal to the norm of v as an element of V for every $v \in V$, by the Hahn–Banach theorem. This implies that E is bounded with respect to the norm on V.

26 Bounded linear mappings, continued

Let V, W be topological vector spaces, both real or both complex. A linear mapping $T : V \to W$ is said to be *bounded* if for every bounded set $E \subseteq V$, $T(E)$ is a bounded set in W. It is easy to see that continuous linear mappings are bounded in this sense, as mentioned at the end of Section 24. Conversely, if the topology on V is determined by a norm and $T : V \to W$ is bounded, then T is continuous. More precisely, if there is an open set $U \subseteq V$ that contains 0 such that $T(U)$ is bounded in W, then it is not difficult to check that T is continuous. In particular, this condition holds when $T : V \to W$ is bounded and there is a bounded neighborhood U of 0 in V. If the topology on V is determined by a norm, then one can simply take U to be the open unit ball in V.

Let V be a vector space over the real or complex numbers equipped with a norm. As in the previous section, the uniform boundedness principle implies that every bounded set in V with respect to the weak topology is also bounded with respect to the norm. Equivalently, the identity mapping on V is bounded as a mapping from V with the weak topology into V with the norm topology. However, the identity mapping on V is not continuous as a mapping from V with the weak topology into V with the norm topology, unless V is finite-dimensional. This is because the open unit ball in V with respect to the norm is not an open set with respect to the weak topology when V is infinite-dimensional, since an open set in V with respect to the weak topology that contains 0 also contains a linear subspace of V of finite codimension.

Similarly, if V is complete, then every bounded set in V^* with respect to the weak* topology is bounded with respect to the dual norm, as in the previous section. This implies that the identity mapping on V^* is bounded as a mapping from V^* with the weak* topology into V^* with the topology determined by the dual norm. As in the preceding paragraph, this mapping is not continuous when V is infinite-dimensional. Note that V^* is infinite-dimensional when V is, by the Hahn–Banach theorem.

Let V, W be topological vector spaces again, both real or both complex. If $T : V \to W$ is a bounded linear mapping and a is a real or complex number, as appropriate, then $a\,T$ is also a bounded linear mapping from V into W. This follows from the fact that a scalar multiple of a bounded set in a topological vector space is bounded as well. Similarly, if $R : V \to W$ is another bounded linear mapping, then the sum $R+T$ is bounded too. This uses the boundedness of the sum of two bounded subsets of a topological vector space. Now let V_1, V_2, and V_3 be topological vector spaces, all real or all complex. If $T_1 : V_1 \to V_2$ and $T_2 : V_2 \to V_3$ are bounded linear mappings, then it is easy to see that their composition $T_2 \circ T_1$ is a bounded linear mapping from V_1 into V_3, directly from the definition of a bounded linear mapping.

27 Bounded sequences

Let V be a topological vector space over the real or complex numbers. A sequence $\{v_j\}_{j=1}^\infty$ of elements of V is said to be bounded if the set of v_j's is bounded in V. If $\{v_j\}_{j=1}^\infty$ converges to an element v of V, then it is easy to see that the set K consisting of the v_j's and v is compact, which works as well in any topological space. This implies that convergent sequences are bounded, since compact sets are bounded. One can also show this more directly from the definitions, which is especially simple when $\{v_j\}_{j=1}^\infty$ converges to 0. Similarly, one can check that Cauchy sequences are bounded in V. If $\{v_j\}_{j=1}^\infty$ is a bounded sequence in V, and $\{t_j\}_{j=1}^\infty$ is a sequence of real or complex numbers, as appropriate, that converges to 0, then it is easy to see that $\{t_j\, v_j\}_{j=1}^\infty$ converges to 0 in V.

Suppose now that there is a countable local base for the topology of V at 0. This means that there is a sequence $U_1, U_2, \ldots$ of open subsets of V that contain 0 with the property that if U is any other open set in V containing 0, then $U_l \subseteq U$ for some l. As in Section 23, we can also take the U_l's to be balanced subsets of V. We may as well ask that $U_{l+1} \subseteq U_l$ for each l too, since otherwise we can replace U_l with $U_1 \cap \cdots \cap U_l$ for each l. Let $\{v_j\}_{j=1}^\infty$ be a sequence of elements of V that converges to 0, and let us show that there is a sequence of positive real numbers $\{r_j\}_{j=1}^\infty$ such that $r_j \to \infty$ as $j \to \infty$ and $\{r_j\, v_j\}_{j=1}^\infty$ converges to 0 in V.

Because $\{v_j\}_{j=1}^\infty$ converges to 0 and $l^{-1}\, U_l$ is an open set in V that contains 0 for each l, there is a positive integer N_l for each l such that

$$v_j \in l^{-1}\, U_l \tag{27.1}$$

when $j \ge N_l$. We may as well ask that $N_{l+1} > N_l$ for every l too, by increasing the N_l's if necessary. Put

$$r_j = l \quad \text{when } N_l \le j < N_{l+1}, \tag{27.2}$$

and $r_j = 1$ when $1 \le j < N_1$ if $N_1 > 1$. Thus $r_j \to \infty$ as $j \to \infty$, and

$$r_j\, v_j \in U_l \quad \text{when } N_l \le j < N_{l+1}. \tag{27.3}$$

This implies that $r_j\, v_j \in U_l$ when $j \ge N_l$, since $U_{l+1} \subseteq U_l$ for each l. It follows that $\{r_j\, v_j\}_{j=1}^\infty$ converges to 0 in V, as desired. In particular, $\{r_j\, v_j\}_{j=1}^\infty$ is a bounded sequence in V.

Let W be another topological vector space, which is real if V is real and complex if V is complex. If T is a bounded linear mapping from V into W and V has a countable local base for its topology at 0, then a well known theorem states that T is continuous. To see this, it suffices to show that if $\{v_j\}_{j=1}^\infty$ is a sequence of elements of V that converges to 0, then $\{T(v_j)\}_{j=1}^\infty$ converges to 0 in W. Let $\{r_j\}_{j=1}^\infty$ be a sequence of positive real numbers such that $r_j \to +\infty$ as $j \to \infty$ and $\{r_j\, v_j\}_{j=1}^\infty$ converges to 0 in V, as in the previous paragraphs. Thus $\{r_j\, v_j\}_{j=1}^\infty$ is bounded in V, which implies that $\{T(r_j\, v_j)\}_{j=1}^\infty$ is bounded in W, since $T : V \to W$ is bounded by hypothesis. It follows that $T(v_j) = r_j^{-1}\, T(r_j\, v_j)$ converges to 0 as $j \to \infty$ in W, because $\{r_j^{-1}\}_{j=1}^\infty$ converges to 0 in the real line. This shows that T is sequentially continuous at 0, and hence that T is continuous at 0, since V has a countable local base for its topology at 0. Of course, a linear mapping between topological vector spaces is continuous at every point as soon as it is continuous at 0.

28 Bounded linear functionals

If V is a topological vector space over the real or complex numbers, then we can restrict our attention in the previous section to the case where W is the one-dimensional vector space of real or complex numbers, as appropriate. Thus a bounded linear functional on V is a linear functional on V that is bounded as a linear mapping into $\mathbf{R}$ or $\mathbf{C}$.

Suppose now that V is equipped with a norm $\|v\|$, so that a linear functional on V is bounded if and only if it is continuous, as in the previous section. Let V^* be the dual space of bounded linear functionals on V, which is equipped with the dual norm $\|\lambda\|_*$, as in Section 11. Let V^{**} be the space of bounded linear functionals on V^*, which is equipped with a dual norm $\|L\|_{**}$ associated to the dual norm $\|\lambda\|_*$ on V^*. As in Section 20, each $v \in V$ determines a bounded linear functional L_v on V^*, defined by

$$L_v(\lambda) = \lambda(v), \tag{28.1}$$

and we also have that

$$\|L_v\|_{**} = \|v\|. \tag{28.2}$$

This defines an isometric linear embedding $v \mapsto L_v$ of V into V^{**}.

A Banach space V is said to be *reflexive* if every bounded linear functional on V^* is of the form L_v for some $v \in V$. It is easy to see that finite-dimensional Banach spaces are automatically reflexive. If E is a nonempty set, then we have seen in Section 14 that the dual of $c_0(E)$ may be identified with $\ell^1(E)$, and we have seen in Section 15 that the dual of $\ell^1(E)$ may be identified with $\ell^\infty(E)$. In this case, the natural embedding of $c_0(E)$ into $c_0(E)^{**}$ described in the previous paragraph corresponds exactly to the standard inclusion of $c_0(E)$ in $\ell^\infty(E)$ as a linear subspace. If E has infinitely many elements, then $c_0(E)$ is a proper linear subspace of $\ell^\infty(E)$, and it follows that $c_0(E)$ is not reflexive.

If V is a real or complex vector space equipped with a norm $\|v\|$, then every subset of V^* that is bounded with respect to the dual norm is also bounded with respect to the weak* topology. This implies that every bounded linear functional on V^* with respect to the weak* topology is also bounded with respect to the dual norm. Conversely, if V is also complete with respect to the norm, then every bounded subset of V^* with respect to the weak* topology is also bounded with respect to the dual norm, as in Section 25. This implies that every bounded linear functional on V^* with respect to the dual norm is also bounded with respect to the weak* topology. However, a linear functional on V^* is continuous with respect to the weak* topology if and only if it is of the form L_v for some $v \in V$, as in Section 9.

29 Uniform boundedness, continued

Let V be a topological vector space over the real or complex numbers, and let Λ be a nonempty collection of continuous linear functionals on V. Suppose that

Λ is bounded pointwise on V, in the sense that

$$\Lambda(v) = \{\lambda(v) : \lambda \in \Lambda\} \tag{29.1}$$

is a bounded set of real or complex numbers, as appropriate, for each $v \in V$. This is equivalent to asking that Λ be bounded with respect to the weak* topology on the dual space V^* of continuous linear functionals on V. If the topology on V is determined by a norm, and if V is complete with respect to this norm, then Λ is bounded with respect to the dual norm on V^*, as in Section 25.

Suppose now that V is metrizable and complete, even if the topology on V may not be determined by a norm. If $\Lambda \subseteq V^*$ is bounded with respect to the weak* topology on V, and hence bounded pointwise on V, then it follows from the Baire category theorem that there is a nonempty open set $U_1 \subseteq V$ on which Λ is uniformly bounded, as before. If $u_1 \in U_1$, then $U = U_1 - u_1$ is an open set in V that contains 0, and Λ is also uniformly bounded on U, because the elements of Λ are linear. This is another version of the theorem of Banach and Steinhaus.

Let us restrict our attention now to the case where the topology on V is determined by a nice collection of seminorms $\mathcal{N}$. More precisely, we ask that $\mathcal{N}$ have only finitely or countably many elements, so that V is metrizable, and we still ask that V be complete. If $\Lambda \subseteq V^*$ is bounded pointwise on V, then Λ is uniformly bounded on a neighborhood of 0, as in the previous paragraph. In this case, this implies that there are finitely many seminorms $N_1, \ldots, N_l \in \mathcal{N}$ and a nonnegative real number C such that

$$|\lambda(v)| \le C \max_{1 \le j \le l} N_j(v) \tag{29.2}$$

for every $v \in V$. This is analogous to the discussion in Section 7.

Of course, if there are finitely many seminorms $N_1, \ldots, N_l \in \mathcal{N}$ and a $C \ge 0$ such that the preceding condition holds for every $\lambda \in \Lambda$ and $v \in V$, then Λ is bounded pointwise on V. In this situation, the choice of $N_1, \ldots, N_l$ is part of the uniform boundedness condition.

30 Another example

Let V be the vector space of real or complex-valued functions on the set $\mathbf{Z}_+$ of positive integers. If $f \in V$ and ρ is a positive real-valued function on $\mathbf{Z}_+$, then put

$$B_\rho(f) = \{g \in V : |f(l) - g(l)| < \rho(l) \text{ for every } l \in \mathbf{Z}_+\}. \tag{30.1}$$

Let us say that a set $U \subseteq V$ is an open set if for every $f \in U$ there is a positive real-valued function ρ on $\mathbf{Z}_+$ such that

$$B_\rho(f) \subseteq U. \tag{30.2}$$

It is easy to see that this defines a topology on V, and that $B_\rho(f)$ is an open set in V with respect to this topology for every $f \in V$ and positive function ρ

on $\mathbf{Z}_+$. Equivalently, V is the same as the Cartesian product of a sequence of copies of the real or complex numbers, and this topology on V corresponds to the "strong product topology", generated by arbitrary products of open subsets of $\mathbf{R}$ or $\mathbf{C}$.

One can also check that

$$(f, g) \mapsto f + g \tag{30.3}$$

defines a continuous mapping from $V \times V$ into V, using the product topology on $V \times V$ determined by the topology just described on V. Similarly,

$$f \mapsto t\, f \tag{30.4}$$

is continuous as a mapping from V into itself for each $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate. However, if $f(l) \neq 0$ for infinitely many $l \in \mathbf{Z}_+$, then

$$t \mapsto t\, f \tag{30.5}$$

is not continuous as a mapping from the real or complex numbers with the standard topology into V, and so V is not a topological vector space.

Let V_0 be the linear subspace of V consisting of functions f such that $f(l) = 0$ for all but finitely many $l \in \mathbf{Z}_+$. It is not difficult to verify that V_0 is a topological vector space with respect to the topology induced by the one just defined on V. In particular, if $f \in V_0$, then (30.5) is continuous as a mapping from the real or complex numbers into V.

Let us check that V_0 is a closed set in V. Let $f \in V \backslash V_0$ be given, and let ρ be defined on $\mathbf{Z}_+$ by

$$\rho(l) = |f(l)| \tag{30.6}$$

when $f(l) \neq 0$, and $\rho(l) = 1$ otherwise. Thus $\rho(l) > 0$ for every $l \in \mathbf{Z}_+$. If $g \in B_\rho(f)$, then

$$|f(l) - g(l)| < \rho(l) = |f(l)| \tag{30.7}$$

when $f(l) \neq 0$, which implies that $g(l) \neq 0$ for infinitely many $l \in \mathbf{Z}_+$. This shows that

$$B_\rho(f) \subseteq V \backslash V_0, \tag{30.8}$$

and hence that $V \backslash V_0$ is an open set in V, as desired.

Let $U_1, U_2, \ldots$, be a sequence of relatively open sets in V_0 containing 0. By construction, there is a sequence $\rho_1, \rho_2, \ldots$ of positive functions on $\mathbf{Z}_+$ such that

$$B_{\rho_j}(0) \cap V_0 \subseteq U_j \tag{30.9}$$

for each j. Put

$$\rho(j) = \frac{\rho_j(j)}{2} \tag{30.10}$$

for each $j \in \mathbf{Z}_+$, so that $\rho(j)$ is another positive function on $\mathbf{Z}_+$. Thus $B_\rho(0) \cap V_0$ is another relatively open set in V_0 that contains 0, and

$$B_{\rho_j}(0) \cap V_0 \not\subseteq B_\rho(0) \cap V_0 \tag{30.11}$$

for each j, because $\rho(j) < \rho_j(j)$ for each j. This implies that

$$U_j \not\subseteq B_\rho(0) \cap V_0 \tag{30.12}$$

for each j, and it follows that V_0 does not have a countable local base for its topology at 0.

Let E be a subset of V_0, and let $L(E)$ be the set of $l \in \mathbf{Z}_+$ for which there is an $f \in E$ such that $f(l) \neq 0$. Also let ρ be a positive function on $\mathbf{Z}_+$ such that

$$\rho(l) = \frac{|f_l(l)|}{l} \tag{30.13}$$

for some $f_l \in E$ with $f_l(l) \neq 0$ when $l \in L(E)$. Thus

$$f_l \not\in t\, B_\rho(0) \tag{30.14}$$

when $l \in L(E)$ and $t \in \mathbf{R}$ or $\mathbf{C}$ satisfies $|t| \le l$. If $L(E)$ has infinitely many elements, then it follows that

$$E \not\subseteq t\, B_\rho(0) \tag{30.15}$$

for any real or complex number t, as appropriate. This shows that E can have only finitely or countably many elements when E is bounded in V.

Let $V_{0,n}$ be the n-dimensional linear subspace of V_0 consisting of functions f on $\mathbf{Z}_+$ such that $f(l) = 0$ when $l > n$, for each positive integer n. Note that

$$\bigcup_{n=1}^{\infty} V_{0,n} = V_0. \tag{30.16}$$

If $E \subseteq V_0$ is bounded, then $E \subseteq V_{0,n}$ for some n, as in the previous paragraph. In this case, E is also bounded as a subset of $V_{0,n}$ in the usual sense, which is to say that

$$E_j = \{f(j) : f \in E\} \tag{30.17}$$

is bounded in $\mathbf{R}$ or $\mathbf{C}$, as appropriate, for each $j \le n$. Conversely, if $E \subseteq V_{0,n}$ and E_j is bounded for each $j \le n$, then E is bounded in $V_{0,n}$, and hence in V_0.

Let τ be a positive real-valued function on $\mathbf{Z}_+$, and consider the norm N_τ on V_0 defined by

$$N_\tau(f) = \max_{j \ge 1} |f(j)|\, \tau(j). \tag{30.18}$$

If $\mathcal{N}$ is the collection of all of these norms N_τ on V_0, then it is not difficult to check that the topology on V_0 associated to $\mathcal{N}$ is the same as the topology on V_0 induced from the one on V as before. To see this, observe that the open unit ball in V_0 with respect to N_τ,

$$\{f \in V_0 : N_\tau(f) < 1\}, \tag{30.19}$$

is the same as the set of $f \in V_0$ for which there is a positive real number $r < 1$ such that

$$|f(j)| < r\, \tau(j)^{-1} \tag{30.20}$$

for each $j \in \mathbf{Z}_+$. This is contained in $B_\rho(0)$ with $\rho = 1/\tau$, and more precisely it is equal to

$$\bigcup_{0<r<1} B_{r\,\rho}(0), \tag{30.21}$$

which is close enough to show that the topologies are the same.

Similarly, if σ is a positive real-valued function on $\mathbf{Z}_+$, then

$$N'_\sigma(f) = \sum_{j=1}^{\infty} |f(j)|\,\sigma(j) \tag{30.22}$$

defines a norm on V_0. Clearly

$$N_\sigma(f) \le N'_\sigma(f) \tag{30.23}$$

for every $f \in V_0$. In the other direction, if we put

$$\tau(j) = j^2\,\sigma(j) \tag{30.24}$$

for each $j \in \mathbf{Z}_+$, then

$$N'_\sigma(f) \le \Big(\sum_{j=1}^{\infty} \frac{1}{j^2}\Big)\, N_\tau(f) \tag{30.25}$$

for every $f \in V_0$. If $\mathcal{N}'$ is the collection of all of these norms N'_σ on V_0, then it follows that $\mathcal{N}'$ determines the same topology on V_0 as $\mathcal{N}$ does. Hence the topology on V_0 associated to $\mathcal{N}'$ is also the same as the one induced on V_0 by the topology on V defined at the beginning of the section.

Let N be any seminorm on V_0, and let $\delta_j(l)$ be the function on $\mathbf{Z}_+$ equal to 1 when $j = l$ and to 0 otherwise. If

$$N(\delta_j) \le \sigma(j) \tag{30.26}$$

for each $j \in \mathbf{Z}_+$, then we get that

$$N(f) \le N'_\sigma(f) \tag{30.27}$$

for every $f \in V_0$. More precisely, if $f \in V_{0,n}$, then $f = \sum_{j=1}^n f(j)\,\delta_j$, and hence

$$N(f) \le \sum_{j=1}^{n} |f(j)|\,N(\delta_j) \le N'_\sigma(f). \tag{30.28}$$

This implies that open balls with respect to N are also open sets in V_0.

Let h be a real or complex-valued function on $\mathbf{Z}_+$, as appropriate, and consider

$$\lambda_h(f) = \sum_{j=1}^{\infty} f(j)\,h(j) \tag{30.29}$$

for $f \in V_0$. This defines a linear functional on V_0, and every linear functional on V_0 is of this form. If

$$|h(j)| \le \sigma(j) \tag{30.30}$$

for each $j \in \mathbf{Z}_+$, then it follows that

$$|\lambda_h(f)| \le N'_\sigma(f) \tag{30.31}$$

for every $f \in V_0$. Thus λ_h is continuous on V_0, and hence every linear functional on V_0 is continuous.

Part II
Algebras of functions

31 Homomorphisms

Let X be a set, and let $\mathcal{F}$ be an ultrafilter on X. If f is a real or complex-valued function on X, then $f_*(\mathcal{F})$ is an ultrafilter on $\mathbf{R}$ or $\mathbf{C}$, as appropriate, as in Section 18. If f is bounded on X, then one can check that $f_*(\mathcal{F})$ converges to an element of $\mathbf{R}$ or $\mathbf{C}$. In this case, it is a bit simpler to think of f as taking values in a compact subset K of $\mathbf{R}$ or $\mathbf{C}$, so that f maps $\mathcal{F}$ to an ultrafilter on K, which therefore converges. Although this is not quite the same as $f_*(\mathcal{F})$ as an ultrafilter on $\mathbf{R}$ or $\mathbf{C}$, they are almost the same, and converge to the same limit, as in Section 21.

Let $L_{\mathcal{F}}(f)$ denote the limit of $f_*(\mathcal{F})$, which may also be described as the limit of f along $\mathcal{F}$. If $p \in X$ and $\mathcal{F}$ is the ultrafilter $\mathcal{F}_p$ based at p as in the previous section, then $L_{\mathcal{F}}(f) = f(p)$ for every bounded function f on X. It is easy to see that every ultrafilter on X is of this type when X has only finitely many elements. Otherwise, if X is an infinite set, then the collection of subsets A of X such that $X \backslash A$ has only finitely many elements is a filter on X. Any ultrafilter on X which is a refinement of this filter is not the same as $\mathcal{F}_p$ for any $p \in X$.

Observe that

$$L_{\mathcal{F}}(f) \in \overline{f(X)} \tag{31.1}$$

for every $f \in \ell^\infty(X)$. In particular,

$$|L_{\mathcal{F}}(f)| \le \|f\|_\infty. \tag{31.2}$$

If f is a constant function on X, then $L_{\mathcal{F}}(f)$ is equal to this constant value. One can also check that

$$L_{\mathcal{F}}(f+g) = L_{\mathcal{F}}(f) + L_{\mathcal{F}}(f) \tag{31.3}$$

and

$$L_{\mathcal{F}}(f\, g) = L_{\mathcal{F}}(f)\, L_{\mathcal{F}}(g) \tag{31.4}$$

for every $f, g \in \ell^\infty(X)$. This is analogous to standard facts about the limits of a sum and product being equal to the corresponding sum or product of limits.

If X is an infinite set and $\mathcal{F} \neq \mathcal{F}_p$ for any $p \in X$, then one can check that $L_\mathcal{F}(f) = 0$ for every $f \in c_0(X)$. Similarly, $L_\mathcal{F}(f)$ is the same as the limit of $f(x)$ at infinity when $f \in c(X)$, as in Section 15. Remember that the limit of $f(x)$ at infinity defines a continuous linear functional on $c(X)$, with dual norm equal to 1 with respect to the ℓ^∞ norm. Thus $L_\mathcal{F}(f)$ is an extension of this linear functional on $c(X)$ to a continuous linear functional on $\ell^\infty(X)$, also with dual norm equal to 1. The existence of such an extension was mentioned before, as a consequence of the Hahn–Banach theorem.

32 Homomorphisms, continued

Let X be a nonempty set, and note that the product of two bounded real or complex-valued functions on X is bounded as well. Suppose that L is a linear functional on $\ell^\infty(X)$ which is a homomorphism with respect to multiplication of functions, in the sense that

$$L(f\,g) = L(f)\,L(g) \tag{32.1}$$

for every $f, g \in \ell^\infty(X)$. If $L(f) = 0$ for every $f \in \ell^\infty(X)$, then L satisfies these conditions trivially, and so we suppose that $L(f) \neq 0$ for at least one $f \in \ell^\infty(X)$. This implies that

$$L(\mathbf{1}_X) = 1, \tag{32.2}$$

where $\mathbf{1}_X$ is the constant function equal to 1 on X, since $\mathbf{1}_X\, f = f$ and hence

$$L(f) = L(\mathbf{1}_X\, f) = L(\mathbf{1}_X)\, L(f). \tag{32.3}$$

We would like to show that L is associated to an ultrafilter on X, as in the previous section.

If $A \subseteq X$, then let $\mathbf{1}_A(x)$ be the indicator function on X associated to A, equal to 1 when $x \in A$ and to 0 when $x \in X \backslash A$. Thus $\mathbf{1}_A^2 = \mathbf{1}_A$, which implies that

$$L(\mathbf{1}_A) = L(\mathbf{1}_A^2) = L(\mathbf{1}_A)^2, \tag{32.4}$$

and hence $L(\mathbf{1}_A) = 0$ or 1. Because $\mathbf{1}_A + \mathbf{1}_{X \backslash A} = \mathbf{1}_X$,

$$L(\mathbf{1}_A) + L(\mathbf{1}_{X \backslash A}) = L(\mathbf{1}_X) = 1, \tag{32.5}$$

so that exactly one of $L(\mathbf{1}_A)$ and $L(\mathbf{1}_{X \backslash A})$ is equal to 1. If $A, B \subseteq X$, then $\mathbf{1}_A\, \mathbf{1}_B = \mathbf{1}_{A \cap B}$, and so

$$L(\mathbf{1}_{A \cap B}) = L(\mathbf{1}_A)\, L(\mathbf{1}_B). \tag{32.6}$$

This shows that $L(\mathbf{1}_{A \cap B}) = 1$ when $L(\mathbf{1}_A) = L(\mathbf{1}_B) = 1$. Similarly, if $A \subseteq B$ and $L(\mathbf{1}_A) = 1$, then $A \cap B = A$, and we get that $L(\mathbf{1}_B) = 1$. Of course, $\mathbf{1}_A = 0$ when $A = \emptyset$, so that $L(\mathbf{1}_A) = 0$. If

$$\mathcal{F}_L = \{A \subseteq X : L(\mathbf{1}_A) = 1\}, \tag{32.7}$$

then it follows that $\mathcal{F}_L$ is a filter on X. More precisely, $\mathcal{F}_L$ is an ultrafilter on X, since A or $X \backslash A$ is in $\mathcal{F}_L$ for each $A \subseteq X$.

It is easy to see that $L(\mathbf{1}_A)$ is the same as the limit of $\mathbf{1}_A$ along $\mathcal{F}_L$ as in the previous section. This implies that $L(f)$ is equal to the limit of f along $\mathcal{F}_L$ when f is a finite linear combination of indicator functions of subsets of X, by linearity. One can also check that finite linear combinations of indicator functions of subsets of X are dense in $\ell^\infty(X)$. We already know that the limit along an ultrafilter defines a continuous linear functional on $\ell^\infty(X)$, as in the previous section, and we would like to check that L is also a continuous linear functional on $\ell^\infty(X)$. This would imply that $L(f)$ is equal to the limit of f along $\mathcal{F}$ for every $f \in \ell^\infty(X)$, by continuity and density.

Suppose that f is a bounded function on X such that $f(x) \neq 0$ for every $x \in X$ and $1/f$ is also bounded. Thus

$$L(f)\,L(1/f) = L(\mathbf{1}_X) = 1, \tag{32.8}$$

and hence $L(f) \neq 0$ in particular. Equivalently, $0 \in \overline{f(X)}$ when $L(f) = 0$. This implies that

$$L(f) \in \overline{f(X)} \tag{32.9}$$

for every $f \in \ell^\infty(X)$, since one can reduce to the case where $L(f) = 0$ by subtracting $L(f)\,\mathbf{1}_X$ from f, using the fact that $L(\mathbf{1}_X) = 1$. In particular,

$$|L(f)| \le \|f\|_\infty \tag{32.10}$$

for every $f \in \ell^\infty$, which implies that L is a continuous linear functional on $\ell^\infty(X)$ with dual norm equal to 1, as desired.

33 Bounded continuous functions

Let X be a topological space, and let $C_b(X)$ be the space of bounded real or complex-valued continuous functions on X. As usual, this may also be denoted $C_b(X, \mathbf{R})$ or $C_b(X, \mathbf{C})$, to indicate whether real or complex-valued functions are being used. Of course, $C_b(X)$ is the same as $\ell^\infty(X)$ when X is equipped with the discrete topology. If X is compact, then continuous functions are automatically bounded on X. Constant functions on X are always continuous, and the existence of nonconstant functions on X depends on the behavior of X.

Remember that sums and products of continuous functions are continuous. Similarly, sums and products of bounded functions are bounded, so that sums and products of bounded continuous functions are bounded and continuous. It follows that $C_b(X)$ is a vector space with respect to pointwise addition and scalar multiplication, and a commutative algebra with respect to multiplication of functions. The supremum norm on $C_b(X)$ is defined by

$$\|f\|_{sup} = \sup_{x \in X} |f(x)|, \tag{33.1}$$

and it is easy to see that this is indeed a norm. Moreover,

$$\|f\,g\|_{sup} \le \|f\|_{sup}\,\|g\|_{sup} \tag{33.2}$$

for every $f, g \in C_b(X)$.

Suppose that ϕ is a linear functional on $C_b(X)$ which is also a homomorphism with respect to multiplication of functions, in the sense that

$$\phi(f\,g) = \phi(f)\,\phi(g) \tag{33.3}$$

for every $f, g \in C_b(X)$. If $\phi(f) = 0$ for each $f \in C_b(X)$, then ϕ satisfies these conditions trivially, and so we also ask that $\phi(f) \ne 0$ for some $f \in C_b(X)$. As before, this implies that

$$\phi(\mathbf{1}_X) = 1, \tag{33.4}$$

where $\mathbf{1}_X$ is the constant function equal to 1 at every point in X. Of course,

$$\phi_p(f) = f(p) \tag{33.5}$$

has these properties for every $p \in X$.

If f is a bounded continuous function on X such that $f(x) \ne 0$ for every $x \in X$, then $1/f$ is also a continuous function on X. If $1/f$ is bounded as well, then

$$\phi(f)\,\phi(1/f) = \phi(\mathbf{1}_X) = 1, \tag{33.6}$$

which implies that $\phi(f) \ne 0$. If f is any bounded continuous function on X such that $\phi(f) = 0$, then it follows that $0 \in \overline{f(X)}$, since otherwise $1/f \in C_b(X)$. This implies that

$$\phi(f) \in \overline{f(X)} \tag{33.7}$$

for every $f \in C_b(X)$, by applying the previous statement to $f - \phi(f)\,\mathbf{1}_X$.

In particular,

$$|\phi(f)| \le \|f\|_{sup} \tag{33.8}$$

for every $f \in C_b(X)$, so that ϕ is a continuous linear functional on $C_b(X)$. The dual norm of ϕ with respect to the sumpremum norm is equal to 1, since $\phi(\mathbf{1}_X) = 1$. In the complex case, (33.7) implies that $\phi(f) \in \mathbf{R}$ when f is real-valued. In both the real and complex cases, we get that

$$\phi(f) \ge 0 \tag{33.9}$$

for every bounded nonnegative real-valued function f on X. If $A \subseteq X$ is both open and closed, then the corresponding indicator function $\mathbf{1}_A$ is continuous on X, and $\phi(\mathbf{1}_A)$ is either 0 or 1.

Let B^* be the closed unit ball in the dual of $C_b(X)$, with respect to the dual norm associated to the supremum norm on $C_b(X)$. Thus multiplicative homomorphisms on $C_b(X)$ are elements of B^*, because of (33.8). It is easy to see that the set of multiplicative homomorphisms on $C_b(X)$ is closed with respect to the weak* topology, since $\phi(f)$, $\phi(g)$, and $\phi(f\,g)$ are continuous functions of $\phi \in C_b(X)^*$ with respect to the weak* topology for every $f, g \in C_b(X)$. The set of nonzero multiplicative homomorphisms on $C_b(X)$ is also closed in the weak* topology, since it can be described by the additional condition $\phi(\mathbf{1}_X) = 1$, and $\phi(\mathbf{1}_X)$ is a continuous function of ϕ with respect to the weak* topology. Hence the set of nonzero multiplicative homomorphisms on $C_b(X)$ is compact with respect to the weak* topology, because it is a closed subset of B^*, which is compact by the Banach–Alaoglu theorem.

If $p \in X$, then $\phi_p(f) = f(p)$ is a nonzero multiplicative homomorphism on $C_b(X)$, as before. Thus $p \mapsto \phi_p$ defines a mapping from X into B^*. It is easy to see that this mapping is continuous with respect to the weak* topology on B^*, since $\phi_p(f) = f(p)$ is continuous on X for every $f \in C_b(X)$.

If X is equipped with the discrete topology, then $p \mapsto \phi_p$ is a one-to-one mapping of X into B^*, and the topology induced on the set

$$\{\phi_p : p \in X\} \tag{33.10}$$

by the weak* topology is the same as the discrete topology. If X is infinite, then of course this set is not compact. Let $\phi \in B^*$ be a limit point of this set with respect to the weak* topology, which is therefore not in the set. If $f \in c(X)$, then one can check that $\phi(f)$ is equal to the limit of $f(x)$ at infinity on X. This is another way to get homomorphisms on $\ell^\infty(X)$ extending the limit at infinity on $c(X)$.

34 Compact spaces

Let X be a compact topological space, and let $C(X)$ be the space of continuous real or complex-valued functions on X. This may also be denoted $C(X, \mathbf{R})$ or $C(X, \mathbf{C})$, to indicate whether real or complex-valued functions are being used. As before, continuous functions on compact spaces are automatically bounded, so that $C(X) = C_b(X)$. Let ϕ be a nonzero multiplicative homomorphism on $C(X)$, as in the previous section. We would like to show that there is a $p \in X$ such that $\phi(f) = f(p)$ for every $f \in C(X)$.

Suppose for the sake of a contradiction that for each $p \in X$ there is a continuous function f_p on X such that $\phi(f_p) \neq f_p(p)$. We may as well ask also that $\phi(f_p) = 0$, since otherwise we can replace f_p with $f_p - \phi(f_p)\, \mathbf{1}_X$, using the fact that $\phi(\mathbf{1}_X) = 1$. Thus $f_p(p) \neq 0$.

Similarly, we may suppose that f_p is a nonnegative real-valued function on X for each $p \in X$, by replacing f_p with $|f_p|^2$ if necessary. More precisely, in the real case,

$$\phi(|f_p|^2) = \phi(f_p^2) = \phi(f_p)^2 = 0, \tag{34.1}$$

while in the complex case,

$$\phi(|f_p|^2) = \phi(f_p \overline{f_p}) = \phi(f_p)\, \phi(\overline{f_p}) = 0. \tag{34.2}$$

Of course, we also get that $f_p(p) > 0$ after this substitution.

Consider

$$U(p) = \{x \in X : f_p(x) > 0\}. \tag{34.3}$$

This is an open set in X for each $p \in X$, because f_p is continuous, and $p \in U(p)$ by construction. Thus $U(p)$, $p \in X$, is an open covering of X, and so there are finitely many elements $p_1, \ldots, p_n$ of X such that

$$X = \bigcup_{j=1}^{n} U(p_j), \tag{34.4}$$

by compactness. If $f = \sum_{j=1}^{n} f_{p_j}$, then f is continuous on X, $\phi(f) = 0$, and $f(x) > 0$ for every $x \in X$. This is a contradiction, because $1/f$ is also a continuous function on X, which implies that $\phi(f) \neq 0$, as in the previous section.

35 Closed ideals

Let X be a topological space, and let $C(X)$ be the space of continuous real or complex-valued functions on X. As usual, this is a vector space with respect to pointwise addition and scalar multiplication, and a commutative algebra with respect to pointwise multiplication of functions. A linear subspace $\mathcal{I}$ of $C(X)$ is said to be an *ideal* if for every $a \in C(X)$ and $f \in \mathcal{I}$ we have that $a\, f \in C(X)$. In this section, we shall restrict our attention to compact Hausdorff spaces X, so that continuous functions on X are automatically bounded. We shall also be especially interested in ideals that are closed subsets of $C(X)$ with respect to the supremum norm.

If $E \subseteq X$, then let $\mathcal{I}_E$ be the collection of $f \in C(X)$ such that $f(x) = 0$ for every $x \in X$. It is easy to see that this is a closed ideal in $C(X)$, directly from the definitions. If $\overline{E}$ is the closure of E in X, then

$$\mathcal{I}_{\overline{E}} = \mathcal{I}_E, \tag{35.1}$$

because any continuous function that vanishes on E automatically vanishes on the closure of E as well. Thus we may as well restrict our attention to closed subsets E of X. We would like to show that any closed ideal $\mathcal{I}$ in $C(X)$ is of the form $\mathcal{I}_E$ for some closed set $E \subseteq X$.

If $\mathcal{I}$ is any subset of $C(X)$, then

$$E = \{x \in X : f(x) = 0\} \tag{35.2}$$

is a closed set in X. This is because the set where a continuous function is equal to 0 is a closed set, and E is the same as the intersection of the zero sets associated to the elements of $\mathcal{I}$. By construction,

$$\mathcal{I} \subseteq \mathcal{I}_E. \tag{35.3}$$

We would like to show that equality holds when $\mathcal{I}$ is a closed ideal in $C(X)$.

If ϕ is a real or complex-valued function on X, then the *support* of ϕ is denoted $\operatorname{supp} \phi$ and is defined to be the closure of the set of $x \in X$ such that

$\phi(x) \neq 0$. Suppose that ϕ is a continuous function on X whose support is contained in the complement of E in X. If $p \in \operatorname{supp} \phi$, so that $p \notin E$, then there is an $f_p \in \mathcal{I}$ such that $f_p(p) \neq 0$. Note that

$$U(p) = \{x \in X : f_p(x) \neq 0\} \tag{35.4}$$

is an open set in X, because f_p is continuous. Thus $U(p)$, $p \in \operatorname{supp} \phi$, is an open covering of $\operatorname{supp} \phi$ in X, since $p \in U(p)$ for each p. We also know that $\operatorname{supp} f$ is compact, because it is a closed set in a compact space. It follows that there are finitely many elements $p_1, \ldots, p_n$ of $\operatorname{supp} \phi$ such that

$$\operatorname{supp} \phi \subseteq \bigcup_{j=1}^{n} U(p_j). \tag{35.5}$$

Observe that

$$\sum_{l=1}^{n} |f_{p_l}(x)|^2 > 0 \tag{35.6}$$

for every $x \in \operatorname{supp} \phi$. Put

$$\psi(x) = \phi(x) \Big(\sum_{l=1}^{n} |f_{p_l}(x)|^2 \Big)^{-1}, \tag{35.7}$$

which is interpreted as being 0 when $x \notin \operatorname{supp} \phi$. This is a continuous function on X, because it is equal to 0 on a neighborhood of every $x \in X \backslash \operatorname{supp} \phi$, and because it is a quotient of continuous functions with nonzero denominator on a neighborhood of every $x \in \operatorname{supp} \phi$. In the real case, we have that

$$\phi(x) = \sum_{j=1}^{n} (\psi(x)\, f_j(x))\, f_j(x) \tag{35.8}$$

for every $x \in X$, and in the complex case we have that

$$\phi(x) = \sum_{j=1}^{n} (\psi(x)\, \overline{f_j(x)})\, f_j(x), \tag{35.9}$$

where $\overline{f_j(x)}$ is the complex conjugate of $f_j(x)$. This implies that $\phi \in \mathcal{I}$, since $f_j \in \mathcal{I}$ for each j, $\psi\, f_j \in C(X)$ in the real case and $\psi\, \overline{f_j} \in C(X)$ in the complex case, and $\mathcal{I}$ is an ideal.

Now let f be a continuous function on X such that $f(x) = 0$ for every $x \in E$, and let $\epsilon > 0$ be given. Thus

$$K(\epsilon) = \{x \in X : |f(x)| \geq \epsilon\} \tag{35.10}$$

is a closed set in X contained in the complement of E. Let $V(\epsilon)$ be an open set in X such that $K(\epsilon) \subseteq V(\epsilon)$ and $\overline{V(\epsilon)} \subseteq X \backslash E$. By Urysohn's lemma, there is a continuous real-valued function θ_ϵ on X such that $\theta(x) = 1$ for every $x \in K(\epsilon)$,

$\theta_\epsilon(x) = 0$ when $x \notin V(\epsilon)$, and $0 \le \theta_\epsilon(x) \le 1$ for every $x \in X$. In particular, the support of θ_ϵ is contained in $\overline{V(\epsilon)}$, which is contained in the complement of E. Of course, the support of $\theta_\epsilon\, f$ is contained in the support of θ_ϵ. Hence

$$\theta_\epsilon\, f \in \mathcal{I} \tag{35.11}$$

for each $\epsilon > 0$, by the discussion in the previous paragraph. Moreover,

$$|\theta_\epsilon(x)\, f(x) - f(x)| = (1 - \theta_\epsilon(x))\, |f(x)| < \epsilon \tag{35.12}$$

for every $x \in X$, because $1 - \theta_\epsilon(x) = 0$ when $x \in K(\epsilon)$, $|f(x)| < \epsilon$ when $x \in X \backslash K(\epsilon)$, and $0 \le \theta_\epsilon(x) \le 1$ for every $x \in X$. This implies that $\theta_\epsilon\, f \to f$ uniformly on X as $\epsilon \to 0$. Thus $f \in \mathcal{I}$ when $\mathcal{I}$ is closed with respect to the supremum norm, since $\theta_\epsilon\, f \in \mathcal{I}$ for each $\epsilon > 0$. This shows that $\mathcal{I} = \mathcal{I}_E$ when $\mathcal{I}$ is a closed ideal in $C(X)$ and E is associated to $\mathcal{I}$ as before.

Let E be any closed set in X, and consider the corresponding closed ideal $\mathcal{I}_E$. In particular, $\mathcal{I}_E$ is a linear subspace of $C(X)$, and the quotient space $C(X)/\mathcal{I}_E$ can be defined as a real or complex vector space, as appropriate. By standard arguments in abstract algebra, there is a natural operation of multiplication on the quotient, so that the quotient mapping from $C(X)$ onto $C(X)/\mathcal{I}_E$ is a multiplicative homomorphism, because $\mathcal{I}_E$ is an ideal in $C(X)$. If $E = \emptyset$, then $\mathcal{I}_E = C(X)$, and $C(X)/\mathcal{I}_E = \{0\}$, and so we suppose from now on that $E \ne \emptyset$. We also have a homomorphism $R_E : C(X) \to C(E)$, defined by sending a continuous function f on X to its restriction $R_E(f)$ to E. The kernel of this homomorphism is equal to $\mathcal{I}_E$, which leads to a one-to-one homomorphism $r_E : C(X)/\mathcal{I}_E \to C(E)$. By the Tietze extension theorem, every continuous function on E has an extension to a continuous function on X. This says exactly that $R_E(C(X)) = C(E)$, and hence that r_E maps $C(X)/\mathcal{I}_E$ onto $C(E)$.

36 Locally compact spaces

Let X be a locally compact Hausdorff topological space, and let $C(X)$ be the space of continuous real or complex-valued functions on X, as usual. If $K \subseteq X$ is nonempty and compact, then the corresponding *supremum seminorm* is defined on $C(X)$ by

$$\|f\|_K = \sup_{x \in K} |f(x)|. \tag{36.1}$$

Of course, every continuous function f on X is bounded on K, because $f(K)$ is a compact set in $\mathbf{R}$ or $\mathbf{C}$, as appropriate. It is easy to see that this is indeed a seminorm on $C(X)$, and that

$$\|f\, g\|_K \le \|f\|_K\, \|g\|_K \tag{36.2}$$

for every $f, g \in C(X)$. It follows that multiplication of functions is continuous as a mapping from $C(X) \times C(X)$ into $C(X)$ with respect to the topology on $C(X)$ determined by the collection of supremum seminorms associated to nonempty compact subsets of X.

If X is compact, then we can take $X = K$, and simply use the supremum norm on X. Thus we shall focus on the case where X is not compact in this section. Suppose that X is *σ-compact*, so that there is a sequence $K_1, K_2, \ldots$ of compact subsets of X such that $X = \bigcup_{l=1}^{\infty} K_l$. We may also ask that $K_l \neq \emptyset$ and $K_l \subseteq K_{l+1}$ for each l, by replacing K_l with the union of $K_1, \ldots, K_l$ if necessary. Moreover, we can enlarge these compact sets in such a way that K_l is contained in the interior of K_{l+1} for each l. This uses the local compactness of X, to get that any compact set in X is contained in the interior of another compact set. In particular, it follows that the union of the interiors of the K_l's is all of X under these conditions. If H is any compact set in X, then the interiors of the K_l's form an open covering of H, for which there is a finite subcovering. This implies that H is contained in the interior of K_l for some l, since the K_l's are increasing. Hence $H \subseteq K_l$ for some l, which implies that the supremum seminorms associated to the K_l's determine the same topology on $C(K)$ as the collection of supremum seminorms corresponding to all nonempty compact subsets of X. Therefore this topology on $C(X)$ is metrizable in this case.

If $E \subseteq X$ and $\mathcal{I}_E$ is the collection of $f \in C(X)$ such that $f(x) = 0$ for every $x \in E$, then $\mathcal{I}_E$ is a closed ideal in $C(X)$, as in the previous section. We also have that $\mathcal{I}_{\overline{E}} = \mathcal{I}_E$, where $\overline{E}$ is the closure of E in X. If $\mathcal{I}$ is any subset of X and E is the set of $x \in X$ such that $f(x) = 0$ for every $f \in \mathcal{I}$, then E is a closed set in X, and $\mathcal{I} \subseteq \mathcal{I}_E$. We would like to show that $\mathcal{I} = \mathcal{I}_E$ when $\mathcal{I}$ is a closed ideal in $C(X)$, as before. Let f be a continuous function on X such that $f(x) = 0$ for every $x \in E$, and let us check that $f \in \mathcal{I}$.

If f has compact support contained in the complement of E, then one can show that $f \in \mathcal{I}$ in the same way as in the previous section. Otherwise, it suffices to show that f can be approximated by continuous functions with compact support contained in $X \backslash E$ in the topology of $C(X)$. Let K be a nonempty compact set in X, and let $\epsilon > 0$ be given. Thus

$$K(\epsilon) = \{x \in K : |f(x)| \geq \epsilon\} \tag{36.3}$$

is a compact set in X, since it is the intersection of the compact set K with the closed set where $|f(x)| \geq \epsilon$. Also, $K(\epsilon) \subseteq X \backslash E$, because $f = 0$ on E by hypothesis. Let $V(\epsilon)$ be an open set in X such that $K(\epsilon) \subseteq V(\epsilon)$, $\overline{V(\epsilon)}$ is compact, and $\overline{V(\epsilon)} \subseteq X \backslash E$. This is possible, because X is locally compact and Hausdorff. By Urysohn's lemma, there is a continuous real-valued function θ_ϵ on X which satisfies $\theta_\epsilon(x) = 1$ when $x \in K(\epsilon)$, $\theta_\epsilon(x) = 0$ when $x \in X \backslash V(\epsilon)$, and $0 \leq \theta_\epsilon \leq 1$ on all of X. In particular, the support of θ_ϵ is contained in $\overline{V(\epsilon)}$, which is a compact subset of $X \backslash E$. Hence $\theta_\epsilon \, f$ is a continuous function with compact support in $X \backslash E$, which implies that $\theta_\epsilon \, f \in \mathcal{I}$. We also have that

$$|\theta_\epsilon(x) \, f(x) - f(x)| = (1 - \theta_\epsilon(x)) \, |f(x)| < \epsilon \tag{36.4}$$

for every $x \in K$, because $1 - \theta_\epsilon(x) = 0$ when $x \in K(\epsilon)$, $|f(x)| < \epsilon$ when $x \in K \backslash K(\epsilon)$, and $0 \leq \theta_\epsilon(x) \leq 1$ for every $x \in X$. This shows that f can be approximated by elements of $\mathcal{I}$ in the topology of $C(X)$, which implies that $f \in \mathcal{I}$, as desired, since $\mathcal{I}$ is supposed to be closed in $C(X)$.

37 Locally compact spaces, continued

Let X be a locally compact Hausdorff space, and let $C_{com}(X)$ be the space of continuous real or complex-valued functions on X with compact support. As usual, this may be denoted $C_{com}(X, \mathbf{R})$ or $C_{com}(X, \mathbf{C})$, to indicate whether real or complex-valued functions are being used. If $K \subseteq X$ is compact, then there is an open set V in X such that $K \subseteq V$ and $\overline{V}$ is compact, because X is locally compact. Urysohn's lemma implies that there is a continuous real-valued function θ on X such that $\theta(x) = 1$ when $x \in K$, $\theta(x) = 0$ when $x \in X \backslash V$, and $0 \le \theta(x) \le 1$ for every $x \in X$. Thus the support of θ is contained in $\overline{V}$, and hence is compact. If f is any continuous function on X, then $\theta\, f$ is a continuous function on X with compact support that is equal to f on K. In particular, this implies that $C_{com}(X)$ is dense in $C(X)$ with respect to the topology determined by supremum seminorms associated to nonempty compact subsets of X.

Suppose that λ is a continuous linear functional on $C(X)$ with respect to this topology. This implies that there is a nonempty compact set $K \subseteq X$ and an nonnegative real number A such that

$$|\lambda(f)| \le A\, \|f\|_K \tag{37.1}$$

for every $f \in C(X)$. In this context, it is not necessary to take the maximum of finitely many seminorms on the right side of this inequality, because the union of finitely many compact subsets of X is also compact. Note that $\lambda(f) = 0$ when $f(x) = 0$ for every $x \in K$, so that $\lambda(f)$ depends only on the restriction of f to K.

Let X^* be the one-point compactification of X. Thus X^* is a compact Hausdorff space consisting of the elements of X and an additional element "at infinity", for which the induced topology on X as a subset of X^* is the same as its given topology. By construction, a set $K \subseteq X$ is closed as a subset of X^* if and only if it is compact in X. In this case, Tietze's extension theorem implies that every continuous function on K can be extended to a continuous function on X^*, and to a continuous function on X in particular. If X is already compact, then one can simply use X instead of X^*.

If λ is a continuous linear functional on $C(X)$ that satisfies (37.1), then it follows that λ corresponds to a continuous linear functional λ_K on $C(K)$ in a natural way. More precisely, if g is a continuous function on K, then there is a continuous function f on X such that $f = g$ on K, and we put

$$\lambda_K(g) = \lambda(f). \tag{37.2}$$

This does not depend on the particular extension f of g, by the earlier remarks. By construction, λ_K satisfies the same continuity condition on $C(K)$ as λ does on $C(X)$, with the same constant A.

Let $C_0(X)$ be the space of continuous functions f on X that vanish at infinity. This means that for every $\epsilon > 0$ there is a compact set $K_\epsilon \subseteq X$ such that

$$|f(x)| < \epsilon \tag{37.3}$$

for every $x \in X \backslash K_\epsilon$. This space may also be denoted $C_0(X, \mathbf{R})$ or $C_0(X, \mathbf{C})$, to indicate whether real or complex-valued functions are being used. If X is compact, then one can take $K_\epsilon = X$ for each ϵ, and $C_0(X) = C(X)$. If X is not compact, then $f \in C_0(X)$ if and only if f has a continuous extension to the one-point compactification X^* of X which is equal to 0 at the point at infinity.

Note that continuous functions on X that vanish at infinity are automatically bounded, so that $C_0(X) \subseteq C_b(X)$. It is not difficult to check that $C_0(X)$ is a closed linear subspace of $C_b(X)$, with respect to the supremum norm. Of course, continuous functions with compact support automatically vanish at infinity, so that $C_{com}(X) \subseteq C_0(X)$. One can also check that $C_0(X)$ is the same as the closure of $C_{com}(X)$ in $C_b(X)$ with respect to the supremum norm, using functions θ as before. This is all trivial when X is compact, in which case these spaces are all the same as $C(X)$.

Suppose that X is not compact, let f be a real or complex-valued continuous function on X, and let a be a real or complex number, as appropriate. We say that $f(x) \to a$ as $x \to \infty$ in X if for each $\epsilon > 0$ there is a compact set $K_\epsilon \subseteq X$ such that

$$|f(x) - a| < \epsilon \tag{37.4}$$

for every $x \in X \backslash K_\epsilon$. Thus f vanishes at infinity if and only if this holds with $a = 0$. If a is any real or complex number, then $f(x) \to a$ as $x \to \infty$ in X if and only if $f(x) - a$ vanishes at infinity. It is easy to see that the limit a is unique when it exists. Similarly, $f(x) \to a$ as $x \to \infty$ in X if and only if f has a continuous extension to the one-point compactification X^* of X which is equal to a at the point at infinity. Note that f is bounded when f has a limit at infinity. One can also check that the collection of continuous functions on X which have a limit at infinity is a closed linear subspace of $C_b(X)$ with respect to the supremum norm.

38 σ-Compactness

Let X be a topological space, and let $\{U_\alpha\}_{\alpha \in A}$ be a collection of open subsets of X such that $\bigcup_{\alpha \in A} U_\alpha = X$, which is to say an open covering of X. Suppose that X is σ-compact, so that there is a sequence $K_1, K_2, \ldots$ of compact subsets of X such that $X = \bigcup_{l=1}^\infty K_l$. Because $\{U_\alpha\}_{\alpha \in A}$ is an open covering of K_l for each l and K_l is compact, there is a finite set of indices $A_l \subseteq A$ such that $K_l \subseteq \bigcup_{\alpha \in A_l} U_\alpha$. If $B = \bigcup_{l=1}^\infty A_l$, then B has only finitely or countably many elements, and $\bigcup_{\alpha \in B} U_\alpha = X$. Conversely, if X is locally compact and every open covering of X can be reduced to a subcovering with only finitely or countably many elements, then X is σ-compact. This follows by using local compactness to cover X by open sets that are contained in compact sets. In particular, if X is locally compact and there is a base for the topology of X with only finitely or countably many elements, then X is σ-compact, since every open covering of X can be reduced to a subcovering with only finitely or countably many elements in this case.

Suppose that the topology on X is determined by a metric. It is well known that there is a base for the topology of X with only finitely or countably many elements if and only if X is separable, in the sense that there is a dense set in X with only finitely or countably many elements. Compact metric spaces are separable, and it follows that X is separable when X is σ-compact. Urysohn's famous metrization theorem states that a regular topological space is metrizable when there is a countable base for its topology. Note that locally compact Hausdorff spaces are automatically regular.

Suppose now that X is a locally compact Hausdorff topological space which is σ-compact. As before, this implies that there is a sequence $K_1, K_2, \ldots$ of compact subsets of X such that $X = \bigcup_{l=1}^{\infty} K_l$ and K_l is contained in the interior of K_{l+1} for each l. By Urysohn's lemma, there is a continuous real-valued function θ_l on X for each positive integer l such that $\theta(x) > 0$ when $x \in K_l$, $0 \le \theta_l(x) \le 1$ for every $x \in X$, and the support of θ_l is contained in K_{l+1}. Let $a_1, a_2, \ldots$ be a sequence of positive real numbers such that $\sum_{l=1}^{\infty} a_l$ converges, and consider

$$f(x) = \sum_{l=1}^{\infty} a_l \, \theta_l(x). \tag{38.1}$$

This series converges everywhere on X, by the comparison test. The partial sums of this series converge uniformly on X, as in Weierstrass' M-test. Thus f is a continuous function on X, which also vanishes at infinity, because θ_l has compact support for each l. Moreover, $f(x) > 0$ for every $x \in X$, by construction.

39 Homomorphisms, revisited

Let X be a locally compact Hausdorff topological space, and let $C(X)$ be the algebra of real or complex-valued continuous functions on X. Also let ϕ be linear functional on $C(X)$ which is a homomorphism with respect to multiplication. If $\phi(f) \ne 0$ for some $f \in C(X)$, then it follows that $\phi(\mathbf{1}_X) = 1$, where $\mathbf{1}_X$ is the constant function equal to 1 on X, as before. Let us suppose from now on that this is the case. If f is a continuous function on X such that $f(x) \ne 0$ for every $x \in X$, then $1/f$ is defines a continuous function on X as well. This implies that $\phi(f) \ne 0$, since

$$\phi(f)\, \phi(1/f) = \phi(\mathbf{1}_X) = 1. \tag{39.1}$$

If f is any continuous function on X and c is a real or complex number, as appropriate, such that $c \notin f(X)$, then $g = f - c\,\mathbf{1}_X$ is a continuous function on X such that $g(x) \ne 0$ for every $x \in X$, so that $\phi(g) \ne 0$. Thus $\phi(f) \ne c$, and hence

$$\phi(f) \in f(X). \tag{39.2}$$

In particular, if $C(X)$ is the algebra of complex-valued continuous functions on X, and f happens to be real-valued, then it follows that $\phi(f) \in \mathbf{R}$.

Suppose now that ϕ is continuous with respect to the topology on $C(X)$ determined by the supremum seminorms corresponding to nonempty compact subsets of X. This means that there is a nonempty compact set $K \subseteq X$ and a nonnegative real number A such that

$$|\phi(f)| \le A \, \|f\|_K \tag{39.3}$$

for every $f \in C(X)$, as in Section 37. In particular, $\phi(f) = 0$ when $f(x) = 0$ for every $x \in K$, so that $\phi(f)$ depends only on the restriction of f to K. As in Section 37 again, every continuous real or complex-valued function on K has a continuous extension to X, so that ϕ determines a continuous linear functional ϕ_K on $C(K)$. It is easy to see that ϕ_K is also a homomorphism with respect to multiplication on $C(K)$. Hence there is a $p \in K$ such that $\phi_K(f) = f(p)$ for every $f \in C(K)$, as in Section 34. This implies that

$$\phi(f) = f(p) \tag{39.4}$$

for every $f \in C(X)$.

Alternatively, consider

$$\mathcal{I}_\phi = \{f \in C(X) : \phi(f) = 0\}. \tag{39.5}$$

It is easy to see that this is a closed ideal in $C(X)$ when ϕ is a continuous homomorphism on $C(X)$. As in Section 36, there is a closed set $E \subseteq X$ such that $\mathcal{I}_\phi = \mathcal{I}_E$, where $\mathcal{I}_E$ consists of $f \in C(X)$ such that $f(x) = 0$ for every $x \in E$. Note that $\mathcal{I}_\phi$ has codimension 1 as a linear subspace of $C(X)$, since it is the same as the kernel of the nonzero linear functional ϕ. Using this, one can check that E has exactly one element, which may be denoted p. Thus $\phi(f) = 0$ for every $f \in C(X)$ such that $f(p) = 0$. If f is any continuous function on X, then $f - f(p)\,\mathbf{1}_X$ is equal to 0 at p, and hence $\phi(f - f(p)\,\mathbf{1}_X) = 0$. This implies that $\phi(f) = f(p)$ for every $f \in C(X)$, since $\phi(\mathbf{1}_X) = 1$.

Remember that the same conclusion holds for every nonzero homomorphism ϕ on $C(X)$ when X is compact, without the additional hypothesis of continuity, as in Section 34. Suppose now that X is a locally compact Hausdorff which is not compact but σ-compact, and that ϕ is a nonzero homomorphism on $C(X)$. Let X^* be the one-point compactification of X, and note that the space $C(X^*)$ of continuous functions on X^* can be identified with the subalgebra of $C(X)$ consisting of functions with a limit at infinity, as in Section 37. The restriction of ϕ to this subalgebra determines a homomorphism on $C(X^*)$, which is nonzero because it sends constant functions to their constant values. It follows that there is a $p \in X^*$ such that $\phi(f) = f(p)$ when $f \in C(X)$ has a limit at infinity, as in Section 34. If p is the point at infinity in X^*, then $f(p)$ refers to the limit of f at infinity on X. Let us check that p cannot be the point at infinity in X^* when X is σ-compact. In this case, there is a continuous real-valued function f on X that vanishes at infinity such that $f(x) > 0$ for every $x \in X$, as in the previous section. Because ϕ is defined on all of $C(X)$, we also have that $\phi(f) \ne 0$, as discussed at the beginning of the section. If p were the point at infinity, then we would have that $\phi(f) = 0$, since $f \in C_0(X)$. Thus $p \in X^*$ is not the point at infinity, which means that $p \in X$. If g is any bounded continuous function on X, then $f\,g \in C_0(X)$, which implies that

$$\phi(f\,g) = f(p)\,g(p), \tag{39.6}$$

and so

$$\phi(f)\,\phi(g) = f(p)\,g(p), \tag{39.7}$$

because ϕ is a homomorphism on $C(X)$. This shows that $\phi(g) = g(p)$ for every bounded continuous function g on X. If h is any continuous function on X and $\epsilon > 0$, then

$$h_\epsilon = \frac{h}{1 + \epsilon\,|h|^2} \tag{39.8}$$

is a bounded continuous function on X, and so $\phi(h_\epsilon) = h_\epsilon(p)$. One can also check that

$$\phi(h_\epsilon) = \frac{\phi(h)}{1 + \epsilon\,|\phi(h)|^2} \tag{39.9}$$

for every $\epsilon > 0$, because ϕ is a homomorphism. Hence

$$\frac{\phi(h)}{1 + \epsilon\,|\phi(h)|^2} = \frac{h(p)}{1 + \epsilon\,|h(p)|^2} \tag{39.10}$$

for every $\epsilon > 0$, which implies that $\phi(h) = h(p)$ for every $h \in C(X)$.

40 σ-Compactness, continued

Let X be a locally compact Hausdorff topological space which is σ-compact, and let $K_1, K_2, \ldots$ be a sequence of compact subsets of X such that $X = \bigcup_{l=1}^\infty K_l$ and K_l is contained in the interior of K_{l+1} for each l. By Urysohn's lemma, there is a continuous real-valued function θ_l on X for each positive integer l such that $\theta_l(x) = 1$ for every x in a neighborhood of K_l, $0 \le \theta_l(x) \le 1$ for every $x \in X$, and the support of θ_l is contained in K_{l+1}. In particular, $\theta_l(x) \le \theta_{l+1}(x)$ for each $x \in X$ and $l \ge 1$. It will be convenient to also put $K_0 = \emptyset$ and $\theta_0 = 0$. Let $b_1, b_2, \ldots$ be a sequence of nonnegative real numbers, and consider

$$B(x) = b_1\,\theta_1(x) + \sum_{l=2}^\infty b_l\,(\theta_l(x) - \theta_{l-2}(x)). \tag{40.1}$$

Note that $\theta_l(x) - \theta_{l-2}(x) = 0$ for every x in a neighborhood of K_{l-2}, and when $x \in X \backslash K_{l+1}$, for $l \ge 2$. This implies that at most three terms on the right side of (40.1) are different from 0 for any $x \in X$, and more precisely that every $x \in X$ has a neighborhood on which at most three terms on the right side of (40.1) are different from 0, so that $B(x)$ is continuous on X. We also have that

$$B(x) \ge b_1\,\theta_1(x) \ge b_1 \tag{40.2}$$

when $x \in K_1$, and

$$B(x) \ge b_l\,(\theta_l(x) - \theta_{l-2}(x)) \ge b_l \tag{40.3}$$

when $x \in K_l \backslash K_{l-1}$, $l \geq 2$.

Suppose that E is a bounded subset of the space $C(X)$ of continuous real or complex-valued continuous functions on X with respect to the collection of supremum seminorms associated to nonempty compact subsets of X. Thus the elements of E are uniformly bounded on compact subsets of X, and so for each positive integer l there is a nonnegative real number b_l such that

$$|f(x)| \leq b_l \tag{40.4}$$

for every $f \in E$ and $x \in K_l$. This implies that

$$|f(x)| \leq B(x) \tag{40.5}$$

for every $f \in E$ and $x \in X$, where B is as in the previous paragraph.

Now let ϕ be a linear functional on $C(X)$ which is a homomorphism with respect to multiplication, and which satisfies $\phi(f) \neq 0$ for some $f \in C(X)$. As in the previous section, $\phi(f) \in f(X)$ for every $f \in C(X)$, and in particular $\phi(f) \geq 0$ when f is a nonnegative real-valued continuous function on X. If $f \in C(X)$ satisfies (40.5), then it follows that

$$|\phi(f)| \leq \phi(B). \tag{40.6}$$

More precisely, if f is real-valued, then $B(x) \pm f(x) \geq 0$ for each $x \in X$, and so

$$\phi(B) \pm \phi(f) = \phi(B \pm f) \geq 0. \tag{40.7}$$

Similarly, if f is complex-valued, then one can use the fact that $\operatorname{Re} \alpha f(x) \leq B$ for each $\alpha \in \mathbf{C}$ with $|\alpha| = 1$ to get that

$$\operatorname{Re} \alpha\, \phi(f) = \phi(\operatorname{Re} \alpha\, f) \leq \phi(B), \tag{40.8}$$

which implies (40.6).

This shows that ϕ is uniformly bounded on every bounded set $E \subseteq C(X)$ with respect to the collection of supremum seminorms associated to nonempty compact subsets of X. There is also a countable local base for the topology at 0 in $C(X)$ with respect to this collection of seminorms, because X is σ-compact. It follows that ϕ is continuous with respect to this topology on $C(X)$, by the result discussed in Section 27. This gives another way to show that there is a point $p \in X$ such that $\phi(f) = f(p)$ for every $f \in C(X)$, by reducing to the case of continuous homomorphisms, as in the previous section.

41 Holomorphic functions

Let U be a nonempty open set in the complex plane $\mathbf{C}$, and let $C(U)$ be the algebra of continuous complex-valued functions on U. Of course, U is locally compact with respect to the topology inherited from the standard topology on $\mathbf{C}$, and it is also σ-compact, because it is a separable metric space, and hence has a countable base for its topology. As usual, $C(U)$ gets a nice topology from the collection of supremum seminorms associated to nonempty compact subsets of U.

Remember that a complex-valued function $f(z)$ on U is said to be complex-analytic or holomorphic if the complex derivative

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \tag{41.1}$$

exists at every point z in U. In particular, the existence of the limit implies that f is continuous, so that the space $\mathcal{H}(U)$ of holomorphic functions on U is contained in $C(U)$. More precisely, $\mathcal{H}(U)$ is a linear subspace of $C(U)$, which is actually a subalgebra, because the product of two holomorphic functions is holomorphic as well. Note that constant functions on U are automatically holomorphic, since they have derivative equal to 0 at every point.

It is well known that $\mathcal{H}(U)$ is closed in $C(U)$, with respect to the topology determined by the collection of supremum seminorms associated to nonempty compact subsets of U. This is equivalent to the statement that if $\{f_j\}_{j=1}^\infty$ is a sequence of holomorphic functions on U that converges uniformly on compact subsets of U to a function f on U, then f is also holomorphic on U. To see this, one can use the Cauchy integral formula to show that the sequence of derivatives $\{f_j'\}_{j=1}^\infty$ converges uniformly on compact subsets of U, and that the limit is equal to the derivative f' of f.

Let ϕ be a linear functional on $\mathcal{H}(U)$ which is a homomorphism with respect to multiplication. As before, if $\phi(f) \neq 0$ for some $f \in \mathcal{H}(U)$, then $\phi(\mathbf{1}_U) = 1$, where $\mathbf{1}_U$ is the constant function on U equal to 1. Let us suppose from now on that this is the case. If f is a holomorphic function on U such that $f(z) \neq 0$ for every $z \in U$, then it is well known that $1/f$ is holomorphic on U too. This implies that

$$\phi(f)\,\phi(1/f) = \phi(\mathbf{1}_U) = 1, \tag{41.2}$$

and hence $\phi(f) \neq 0$. If c is a complex number such that $c \notin f(U)$, then we can apply this to $f - c\,\mathbf{1}_U$ to get that $\phi(f) \neq c$. Thus $\phi(f) \in f(U)$, as in the context of continuous functions. In particular, this holds when $f(z) = z$ for every $z \in U$, which is holomorphic with derivative equal to 1 at every point. If $\phi(f)$ is denoted p when $f(z) = z$ for every $z \in U$, then it follows that $p \in U$. We would like to show that

$$\phi(g) = g(p) \tag{41.3}$$

for every $g \in \mathcal{H}(U)$. If $g(p) = 0$, then g can be expressed as

$$g(z) = (z - p)\,h(z) \tag{41.4}$$

for some $h \in \mathcal{H}(U)$, by standard results in complex analysis. This implies that $\phi(g) = 0$, by the definition of p and the fact that ϕ is a homomorphism. If $g(p) \neq 0$, then one can reduce to the case where $g(p) = 0$ by subtracting a constant from g.

42 The disk algebra

Let U be the open unit disk in the complex plane $\mathbf{C}$,

$$U = \{z \in \mathbf{C} : |z| < 1\}. \tag{42.1}$$

Thus the closure $\overline{U}$ of U is the closed unit disk,

$$\overline{U} = \{z \in \mathbf{C} : |z| \le 1\}, \tag{42.2}$$

and the boundary ∂U of U is the same as the unit circle,

$$\partial U = \{z \in \mathbf{C} : |z| = 1\}. \tag{42.3}$$

Let $C(\overline{U})$ be the algebra of continuous complex-valued functions on $\overline{U}$, equipped with the supremum norm.

Let $\mathcal{A}$ be the collection of $f \in C(\overline{U})$ such that the restriction of f to U is holomorphic. Thus $\mathcal{A}$ is a subalgebra of $C(\overline{U})$, since sums and products of holomorphic functions are also holomorphic, which is known as the *disk algebra*. Note that constant functions on $\overline{U}$ are elements of $\mathcal{A}$, and that $\mathcal{A}$ is a closed set in $C(\overline{U})$ with respect to the supremum norm, for the same reasons as in the previous section. If $f \in \mathcal{A}$ and $f(z) \ne 0$ for every $z \in \overline{U}$, then $1/f$ is continuous on $\overline{U}$ and holomorphic on U, and hence is in $\mathcal{A}$ too.

If $f \in C(\overline{U})$ and $0 \le r < 1$, then

$$f_r(z) = f(r\,z) \tag{42.4}$$

is an element of $C(\overline{U})$ as well. Note that f is automatically uniformly continuous on $\overline{U}$, because f is continuous on $\overline{U}$ and $\overline{U}$ is a compact set in a metric space. Using this, it is easy to see that $f_r \to f$ uniformly on $\overline{U}$ as $r \to 1$.

If f is a holomorphic function on the open unit disk U, then

$$f(z) = \sum_{j=0}^{\infty} a_j\, z^j \tag{42.5}$$

for some complex numbers $a_0, a_1, \ldots$ and every $z \in U$. More precisely, z^j is interpreted as being equal to 1 for every z when $j = 0$, and the convergence of the series when $|z| < 1$ is part of the conclusion. The series actually converges absolutely for every $z \in U$, and the partial sums converge uniformly on compact subsets of U.

If $0 \le r < 1$, then

$$f_r(z) = f(r\,z) = \sum_{j=0}^{\infty} a_j r^j\, z^j \tag{42.6}$$

for every $z \in \overline{U}$. Under these conditions, the series converges absolutely when $|z| \le 1$, and the partial sums converge uniformly on $\overline{U}$, by the remarks in the previous paragraph. If $f \in \mathcal{A}$, then f can be approximated uniformly by f_r as

$r \to 1$, and f_r is approximated uniformly by partial sums of its series expansion for each $r < 1$. It follows that f can be approximated uniformly by polynomials in z on $\overline{U}$ when $f \in \mathcal{A}$.

Let ϕ be a linear functional on $\mathcal{A}$ which is a homomorphism with respect to multiplication. As usual, we suppose that $\phi(f) \neq 0$ for some $f \in \mathcal{A}$, so that ϕ sends constant functions on $\overline{U}$ to their constant values. If $f \in \mathcal{A}$ and $f(z) \neq 0$ for every $z \in \overline{U}$, then $1/f \in \mathcal{A}$, and we get that $\phi(f) \neq 0$. This implies that

$$\phi(f) \in f(\overline{U}) \tag{42.7}$$

for every $f \in \mathcal{A}$, as before. In particular,

$$|\phi(f)| \le \sup_{|z| \le 1} |f(z)| \tag{42.8}$$

for every $f \in \mathcal{A}$, so that ϕ is continuous with respect to the supremum norm on $\mathcal{A}$. Of course, $f(z) = z$ defines an element of $\mathcal{A}$, and we can put $\phi(f) = p$ for this choice of f. Note that $p \in \overline{U}$, by the previous remarks. If g is a polynomial in z, then

$$\phi(g) = g(p), \tag{42.9}$$

because ϕ is a homomorphism. This also works for every $g \in \mathcal{A}$, because polynomials are dense in $\mathcal{A}$ with respect to the supremum norm, and because ϕ is continuous on $\mathcal{A}$ with respect to the supremum norm.

If $f \in \mathcal{A}$, then

$$\sup_{|z|=1} |f(z)| = \sup_{|z| \le 1} |f(z)|, \tag{42.10}$$

by the maximum modulus principle. In particular, if $f(z) = 0$ for every $z \in \partial U$, then $f(z) = 0$ for every $z \in \overline{U}$. This implies that f is determined on the closed disk $\overline{U}$ by its restriction to the unit circle ∂U. Using this, one can identify the disk algebra with a closed subalgebra of the algebra of continuous complex-valued functions on the unit circle.

43 Bounded holomorphic functions

Let U be the open unit disk in the complex plane again, and let $C_b(U)$ be the algebra of bounded continuous complex-valued functions on U, equipped with the supremum norm. Also let $\mathcal{B}$ be the collection of bounded holomorphic functions on U, which is the same as the intersection of $C_b(U)$ with $\mathcal{H}(U)$. As usual, this is a closed subalgebra of $C_b(U)$ with respect to the supremum norm.

Let ϕ be a linear functional on $\mathcal{B}$ which is a homomorphism with respect to multiplication. Suppose also that $\phi(f) \neq 0$ for some $f \in \mathcal{B}$, which implies that ϕ sends constant functions on U to their constant values. If $f \in \mathcal{B}$ and $|f(z)| \ge \delta$ for some $\delta > 0$ and every $z \in U$, then $1/f$ is also a bounded holomorphic function on U, and it follows that $\phi(f) \neq 0$, because $\phi(f)\,\phi(1/f) = 1$. This implies that

$$\phi(f) \in \overline{f(U)} \tag{43.1}$$

for every $f \in \mathcal{B}$, as in the previous situations, and hence that

$$|\phi(f)| \le \sup_{|z|<1} |f(z)|. \tag{43.2}$$

Thus ϕ is a continuous linear functional on $\mathcal{B}$ with respect to the supremum norm, with dual norm equal to 1, since ϕ sends constants to themselves.

Each element p of U determines a nonzero homomorphism ϕ_p on $\mathcal{B}$, given by evaluation at p, or

$$\phi_p(f) = f(p). \tag{43.3}$$

The collection of nonzero homomorphisms on $\mathcal{B}$ is contained in the unit ball of the dual of $\mathcal{B}$ with respect to the supremum norm, as in the previous paragraph, and it is also a closed set with respect to the weak* topology, as in Section 33. Hence the collection of nonzero homomorphisms on $\mathcal{B}$ is compact with respect to the weak* topology on the dual of $\mathcal{B}$, by the Banach–Alaoglu theorem. Of course, the restriction of any nonzero homomorphism on $C_b(U)$ is a nonzero homomorphism on $\mathcal{B}$, which includes evaluation at elements of U.

Suppose that $z_1, z_2, \ldots$ is a sequence of elements of U such that $|z_j| \to 1$ as $j \to \infty$. Also let L be a nonzero homomorphism on $\ell^\infty(\mathbf{Z}_+)$ which is equal to 0 on $c_0(\mathbf{Z}_+)$. This determines a nonzero homomorphism on $C_b(U)$, by applying L to $f(z_j)$ as a bounded function on $\mathbf{Z}_+$ for each $f \in C_b(U)$. If $w_1, w_2, \ldots$ is another sequence of elements of U such that $|w_j| \to 1$ as $j \to \infty$, then we can apply L to $f(w_j)$ to get another homomorphism on $C_b(U)$. If $z_j \ne w_l$ for every $j, l \ge 1$, then it is easy to see that these are distinct homomorphisms on $C_b(U)$, because one can choose a bounded continuous function f on U such that $f(z_j) = 0$ and $f(w_l) = 1$ for each j, l.

If f is a bounded holomorphic function on U, then one can check that there is a $C \ge 0$ such that

$$\sup_{|z|<1} (1 - |z|)\, |f'(z)| \le C \sup_{|z|<1} |f(z)|. \tag{43.4}$$

This follows from the Cauchy integral formula for $f'(z)$ applied to the disk centered at z with radius $(1 - |z|)/2$, for instance.

Suppose that z_j, w_l are as before, and satisfy the additional property that

$$\lim_{j\to\infty} \frac{|z_j - w_j|}{(1 - |z_j|)} = 0. \tag{43.5}$$

If f is a bounded holomorphic function on U, then

$$\lim_{j\to\infty} (f(z_j) - f(w_j)) = 0. \tag{43.6}$$

This follows from the fact that $(1-|z|)|f'(z)|$ is bounded on U, as in the previous paragraph. If L is a nonzero homomorphism on $\ell^\infty(\mathbf{Z}_+)$ that vanishes on $c_0(\mathbf{Z}_+)$, then L applied to $f(z_j) - f(w_j)$ is equal to 0, so that L applied to $f(z_j)$ is the same as L applied to $f(w_j)$. This shows that distinct homomorphisms on $C_b(U)$ may determine the same homomorphism on $\mathcal{B}$.

A sequence $\{z_j\}_{j=1}^\infty$ of points in U is said to be an *interpolating sequence* if for every bounded sequence of complex numbers $\{a_j\}_{j=1}^\infty$ there is a bounded holomorphic function f on U such that $f(z_j) = a_j$ for each j. Equivalently, $\{z_j\}_{j=1}^\infty$ is an interpolating sequence in U if

$$f \mapsto \{f(z_j)\}_{j=1}^\infty \tag{43.7}$$

maps $\mathcal{B}$ onto $\ell^\infty(\mathbf{Z}_+)$. Of course, (43.7) defines a bounded linear mapping from $\mathcal{B}$ into $\ell^\infty(\mathbf{Z}_+)$ for any sequence $\{z_j\}_{j=1}^\infty$ of elements of U, and is also a homomorphism with respect to pointwise multiplication. A famous theorem of Carleson characterizes interpolating sequences in U. In particular, there are plenty of them.

44 Density

Let X be a topological space, and let ψ be a nonzero homomorphism from $C_b(X)$ into the real or complex numbers, as appropriate. As in Section 33, ψ is automatically a bounded linear functional on $C_b(X)$, and thus an element of the dual space $C_b(X)^*$. If $p \in X$, then let $\phi_p(f) = f(p)$ be the corresponding point evaluation homomorphism on $C_b(X)$, as usual. We would like to show that ψ can be approximated by point evealuations with respect to the weak* topology on $C_b(X)^*$, so that point evaluations are dense in the set of nonzero homomorphisms on $C_b(X)$ with respect to the weak* topology on $C_b(X)^*$.

More precisely, we would like to show that for any finite collection of bounded continuous functions $f_1, \ldots, f_n$ on X and any $\epsilon > 0$ there is a $p \in X$ such that

$$|\psi(f_j) - \phi_p(f)| = |\psi(f_j) - f_j(p)| < \epsilon \tag{44.1}$$

for $j = 1, \ldots, n$. Otherwise, there are $f_1, \ldots, f_n \in C_b(X)$ and $\epsilon > 0$ such that

$$\max_{1 \le j \le n} |\psi(f_j) - f_j(p)| \ge \epsilon \tag{44.2}$$

for every $p \in X$. We may as well ask also that $\psi(f_j) = 0$ for each j, since this can always be arranged by subtracting $\psi(f_j)$ as a constant function on U from f_j. In this case, (44.2) reduces to

$$\max_{1 \le j \le n} |f_j(p)| \ge \epsilon \tag{44.3}$$

for each $p \in X$.

If

$$g(p) = \sum_{j=1}^n |f_j(p)|^2, \tag{44.4}$$

then g is a bounded continuous function on X, and $g(p) \ge \epsilon^2$ for each $p \in U$, by (44.3). Thus $1/g$ is also a bounded continuous function on X, which implies that $\psi(g) \ne 0$, as in Section 33. Of course, g can also be expressed as

$$g = \sum_{j=1}^n f_j^2 \tag{44.5}$$

in the real case, and as

$$g = \sum_{j=1}^{n} f_j \, \overline{f_j} \tag{44.6}$$

in the complex case, where $\overline{f_j}$ is the complex conjugate of f_j. In both cases, this implies that $\psi(g) = 0$, a contradiction, because $\psi(f_j) = 0$ for each j, and ψ is a homomorphism.

Let $\mathcal{B}$ be the algebra of bounded holomorphic functions on the open unit disk U, as in the preceding section. Carleson's corona theorem states that every nonzero homomorphism ψ on $\mathcal{B}$ can be approximated by point evaluations $\phi_p(f) = f(p)$, $p \in U$, with respect to the weak* topology on the dual of $\mathcal{B}$. As before, if this were not the case, then there would be bounded holomorphic functions $f_1, \ldots, f_n$ on U and $\epsilon > 0$ such that $\psi(f_j) = 0$ for $j = 1, \ldots, n$ and (44.3) holds. However, the previous argument does not work, because $\overline{f_j}$ is not holomorphic on U unless f_j is constant. Instead, one can try to show that there are bounded holomorphic functions $g_1, \ldots, g_n$ on U such that

$$\sum_{j=1}^{n} f_j(p) \, g_j(p) = 1 \tag{44.7}$$

for every $p \in U$, which would give a contradiction as before.

45 Mapping properties

Let X be a topological space, and let us use $\mathrm{Hom}(X)$ to denote the set of nonzero homomorphisms from $C_b(X)$ into $\mathbf{R}$ or $\mathbf{C}$, as appropriate. In situations in which other types of algebras are considered as well, this may be denoted more precisely as $\mathrm{Hom}(C_b(X))$, to avoid confusion. As in Section 33, $\mathrm{Hom}(X)$ is a compact subset of $C_b(X)^*$ with respect to the weak* topology.

If $p \in X$, then $\phi_p(f) = f(p)$ is an element of $\mathrm{Hom}(X)$, and we let $\mathrm{Hom}_1(X)$ be the subset of $\mathrm{Hom}(X)$ consisting of homomorphisms on $C_b(X)$ of this form. Thus $\mathrm{Hom}(X) = \mathrm{Hom}_1(X)$ when X is compact, as in Section 34. Otherwise, $\mathrm{Hom}_1(X)$ is dense in $\mathrm{Hom}(X)$ with respect to the weak* topology on $C_b(X)^*$ for any X, as in the previous section.

We have also seen in Section 33 that $p \mapsto \phi_p$ is continuous as a mapping from X into $C_b(X)^*$ with the weak* topology. By definition, this mapping sends X onto $\mathrm{Hom}_1(X)$ in $C_b(X)^*$. If X is compact, then it follows that $\mathrm{Hom}_1(X)$ is compact with respect to the weak* topology on $C_b(X)^*$, and hence closed. This gives another way to show that $\mathrm{Hom}_1(X) = \mathrm{Hom}(X)$ when X is compact, since $\mathrm{Hom}(X)$ is the same as the closure of $\mathrm{Hom}_1(X)$ with respect to the weak* topology on $C_b(X)^*$ for any X.

Note that $p \mapsto \phi_p$ is a one-to-one mapping of X into $C_b(X)^*$ exactly when continuous functions separate points on X. If X is completely regular, then it is easy to see that $p \mapsto \phi_p$ is a homeomorphism from X onto $\mathrm{Hom}_1(X)$ with respect to the topology on $\mathrm{Hom}_1(X)$ induced by the weak* topology on $C_b(X)^*$.

In particular, if X is compact and Hausdorff, then $p \mapsto \phi_p$ is a homeomorphism from X onto $\mathrm{Hom}(X)$ with respect to the topology on $\mathrm{Hom}(X)$ induced by the weak* topology on $C_b(X)^*$. Remember that compact Hausdorff topological spaces are normal and hence completely regular.

Now let Y be another topological space, and let ρ be a continuous mapping from X into Y. This leads to a linear mapping $T_\rho : C_b(Y) \to C_b(X)$, defined by

$$T_\rho(f) = f \circ \rho \tag{45.1}$$

for each $f \in C_b(Y)$. Observe that

$$\|T_\rho(f)\|_{sup,X} \le \|f\|_{sup,Y} \tag{45.2}$$

for every $f \in C_b(Y)$, where the subscripts X, Y indicate on which space the supremum norm is taken. This shows that T_ρ is a bounded linear mapping from $C_b(Y)$ into $C_b(X)$ with respect to the supremum norm, with operator norm less than or equal to 1, and the operator norm is actually equal to 1, because $T_\rho(\mathbf{1}_Y) = \mathbf{1}_X$. If $\rho(X)$ is dense in Y, then T_ρ is an isometric embedding of $C_b(Y)$ into $C_b(X)$ with respect to their supremum norms.

Let $T_\rho^* : C_b(X)^* \to C_b(Y)^*$ be the dual mapping associated to T_ρ. This sends a bounded linear functional λ on $C_b(X)$ to the bounded linear functional $\mu = T_\rho^*(\lambda)$ defined by

$$\mu(f) = \lambda(T_\rho(f)) = \lambda(f \circ \rho) \tag{45.3}$$

for each $f \in C_b(Y)$. The fact that $\mu = T_\rho^*(\lambda)$ is a bounded linear functional on $C_b(Y)$ uses the fact that T_ρ is a bounded linear mapping from $C_b(Y)$ into $C_b(X)$, as well as the boundedness of λ on $C_b(X)$. Similarly, it is easy to see that T_ρ^* is bounded as a linear mapping from $C_b(X)^*$ into $C_b(Y)^*$ with respect to the corresponding dual norms. It is also easy to see that T_ρ^* is continuous as a mapping from $C_b(X)^*$ into $C_b(Y)^*$ with respect to their corresponding weak* topologies.

Observe that T_ρ is a homomorphism from $C_b(Y)$ into $C_b(X)$, in the sense that

$$T_\rho(f\,g) = T_\rho(f)\,T_\rho(g) \tag{45.4}$$

for every $f, g \in C_b(Y)$. If λ is a homomorphism from $C_b(X)$ into the real or complex numbers, as appropriate, then it follows that $T_\rho^*(\lambda)$ is a homomorphism on $C_b(Y)$ too. If λ is a nonzero homomorphism on $C_b(X)$, so that $\lambda(\mathbf{1}_X) = 1$, then $T_\rho^*(\lambda)$ is nonzero on $C_b(Y)$ too, because

$$T_\rho^*(\lambda)(\mathbf{1}_Y) = \lambda(T_\rho(\mathbf{1}_Y) = \lambda(\mathbf{1}_Y \circ \rho) = \lambda(\mathbf{1}_X) = 1. \tag{45.5}$$

Thus $T_\rho^*(\mathrm{Hom}(X)) \subseteq \mathrm{Hom}(Y)$.

If $q \in Y$, then let $\psi_q(f) = f(q)$ be the corresponding point evaluation on $C_b(Y)$. Observe that

$$T_\rho^*(\phi_p) = \psi_{\rho(p)} \tag{45.6}$$

for each $p \in X$, since

$$T_\rho^*(\phi_p)(f) = \phi_p(T_\rho(f)) = \phi_p(f \circ \rho) = f(\rho(p)) = \psi_{\rho(p)}(f) \tag{45.7}$$

for every $f \in C_b(Y)$. Thus $T_\rho^*(\mathrm{Hom}_1(X)) \subseteq \mathrm{Hom}_1(Y)$. If $\rho(X)$ is dense in Y, then it follows that $T_\rho^*(\mathrm{Hom}_1(X))$ is dense in $\mathrm{Hom}_1(Y)$ with respect to the weak* topology on $C_b(Y)^*$, because $q \mapsto \psi_q$ is a continuous mapping from Y into $C_b(Y)^*$ with respect to the weak* topology on $C_b(Y)^*$. This implies that $T_\rho^*(\mathrm{Hom}_1(X))$ is dense in $\mathrm{Hom}(Y)$ with respect to the weak* topology on $C_b(Y)^*$ when $\rho(X)$ is dense in Y, since $\mathrm{Hom}_1(Y)$ is dense in $\mathrm{Hom}(Y)$ with respect to the weak* topology on $C_b(Y)^*$.

If $\rho(X)$ is dense in Y, then we also get that

$$T_\rho^*(\mathrm{Hom}(X)) = \mathrm{Hom}(Y). \tag{45.8}$$

Remember that $\mathrm{Hom}(X)$ is compact in $C_b(X)^*$ with respect to the weak* topology, which implies that $T_\rho^*(\mathrm{Hom}(X))$ is compact in $C_b(Y)^*$ with respect to the weak* topology, because T_ρ^* is a continuous mapping from $C_b(X)^*$ into $C_b(Y)^*$ with respect to their weak* topologies. Hence $T_\rho^*(\mathrm{Hom}(X))$ is a closed set in $C_b(Y)^*$ with respect to the weak* topology. This implies that $\mathrm{Hom}(Y)$ is contained in $T_\rho^*(\mathrm{Hom}(X))$, because $T_\rho^*(\mathrm{Hom}_1(X)) \subseteq T_\rho^*(\mathrm{Hom}(X))$ is dense in $\mathrm{Hom}(Y)$ with respect to the weak* topology on $C_b(Y)^*$ when $\rho(X)$ is dense in Y, as in the previous paragraph. Therefore (45.8) holds, since $T_\rho^*(\mathrm{Hom}(X))$ is contained in $\mathrm{Hom}(Y)$ automatically.

Suppose now that Y is compact and Hausdorff, so that $\mathrm{Hom}_1(Y) = \mathrm{Hom}(Y)$, and $q \mapsto \psi_q$ defines a homeomorphism from Y onto $\mathrm{Hom}(Y)$ with respect to the topology on $\mathrm{Hom}(Y)$ induced by the weak* topology on $C_b(Y)^*$. In this case, the restriction of T_ρ^* to $\mathrm{Hom}(X)$ can be identified with a mapping into Y. If $\rho(X)$ is dense in Y, then we get a mapping from $\mathrm{Hom}(X)$ onto Y, as in the previous paragraph. If X is completely regular, so that $p \mapsto \phi_p$ defines a homeomorphism from X onto $\mathrm{Hom}_1(X)$ with respect to the topology induced on $\mathrm{Hom}_1(X)$ by the weak* topology on $C_b(X)^*$, then the restriction of T_ρ^* to $\mathrm{Hom}(X)$ is basically an extension of ρ. If X is compact and Hausdorff, then the restriction of ρ to $\mathrm{Hom}(X)$ is essentially the same as ρ itself.

46 Discrete sets

Let X be a nonempty set, and let βX be the set of all untrafilters on X. As in Sections 31 and 32, there is a natural one-to-one correspondence between βX and the set of all nonzero homomorphisms on $\ell^\infty(X)$. If X is equipped with the discrete topology, then $\ell^\infty(X)$ is the same as $C_b(X)$, and the set of nonzero homomorphisms on $\ell^\infty(X)$ is the same as the set $\mathrm{Hom}(X)$ discussed in the previous section. In this section, we shall see how properties of $\mathrm{Hom}(X)$ can be described more directly in terms of ultrafilters on X.

If $A \subseteq X$, then let $\widehat{A} \subseteq \beta X$ be the set of ultrafilters $\mathcal{F}$ on X such that $A \in \mathcal{F}$. Thus $\widehat{X} = \beta X$, and there is a natural one-to-one correspondence between $\widehat{A}$ and βA for any A, in which an ultrafilter on A is extended to an ultrafilter on X that contains A as an element, as in Section 21. It is easy to see that

$$\widehat{A \cap B} = \widehat{A} \cap \widehat{B} \tag{46.1}$$

for every $A, B \subseteq X$. Moreover,

$$\widehat{X \backslash A} = \widehat{X} \backslash \widehat{A} = \beta X \backslash \widehat{A} \tag{46.2}$$

for every $A \subseteq X$, because any ultrafilter $\mathcal{F}$ on X contains exactly one of A and $X \backslash A$ as an element. It follows that

$$\widehat{A \cup B} = \widehat{A} \cup \widehat{B} \tag{46.3}$$

for every $A, B \subseteq X$.

Let us define a topology on βX by saying that a subset of βX is an open set if it can be expressed as a union of subsets of the form $\widehat{A}$, $A \subseteq X$. Equivalently, $\widehat{A}$ is an open set in βX for each $A \subseteq X$, and these open subsets of βX form a base for the topology of βX. It is easy to see that the intersection of two open subsets of βX is also open, so that this does define a topology on βX, because of the fact about intersections mentioned in the previous paragraph. The fact about complements mentioned in the previous paragraph implies that $\widehat{A}$ is both open and closed for every $A \subseteq X$.

If $\mathcal{F}$ is an ultrafilter on X, then let $L_{\mathcal{F}}$ be the corresponding homomorphism on $\ell^\infty(X)$, as in Section 31. Let A be a subset of X, and let $\mathbf{1}_A$ be the indicator function on X corresponding to A, so that $\mathbf{1}_A(x) = 1$ when $x \in A$ and $\mathbf{1}_A(x) = 0$ when $x \in X \backslash A$. It is easy to check that

$$\begin{aligned} L_{\mathcal{F}}(\mathbf{1}_A) &= 1 \text{ when } A \in \mathcal{F} \\ &= 0 \text{ when } X \backslash A \in \mathcal{F}, \end{aligned} \tag{46.4}$$

directly from the definition of $L_{\mathcal{F}}$. Remember that $\mathcal{F} \mapsto L_{\mathcal{F}}$ defines a one-to-one correspondence between βX and the set $\mathrm{Hom}(X)$ of nonzero homomorphisms on $\ell^\infty(X) = C_b(X)$. Using (46.4), one can check that $\widehat{A}$ corresponds to a relatively open subset of $\mathrm{Hom}(X)$ with respect to the weak* topology on $\ell^\infty(X)^*$ for each $A \subseteq X$. This implies that every open set in βX with respect to the topology described earlier corresponds to a relatively open set in $\mathrm{Hom}(X)$ with respect to the weak* topology on $\ell^\infty(X)$. Conversely, one can show that relatively open subsets of $\mathrm{Hom}(X)$ with respect to the weak* topology on $\ell^\infty(X)^*$ correspond to open subsets of βX. This uses the facts that finite linear combinations of indicator functions of subsets of X are dense in $\ell^\infty(X)$, and that homomorphisms on $\ell^\infty(X)$ have bounded dual norm.

In particular, βX should be compact and Hausdorff with respect to the topology defined before, because of the corresponding properties of $\mathrm{Hom}(X)$ with respect to the topology induced by the weak* topology on $\ell^\infty(X)^*$. Let us check these properties directly from the definition of the topology on βX. If $\mathcal{F}$, $\mathcal{F}'$ are distinct ultrafilters on X, then there is a set $A \subseteq X$ such that $A \in \mathcal{F}$ and $X \backslash A \in \widehat{F}'$. Hence $\mathcal{F} \in \widehat{A}$ and $\mathcal{F}' \in \widehat{X \backslash A}$, so that $\mathcal{F}$, $\mathcal{F}'$ are contained in disjoint open subsets of βX, which implies that βX is Hausdorff.

To show that βX is compact, let $\mathcal{U}$ be an arbitrary ultrafilter on βX, and let us show that $\mathcal{U}$ converges to an element of βX. Let $\mathcal{F}$ be the collection of subsets A of X such that $\widehat{A} \in \mathcal{U}$. It is easy to see that $\mathcal{F}$ is a filter on X, because $\mathcal{U}$ is a filter on βX. If $A \subseteq X$, then either $\widehat{A}$ or $\widehat{X \backslash A} = \beta X \backslash \widehat{A}$ is an element of $\mathcal{U}$, because $\mathcal{U}$ is an ultrafilter on βX. This implies that either A or $X \backslash A$ is an element of $\mathcal{F}$ for every $A \subseteq X$, and hence that $\mathcal{F}$ is an ultrafilter on X. It remains to check that $\mathcal{U}$ converges to $\mathcal{F}$ as an element of βX. By definition, this means that every neighborhood of $\mathcal{F}$ in βX should be an element of $\mathcal{U}$. Because the sets $\widehat{A}$, $A \subseteq X$, form a base for the topology of βX, it suffices to have $\widehat{A} \in \mathcal{U}$ for every $A \subseteq X$ such that $A \in \mathcal{F}$, which follows from the definition of $\mathcal{F}$.

If $p \in X$, then the collection $\mathcal{F}_p$ of $A \subseteq X$ with $p \in A$ is an ultrafilter on X. Thus $p \mapsto \mathcal{F}_p$ defines a natural embedding of X into βX. It is easy to see that the set of ultrafilters $\mathcal{F}_p$, $p \in X$, is dense in βX with respect to the topology defined earlier. One can also check that the homomorphism $L_{\mathcal{F}_p}$ on $\ell^\infty(X)$ corresponding to $\mathcal{F}_p$ is the same as evaluation at p.

Let Y be a compact Hausdorff topological space, and let ρ be a mapping from X into Y. If $\mathcal{F}$ is an ultrafilter on X, then we can define $\rho_*(\mathcal{F})$ as usual as the collection of sets $E \subseteq Y$ such that $\rho^{-1}(E) \in \mathcal{F}$. In particular, we have seen that $\rho_*(\mathcal{F})$ is an ultrafilter on Y. It follows that $\rho_*(\mathcal{F})$ converges to a unique element of Y, because Y is compact and Hausdorff. Let $\widehat{\rho}(\mathcal{F})$ be the limit of $\rho_*(\mathcal{F})$ in Y, which defines $\widehat{\rho}$ as a mapping from βX into Y. If $p \in X$, then it is easy to see that $\widehat{\rho}(\mathcal{F}_p) = \rho(p)$. Thus $\widehat{\rho}$ is basically an extension of ρ to a mapping from βX into Y.

Let us check that $\widehat{\rho}$ is continuous as a mapping from βX into Y. Let $\mathcal{F}$ be an ultrafilter on X, and let W be an open set in Y that contains $\widehat{\rho}(\mathcal{F})$ as an element. Because Y is compact and Hausdorff, it is regular, which implies that there is an open set V in Y such that $\widehat{\rho}(\mathcal{F}) \in V$ and the closure $\overline{V}$ of V in Y is contained in W. Remember that $\rho_*(\mathcal{F})$ converges to $\widehat{\rho}(\mathcal{F})$ in Y, which implies that $V \in \rho_*(\mathcal{F})$. This implies in turn that $\rho^{-1}(V) \in \mathcal{F}$, by the definition of $\rho_*(\mathcal{F})$. Put $A = \rho^{-1}(V)$, so that $\widehat{A}$ is an open set in βX that contains $\mathcal{F}$ as an element. Let $\mathcal{F}'$ be any other ultrafilter on X that is an element of $\widehat{A}$. This means that $\rho^{-1}(V) = A \in \mathcal{F}'$, and hence that $A \in \rho_*(\mathcal{F}')$. By construction, $\rho_*(\mathcal{F}')$ converges to $\widehat{\rho}(\mathcal{F}')$ in Y, which implies that $\widehat{\rho}(\mathcal{F}') \in \overline{V}$. This shows that $\widehat{\rho}(\mathcal{F}') \in \overline{V} \subseteq W$ for every $\mathcal{F}' \in \widehat{A}$, and hence that $\widehat{\rho}$ is continuous at $\mathcal{F}$ for every $\mathcal{F} \in \beta X$, as desired.

47 Locally compact spaces, revisited

Let X be a locally compact Hausdorff topological space which is not compact, and let X^* be the one-point compactification of X, as in Section 37. Also let $C_{lim}(X)$ be the space of continuous real or complex-valued functions on X which have a limit at infinity, as in Section 37. As usual, this may also be denoted $C_{lim}(X, \mathbf{R})$ or $C_{lim}(X, \mathbf{C})$, to indicate whether real or complex-valued functions are being used. As in Section 37, $C_{lim}(X)$ is a closed subalgebra of the algebra $C_b(X)$ of bounded continuous functions on X with respect to the supremum norm, and $C_{lim}(X)$ is the same as the linear span in $C_b(X)$ of the subspace

$C_0(X)$ of functions that vanish at infinity on X and the constant functions on X. Equivalently, $C_{lim}(X)$ is the same as the space of continuous functions on X that have a continuous extension to X^*.

Thus a nonzero homomorphism ϕ from $C_{lim}(X)$ into the real or complex numbers, as appropriate, is basically the same as a nonzero homomorphism on $C(X^*)$. As in Section 34, every nonzero homomorphism on $C(X^*)$ can be represented by evaluation at a point in X^*, because X^* is compact. This point in X^* is either an element of X, or the point at infinity in X^*. This implies that either there is a $p \in X$ such that

$$\phi(f) = f(p) \tag{47.1}$$

for every $f \in C_{lim}(X)$, or that

$$f(x) \to \phi(f) \text{ as } x \to \infty \tag{47.2}$$

for every $f \in C_{lim}(X)$.

Suppose now that ϕ is a nonzero homomorphism on $C_b(X)$. The restriction of ϕ to $C_{lim}(X)$ is a nonzero homomorphism on $C_{\mathrm{lim}}(X)$, since $\phi(\mathbf{1}_X) = 1$. If $\phi(f) = f(p)$ for some $p \in X$ and every $f \in C_{lim}(X)$, then we would like to check that this also holds for every $f \in C_b(X)$. To see this, we can use Urysohn's lemma to get a continuous function θ with compact support on X such that $\theta(p) = 1$. Let f be a bounded continuous function on X, and observe that $\theta\, f \in C_{lim}(X)$, because it has compact support on X. This implies that

$$\phi(\theta\, f) = (\theta\, f)(p) = \theta(p)\, f(p) = f(p), \tag{47.3}$$

since $\theta(p) = 1$. Similarly,

$$\phi((1-\theta)\, f) = \phi(1-\theta)\, \phi(f) = (1-\theta(p))\, \phi(f) = 0. \tag{47.4}$$

More precisely, this uses the hypothesis that ϕ is a homomorphism on $C_b(X)$ in the first step, and then the fact that $1 - \theta \in C_{lim}(X)$ to get that $\phi(1-\theta)$ is equal to $1 - \theta(p)$. Combining these two equations, we get that $\phi(f) = f(p)$, as desired.

Let ρ be the standard embedding of X into X^*, which sends each $p \in X$ to itself as an element of X^*. As in Section 45, this leads to a mapping T_ρ from $C(X^*)$ into $C_b(X)$, which sends $C(X^*)$ onto $C_{lim}(X)$ in this case. The corresponding dual mapping T_ρ^* sends the set $\mathrm{Hom}(X)$ of nonzero homomorphisms on $C_b(X)$ into the analogous set $\mathrm{Hom}(X^*)$ for X^*, which can be identified with X^*, because X^* is compact and Hausdorff. Remember that $\mathrm{Hom}_1(X) \subseteq \mathrm{Hom}(X)$ is the set of homomorphisms on $C_b(X)$ defined by evaluation at elements of X, and that T_ρ^* maps $\mathrm{Hom}_1(X)$ to the point evaluations on $C(X^*)$ that correspond to elements of X. The discussion in the previous paragraph implies that T_ρ^* sends every other element of $\mathrm{Hom}(X)$ to the point evaluation on $C(X^*)$ that corresponds to the point at infinity in X^*.

48 Mapping properties, continued

Let U be the open unit disk in the complex plane, so that $\overline{U}$ is the closed unit disk. Also let ρ be the standard embedding of U into $\overline{U}$, which sends each $z \in U$ to itself as an element of $\overline{U}$. This leads to a mapping T_ρ from $C(\overline{U})$ into $C_b(U)$, as in Section 45, which sends a continuous function f on $\overline{U}$ to its restriction to U. The dual mapping $T_\rho^* : C_b(U)^* \to C(\overline{U})^*$ sends the set $\mathrm{Hom}(U)$ of nonzero homomorphisms on $C_b(U)$ into the analogous set $\mathrm{Hom}(\overline{U})$ for $\overline{U}$, as before. If ϕ is a nonzero homomorphism on $C_b(U)$, then $T_\rho^*(\phi)$ is basically the same as the restriction of ϕ to $C(\overline{U})$, which is identified with a subalgebra of $C_b(U)$. Each nonzero homomorphism on $C(\overline{U})$ can be represented as a point evaluation, as in Section 34. If there is a $p \in U$ such that $\phi(f) = f(p)$ for every $f \in C(\overline{U})$, then the same relation holds for every $f \in C_b(U)$, as in the previous section. If $\phi \in \mathrm{Hom}(U)$ does not correspond to evaluation at a point in U, then it follows that the restriction of ϕ to $C(\overline{U})$ corresponds to evaluation at a point in ∂U.

Let $\mathcal{A}$ be the algebra of continuous complex-valued functions on $\overline{U}$ that are holomorphic on U, as in Section 42, and let $\mathcal{B}$ be the algebra of bounded holomorphic functions on U, as in Section 43. If $f \in \mathcal{A}$, then the restriction of f to U is an element of $\mathcal{B}$, and f is determined on $\overline{U}$ by its restriction to U, by continuity. Thus we can identify $\mathcal{A}$ with a subalgebra of $\mathcal{B}$.

Let $\mathrm{Hom}(\mathcal{A})$, $\mathrm{Hom}(\mathcal{B})$ denote the sets of nonzero homomorphisms from $\mathcal{A}$, $\mathcal{B}$ into the complex numbers, respectively. As in Sections 42 and 43, these are subsets of the duals of $\mathcal{A}$, $\mathcal{B}$, and we are especially interested in the topologies induced on $\mathrm{Hom}(\mathcal{A})$, $\mathrm{Hom}(\mathcal{B})$ by the weak* topologies on the corresponding dual spaces.

If $p \in \overline{U}$, then $\phi_p(f) = f(p)$ defines a homomorphism on $\mathcal{A}$, and we have seen in Section 42 that every nonzero homomorphism on $\mathcal{A}$ is of this form. Of course, $\phi_p(f) = f(p)$ is a continuous function on $\overline{U}$ for every $f \in \mathcal{A}$, by definition of $\mathcal{A}$, which implies that $p \mapsto \phi_p$ is continuous as a mapping from $\overline{U}$ into $\mathrm{Hom}(\mathcal{A})$ with respect to the weak* topology on $\mathcal{A}$. If $f_1(z)$ is the element of $\mathcal{A}$ defined by $f_1(z) = z$ for each $z \in \overline{U}$, then $\phi_p(f_1) = p$ for each $p \in \overline{U}$. This shows that $p \mapsto \phi_p$ is actually a homeomorphism from $\overline{U}$ onto $\mathrm{Hom}(\mathcal{A})$ with respect to the topology induced on $\mathrm{Hom}(\mathcal{A})$ by the weak* topology on $\mathcal{A}^*$.

Similarly, if $p \in U$, then $\phi_p(f) = f(p)$ defines a nonzero homomorphism on $\mathcal{B}$, and $p \mapsto \phi_p$ defines a continuous mapping from U into $\mathrm{Hom}(\mathcal{B})$ with respect to the weak* topology on $\mathcal{B}^*$. Let $\mathrm{Hom}_1(\mathcal{B})$ be the set of homomorphisms on $\mathcal{B}$ of this form. If $f_1(z)$ is the element of $\mathcal{B}$ defined by $f_1(z) = z$ for each $z \in U$, then $\phi_p(f_1) = p$ for each $p \in U$. This implies that $p \mapsto \phi_p$ is a homeomorphism from U onto $\mathrm{Hom}_1(\mathcal{B})$ with respect to the topology induced on $\mathrm{Hom}_1(\mathcal{B})$ by the weak* topology on $\mathcal{B}^*$.

If ϕ is a nonzero homomorphism on $\mathcal{B}$, then the restriction of ϕ to $\mathcal{A}$ is a nonzero homomorphism on $\mathcal{A}$. This defines a natural mapping from $\mathrm{Hom}(\mathcal{B})$ into $\mathrm{Hom}(\mathcal{A})$. It is easy to see that this mapping is continuous with respect to the topologies induced on $\mathrm{Hom}(\mathcal{A})$, $\mathrm{Hom}(\mathcal{B})$ by the weak* topologies on $\mathcal{A}^*$, $\mathcal{B}^*$, respectively.

Let f_1 be the element of $\mathcal{B}$ defined by $f_1(z) = z$ for each $z \in U$ again. Also let ϕ be a nonzero homomorphism on $\mathcal{B}$, and put $p = \phi(f_1)$. Note that $p \in \overline{U}$, since ϕ has dual norm equal to 1 with respect to the supremum norm on $\mathcal{B}$, as in Section 43. If $f \in \mathcal{A}$, then $\phi(f) = f(p)$, by the arguments in Section 42 applied to the restriction of ϕ to $\mathcal{A}$.

Suppose that $p \in U$, and let us check that $\phi(f) = f(p)$ for every $f \in \mathcal{B}$. Any holomorphic function f on U can be expressed as

$$f(z) = f(p) + (z - p)\, g(z) \tag{48.1}$$

for some holomorphic function g on U, and g is also bounded on U when f is. This implies that $\phi(f) = f(p)$ for every $f \in \mathcal{B}$, because ϕ applied to $z - p$ is equal to 0, by definition of p. If $\phi \in \mathrm{Hom}(\mathcal{B}) \setminus \mathrm{Hom}_1(\mathcal{B})$, then it follows that $p \in \partial U$. This is analogous to the situation for bounded continuous functions on U mentioned at the beginning of the section.

Let us take $C_b(U)$ to be the algebra of bounded continuous complex-valued functions on U, so that $\mathcal{B}$ is a subalgebra of $C_b(U)$. Let us also use $\mathrm{Hom}(C_b(U))$ to denote the set of nonzero homomorphisms on $C_b(U)$, to be more consistent with the notation for $\mathcal{B}$. If ϕ is a nonzero homomorphism on $C_b(U)$, then the restriction of ϕ to $\mathcal{B}$ is a nonzero homomorphism on $\mathcal{B}$. This defines a natural mapping R from $\mathrm{Hom}(C_b(U))$ into $\mathrm{Hom}(\mathcal{B})$, which is easily seen to be continuous with respect to the topologies induced by the weak* topologies on $C_b(U)^*$ and $\mathcal{B}^*$, respectively.

By construction, R sends $\mathrm{Hom}_1(C_b(U))$ onto $\mathrm{Hom}_1(\mathcal{B})$. We also know that $\mathrm{Hom}(C_b(U))$ is compact with respect to the weak* topology on $C_b(U)^*$, which implies that $R(\mathrm{Hom}(C_b(U)))$ is compact with respect to the weak* topology on $\mathcal{B}^*$. In particular, $R(\mathrm{Hom}(C_b(U)))$ is closed with respect to the weak* topology on $\mathcal{B}^*$. As in Section 44, Carleson's corona theorem states that $\mathrm{Hom}_1(\mathcal{B})$ is dense in $\mathrm{Hom}(\mathcal{B})$ with respect to the weak* topology on $\mathcal{B}^*$. It follows that R maps $\mathrm{Hom}(C_b(U))$ onto $\mathrm{Hom}(\mathcal{B})$, so that every nonzero homomorphism on $\mathcal{B}$ is the restriction to $\mathcal{B}$ of a nonzero homomorphism on $C_b(U)$.

49 Banach algebras

A vector space $\mathcal{A}$ over the real or complex numbers is said to be an (associative) algebra if every $a, b \in \mathcal{A}$ has a well-defined product $a\, b \in \mathcal{A}$ which is linear in a and b separately and satisfies the associative law

$$(a\, b)\, c = a\, (b\, c) \quad \text{for every } a, b, c \in \mathcal{A}. \tag{49.1}$$

We shall be primarily concerned here with commutative algebras, so that

$$a\, b = b\, a \tag{49.2}$$

for each $a, b \in \mathcal{A}$. We also ask that there be a nonzero multiplicative identity element e in $\mathcal{A}$, which means that $e \neq 0$ and

$$e\, a = a\, e = a \tag{49.3}$$

for every $a \in \mathcal{A}$. We have seen several examples of algebras of functions in the previous sections, for which the multiplicative identity element is the constant function equal 1.

Suppose that $\mathcal{A}$ is equipped with a norm $\|a\|$. This norm should also be compatible with multiplication on $\mathcal{A}$, in the sense that $\|e\| = 1$ and

$$\|a\, b\| \le \|a\|\, \|b\| \tag{49.4}$$

for every $a, b \in \mathcal{A}$. We say that $\mathcal{A}$ is a Banach algebra if it is also complete as a metric space with respect to the metric $d(a, b) = \|a - b\|$ associated to the norm. The algebra of bounded continuous functions on any topological space is a Banach algebra with respect to the supremum norm. Closed subalgebras of Banach algebras are also Banach algebras, such as the disk algebra and the algebra of bounded holomorphic functions on the unit disk.

Suppose that $\mathcal{A}$ is any Banach algebra, and let a be an element of $\mathcal{A}$. If n is a positive integer, then a^n is the product $a\, a \cdots a$ of n a's in $\mathcal{A}$, which can also be described by $a^n = a$ when $n = 1$, and $a^{n+1} = a\, a^n$ for every n. This is interpreted as being equal to the multiplicative identity element e when $n = 0$. Observe that

$$\|a^n\| \le \|a\|^n \tag{49.5}$$

for each $n \ge 0$, where again the right side is interpreted as being equal to 1 when $n = 0$.

An element a of $\mathcal{A}$ is said to be *invertible* if there is another element a^{-1} of $\mathcal{A}$ such that

$$a\, a^{-1} = a^{-1}\, a = e. \tag{49.6}$$

It is easy to see that the inverse a^{-1} of a is unique when it exists. If a, b are invertible elements of $\mathcal{A}$, then their product $a\, b$ is also invertible, with

$$(a\, b)^{-1} = b^{-1}\, a^{-1}. \tag{49.7}$$

If x is an invertible element of $\mathcal{A}$ and y is another element of $\mathcal{A}$ that commutes with x, so that $x\, y = y\, x$, then y also commutes with x^{-1},

$$y\, x^{-1} = x^{-1}\, y. \tag{49.8}$$

If a, b are commuting elements of $\mathcal{A}$ whose product $a\, b$ is invertible, then a, b are also invertible, with

$$a^{-1} = b\, (a\, b)^{-1}, \quad b^{-1} = (a\, b)^{-1}\, a. \tag{49.9}$$

This uses the fact that a, b commute with $(a\, b)^{-1}$, since they commute with $a\, b$. Note that these statements do not involve the norm on $\mathcal{A}$.

If $a \in \mathcal{A}$ and n is a positive integer, then

$$(e - a) \Big(\sum_{j=0}^{n} a^j\Big) = \Big(\sum_{j=0}^{n} a^j\Big) (e - a) = e - a^{n+1}. \tag{49.10}$$

This is basically the same as for real or complex numbers. If $\|a\| < 1$, then

$$\lim_{n \to \infty} a^n = 0 \tag{49.11}$$

in $\mathcal{A}$, since $\|a^n\| \le \|a\|^n \to 0$ as $n \to \infty$. Similarly,

$$\sum_{j=0}^{\infty} \|a^j\| \le \sum_{j=0}^{\infty} \|a\|^j = \frac{1}{1 - \|a\|}. \tag{49.12}$$

As in the context of real or complex numbers, the convergence of $\sum_{j=0}^{\infty} \|a^j\|$ means that $\sum_{j=0}^{\infty} a^j$ converges absolutely. More precisely, this implies that the partial sums $\sum_{j=0}^{n} a^j$ of $\sum_{j=0}^{\infty} a^j$ form a Cauchy sequence in $\mathcal{A}$, which converges when $\mathcal{A}$ is complete. It follows that

$$(e - a) \Big(\sum_{j=0}^{\infty} a^j \Big) = \Big(\sum_{j=0}^{\infty} a^j \Big) (e - a) = e \tag{49.13}$$

when $a \in \mathcal{A}$, $\|a\| < 1$, and $\mathcal{A}$ is a Banach algebra. Thus $e - a$ is invertible in $\mathcal{A}$ under these conditions, with

$$(e - a)^{-1} = \sum_{j=0}^{\infty} a^j. \tag{49.14}$$

We also get that

$$\|(e - a)^{-1}\| \le \frac{1}{1 - \|a\|}. \tag{49.15}$$

If b is any invertible element of $\mathcal{A}$ and $\|a\| \, \|b^{-1}\| < 1$, then $b - a$ is also invertible in $\mathcal{A}$, because

$$b - a = (e - a\, b^{-1})\, b \tag{49.16}$$

and $e - a\, b^{-1}$ is invertible by the previous argument. This shows that the invertible elements in a Banach algebra $\mathcal{A}$ form an open set in $\mathcal{A}$ with respect to the metric associated to the norm.

Let $\mathcal{A}$ be a real or complex algebra, and let ϕ be a linear functional on $\mathcal{A}$, which is to say a linear mapping from $\mathcal{A}$ into the real or complex numbers, as appropriate. We say that ϕ is a *homomorphism* on $\mathcal{A}$ if

$$\phi(a\, b) = \phi(a)\, \phi(b) \tag{49.17}$$

for every $a, b \in \mathcal{A}$. Of course, ϕ satisfies this condition trivially when $\phi(a) = 0$ for every $a \in \mathcal{A}$, and we are primarily interested in the nonzero homomorphisms ϕ, which means that $\phi(a) \ne 0$ for some $a \in \mathcal{A}$. This implies that

$$\phi(e) = 1, \tag{49.18}$$

because $\phi(a) = \phi(e)\, \phi(a)$, since $a = e\, a$. If b is any invertible element of $\mathcal{A}$, then we get that

$$\phi(b)\, \phi(b^{-1}) = \phi(b\, b^{-1}) = \phi(e) = 1, \tag{49.19}$$

and hence $\phi(b) \neq 0$.

Suppose now that $\mathcal{A}$ is a Banach algebra again, and let ϕ be a nonzero homomorphism on $\mathcal{A}$. If $a \in \mathcal{A}$ and $\|a\| < 1$, then $e - a$ is invertible, and so

$$\phi(e - a) \neq 0, \tag{49.20}$$

which means that $\phi(a) \neq 1$. By the same argument, $\phi(t\,a) = t\,\phi(a) \neq 1$ for every $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, such that $|t| < 1$. This implies that $|\phi(a)| < 1$ when $a \in \mathcal{A}$ satisfies $\|a\| < 1$. Hence

$$|\phi(a)| \leq \|a\| \tag{49.21}$$

for every $a \in \mathcal{A}$, which shows that ϕ is a continuous linear functional on $\mathcal{A}$ with dual norm less than or equal to 1. The dual norm of ϕ is actually equal to 1, because $\phi(e) = 1$. It is easy to see that the collection of nonzero homomorphisms on $\mathcal{A}$ is closed with respect to the weak* topology on the dual of $\mathcal{A}$. It follows that the collection of nonzero homomorphisms on $\mathcal{A}$ is compact with respect to the weak* topology, by the Banach–Alaoglu theorem.

A linear subspace $\mathcal{I}$ of a real or complex algebra $\mathcal{A}$ is said to be an *ideal* in $\mathcal{A}$ if $a\,x$ and $x\,a$ are contained in $\mathcal{I}$ for every $a \in \mathcal{A}$ and $x \in \mathcal{I}$. Of course, $\mathcal{A}$ itself and the trivial subspace $\{0\}$ are ideals in $\mathcal{A}$, and an ideal $\mathcal{I}$ in $\mathcal{A}$ is said to be proper if $\mathcal{I} \neq \mathcal{A}$. If $\mathcal{I}$ is an ideal in $\mathcal{A}$ and $\mathcal{I}$ contains the identity element e, or any invertible element x, then $\mathcal{I} = \mathcal{A}$. If $\mathcal{A}$ is a Banach algebra and $\mathcal{I}$ is an ideal in $\mathcal{A}$, then it is easy to see that the closure $\overline{\mathcal{I}}$ of $\mathcal{I}$ with respect to the norm on $\mathcal{A}$ is also an ideal in $\mathcal{A}$. If $\mathcal{I}$ is a proper ideal in a Banach algebra $\mathcal{A}$, then $e \not\in \overline{\mathcal{I}}$. This is because elements of $\mathcal{A}$ sufficiently close to e are invertible, as before. Thus the closure of a proper ideal in a Banach algebra is still proper.

A proper ideal $\mathcal{I}$ in an algebra $\mathcal{A}$ is said to be maximal if $\mathcal{A}$ and $\mathcal{I}$ are the only ideals that contain $\mathcal{I}$. It is easy to see that the kernel of a nonzero homomorphism on $\mathcal{A}$ is maximal, since it has codimension 1. A maximal ideal $\mathcal{I}$ in a Banach algebra $\mathcal{A}$ is automatically closed, because its closure $\overline{I}$ is a proper ideal that contains $\mathcal{I}$, and hence is equal to $\mathcal{I}$.

Using the axiom of choice, one can show that every proper ideal in an algebra with nonzero multiplicative identity element is contained in a maximal ideal. More precisely, one can use Zorn's lemma or the Hausdorff maximality principle, by checking that the union of a chain of proper ideals is a proper ideal. To get properness, one uses the fact that the ideals do not contain the identity element.

If $\mathcal{A}$ is a commutative algebra and $a \in \mathcal{A}$, then

$$\mathcal{I}_a = \{a\,b : b \in \mathcal{A}\} \tag{49.22}$$

is an ideal in $\mathcal{A}$. Moreover, $\mathcal{I}_a$ is a proper ideal in $\mathcal{A}$ if and only if a is not invertible in $\mathcal{A}$.

Suppose from now on that $\mathcal{A}$ is a complex Banach algebra. Let $a \in \mathcal{A}$ be given, and suppose that $t\,e - a$ is invertible in $\mathcal{A}$ for every $t \in \mathbf{C}$. If λ is a continuous linear functional on $\mathcal{A}$, then one can show that

$$f_\lambda(t) = \lambda((t\,e - a)^{-1}) \tag{49.23}$$

is a holomorphic function on the complex plane $\mathbf{C}$. One can also check that $(t\,e - a)^{-1} \to 0$ in $\mathcal{A}$ as $|t| \to \infty$, so that $f_\lambda(t) \to 0$ as $|t| \to \infty$ for each λ. This implies that $f_\lambda(t) = 0$ for every $t \in \mathbf{C}$ and continuous linear functional λ on $\mathcal{A}$, by standard results in complex analysis. Using the Hahn–Banach theorem, it follows that $(t\,e - a)^{-1} = 0$ for every $t \in \mathbf{C}$, contradicting the fact that invertible elements of $\mathcal{A}$ are not zero. This is a brief sketch of the well-known fact that for each $a \in \mathcal{A}$ there is a $t \in \mathbf{C}$ such that $t\,e - a$ is not invertible.

Suppose that every nonzero element of $\mathcal{A}$ is invertible. If $a \in \mathcal{A}$, then there is a $t \in \mathbf{C}$ such that $t\,e - a$ is not invertible, as in the previous paragraph. In this case, it follows that $a = t\,e$, so that $\mathcal{A}$ is isomorphically equivalent to the complex numbers.

If $\mathcal{A}$ is an algebra $\mathcal{I}$ is an ideal in $\mathcal{A}$, then the quotient $\mathcal{A}/\mathcal{I}$ defines an algebra in a natural way, so that the corresponding quotient mapping is a homomorphism from $\mathcal{A}$ onto $\mathcal{A}/\mathcal{I}$ with kernel equal to $\mathcal{I}$. If $\mathcal{A}$ has a nonzero multiplicative identity element and $\mathcal{I}$ is proper, then $\mathcal{A}/\mathcal{I}$ also has a nonzero multiplicative identity element. If $\mathcal{I}$ is a maximal ideal, then $\mathcal{A}/\mathcal{I}$ contains no nontrivial proper ideals. If $\mathcal{A}$ is commutative and $\mathcal{I}$ is maximal, then it follows that every nonzero element of $\mathcal{A}/\mathcal{I}$ is invertible in the quotient. If $\mathcal{A}$ is a Banach algebra and $\mathcal{I}$ is a proper closed ideal in $\mathcal{A}$, then $\mathcal{A}/\mathcal{I}$ is also a Banach algebra, with respect to the usual quotient norm. If $\mathcal{A}$ is a complex commutative Banach algebra and $\mathcal{I}$ is a maximal ideal in $\mathcal{A}$, then it follows that $\mathcal{A}/\mathcal{I}$ is isomorphic to the complex numbers. This implies that every maximal ideal in a commutative complex Banach algebra $\mathcal{A}$ is the kernel of a homomorphism from $\mathcal{A}$ onto the complex numbers. If $\mathcal{A}$ is a commutative complex Banach algebra and $a \in \mathcal{A}$ is not invertible, then a is contained in a maximal ideal in $\mathcal{A}$, and hence there is a nonzero homomorphism $\phi : \mathcal{A} \to \mathbf{C}$ such that $\phi(a) = 0$.

50 Ideals and filters

Let E be a nonempty set, and let $\mathcal{A}$ be the algebra of all real or complex-valued functions on E. Put

$$Z(f) = \{x \in E : f(x) = 0\} \tag{50.1}$$

for each $f \in \mathcal{A}$. Thus

$$Z(f) \cap Z(g) \subseteq Z(f + g) \tag{50.2}$$

and

$$Z(f) \cup Z(g) = Z(f\,g) \tag{50.3}$$

for every $f, g \in \mathcal{A}$. If f, g are nonnegative real-valued functions on E, then

$$Z(f) \cap Z(g) = Z(f + g). \tag{50.4}$$

If $\mathcal{F}$ is a filter on E, then put

$$\mathcal{I}(\mathcal{F}) = \{f \in \mathcal{A} : Z(f) \in \mathcal{F}\}. \tag{50.5}$$

It is easy to see that this is an ideal in $\mathcal{A}$, using the properties of the zero sets of sums and products of functions mentioned in the previous paragraph. More precisely, $\mathcal{I}(\mathcal{F})$ is a proper ideal in $\mathcal{A}$, since the elements of a filter are nonempty sets. As a special case, suppose that $A \subseteq E$ is not empty, and let $\mathcal{F}^A$ be the collection of subsets B of E such that $A \subseteq B$. This is a filter on E, and the corresponding ideal $\mathcal{I}(\mathcal{F}^A)$ is the same as

$$\mathcal{I}_A = \{f \in \mathcal{A} : f(x) = 0 \text{ for every } x \in A\}. \tag{50.6}$$

In this case, the quotient $\mathcal{A}/\mathcal{I}_A$ can be identified with the algebra of real or complex-valued functions on A, as appropriate. In particular, if A consists of a single point, then the quotient is isomorphic to the real or complex numbers, as appropriate.

Conversely, if $\mathcal{I}$ is a proper ideal in $\mathcal{A}$, then put

$$\mathcal{F}(\mathcal{I}) = \{Z(f) : f \in \mathcal{I}\}. \tag{50.7}$$

It is easy to check that this is a filter on E. In connection with this, note that

$$Z(|f|) = Z(f) \tag{50.8}$$

for every $f \in \mathcal{A}$, and that $|f| \in \mathcal{I}$ when $f \in \mathcal{I}$. This implies that $\mathcal{F}(\mathcal{I})$ is the same as the collection of zero sets of nonnegative real-valued functions on E in $\mathcal{I}$. Observe also that

$$\mathcal{F}(\mathcal{I}(\mathcal{F})) = \mathcal{F} \tag{50.9}$$

for every filter $\mathcal{F}$ on E, and that

$$\mathcal{I}(\mathcal{F}(\mathcal{I})) = \mathcal{I} \tag{50.10}$$

for every proper ideal $\mathcal{I}$ in $\mathcal{A}$. This shows that every proper ideal $\mathcal{I}$ in $\mathcal{A}$ is of the form $\mathcal{I}(\mathcal{F})$ for some filter $\mathcal{F}$ on E.

If $\mathcal{F}$, $\mathcal{F}'$ are filters on E, then it is easy to see that

$$\mathcal{I}(\mathcal{F}) \subseteq \mathcal{I}(\mathcal{F}') \tag{50.11}$$

if and only if $\mathcal{F} \subseteq \mathcal{F}'$, which is to say that $\mathcal{F}'$ is a refinement of $\mathcal{F}$. It follows that ultrafilters on E correspond exactly to maximal ideals in $\mathcal{A}$. In particular, if $\mathcal{F}$ is an ultrafilter on E, then $\mathcal{A}/\mathcal{I}(\mathcal{F})$ is a field. One can also see this more directly, as follows. Suppose that $f \in \mathcal{A}$ and $f \not\in \mathcal{I}(\mathcal{F})$, so that the element of the quotient $\mathcal{A}/\mathcal{I}(\mathcal{F})$ corresponding to f is not zero. Thus $Z(f) \not\in \mathcal{F}$, by definition of $\mathcal{I}(\mathcal{F})$, and so $E \backslash Z(f) \in \mathcal{F}$, because $\mathcal{F}$ is an ultrafilter. If $g \in \mathcal{A}$ satisfies $f(x)\, g(x) = 1$ for every $x \in E \backslash Z(f)$, then $f\, g - 1 \in \mathcal{I}(\mathcal{F})$, which means that the product of the elements of the quotient $\mathcal{A}/\mathcal{I}(\mathcal{F})$ corresponding to f, g is equal to the multiplicative identity element in the quotient, as desired.

51 Closure

Let X be a topological space, and remember that $C_b(X)$ is the algebra of bounded continuous real or complex-valued functions on X, equipped with the supremum norm. Put

$$Z_\epsilon(f) = \{x \in X : |f(x)| \le \epsilon\} \tag{51.1}$$

for every $f \in C_b(X)$ and $\epsilon > 0$, which is a closed set in X, since f is continuous. Note that $Z_\epsilon(f) = \emptyset$ for some $\epsilon > 0$ if and only if f is invertible in $C_b(X)$. If $\mathcal{F}$ is a filter on X, then let $\mathcal{I}(\mathcal{F})$ be the collection of $f \in C_b(X)$ such that $f_*(\mathcal{F})$ converges to 0 in $\mathbf{R}$ or $\mathbf{C}$, as appropriate. Equivalently,

$$\mathcal{I}(\mathcal{F}) = \{f \in C_b(X) : Z_\epsilon(f) \in \mathcal{F} \text{ for every } \epsilon > 0\}. \tag{51.2}$$

This is analogous to, but different from, the definition in the previous section. It is not difficult to check that $\mathcal{I}(\mathcal{F})$ is a proper closed ideal in $C_b(X)$ under these conditions. This uses the fact that

$$Z_{\epsilon/2}(f) \cap Z_{\epsilon/2}(g) \subseteq Z_\epsilon(f + g) \tag{51.3}$$

for every $f, g \in C_b(X)$ and $\epsilon > 0$, and that

$$Z_{\epsilon/k}(f) \subseteq Z_\epsilon(f\, g) \tag{51.4}$$

when $|g(x)| \le k$ for each $x \in X$ and $k > 0$.

Let $\overline{\mathcal{F}}$ be the collection of subsets B of X for which there is an $A \in \mathcal{F}$ such that $\overline{A} \subseteq B$. One can check that $\overline{\mathcal{F}}$ is also a filter on X, and that $\mathcal{I}(\overline{\mathcal{F}}) = \mathcal{I}(\mathcal{F})$. Thus one might as well restrict one's attention to filters on X generated by closed subsets of X. As a special case, if $A \subseteq X$ is nonempty and $\mathcal{F}^A$ is the filter consisting of $B \subseteq X$ such that $A \subseteq B$, then $\overline{\mathcal{F}^A} = \mathcal{F}^{\overline{A}}$.

Now let $\mathcal{I}$ be a proper ideal in $C_b(X)$, and put

$$\mathcal{F}(\mathcal{I}) = \{A \subseteq X : Z_\epsilon(f) \subseteq A \text{ for some } f \in \mathcal{I} \text{ and } \epsilon > 0\}. \tag{51.5}$$

Again this is analogous to, but different from, the corresponding definition in the previous section. One can also check that $\mathcal{F}(\mathcal{I})$ is a filter on X under these conditions. This uses the fact that $Z_\epsilon(f) \ne \emptyset$ for each $f \in \mathcal{I}$ and $\epsilon > 0$, because $\mathcal{I}$ is proper. If $f \in \mathcal{I}$, then $|f|^2 \in \mathcal{I}$, and

$$Z_{\epsilon^2}(|f|^2) = Z_\epsilon(f), \tag{51.6}$$

which means that one can restrict one's attention to nonnegative real-valued functions in $\mathcal{I}$. If f, g are nonnegative real-valued functions on X and $\epsilon > 0$, then

$$Z_\epsilon(f + g) \subseteq Z_\epsilon(f) \cap Z_\epsilon(g). \tag{51.7}$$

This implies that $A \cap B \in \mathcal{F}(\mathcal{I})$ for every $A, B \in \mathcal{F}(\mathcal{I})$. Note that $\mathcal{F}(\mathcal{I})$ is automatically generated by closed subsets of X. One can also check that $\mathcal{F}(\mathcal{I})$

is the same as the filter associated to the closure of $\mathcal{I}$ in $C_b(X)$, with respect to the supremum norm. This uses the fact that

$$Z_{\epsilon/2}(f) \subseteq Z_\epsilon(g) \tag{51.8}$$

when $|f(x) - g(x)| \le \epsilon/2$ for every $x \in X$.

By construction, $\mathcal{I} \subseteq \mathcal{I}(\mathcal{F}(\mathcal{I}))$. We have seen that $\mathcal{I}(\mathcal{F})$ is closed in $C_b(X)$ for any filter $\mathcal{F}$ on X, and so $\overline{\mathcal{I}} \subseteq \mathcal{I}(\mathcal{F}(\mathcal{I}))$. In order to show that

$$\overline{\mathcal{I}} = \mathcal{I}(\mathcal{F}(\mathcal{I})), \tag{51.9}$$

let $f \in \mathcal{I}(\mathcal{F}(\mathcal{I}))$ and $\epsilon > 0$ be given. By definition of $\mathcal{I}(\mathcal{F}(\mathcal{I}))$, there are a $g \in \mathcal{I}$ and a $\delta > 0$ such that

$$Z_\delta(g) \subseteq Z_\epsilon(f). \tag{51.10}$$

Put

$$f_\eta = f \, \frac{|g|^2}{|g|^2 + \eta^2} \tag{51.11}$$

for each $\eta > 0$. Thus $f_\eta \in C_b(X)$ for each η, and in fact $f_\eta \in \mathcal{I}$, because $g \in \mathcal{I}$. We would like to check that

$$|f(x) - f_\eta(x)| = |f(x)| \, \frac{\eta^2}{|g(x)|^2 + \eta^2} \le \epsilon \tag{51.12}$$

for every $x \in X$ when η is sufficiently small. If $x \in Z_\epsilon(f)$, then this holds for every $\eta > 0$, since $|f(x)| \le \epsilon$ and $\eta^2/(|g(x)|^2 + \eta^2) \le 1$. If $x \not\in Z_\epsilon(f)$, then $x \not\in Z_\delta(g)$, so that $|g(x)| > \delta$, and the desired estimate holds when η is sufficiently small, because f is bounded.

52 Regular topological spaces

Remember that a topological space X is said to be *regular*, or equivalently to satisfy the third separation condition, if it has the following two properties. First, X should satisfy the first separation condition, so that subsets of X with exactly one element are closed. Second, for each $x \in X$ and closed set $E \subseteq X$ with $x \not\in E$, there should be disjoint open subsets U, V of X such that $x \in U$ and $E \subseteq V$. In particular, this implies that X is Hausdorff, since one can take $E = \{y\}$ when $y \in X$ and $y \neq x$. Sometimes the term "regular" is used for topological spaces with the second property just mentioned, and then the third separation condition is defined to be the combination of regularity with the first separation condition. We shall include the first separation condition in the definition of regularity here for the sake of simplicity. As in Section 38, it is well known that locally compact Hausdorff topological spaces are regular.

Equivalently, X is regular if it satisfies the first separation condition and for each $x \in X$ and open set $W \subseteq X$ with $x \in W$ there is an open set $U \subseteq X$ such that $x \in U$ and $\overline{U} \subseteq W$. This corresponds to the previous definition with $W = X \backslash E$. Let $\mathcal{F}$ be a filter on X, and let $\overline{\mathcal{F}}$ be the filter on X generated by the closures of the elements of $\mathcal{F}$, as in the previous section. If $\mathcal{F}$ converges to a point $x \in X$ and X is regular, then it is easy to see that $\overline{\mathcal{F}}$ also converges to x in X. For if W is an open set in X that contains x and U is an open set in X that contains x and satisfies $\overline{U} \subseteq W$, then $U \in \mathcal{F}$, because $\mathcal{F}$ converges to x, and hence $W \in \overline{\mathcal{F}}$.

Now let $x \in X$ be given, and let $\mathcal{F}(x)$ be the collection of subsets A of X for which there is an open set $U \subseteq X$ such that $x \in U$ and $U \subseteq A$. This is a filter on X that converges to x, by construction. The filter $\overline{\mathcal{F}(x)}$ generated by the closed subsets of X is the same as the collection of subsets B of X for which there is an open set $U \subseteq X$ such that $x \in U$ and $\overline{U} \subseteq B$. If $\overline{\mathcal{F}(x)}$ converges to x, then for each open set $W \subseteq X$ with $x \in W$ there is an open set $U \subseteq X$ such that $x \in U$ and $\overline{U} \subseteq W$. It follows that X is regular if it satisfies the first separation condition and $\overline{\mathcal{F}(x)}$ converges to x for every $x \in X$.

Of course, metric spaces are regular as topological spaces. Real and complex topological vector spaces are also regular as topological spaces. To see this, remember that if U is an open set in a topological vector space V that contains 0, then there are open subsets U_1, U_2 of V that contain 0 and satisfy

$$U_1 + U_2 \subseteq U, \tag{52.1}$$

as in Section 23. Moreover,

$$\overline{U_1} \subseteq U_1 + U_2, \tag{52.2}$$

as in (23.14). Hence $\overline{U_1} \subseteq U$, which implies that V is regular, because of the translation-invariance of the topology on V.

53 Closed sets

Let X be a topological space, and let us say that a nonempty collection $\mathcal{E}$ of nonempty closed subsets of X is a *C-filter* if $A \cap B \in \mathcal{E}$ for every $A, B \in \mathcal{E}$, and if $E \in \mathcal{E}$ whenever $E \subseteq X$ is a closed set such that $A \subseteq E$ for some $A \in \mathcal{E}$. This is the same as a filter on X, except that we restrict our attention to closed subsets of X. If $\mathcal{F}$ is a filter on X and $\mathcal{E}(\mathcal{F})$ is the collection of closed subsets of X that are elements of X, then $\mathcal{E}(\mathcal{F})$ is a C-filter. This can also be described as the collection of closures of elements of $\mathcal{F}$, since the closure of an element of $\mathcal{F}$ is a closed set in X that is contained in $\mathcal{F}$.

A C-filter $\mathcal{E}$ on X also generates an ordinary filter $\mathcal{F}(\mathcal{E})$ on X, consisting of the subsets B of X that contain an element of $\mathcal{E}$ as a subset. If $\mathcal{F}$ is any filter on X, and $\mathcal{E}(\mathcal{F})$ is the C-filter obtained from it as in the preceding paragraph, then the filter generated by $\mathcal{E}(\mathcal{F})$ is the same as the filter $\overline{\mathcal{F}}$ defined previously. However, if $\mathcal{E}$ is any C-filter on X, and $\mathcal{F}(\mathcal{E})$ is the ordinary filter generated by $\mathcal{E}$, then the C-filter of closed sets in $\mathcal{F}(\mathcal{E})$ is the same as $\mathcal{E}$.

Let us say that a C-filter $\mathcal{E}$ on X converges to a point $x \in X$ if for every open set $U \subseteq X$ with $x \in U$ there is an $E \in \mathcal{E}$ such that $E \subseteq U$. This is equivalent to saying that $U \in \mathcal{F}(\mathcal{E})$ for every open set $U \subseteq X$ with $x \in U$, so that $\mathcal{E}$ converges to x if and only if $\mathcal{F}(\mathcal{E})$ converges to x. If X is Hausdorff, then the limit of a convergent C-filter on X is unique, for the same reasons as for ordinary filters. If $\mathcal{F}$ is an ordinary filter on X that converges to a point $x \in X$ and X is regular, then the corresponding C-filter $\mathcal{E}(\mathcal{F})$ also converges to x, for the same reasons as in the preceding section.

Let A be a nonempty subset of X, and let $\mathcal{E}^A$ be the collection of closed sets $B \subseteq X$ that contain A. This is a C-filter on X, and it is easy to see that $\mathcal{E}^{\overline{A}} = \mathcal{E}^A$ for every $A \subseteq X$. Note that $A \in \mathcal{E}^A$ if and only if A is a closed set in X. If $A = \{p\}$ for some $p \in X$ and X satisfies the first separation condition, then $\{p\} \in \mathcal{E}^A$, and $\mathcal{E}^A$ converges to p in X.

Suppose that $\mathcal{E}$ is a C-filter on X that converges to a point $p \in X$, and let $A \in \mathcal{E}$ be given. If U is an open set in X that contains p, then there is an $E \in \mathcal{E}$ such that $E \subseteq U$, by definition of convergence. This implies that $A \cap U \neq \emptyset$, because $A \cap E$ is contained in $A \cap U$ and nonempty, since it is an element of $\mathcal{E}$. It follows that $p \in A$ for every $A \in \mathcal{E}$, because every $A \in \mathcal{E}$ is a closed set in X.

Let $\mathcal{E}$ be a C-filter on X, and suppose that $B \subseteq X$ satisfies $A \cap B \neq \emptyset$ for every $A \in \mathcal{E}$. Let $\mathcal{E}_B$ be the collection of closed subsets E of X such that $A \cap B \subseteq E$ for some $A \in \mathcal{E}$. It is easy to see that this is also a C-filter on X, which is a refinement of $\mathcal{E}$ in the sense that $\mathcal{E} \subseteq \mathcal{E}_B$ as collections of subsets of X. If B is a closed set in X, then $A \cap B \in \mathcal{E}_B$ for every $A \in \mathcal{E}$.

A C-filter $\mathcal{E}$ on X may be described as a *C-ultrafilter* if it is maximal with respect to inclusion. More precisely, $\mathcal{E}$ is a C-ultrafilter if for every C-filter $\mathcal{E}'$ such that $\mathcal{E} \subseteq \mathcal{E}'$, we have that $\mathcal{E} = \mathcal{E}'$. Using Zorn's lemma or the Hausdorff maximality principle, one can show that every C-filter has a refinement which is a C-ultrafilter, just as for ordinary ultrafilters.

For each $p \in X$, let $\mathcal{E}_p$ be the C-filter consisting of all closed subsets of X that contain p as an element. This is the same as $\mathcal{E}^A$ with $A = \{p\}$, as before. If X satisfies the first separation condition, then $\{p\}$ is a closed set in X, $\{p\} \in \mathcal{E}_p$, and it is easy to see that $\mathcal{E}_p$ is a C-ultrafilter on X. If $\mathcal{E}$ is any C-filter on X and $p \in E$ for each $E \in \mathcal{E}$, then $\mathcal{E} \subseteq \mathcal{E}_p$, and hence $\mathcal{E} = \mathcal{E}_p$ when $\mathcal{E}$ is a C-ultrafilter. In particular, this holds when $\mathcal{E}$ converges to p. If $\mathcal{E}$ is a C-filter on X and X is compact, then $\bigcap_{E \in \mathcal{E}} E \neq \emptyset$, because $\mathcal{E}$ has the finite intersection property. If $\mathcal{E}$ is a C-ultrafilter, then it follows that $\mathcal{E} = \mathcal{E}_p$ for some $p \in X$.

Let $\mathcal{E}$ be a C-filter on X, and suppose that B is a closed set in X such that $A \cap B \neq \emptyset$ for every $A \in \mathcal{E}$. This implies that $\mathcal{E} \subseteq \mathcal{E}_B$, where $\mathcal{E}_B$ is the C-filter generated by the intersections $A \cap B$ with $A \in \mathcal{E}$, as before. If $\mathcal{E}$ is a C-ultrafilter, then it follows that $\mathcal{E} = \mathcal{E}_B$, and hence $B \in \mathcal{E}$. Conversely, a C-filter $\mathcal{E}$ is a C-ultrafilter when $B \in \mathcal{E}$ for every closed set $B \subseteq X$ such that $A \cap B \neq \emptyset$ for every $A \in \mathcal{E}$. For if $\mathcal{E}'$ is a C-filter on X such that $\mathcal{E} \subseteq \mathcal{E}'$, then $A \cap B$ is contained in $\mathcal{E}'$ and is therefore nonempty for every $A \in \mathcal{E}$ and $B \in \mathcal{E}'$.

Let X, Y be topological spaces, and let f be a continuous mapping from X into Y. Thus $f^{-1}(B)$ is a closed set in X for every closed set B in Y. Also let $\mathcal{E}$ be a C-filter on X, and let $f_*(\mathcal{E})$ be the collection of closed sets $B \subseteq Y$ such that $f^{-1}(B) \in \mathcal{E}$. It is easy to see that $f_*(\mathcal{E})$ is a C-filter on Y. Note that the closure of $f(A)$ in Y is an element of $f_*(\mathcal{E})$ for each $A \in \mathcal{E}$.

Suppose that Y is compact, so that $\bigcap_{B \in f_*(\mathcal{E})} B \neq \emptyset$, and let q be an element of the intersection. Thus q is contained in the closure of $f(A)$ in Y for every $A \in \mathcal{E}$. If V is any open set in Y that contains q, then $f(A) \cap V \neq \emptyset$ for every $A \in \mathcal{E}$, and hence $A \cap f^{-1}(V) \neq \emptyset$. Let $\mathcal{E}'$ be the collection of closed sets E in X such that $A \cap f^{-1}(V) \subseteq E$ for some $A \in \mathcal{E}$ and open set $V \subseteq Y$ with $q \in V$. It is easy to see that $\mathcal{E}'$ is a C-filter on X that is a refinement of $\mathcal{E}$, and that $A \cap f^{-1}(\overline{V}) \in \mathcal{E}'$ for every open set $V \subseteq Y$ with $q \in V$. In particular, $f^{-1}(\overline{V}) \in \mathcal{E}'$ under these conditions, which means that $\overline{V} \in f_*(\mathcal{E}')$. If Y is also Hausdorff, and hence regular, then it follows that $f_*(\mathcal{E}')$ converges to q in Y. If $\mathcal{E}$ is a C-ultrafilter on X, then $\mathcal{E} = \mathcal{E}'$, and $f_*(\mathcal{E})$ converges to q in Y.

54 Multi-indices

Let n be a positive integer, which will be kept fixed throughout this section. A *multi-index* $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of nonnegative integers. The sum of two multi-indices is defined coordinatewise, and we put

$$|\alpha| = \alpha_1 + \cdots + \alpha_n. \tag{54.1}$$

If α is a multi-index and $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, then the corresponding monomial x^α is defined by the product

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}. \tag{54.2}$$

More precisely, $x_j^{\alpha_j}$ is interpreted as being equal to 1 for every $x_j \in \mathbf{R}$ when $\alpha_j = 0$, so that $x^\alpha = 1$ for every $x \in \mathbf{R}^n$ when $\alpha = 0$. Note that $|\alpha|$ is the same as the degree of the monomial x^α, and a polynomial on $\mathbf{R}^n$ is the same as a linear combination of finitely many monomials. Moreover,

$$x^{\alpha+\beta} = x^\alpha \, x^\beta \tag{54.3}$$

for all multi-indices α, β and $x \in \mathbf{R}^n$.

If l is a positive integer, then $l!$ is "l factorial", the product of $1, \ldots, l$. It is customary to include $l = 0$ by setting $0! = 1$. If α is a multi-index, then we put

$$\alpha! = \alpha_1! \cdots \alpha_n!. \tag{54.4}$$

If α is a multi-index and $x, y \in \mathbf{R}^n$, then

$$(x+y)^\alpha = \sum_{\alpha = \beta + \gamma} \frac{\alpha!}{\beta! \, \gamma!} \, x^\beta \, y^\gamma, \tag{54.5}$$

where the sum is taken over all multi-indices β, γ such that $\alpha = \beta + \gamma$. This follows from the binomial theorem applied to $(x_j + y_j)^{\alpha_j}$ for $j = 1, \ldots, n$.

Let $\partial_j = \partial / \partial x_j$ be the usual partial derivative in x_j, $1 \le j \le n$. If α is a multi-index, then the corresponding differential operator ∂^α is defined by

$$\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}. \tag{54.6}$$

Here $\partial_j^{\alpha_j}$ is interpreted as being the identity operator when $\alpha_j = 0$, so that ∂^α reduces to the identity operator when $\alpha = 0$. Observe that

$$\partial^{\alpha+\beta} = \partial^\alpha \, \partial^\beta \tag{54.7}$$

for all multi-indices α, β.

55 Smooth functions

Let U be a nonempty open set in $\mathbf{R}^n$ for some positive integer n, and let $C^\infty(U)$ be the space of real or complex-valued functions on U that are smooth in the sense that they are continuously-differentiable of all orders. As usual, this may also be denoted $C^\infty(U, \mathbf{R})$ or $C^\infty(U, \mathbf{C})$, to indicate whether real or complex-valued functions are being used. It is well known that $C^\infty(U)$ is a commutative algebra with respect to pointwise addition and multiplication, since sums and products of smooth functions are smooth.

If α is a multi-index and $K \subseteq U$ is a nonempty compact set, then

$$\|f\|_{\alpha,K} = \sup_{x \in K} |\partial^\alpha f(x)| \tag{55.1}$$

defines a seminorm on $C^\infty(U)$. This is the same as the supremum seminorm $\|f\|_K$ of f over K when $\alpha = 0$, and otherwise this is the same as the supremum seminorm of $\partial^\alpha f$ over K. The collection of all of these seminorms defines a topology on $C(U)$, as in Section 3. Of course, U is a locally compact Hausdorff topological space with respect to the topology induced by the standard topology on $\mathbf{R}^n$, and one can also check that U is σ-compact. As in Section 36, there is a sequence of compact subsets $K_1, K_2, \ldots$ of U such that every compact set $H \subseteq U$ is contained in K_l for some l. It follows that the seminorms $\|f\|_{\alpha,K_l}$ are sufficient to determine the same topology on $C^\infty(U)$ as the one that was just described, where α is a multi-index and l is a positive integer. In particular, this collection of seminorms on $C^\infty(U)$ is countable, since there are only countably many multi-indices.

If f, g are smooth functions on U and α is a multi-index, then

$$\partial^\alpha (f\,g) = \sum_{\alpha = \beta + \gamma} \frac{\alpha!}{\beta!\,\gamma!} \, (\partial^\beta f) \, (\partial^\gamma g), \tag{55.2}$$

where the sum is taken over all multi-indices β, γ such that $\alpha = \beta + \gamma$. This can be derived from the usual product rule for first derivatives, starting with the $n = 1$ case. Using this identity, it is easy to check that multiplication of functions is continuous as a mapping from $C^\infty(U) \times C^\infty(U)$ into $C^\infty(U)$, with respect to the topology on $C^\infty(U)$ defined in the previous paragraph.

Let ϕ be a homomorphism from $C^\infty(U)$ into the real or complex numbers, as appropriate. As usual, we suppose also that ϕ is nontrivial in the sense that $\phi(f) \neq 0$ for some $f \in C^\infty(U)$. This implies that $\phi(\mathbf{1}_U) = 1$, where $\mathbf{1}_U$ is the constant function on U equal to 1 at every point. If f is a smooth function on

U such that $f(x) \neq 0$ for every $x \in U$, then $1/f(x)$ is also a smooth function on U, and it follows that

$$\phi(f)\,\phi(1/f) = \phi(\mathbf{1}_U) = 1. \tag{55.3}$$

In particular, $\phi(f) \neq 0$ when $f(x) \neq 0$ for every $x \in U$. Equivalently, if f is any smooth function on U and $\phi(f) = 0$, then $f(x) = 0$ for some $x \in U$. If f is any smooth function on U and $\phi(f) = c$, then there is an $x \in U$ such that $f(x) = c$, since one can apply the previous statement to $f - c\,\mathbf{1}_U$.

Let f_j be the smooth function on U defined by $f_j(x) = x_j$, $j = 1, \ldots, n$, and put $p_j = \phi(f_j)$. We would like to check that

$$p = (p_1, \ldots, p_n) \in U. \tag{55.4}$$

Consider the smooth function on U given by

$$f(x) = \sum_{j=1}^{n} (x_j - p_j)^2. \tag{55.5}$$

Equivalently,

$$f = \sum_{j=1}^{n} (f_j - p_j\,\mathbf{1}_U)^2, \tag{55.6}$$

and so

$$\phi(f) = \sum_{j=1}^{n} (\phi(f_j) - p_j)^2 = 0, \tag{55.7}$$

because ϕ is a homomorphism. Hence $f(x) = 0$ for some $x \in U$, as in the previous paragraph, which is only possible if $x = p$, in which case $p \in U$.

If g is a smooth function on U and U is convex, then

$$\begin{aligned} g(x) - g(p) &= \int_0^1 (\partial/\partial t) g(t\,x + (1-t)\,p)\,dt \\ &= \sum_{j=1}^{n} (x_j - p_j) \int_0^1 (\partial_j g)(t\,x + (1-t)\,p)\,dt. \end{aligned} \tag{55.8}$$

Hence there are smooth functions $g_1, \ldots, g_n$ on U such that

$$g(x) = g(p) + \sum_{j=1}^{n} (x_j - p_j)\,g_j(x). \tag{55.9}$$

This also works when g is the restriction to U of a smooth function on a convex open set that contains U, such as $\mathbf{R}^n$ itself. In particular, this works when $g(x) = 0$ on the complement of a closed ball contained in U. Otherwise, if $g(x) = 0$ for every x in a neighborhood of p, then we can simply take $g_j(x)$ to be $(x_j - p_j)/|x - p|^2$ times $g(x)$, where $|x - p|^2 = \sum_{l=1}^{n} (x_l - p_l)^2$, and $g_j(p) = 0$. Any smooth function on U can be expressed as the sum of a smooth function supported on a closed ball in U and a smooth function that vanishes on a neighborhood of p, using standard cut-off functions. It follows that every smooth function g on U can be expressed as in (55.9) for some smooth functions $g_1, \ldots, g_n$ on U.

Using this representation, we get that $\phi(g) = g(p)$ for every $g \in C^\infty(U)$. Of course, $\phi_p(g) = g(p)$ defines a homomorphism on $C^\infty(U)$ for every $p \in U$.

56 Polynomials

Let $\mathcal{P}(\mathbf{R}^n)$ be the space of polynomials on $\mathbf{R}^n$ with real coefficients, which can be expressed as finite linear combinations of the monomials x^α, where α is a multi-index. This is an algebra in a natural way, corresponding to pointwise addition and multiplication of functions. If $p \in \mathbf{R}^n$, then $\phi_p(f) = f(p)$ defines a homomorphism on $\mathcal{P}(\mathbf{R}^n)$, as usual. Conversely, if ϕ is a homomorphism on $\mathcal{P}(\mathbf{R}^n)$ which is not identically 0, then $\phi = \phi_p$ for some $p \in \mathbf{R}^n$. As in the previous section, $p = (p_1, \ldots, p_n)$ is given by $p_j = \phi(f_j)$, where $f_j(x) = x_j$. In this case, the fact that $\phi(f) = f(p)$ for every polynomial f on $\mathbf{R}^n$ follows from simple algebra. There are analogous statements for polynomials on $\mathbf{C}^n$ with complex coefficients, which can be expressed as finite linear combinations of monomials $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$.

57 Continuously-differentiable functions

A real or complex-valued function f on the closed unit interval $[0, 1]$ is said to be *continuously differentiable* if it satisfies the following three conditions. First, the derivative $f'(x)$ of f should exist at every x in the open unit interval $(0, 1)$. Second, the appropriate one-sided derivatives should exist at the endpoints 0, 1, which will also be denoted $f'(0)$, $f'(1)$ for simplicity. Third, the resulting function $f'(x)$ should be continuous on $[0, 1]$. Of course, differentiability of f implies that f is continuous on $[0, 1]$.

Equivalently, a continuous function f on $[0, 1]$ is continuously differentiable if it is differentiable on $(0, 1)$, and if the derivative can be extended to a continuous function on $[0, 1]$, also denoted f'. More precisely, one can check that the one-sided derivatives of f exist at the endpoints, and are given by the extension of f' to 0, 1. This follows from the fact that

$$f(y) - f(x) = \int_x^y f'(t)\, dt \tag{57.1}$$

when $0 \le x \le y \le 1$.

The space of continuously-differentiable functions on $[0, 1]$ may be denoted $C^1([0, 1])$, or by $C^1([0, 1], \mathbf{R})$, $C^1([0, 1], \mathbf{C})$ to indicate whether real or complex-valued functions are being used. As usual, $C^1([0, 1])$ is an algebra with respect to pointwise addition and scalar multiplication of functions. If

$$\|f\|_{sup} = \sup_{0 \le x \le 1} |f(x)| \tag{57.2}$$

is the supremum norm of a bounded function on $[0, 1]$, then

$$\|f\|_{C^1} = \|f\|_{C^1([0,1])} = \|f\|_{sup} + \|f'\|_{sup} \tag{57.3}$$

is a natural choice of norm on $C^1([0, 1])$. In particular,

$$\|f\,g\|_{C^1} \le \|f\|_{C^1}\,\|f\|_{C^1} \tag{57.4}$$

for every $f, g \in C^1([0, 1])$. To see this, remember that

$$\|f\,g\|_{sup} \le \|f\|_{sup}\,\|g\|_{sup}, \tag{57.5}$$

so that

$$\|f\,g\|_{C^1} = \|f\,g\|_{sup} + \|(f\,g)'\|_{sup} \le \|f\|_{sup}\,\|g\|_{sup} + \|(f\,g)'\|_{sup}. \tag{57.6}$$

The product rule implies that

$$\|(f\,g)'\|_{sup} = \|f'\,g + f\,g'\|_{sup} \le \|f'\|_{sup}\|g\|_{sup} + \|f\|_{sup}\,\|g'\|_{sup}, \tag{57.7}$$

and hence

$$\begin{aligned} \|f\,g\|_{C^1} &\le \|f\|_{sup}\,\|g\|_{sup} + \|f'\|_{sup}\,\|g\|_{sup} + \|f\|_{sup}\,\|g'\|_{sup} \\ &\le (\|f\|_{sup} + \|f'\|_{sup})\,(\|g\|_{sup} + \|g'\|_{sup}) \\ &= \|f\|_{C^1}\,\|g\|_{C^1}. \end{aligned} \tag{57.8}$$

Note that the C^1 norm of a constant function is the same as the absolute value or modulus of the corresponding real or complex number. One can also check that $C^1([0, 1])$ is complete with respect to the C^1 norm, so that $C^1([0, 1])$ is a Banach algebra.

Remember that continuous functions on $[0, 1]$ can be approximated uniformly by polynomials, by Weierstrass' approximation theorem. Using this, one can show that continuously-differentiable functions on $[0, 1]$ can be approximated by polynomials in the C^1 norm. More precisely, in order to approximate a continuously-differentiable function f on $[0, 1]$ by polynomials in the C^1 norm, one can integrate polynomials that approximate f' uniformly on $[0, 1]$. One can choose the constant terms of these approximations to f to be equal to $f(0)$, so that the approximation of f follows from the approximation of f'.

Let ϕ be a homomorphism from $C^1([0, 1])$ into the real or complex numbers, as appropriate. Suppose also that $\phi(f) \ne 0$ for some $f \in C^1([0, 1])$, so that ϕ takes the constant function equal to 1 on $[0, 1]$ to 1, by the usual argument. If f is a continuously-differentiable function on $[0, 1]$ such that $f(x) \ne 0$ for every $x \in [0, 1]$, then $1/f$ is also a continuously-differentiable function on $[0, 1]$, and hence $\phi(f) \ne 0$. This implies that $\phi(f) \in f([0, 1])$ for every $f \in C^1([0, 1])$, as in previous situations. In particular, it follows that

$$|\phi(f)| \le \|f\|_{sup} \le \|f\|_{C^1} \tag{57.9}$$

for every $f \in C^1([0, 1])$.

Of course, $f_0(x) = x$ is a continuously-differentiable function on $[0,1]$. Put $p = \phi(f_0)$, so that $p \in f_0([0,1]) = [0,1]$. It follows that

$$\phi(f) = f(p) \tag{57.10}$$

when f is a polynomial, by simple algebra. The same relation holds for every $f \in C^1([0,1])$, because polynomials are dense in $C^1([0,1])$ with respect to the supremum norm. We do not need the stronger fact that polynomials are dense in $C^1([0,1])$ with respect to the C^1 norm here, because ϕ is continuous with respect to the supremum norm, by (57.9).

Alternatively, we can use the continuity of ϕ with respect to the supremum norm to extend ϕ to a homomorphism on $C([0,1])$, since $C^1([0,1])$ is dense in $C([0,1])$ with respect to the supremum norm. This permits us to use the results about homomorphisms on $C(X)$ when X is compact, as in Section 34. This approach has the advantage of working in more abstract situations, such as on compact manifolds. The same type of arguments as in Section 34 can also be used directly in these situations.

At any rate, every nonzero homomorphism on $C^1([0,1])$ can be represented as $\phi(f) = f(p)$ for some $p \in [0,1]$. Of course, $\phi_p(f) = f(p)$ is a homomorphism on $C^1([0,1])$ for every $p \in [0,1]$.

58 Spectral radius

Let $(\mathcal{A}, \|\cdot\|)$ be a Banach algebra over the real or complex numbers with nonzero multiplicative identity element e. If $x \in \mathcal{A}$ satisfies $\|x\| < 1$, then $e - x$ is invertible in $\mathcal{A}$, as in Section 49. The same conclusion also holds when $\|x^n\| < 1$ for any positive integer n. One way to see this is to use the previous result to get that $e - x^n$ is invertible, and then observe that

$$(e - x)\Big(\sum_{j=1}^{n-1} x^j\Big) = \Big(\sum_{j=1}^{n-1} x^j\Big)(e - x) = e - x^n. \tag{58.1}$$

This shows that the product of $e - x$ with an element of $\mathcal{A}$ that commutes with it is invertible, which implies that $e - x$ is invertible too, as in Section 49. Alternatively, one can check that $\sum_{j=1}^\infty \|x^j\|$ converges when $\|x^n\| < 1$ for some n, and then argue as in Section 49 that $\sum_{j=1}^\infty x^j$ converges in $\mathcal{A}$, and that the sum is the inverse of $e - x$. To do this, note first that every positive integer j can be represented as $l\,n + r$ for some nonnegative integers l, r with $r < n$. This leads to the estimate

$$\|x^j\| \le \|x^n\|^l\, \|x\|^r, \tag{58.2}$$

which implies the convergence of $\sum_{j=1}^\infty \|x^j\|$ when $\|x^n\| < 1$.

If x is any element of $\mathcal{A}$, then put

$$r(x) = \inf_{n \ge 1} \|x^n\|^{1/n}, \tag{58.3}$$

where more precisely the infimum is taken over all positive integers n. Thus $e - x$ is invertible in $\mathcal{A}$ when $r(x) < 1$, as in the previous paragraph. Observe also that

$$r(t\,x) = |t|\,r(x) \tag{58.4}$$

for every real or complex number t, as appropriate. It follows that $e - t\,x$ is invertible in $\mathcal{A}$ when $|t|\,r(x) < 1$. Equivalently, $t\,e - x$ is invertible in $\mathcal{A}$ when $|t| > r(x)$.

Let us check that

$$\lim_{j \to \infty} \|x^j\|^{1/j} = r(x), \tag{58.5}$$

where the existence of the limit is part of the conclusion. Because of the way that $r(x)$ is defined, it suffices to show that

$$\limsup_{j \to \infty} \|x^j\|^{1/j} \le r(x), \tag{58.6}$$

which is the same as saying that

$$\limsup_{j \to \infty} \|x^j\|^{1/j} \le \|x^n\|^{1/n} \tag{58.7}$$

for each $n \ge 1$. As before, each positive integer j can be represented as $l\,n + r$ for some nonnegative integers l, r with $r < n$, and (58.2) implies that

$$\|x^j\|^{1/j} \le (\|x^n\|^{1/n})^{ln/j}\,\|x\|^{r/j} = (\|x^n\|^{1/n})^{1-(r/j)}\,\|x\|^{r/j}. \tag{58.8}$$

It is not difficult to derive (58.7) from this estimate, using the fact that $a^{1/j} \to 1$ as $j \to \infty$ for every positive real number a. This is trivial when $x^n = 0$, since x^j is then equal to 0 for each $j \ge n$.

As a basic class of examples, suppose that $\mathcal{A}$ is the algebra $C_b(X)$ of bounded continuous functions on a topological space X, equipped with the supremum norm. In this case, it is easy to see that

$$\|f^n\|_{sup} = \|f\|_{sup}^n \tag{58.9}$$

for every $f \in C_b(X)$ and $n \ge 1$, and hence that

$$r(f) = \|f\|_{sup}. \tag{58.10}$$

Suppose now that $\mathcal{A}$ is the algebra $C^1([0,1])$ of continuously-differentiable functions on the unit interval, as in the previous section. Thus $\|f\|_{C^1} \ge \|f\|_{sup}$, and hence

$$r(f) \ge \|f\|_{sup} \tag{58.11}$$

for every $f \in C^1([0,1])$. In the other direction,

$$\begin{aligned} \|f^n\|_{C^1} &= \|f^n\|_{sup} + \|(f^n)'\|_{sup} \\ &= \|f\|_{sup}^n + \|n\,f'\,f^{n-1}\|_{sup} \\ &\le \|f\|_{sup}^n + n\,\|f'\|_{sup}\,\|f\|_{sup}^{n-1}. \end{aligned} \tag{58.12}$$

for each n. Using this, it is not too difficult to show that

$$r(f) = \lim_{n \to \infty} \|f^n\|_{C^1}^{1/n} = \|f\|_{sup}. \tag{58.13}$$

This also uses the fact that $(a + b\,n)^{1/n} \to 1$ as $n \to \infty$ for any two positive real numbers a, b.

Let $\mathcal{A}$ be a complex Banach algebra, and put

$$R(x) = \sup\{|t| : t \in \mathbf{C} \text{ and } t\,e - x \text{ is not invertible}\} \tag{58.14}$$

for every $x \in \mathcal{A}$. We have already seen that $t\,e - x$ is invertible in $\mathcal{A}$ when $|t| > r(x)$, which works for both real and complex Banach algebras. If $\mathcal{A}$ is a complex Banach algebra, then for each $x \in \mathcal{A}$ there is a $t \in \mathbf{C}$ such that $t\,e - x$ is not invertible, as in Section 49. Thus the supremum in the definition of $R(x)$ makes sense, and $R(x) \le r(x)$. A well-known theorem states that $r(x) \le R(x)$ for every $x \in \mathcal{A}$ when $\mathcal{A}$ is a complex Banach algebra, and hence $r(x) = R(x)$.

To see this, note that $t\,e - x$ is invertible when $t \in \mathbf{C}$ satisfies $|t| > R(x)$, which implies that $e - t\,x$ is invertible when $|t|\,R(x) < 1$. As in Section 49, the basic idea is to look at

$$f(t) = (e - t\,x)^{-1} \tag{58.15}$$

as a holomorphic function on the disk where $|t|\,R(x) < 1$ with values in $\mathcal{A}$. In particular, the composition of f with a continuous linear functional on $\mathcal{A}$ defines a complex-valued function on this disk which is holomorphic in the usual sense. We also know that $f(t)$ is given by the power series $\sum_{j=0}^\infty t^j\,x^j$ when $|t|$ is sufficiently small, as in Section 49. By standard arguments in complex analysis, one can estimate the size of the coefficients of this power series in t in terms of the behavior of $f(t)$ on any circle $|t| = a$ with $a\,R(x) < 1$. Note that $f(t)$ is bounded on any circle of this type, because the circle is compact and $f(t)$ is continuous on it. More precisely, one can show that for each positive real number a with $a\,R(x) < 1$, there is a $C(a) \ge 0$ such that

$$a^j\,\|x^j\| \le C(a) \tag{58.16}$$

for every $j \ge 1$. Equivalently, $a\,\|x^j\|^{1/j} \le C(a)^{1/j}$ for each j, which implies that $a\,r(x) \le 1$ when $a\,R(x) < 1$, by taking the limit as $j \to \infty$. Thus $r(x) \le R(x)$, as desired.

59 Topological algebras

Let $\mathcal{A}$ be an associative algebra over the real or complex numbers, as in Section 49. Suppose that $\mathcal{A}$ is also equipped with a topology which makes it into a topological vector space, as in Section 12. In the same way, one can ask that multiplication in $\mathcal{A}$ be continuous as a mapping from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{A}$, using the product topology on $\mathcal{A} \times \mathcal{A}$ associated to the given topology on $\mathcal{A}$. Under these conditions, we can say that $\mathcal{A}$ is a *topological algebra*. As before, we are especially interested here in the case where multiplication on $\mathcal{A}$ is commutative.

Of course, Banach algebras are topological algebras, with respect to the topology associated to the norm. If X is a locally compact Hausdorff topological space, then the algebra of continuous functions on X is a topological algebra with respect to the topology determined by the collection of supremum seminorms corresponding to nonempty compact subsets of X, as in Section 36. If U is a nonempty open set in $\mathbf{R}^n$, then the algebra of smooth functions on U is a topological algebra with respect to the collection of supremum seminorms of derivatives of f over nonempty compact subsets of U, as in Section 55.

As in the case of Banach algebras, one may wish to look at topological algebras $\mathcal{A}$ that are complete as topological vector spaces. If $\mathcal{A}$ has a countable local base for its topology at 0, then this can be defined in terms of convergence of Cauchy sequences, as usual. Otherwise, one can consider more general Cauchy conditions for nets or filters on $\mathcal{A}$. It is not too difficult to show that the examples of topological algebras of continuous and smooth functions mentioned in the previous paragraph are complete.

If U is a nonempty open set in the complex plane, then the algebra $\mathcal{H}(U)$ of holomorphic functions on U may be considered as a subalgebra of the algebra $C(U)$ of continuous complex-valued functions on U. More precisely, we have seen that $\mathcal{H}(U)$ is a closed subalgebra of $C(U)$ with respect to the topology associated to the collection of supremum seminorms over nonempty compact subsets of U. Of course, $\mathcal{H}(U)$ is also a topological algebra with respect to the topology determined by this collection of seminorms, and it follows that $\mathcal{H}(U)$ is complete as well, because $C(U)$ is complete.

60 Fourier series

Let $\mathbf{T}$ be the unit circle in the complex plane, consisting of the $z \in \mathbf{C}$ with $|z| = 1$. It is well known that

$$\int_{\mathbf{T}} z^j \, |dz| = 0 \tag{60.1}$$

for each nonzero integer j, where $|dz|$ is the element of arc length along $\mathbf{T}$. This integral is the same as $-i$ times the line integral

$$\oint_{\mathbf{T}} z^{j-1} \, dz, \tag{60.2}$$

the vanishing of which when $j \neq 0$ is a basic fact in complex analysis. More precisely, the relationship between these two integrals follows from identifying $i\,z$ with the unit tangent vector to $\mathbf{T}$ at z in the positive orientation. Note that

$$\overline{\Big(\int_{\mathbf{T}} z^j \, |dz|\Big)} = \int_{\mathbf{T}} z^{-j} \, |dz|, \tag{60.3}$$

since $\overline{z} = z^{-1}$ when $|z| = 1$, and so it suffices to verify (60.1) when j is a positive integer. If $j = 0$, then z^j is interpreted as being equal to 1 for each z, so that the integral in (60.1) is equal to the length $2\,\pi$ of $\mathbf{T}$.

If f is a continuous complex-valued function on $\mathbf{T}$ and j is an integer, then the jth Fourier coefficient of f is defined by

$$\widehat{f}(j) = \frac{1}{2\pi} \int_{\mathbf{T}} f(w)\, w^{-j}\, |dw|. \tag{60.4}$$

The corresponding Fourier series is given by

$$\sum_{j=-\infty}^{\infty} \widehat{f}(j)\, z^j. \tag{60.5}$$

For the moment, this should be considered as a formal sum, without regard to convergence. If $f(z) = z^l$ for some integer l, then $\widehat{f}(j)$ is equal to 1 when $j = l$ and to 0 when $j \neq l$, as in the previous paragraph. Thus the Fourier series (60.5) reduces to $f(z)$ in this case, and also when $f(z)$ is a linear combination of z^l for finitely many integers l.

Suppose that $f(z)$ is a continuous complex-valued function on the closed unit disk in $\mathbf{C}$ which is holomorphic on the open unit disk. By standard results in complex analysis, $f(z)$ can be represented by an absolutely convergent power series

$$f(z) = \sum_{j=0}^{\infty} a_j\, z^j \tag{60.6}$$

on the open unit disk, which is to say for $z \in \mathbf{C}$ with $|z| < 1$. In this case,

$$a_j = \widehat{f}(j) \tag{60.7}$$

for each $j \geq 0$, where $\widehat{f}(j)$ is the jth Fourier coefficient of the restriction of f to the unit circle. This follows from the usual Cauchy integral formulae, where one integrates over the unit circle. Normally one might integrate over circles of radius $r < 1$ when dealing with holomorphic functions on the open unit disk, but one can pass to the limit $r \to 1$ when f extends to a continuous function on the closed unit disk.

Under these conditions, we also have that $\widehat{f}(j) = 0$ when $j < 0$. This can be derived from Cauchy's theorem for line integrals of holomorphic functions, starting with integrals over circles of radius $r < 1$, and then passing to the limit $r \to 1$ as in the previous paragraph. Conversely, if f is a continuous function on the unit circle with $\widehat{f}(j) = 0$ when $j < 0$, then it can be shown that f has a continuous extension to the closed unit disk which is holomorphic on the open unit disk. More precisely, the holomorphic function on the open unit disk is given by the power series defined by the Fourier coefficients of f, as before. The remaining point is to show that the combination of this holomorphic function on the open unit disk with the given function f on the unit circle is continuous on the closed unit disk, which will be discussed in Section 62.

61 Absolute convergence

Let $\ell^1(\mathbf{Z})$ be the space of doubly-infinite sequences $a = \{a_j\}_{j=-\infty}^{\infty}$ of complex numbers such that

$$\|a\|_1 = \sum_{j=-\infty}^{\infty} |a_j| \tag{61.1}$$

converges. This is equivalent to the definition in Section 13 with $E = \mathbf{Z}$, but in this case it is a bit simpler to think of a sum over $\mathbf{Z}$ as a combination of two ordinary infinite series, corresponding to sums over $j \geq 0$ and $j < 0$. In particular, if $a \in \ell^1(\mathbf{Z})$, then $\sum_{j=0}^{\infty} a_j$ and $\sum_{j=1}^{\infty} a_{-j}$ converge absolutely, so that their sum $\sum_{j=-\infty}^{\infty} a_j$ is well-defined, and satisfies

$$\Big|\sum_{j=-\infty}^{\infty} a_j\Big| \leq \|a\|_1. \tag{61.2}$$

As before, it is easy to see that $\|a\|_1$ defines a norm on $\ell^1(\mathbf{Z})$.

If $a \in \ell^1(\mathbf{Z})$, $z \in \mathbf{C}$, and $|z| = 1$, then put

$$\widehat{a}(z) = \sum_{j=-\infty}^{\infty} a_j\, z^j, \tag{61.3}$$

which is the Fourier transform of a. This makes sense, because

$$\sum_{j=-\infty}^{\infty} |a_j\, z^j| = \sum_{j=-\infty}^{\infty} |a_j| \tag{61.4}$$

converges. Moreover,

$$\sup_{z \in \mathbf{T}} |\widehat{a}(z)| \leq \|a\|_1. \tag{61.5}$$

The partial sums $\sum_{j=-n}^{n} a_j\, z^j$ are continuous functions that converge to $\widehat{a}(z)$ uniformly on the unit circle, by Weierstrass' M-test, and so $\widehat{a}(z)$ is a continuous function on $\mathbf{T}$. It is easy to see that

$$\widehat{(\widehat{a})}(j) = \frac{1}{2\pi} \int_{\mathbf{T}} \widehat{a}(z)\, z^{-j}\, |dz| = a_j \tag{61.6}$$

for each j, using the uniform convergence of the partial sums to reduce to the identities discussed in the previous section.

The convolution $a * b$ of $a, b \in \ell^1(\mathbf{Z})$ is defined by

$$(a * b)_j = \sum_{l=-\infty}^{\infty} a_{j-l}\, b_l. \tag{61.7}$$

The sum on the right converges absolutely as soon as one of a, b is summable and the other is bounded, and in particular when both a, b are summable. We also have that

$$|(a * b)_j| = \sum_{l=-\infty}^{\infty} |a_{j-l}|\, |b_l|, \tag{61.8}$$

which implies that

$$\sum_{j=-\infty}^{\infty} |(a * b)_j| \le \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} |a_{j-l}|\,|b_l|. \tag{61.9}$$

Interchanging the order of summation, we get that

$$\sum_{j=-\infty}^{\infty} |(a * b)_j| \le \sum_{l=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |a_{j-l}|\,|b_l|. \tag{61.10}$$

Of course,

$$\sum_{j=-\infty}^{\infty} |a_{j-l}| = \sum_{j=-\infty}^{\infty} |a_j| \tag{61.11}$$

for each l, by making the change of variables $j \mapsto j + l$. Thus

$$\sum_{j=-\infty}^{\infty} |(a * b)_j| \le \Big(\sum_{j=-\infty}^{\infty} |a_j|\Big)\Big(\sum_{l=-\infty}^{\infty} |b_l|\Big), \tag{61.12}$$

so that $a * b \in \ell^1(\mathbf{Z})$ when $a, b \in \ell^1(\mathbf{Z})$. Equivalently,

$$\|a * b\|_1 \le \|a\|_1\,\|b\|_1. \tag{61.13}$$

If $a, b \in \ell^1(\mathbf{Z})$, $z \in \mathbf{C}$, and $|z| = 1$, then

$$\widehat{(a * b)}(z) = \sum_{j=-\infty}^{\infty} \Big(\sum_{l=-\infty}^{\infty} a_{j-l}\, b_l\Big) z^j. \tag{61.14}$$

This is the same as

$$\sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} a_{j-l}\, z^{j-l}\, b_l\, z^l, \tag{61.15}$$

which is equal to

$$\sum_{l=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} a_{j-l}\, z^{j-l}\, b_l\, z^l, \tag{61.16}$$

by interchanging the order of summation. This uses the absolute summability shown in the previous paragraph. As before, we can make the change of variables $j \mapsto j + l$, to get that

$$\sum_{j=-\infty}^{\infty} a_{j-l}\, z^{j-l} = \sum_{j=-\infty}^{\infty} a_j\, z^j = \widehat{a}(z) \tag{61.17}$$

for each l. Substituting this into the previous double sum, we get that

$$\widehat{(a * b)}(z) = \widehat{a}(z)\,\widehat{b}(z) \tag{61.18}$$

for every $z \in \mathbf{T}$.

Let $\delta(n) = \{\delta_j(n)\}_{j=-\infty}^{\infty}$ be defined for each integer n by putting $\delta_j(n) = 1$ when $j = n$ and $\delta_j(n) = 0$ when $j \neq n$, so that $\|\delta(n)\|_1 = 1$ for each n. It is easy to see that

$$\delta(n) * \delta(r) = \delta(n + r) \tag{61.19}$$

for every $n, r \in \mathbf{Z}$, and that

$$\delta(0) * a = a * \delta(0) = a \tag{61.20}$$

for every $a \in \ell^1(\mathbf{Z})$. One can also check that

$$a * b = b * a \tag{61.21}$$

and

$$(a * b) * c = a * (b * c) \tag{61.22}$$

for every $a, b, c \in \ell^1(\mathbf{Z})$, directly from the definition of convolution, or using the fact that linear combinations of the $\delta(n)$'s are dense in $\ell^1(\mathbf{Z})$. It is well known and not too difficult to show that $\ell^1(\mathbf{Z})$ is complete with respect to the ℓ^1 norm $\|a\|_1$. It follows that $\ell^1(\mathbf{Z})$ is a commutative Banach algebra, with convolution as multiplication and $\delta(0)$ as the multiplicative identity element.

Suppose that ϕ is a linear functional on $\ell^1(\mathbf{Z})$ that is also a homomorphism with respect to convolution, so that $\phi(a * b) = \phi(a)\,\phi(b)$ for every $a, b \in \ell^1(\mathbf{Z})$. If $\phi(a) \neq 0$ for some $a \in \ell^1(\mathbf{Z})$, then $\phi(\delta(0)) = 1$, and ϕ is a continuous linear functional on $\ell^1(\mathbf{Z})$ with dual norm 1, as in Section 49. We would like to show that

$$\phi(a) = \widehat{a}(z) \tag{61.23}$$

for some $z \in \mathbf{T}$ and every $a \in \ell^1(\mathbf{Z})$. Of course, we have already seen that $\phi_z(a) = \widehat{a}(z)$ defines a homomorphism on $\ell^1(\mathbf{Z})$ for every $z \in \mathbf{T}$.

If $z = \phi(\delta(1))$, then $|z| \le 1$, because $\|\delta(1)\|_1 = 1$ and ϕ has dual norm 1. We also know that $\delta(-1) * \delta(1) = \delta(0)$, which implies that $\phi(\delta(-1))\,\phi(\delta(1)) = 1$. Thus $z \neq 0$, $z^{-1} = \phi(\delta(-1))$, and hence $|z^{-1}| \le 1$, because $\|\delta(-1)\|_1 = 1$ and ϕ has dual norm 1. It follows that $|z| = 1$, and that $\phi(\delta(n)) = z^n$ for each $n \in \mathbf{Z}$. Equivalently, $\phi(a) = \widehat{a}(z)$ when $a = \delta(n)$ for some n. This also works when a is a finite linear combination of $\delta(n)$'s, by linearity. Therefore $\phi(a) = \widehat{a}(z)$ for every $a \in \ell^1(\mathbf{Z})$, because linear combinations of the $\delta(n)$'s are dense in $\ell^1(\mathbf{Z})$.

62 The Poisson kernel

Let $f(z)$ be a continuous complex-valued function on the unit circle $\mathbf{T}$. Note that the Fourier coefficients of f are bounded, with

$$|\widehat{f}(j)| \le \frac{1}{2\pi} \int_{\mathbf{T}} |f(w)|\,|dw| \le \sup_{w \in \mathbf{T}} |f(w)| \tag{62.1}$$

for each $j \in \mathbf{Z}$. Put

$$\phi(z) = \sum_{j=0}^{\infty} \widehat{f}(j)\, z^j + \sum_{j=1}^{\infty} \widehat{f}(-j)\, \overline{z}^j \tag{62.2}$$

for each $z \in \mathbf{C}$ with $|z| < 1$, where z^j is interpreted as being equal to 1 for each z when $j = 0$, as usual. These two infinite series converge absolutely when $|z| < 1$, because $\widehat{f}(j)$ is bounded. If $|z| = 1$, then $\overline{z} = z^{-1}$, and the sum of these two series is formally the same as the Fourier series (60.5) associated to f.

Equivalently, $\phi = \phi_1 + \phi_2$, where

$$\phi_1(z) = \sum_{j=0}^{\infty} \widehat{f}(j)\, z^j, \quad \phi_2(z) = \sum_{j=1}^{\infty} \widehat{f}(-j)\, \overline{z}^j. \tag{62.3}$$

Of course, ϕ_1 is a holomorphic function on the open unit disk, and ϕ_2 is the complex conjugate of a holomorphic function on the open unit disk. It is well known that a holomorphic function $h(z)$ is harmonic, meaning that it satisfies Laplace's equation

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \tag{62.4}$$

when we identify the complex plane $\mathbf{C}$ with $\mathbf{R}^2$, and where x, y correspond to the real and imaginary parts of $z \in \mathbf{C}$. More precisely, Laplace's equation applies to the real and imaginary parts of $h(z)$ separately, both of which are harmonic. Thus the complex conjugate of a holomorphic function is also harmonic, and hence ϕ is a harmonic function on the open unit disk.

The Poisson kernel is defined by

$$P(z, w) = \frac{1}{2\pi}\Big(\sum_{j=0}^{\infty} z^j\, \overline{w}^j + \sum_{j=1}^{\infty} \overline{z}^j\, w^j\Big) \tag{62.5}$$

for $z, w \in \mathbf{C}$ with $|z| < 1$ and $|w| = 1$. Of course, these series converge absolutely under these conditions, and their partial sums converge uniformly on the set where $|z| \le r$ and $|w| = 1$ for every $r < 1$. This implies that

$$\phi(z) = \int_{\mathbf{T}} P(z, w)\, f(w)\, |dw| \tag{62.6}$$

for every z in the open unit disk, using uniform convergence for $w \in \mathbf{T}$ to interchange the order of summation and integration. In particular,

$$\int_{\mathbf{T}} P(z, w)\, |dw| = 1 \tag{62.7}$$

for every z in the open unit disk, because $\phi(z) = 1$ for each z when f is the constant function equal to 1 on the unit circle.

Observe that

$$\sum_{j=1}^{\infty} \overline{z}^j\, w^j = \overline{\Big(\sum_{j=1}^{\infty} z^j\, \overline{w}^j\Big)}, \tag{62.8}$$

and hence

$$P(z,w) = \frac{1}{2\pi}\Big(2\operatorname{Re}\sum_{j=0}^{\infty} z^j\, \overline{w}^j - 1\Big) \tag{62.9}$$

for all z, w as before. Here $\operatorname{Re} a$ denotes the real part of a complex number a, and we are using the simple fact that $a + \overline{a} = 2\operatorname{Re} a$. Summing the geometric series, we get that

$$\sum_{j=0}^{\infty} z^j\, \overline{w}^j = \frac{1}{1 - z\,\overline{w}} = \frac{1 - \overline{z}\, w}{|1 - z\,\overline{w}|^2} \tag{62.10}$$

when $|z| < 1$ and $|w| = 1$. Thus

$$P(z,w) = \frac{1}{2\pi}|1 - z\,\overline{w}|^{-2}(2 - 2\operatorname{Re} z\,\overline{w} - |1 - z\,\overline{w}|^2). \tag{62.11}$$

We can expand $|1-z\,\overline{w}|^2$ into $(1-z\,\overline{w})(1-\overline{z}\,w)$, which reduces to $1-2\operatorname{Re} z\,\overline{w}-|z|^2$ when $|w| = 1$. It follows that

$$P(z,w) = \frac{1}{2\pi}\frac{1 - |z|^2}{|1 - z\,\overline{w}|^2} = \frac{1}{2\pi}\frac{1 - |z|^2}{|w - z|^2}, \tag{62.12}$$

using $|w| = 1$ again in the second step. In particular, $P(z,w) > 0$.

If $z_0, w \in \mathbf{T}$ and $z_0 \neq w$, then $P(z,w) \to 0$ as $z \to z_0$, where the limit is restricted to z in the open unit disk. This is an immediate consequence of (62.12), which also shows that we have uniform convergence for $w \in \mathbf{T}$ that satisfy $|w - z_0| \geq \delta$ for some $\delta > 0$.

Note that

$$\phi(z) - f(z_0) = \int_{\mathbf{T}} P(z,w)\,(f(w) - f(z_0))\,|dw| \tag{62.13}$$

for every $z_0 \in \mathbf{T}$ and z in the open unit disk, because of (62.7), and hence

$$|\phi(z) - f(z_0)| \leq \int_{\mathbf{T}} P(z,w)\,|f(w) - f(z_0)|\,|dw|. \tag{62.14}$$

Using this and the continuity of f, one can check that $\phi(z) \to f(z_0)$ as $z \to z_0$ in the open unit disk. More precisely, $f(w) - f(z_0)$ is small when w is close to z_0, while $P(z,w)$ is small when w is not too close to z_0 and z is very close to z_0.

It follows that the function defined on the closed unit disk by taking ϕ on the open unit disk and f on the unit circle is continuous. In particular, if $\widehat{f}(j) = 0$ when $j < 0$, then $\phi = \phi_1$ is holomorphic, as mentioned at the end of Section 60.

63 Cauchy products

If $\sum_{j=0}^{\infty} a_j\, z^j$, $\sum_{j=0}^{\infty} b_l\, z^l$ are power series with complex coefficients, then

$$\Big(\sum_{j=0}^{\infty} a_j\, z^j\Big)\Big(\sum_{l=0}^{\infty} b_l\, z^l\Big) = \sum_{n=0}^{\infty} c_n\, z^n \tag{63.1}$$

formally, where

$$c_n = \sum_{j=0}^{n} a_j\, b_{n-j}. \tag{63.2}$$

In particular,

$$\Big(\sum_{j=0}^{\infty} a_j\Big)\Big(\sum_{l=0}^{\infty} b_l\Big) = \sum_{n=0}^{\infty} c_n \tag{63.3}$$

formally. These identities clearly hold when $a_j = b_l = 0$ for all but finitely many j, l, for instance.

If a_j, b_l are nonnegative real numbers, then it is easy to see that

$$\sum_{n=0}^{N} c_n \le \Big(\sum_{j=0}^{N} a_j\Big)\Big(\sum_{l=0}^{N} b_l\Big) \tag{63.4}$$

for every nonnegative integer N. Similarly,

$$\Big(\sum_{j=0}^{N} a_j\Big)\Big(\sum_{l=0}^{N} b_l\Big) \le \sum_{n=0}^{2N} c_n. \tag{63.5}$$

Hence $\sum_{n=0}^{\infty} c_n$ converges and satisfies (63.3) when $\sum_{j=0}^{\infty} a_j$, $\sum_{l=0}^{\infty} b_l$ converge.

If a_j, b_l are arbitrary real or complex numbers, then

$$|c_n| \le \sum_{j=0}^{n} |a_j|\, |b_{n-j}| \tag{63.6}$$

for each n. If $\sum_{j=0}^{\infty} a_j$, $\sum_{l=0}^{\infty} b_l$ converge absolutely, then it follows that $\sum_{n=0}^{\infty} c_n$ converges absolutely too, by the remarks in the previous paragraph. In this case, one can check that (63.3) holds, by expressing these series as linear combinations of convergent series of nonnegative real numbers, and using the remarks in the previous paragraph. Alternatively, one can approximate these series by ones with only finitely many nonzero terms, and estimate the remainders using absolute convergence.

Suppose now that $\sum_{j=0}^{\infty} a_j\, z^j$, $\sum_{l=0}^{\infty} b_l\, z^l$ are power series that converge when $|z| < 1$, and hence converge absolutely when $|z| < 1$, by standard results. Thus $\sum_{n=0}^{\infty} c_n\, z^n$ converges absolutely when $|z| < 1$, and is equal to the product of the other two series. The partial sums of these series also converge uniformly for $|z| \le r$ when $r < 1$, by standard results.

Put $f(z) = \sum_{j=0}^{\infty} a_j\, z^j$, $g(z) = \sum_{l=0}^{\infty} b_l\, z^l$, and $h(z) = \sum_{n=0}^{\infty} c_n\, z^n$ when $|z| < 1$, so that

$$f(z)\, g(z) = h(z), \tag{63.7}$$

as in the preceding paragraph. If $f(z)$, $g(z)$ have continuous extensions to the closed unit disk, then it follows that $h(z)$ does as well.

Note that

$$a_j\, r^j = \frac{1}{2\pi} \int_{\mathbf{T}} f(r\, z)\, z^{-j}\, |dz| \tag{63.8}$$

for each $j \ge 0$ and $0 < r < 1$, and similarly for g, h. This is because $f(r\, z)$ is defined by an absolutely convergent Fourier series, so that we can reduce to the usual identities for the integral of a power of z on the unit circle by interchaning the order of integration and summation. If f extends continuously to the closed unit disk, then this formula also holds with $r = 1$.

If $\sum_{j=-\infty}^{\infty} a_j$, $\sum_{l=-\infty}^{\infty} b_l$ are doubly-infinite series of complex numbers, then we have again that

$$\Big(\sum_{j=-\infty}^{\infty} a_j\Big) \Big(\sum_{l=-\infty}^{\infty} b_l\Big) = \sum_{n=-\infty}^{\infty} c_n \tag{63.9}$$

with $c_n = \sum_{j=-\infty}^{\infty} a_j\, b_{n-j}$, and similarly

$$\Big(\sum_{j=-\infty}^{\infty} a_j\, z^j\Big) \Big(\sum_{l=-\infty}^{\infty} b_l\, z^l\Big) = \sum_{n=-\infty}^{\infty} c_n\, z^n, \tag{63.10}$$

at least formally. As before, there is no problem with these identities when $a_j = b_l = 0$ for all but finitely many j, l. Otherwise, even the definition of c_n requires some convergence conditions. If the a_j's are absolutely summable and the b_l's are bounded, or vice-versa, then the series defining c_n converges absolutely, and

$$|c_n| \le \sum_{j=-\infty}^{\infty} |a_j|\, |b_{n-j}| \tag{63.11}$$

for each n. If both the a_j's and b_l's are absolutely summable, then it is easy to see that c_n's are absolutely summable too, with

$$\sum_{n=-\infty}^{\infty} |c_n| \le \Big(\sum_{j=-\infty}^{\infty} |a_j|\Big) \Big(\sum_{l=-\infty}^{\infty} |b_l|\Big). \tag{63.12}$$

This follows from the previous estimate for $|c_n|$ by interchanging the order of summation. One can also check that (63.9) holds under these conditions, in the same way as in the earlier situation for sums over nonnegative integers. Of course, this implies that (63.10) holds as well when $|z| = 1$, which is basically the same as (61.18).

64 Inner product spaces

Let V be a vector space over the real or complex numbers. An *inner product* on V is a function $\langle v, w\rangle$ defined for $v, w \in V$ with values in $\mathbf{R}$ or $\mathbf{C}$, as appropriate, that satisfies the following three conditions. First,

$$\lambda_w(v) = \langle v, w\rangle \tag{64.1}$$

is linear as a function of v for each $w \in V$. Second,

$$\langle w, v\rangle = \langle v, w\rangle \tag{64.2}$$

for every $v, w \in V$ in the real case, and

$$\langle w, v\rangle = \overline{\langle v, w\rangle} \tag{64.3}$$

for every $v, w \in V$ in the complex case. This implies that $\langle v, w\rangle$ is linear in w in the real case, and conjugate-linear in w in the complex case. It also implies that

$$\langle v, w\rangle = \overline{\langle v, v\rangle} \in \mathbf{R} \tag{64.4}$$

for every $v \in V$ in the complex case. The third condition is that $\langle v, v\rangle \geq 0$ for every $v \in V$ in both the real and complex cases, with equality only when $v = 0$.

Put

$$\|v\| = \langle v, v\rangle^{1/2} \tag{64.5}$$

for every $v \in V$. This satisfies the positivity and homogeneity requirements of a norm, and we would like to show that it also satisfies the triangle inequality. Observe that

$$\begin{aligned} 0 \leq \|v + t\,w\|^2 &= \langle v, v\rangle + t\,\langle v, w\rangle + t\,\langle w, v\rangle + t^2\langle w, w\rangle \\ &= \|v\|^2 + 2\,t\,\langle v, w\rangle + t^2\,\|w\|^2 \end{aligned} \tag{64.6}$$

for every $v, w \in V$ and $t \in \mathbf{R}$ in the real case, and similarly

$$\begin{aligned} 0 \leq \|v + t\,w\|^2 &= \langle v, v\rangle + t\,\langle v, w\rangle + \overline{t}\,\langle w, v\rangle + |t|^2\,\langle w, w\rangle \\ &= \|v\|^2 + t\,\langle v, w\rangle + \overline{t}\,\overline{\langle v, w\rangle} + |t|^2\,\|w\|^2 \\ &= \|v\|^2 + 2\,\mathrm{Re}\,t\,\langle v, w\rangle + |t|^2\,\|w\|^2 \end{aligned} \tag{64.7}$$

for every $v, w \in V$ and $t \in \mathbf{C}$ in the complex case. In both cases, we get that

$$0 \leq \|v\|^2 - 2\,r\,|\langle v, w\rangle| + r^2\|w\|^2 \tag{64.8}$$

for every $v, w \in V$ and $r \geq 0$, by taking $t = -r\,\alpha$, where $|\alpha| = 1$ and

$$\alpha\,\langle v, w\rangle = |\langle v, w\rangle|. \tag{64.9}$$

Equivalently,

$$2\,r\,|\langle v, w\rangle| \leq \|v\|^2 + r^2\,\|w\|^2 \tag{64.10}$$

for every $v, w \in V$ and $r \geq 0$, and hence

$$|\langle v, w \rangle| \leq \frac{1}{2}\,(r^{-1}\,\|v\|^2 + r\,\|w\|^2) \tag{64.11}$$

when $r > 0$. If $v, w \neq 0$, then we can take $r = \|v\|/\|w\|$ to get that

$$|\langle v, w \rangle| \leq \|v\|\,\|w\|. \tag{64.12}$$

This is the *Cauchy–Schwarz inequality*, which also holds trivially when $v = 0$ or when $w = 0$.

As before,

$$\|v + w\|^2 = \|v\|^2 + 2\,\langle v, w \rangle + \|w\|^2 \tag{64.13}$$

for every $v, w \in V$ in the real case, and

$$\|v + w\|^2 = \|v\|^2 + 2\ \mathrm{Re}\langle v, w \rangle + \|w\|^2 \tag{64.14}$$

for every $v, w \in V$ in the complex case. In both case,

$$\begin{aligned} \|v + w\|^2 &\leq \|v\|^2 + 2\,|\langle v, w \rangle| + \|w\|^2 \\ &\leq \|v\|^2 + 2\,\|v\|\,\|w\| + \|w\|^2 = (\|v\| + \|w\|)^2, \end{aligned} \tag{64.15}$$

using the Cauchy–Schwarz inequality in the second step. This implies that

$$\|v + w\| \leq \|v\| + \|w\| \tag{64.16}$$

for every $v, w \in V$, so that $\|v\|$ defines a norm on V, as desired.

The standard inner products on $\mathbf{R}^n$ and $\mathbf{C}^n$ are given by

$$\langle v, w \rangle = \sum_{j=1}^{n} v_j\, w_j \tag{64.17}$$

and

$$\langle v, w \rangle = \sum_{j=1}^{n} v_j\, \overline{w_j}, \tag{64.18}$$

respectively. In both cases, the corresponding norm is given by

$$\|v\| = \Big(\sum_{j=1}^{n} |v_j|^2\Big)^{1/2}. \tag{64.19}$$

This is the standard Euclidean norm on $\mathbf{R}^n$, $\mathbf{C}^n$, for which the corresponding topology is the standard topology.

65 $\ell^2(E)$

Let E be a nonempty set, and let $\ell^2(E)$ be the space of real or complex-valued functions $f(x)$ on E such that $|f(x)|^2$ is a summable function on E, as in Section 13. As usual, this may also be denoted $\ell^2(E, \mathbf{R})$ or $\ell^2(E, \mathbf{C})$, to indicate whether real or complex-valued functions are being used. Remember that

$$a\,b \le \frac{a^2 + b^2}{2} \tag{65.1}$$

for every $a, b \ge 0$, since

$$0 \le (a - b)^2 = a^2 - 2\,a\,b + b^2. \tag{65.2}$$

If $f, g \in \ell^2(E)$, then it follows that

$$\begin{aligned} |f(x) + g(x)|^2 &\le (|f(x)| + |g(x)|)^2 \\ &= |f(x)|^2 + 2\,|f(x)|\,|g(x)| + |g(x)|^2 \\ &\le 2\,|f(x)|^2 + 2\,|g(x)|^2 \end{aligned} \tag{65.3}$$

for every $x \in E$. Hence $f + g \in \ell^2(E)$, because $|f(x)|^2$, $|g(x)|^2$ are summable on E by hypothesis.

Similarly,

$$|f(x)|\,|g(x)| \le \frac{1}{2}\,|f(x)|^2 + \frac{1}{2}\,|g(x)|^2 \tag{65.4}$$

is a summable function on E when $f, g \in \ell^2(E)$. Put

$$\langle f, g \rangle = \sum_{x \in E} f(x)\,g(x) \tag{65.5}$$

in the real case, and

$$\langle f, g \rangle = \sum_{x \in E} f(x)\,\overline{g(x)} \tag{65.6}$$

in the complex case. Thus

$$\langle f, f \rangle = \sum_{x \in E} |f(x)|^2 \tag{65.7}$$

in both cases. It is easy to see that $\ell^2(E)$ is a vector space with respect to pointwise addition and scalar multiplication, and that $\langle f, g \rangle$ defines an inner product on $\ell^2(E)$. The norm associated to this inner product is denoted $\|f\|_2$.

If $f \in \ell^1(E)$, then f is bounded, and $\|f\|_\infty \le \|f\|_1$. This implies that

$$\sum_{x \in E} |f(x)|^2 \le \|f\|_\infty \sum_{x \in E} |f(x)| = \|f\|_\infty\,\|f\|_1 \le \|f\|_1^2, \tag{65.8}$$

so that $f \in \ell^2(E)$ and

$$\|f\|_2 \le \|f\|_1. \tag{65.9}$$

Similarly, if $f \in \ell^2(E)$, then f is bounded on E, and

$$\|f\|_\infty \le \|f\|_2. \tag{65.10}$$

One can also check that $f \in c_0(E)$, for the same reasons as for summable functions, and hence

$$\ell^1(E) \subseteq \ell^2(E) \subseteq c_0(E). \tag{65.11}$$

As in the case of $\ell^1(E)$, one can show that functions with finite support on E are dense in $\ell^2(E)$.

If $(V, \langle v, w\rangle)$ is a real or complex inner product space, then $\lambda_w(v) = \langle v, w\rangle$ defines a continuous linear functional on V for every $w \in V$. This uses the Cauchy–Schwarz inequality, which implies that the dual norm of λ_w is less than or equal to the norm of w. The dual norm of λ_w is actually equal to the norm of w, as one can check by taking $v = w$. If $V = \ell^2(E)$ with the inner product defined before, then one can show that every continuous linear functional is of this form, using arguments like those in Sections 14 and 15. An inner product space $(V, \langle v, w\rangle)$ is said to be a *Hilbert space* if V is complete as a metric space with respect to the metric determined by the norm associated to the inner product. It is well known that $\ell^2(E)$ is complete with respect to the ℓ^2 norm, and hence is a Hilbert space. Conversely, it can be shown that every Hilbert space is isometrically equivalent to $\ell^2(E)$ for some set E. This is simpler when V is separable, in the sense that it has a countable dense set, in which case E has only finitely or countably many elements. One can also show more directly that every continuous linear functional on a Hilbert space can be expressed as $\lambda_w(v)$ for some $w \in V$.

66 Orthogonality

Let $(V, \langle v, w\rangle)$ be a real or complex inner product space. We say that $v, w \in V$ are *orthogonal* if

$$\langle v, w\rangle = 0, \tag{66.1}$$

which implies that

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2. \tag{66.2}$$

A collection of vectors $v_1, \ldots, v_n \in V$ is said to be *orthonormal* if v_j is orthogonal to v_l when $j \ne l$, and $\|v_j\| = 1$ for each j. This implies that

$$\Big\langle \sum_{j=1}^n a_j\, v_j, \sum_{l=1}^n b_l\, v_l \Big\rangle = \sum_{j=1}^n a_j\, b_j \tag{66.3}$$

for every $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbf{R}$ in the real case, and

$$\Big\langle \sum_{j=1}^n a_j\, v_j, \sum_{l=1}^n b_l\, v_l \Big\rangle = \sum_{j=1}^n a_j\, \overline{b_l} \tag{66.4}$$

for every $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbf{C}$ in the complex case.

Suppose that $v_1, \ldots, v_n \in V$ are orthonormal, and put

$$P(v) = \sum_{j=1}^{n} \langle v, v_j \rangle v_j \tag{66.5}$$

for each $v \in V$. Thus $P(v)$ is an element of the linear span of $v_1, \ldots, v_n$ for each $v \in V$, and $P(v) = v$ when v is in the linear span of $v_1, \ldots, v_n$. Moreover,

$$\langle P(v), v_l \rangle = \langle v, v_l \rangle \tag{66.6}$$

for every $v \in V$ and $l = 1, \ldots, n$, which implies that

$$\langle v - P(v), v_l \rangle = 0 \tag{66.7}$$

for $l = 1, \ldots, n$. Hence $v - P(v)$ is orthogonal to every element of the linear span of $v_1, \ldots, v_n$. In particular, $v - P(v)$ is orthogonal to $P(v)$, which implies that

$$\|v\|^2 = \|v - P(v)\|^2 + \|P(v)\|^2 = \|v - P(v)\|^2 + \sum_{j=1}^{n} |\langle v, v_j \rangle|^2. \tag{66.8}$$

Let w be any element of the linear span of $v_1, \ldots, v_n$. Thus $v - P(v)$ is orthogonal to w, and hence $v - P(v)$ is orthogonal to $P(v) - w$. This implies that

$$\|v - w\|^2 = \|v - P(v)\|^2 + \|P(v) - w\|^2 \ge \|v - P(v)\|^2, \tag{66.9}$$

so that $P(v)$ is the element of the linear span of $v_1, \ldots, v_n$ closest to v.

Let A be a nonempty set, and suppose that for each $\alpha \in A$ we have a vector $v_\alpha \in V$ such that $\|v_\alpha\| = 1$ and v_α is orthogonal to v_β when $\beta \in A$ and $\alpha \neq \beta$. Thus v_α, $\alpha \in A$, is an orthonormal family of vectors in V. If $v \in V$ and $\alpha_1, \ldots, \alpha_n$ are distinct elements of A, then (66.8) implies that

$$\sum_{j=1}^{n} |\langle v, v_{\alpha_j} \rangle|^2 \le \|v\|^2. \tag{66.10}$$

It follows that $\langle v, v_\alpha \rangle$ is an element of $\ell^2(A)$ as a function of α, with

$$\sum_{\alpha \in A} |\langle v, v_\alpha \rangle|^2 \le \|v\|^2. \tag{66.11}$$

If v is in the closure of the linear span of the v_α's, $\alpha \in A$, with respect to the norm associated to the inner product on V, then one can check that

$$\sum_{\alpha \in A} |\langle v, v_\alpha \rangle|^2 = \|v\|^2. \tag{66.12}$$

67 Parseval's formula

Let $C(\mathbf{T})$ be the space of continuous complex-valued functions on the unit circle. It is easy to see that

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{\mathbf{T}} f(z)\,\overline{g(z)}\,|dz| \tag{67.1}$$

defines an inner product on $C(\mathbf{T})$, for which the corresponding norm is given by

$$\|f\| = \Big(\frac{1}{2\pi}\int_{\mathbf{T}} |f(z)|^2\,|dz|\Big)^{1/2}. \tag{67.2}$$

As in Section 60, the functions on $\mathbf{T}$ of the form z^j, $j \in \mathbf{Z}$, are orthonormal with respect to this inner product. The Fourier coefficients of a continuous function f on $\mathbf{T}$ can also be expressed as

$$\widehat{f}(j) = \langle f, z^j\rangle. \tag{67.3}$$

Parseval's formula states that

$$\sum_{j=-\infty}^{\infty} |\widehat{f}(j)|^2 = \frac{1}{2\pi}\int_{\mathbf{T}} |f(z)|^2\,|dz|. \tag{67.4}$$

That the sum on the left is less than or equal to the integral on the right follows immediately from the orthonormality of z^j, $j \in \mathbf{Z}$, as in the previous section. In order to show that equality holds, it suffices to check that f can be approximated by finite linear combinations of the z^j's with respect to the norm associated to the inner product. In fact, a continuous function f on the unit circle can be approximated uniformly by a finite linear combinations of the z^j's, $j \in \mathbf{Z}$. To see this, one can use the function $\phi(z)$ on the open unit disk discussed in Section 62. Remember that ϕ extends to a continuous function on the closed unit disk, which is equal to f on the unit circle. It follows that $\phi(r\,z)$ converges uniformly to $f(z)$ for $z \in \mathbf{T}$ as $r \to 1$, because continuous functions on compact sets are uniformly continuous. It is easy to see that $\phi(r\,z)$ can be approximated uniformly on $\mathbf{T}$ by a finite linear combination of the z^j's for each $r < 1$, because of the absolute convergence of the series defining $\phi(r\,z)$ when $r < 1$. This implies that f can be approximated uniformly by finite linear combinations of the z^j's on $\mathbf{T}$, as desired.

68 $\ell^p(E)$

Let E be a nonempty set, and let p be a positive real number. A real or complex-valued function $f(x)$ on E is said to be *p-summable* if $|f(x)|^p$ is a summable function on E. The space of p-summable functions on E is denoted $\ell^p(E)$, or $\ell^p(E, \mathbf{R})$, $\ell^p(E, \mathbf{C})$ to indicate whether real or complex-valued functions are being used. This is consistent with previous definitions when $p = 1, 2$.

Observe that

$$(a+b)^p \le (2\max(a,b))^p = 2^p \max(a^p, b^p) \le 2^p\,(a^p + b^p) \tag{68.1}$$

for any pair of nonnegative real numbers a, b. If f, g are p-summable functions on E, then it follows that $f+g$ is also p-summable, with

$$\begin{aligned} \sum_{x\in E} |f(x)+g(x)|^p &\le \sum_{x\in E} (|f(x)| + |g(x)|)^p \\ &\le 2^p \sum_{x\in E} |f(x)|^2 + 2^p \sum_{x\in E} |g(x)|^p. \end{aligned} \tag{68.2}$$

This implies that $\ell^p(E)$ is a vector space with respect to pointwise addition and scalar multiplication over the real or complex numbers, as appropriate.

If f is a p-summable function on E, then we put

$$\|f\|_p = \Big(\sum_{x\in E} |f(x)|^p\Big)^{1/p}. \tag{68.3}$$

It is easy to see that f vanishes at infinity on E, as in the $p=1$ case. In particular, f is bounded, and we have that

$$\|f\|_\infty \le \|f\|_p. \tag{68.4}$$

This implies that f is q-summable when $p \le q < \infty$, since

$$\sum_{x\in E} |f(x)|^q \le \|f\|_\infty^{q-p} \sum_{x\in E} |f(x)|^p. \tag{68.5}$$

More precisely, we get that

$$\|f\|_q^q \le \|f\|_\infty^{q-p}\,\|f\|_p^p \le \|f\|_p^q, \tag{68.6}$$

and hence

$$\|f\|_q \le \|f\|_p. \tag{68.7}$$

If $0 < p \le 1$, then

$$a + b \le (a^p + b^p)^{1/p} \tag{68.8}$$

for every $a, b \ge 0$. This follows from (68.7) with $q = 1$, using a set E with two elements. Equivalently,

$$(a+b)^p \le a^p + b^p. \tag{68.9}$$

If f, g are p-summable functions on E, then we get that

$$\begin{aligned} \sum_{x\in E} |f(x)+g(x)|^p &\le \sum_{x\in E} (|f(x)| + |g(x)|)^p \\ &\le \sum_{x\in E} |f(x)|^p + \sum_{x\in E} |g(x)|^p. \end{aligned} \tag{68.10}$$

Thus

$$\|f+g\|_p^p \le \|f\|_p^p + \|g\|_p^p. \tag{68.11}$$

This is a bit better than what we had before, since there is no longer an extra factor of 2^p. Note that $\|f\|_p$ does not satisfy the ordinary triangle inequality when $0 < p < 1$ and E has at least two elements, and hence is not a norm on $\ell^p(E)$. However, $\|f-g\|_p^p$ defines a metric on $\ell^p(E)$ when $0 < p \le 1$, by (68.11).

69 Convexity

It is well known that $\phi_p(r) = r^p$ defines a convex function of $r \ge 0$ when $p \ge 1$. Therefore

$$(t\,a + (1-t)\,b)^p \le t\,a^p + (1-t)\,b^p \tag{69.1}$$

for every $a, b \ge 0$ and $0 \le t \le 1$ when $p \ge 1$. In particular, if we take $t = 1/2$, then we get that

$$(a+b)^p \le 2^{p-1}\,(a^p + b^p). \tag{69.2}$$

This improves an inequality in the previous section by a factor of 2.

If f, g are p-summable functions on a set E, $0 \le t \le 1$, and $p \ge 1$, then it follows that

$$\begin{aligned} \sum_{x \in E} |t\,f(x) + (1-t)\,g(x)|^p &\le \sum_{x \in E} (t\,|f(x)| + (1-t)\,|g(x)|)^p \\ &\le t \sum_{x \in E} |f(x)|^p + (1-t) \sum_{x \in E} |g(x)|^p. \end{aligned} \tag{69.3}$$

Equivalently,

$$\|t\,f + (1-t)\,g\|_p^p \le t\,\|f\|_p^p + (1-t)\,\|g\|_p^p. \tag{69.4}$$

Minkowski's inequality states that

$$\|f+g\|_p \le \|f\|_p + \|g\|_p \tag{69.5}$$

for every $f, g \in \ell^p(E)$ when $p \ge 1$. This implies that $\|f\|_p$ is a norm on $\ell^p(E)$ when $p \ge 1$, because $\|f\|_p$ satisfies the positivity and homogeneity conditions of a norm for every $p > 0$.

To prove Minkowski's inequality, we may as well suppose that neither f nor g is identically 0 on E, since it is trivial otherwise. Put $f' = f/\|f\|_p$, $g' = g/\|g\|_p$, so that $\|f'\|_p = \|g'\|_p = 1$. Thus

$$\|t\,f' + (1-t)\,g'\|_p \le 1 \tag{69.6}$$

when $0 \le t \le 1$, by (69.4). If

$$t = \frac{\|f\|_p}{(\|f\|_p + \|g\|_p)}, \tag{69.7}$$

then $1-t = \|g\|_p/(\|f\|_p + \|g\|_p)$, and Minkowski's inequality follows from (69.6).

Remember that a subset A of a vector space V is said to be *convex* if

$$t\,v + (1-t)\,w \in A \tag{69.8}$$

for every $v, w \in A$ and $0 \le t \le 1$. If $N(v)$ is a seminorm on V, then it is easy to see that the corresponding closed unit ball

$$B = \{v \in V : N(v) \le 1\} \tag{69.9}$$

is a convex set in V. Conversely, if a nonnegative real-valued function $N(v)$ on V satisfies the homogeneity condition of a seminorm and B is convex, then one can check $N(v)$ is a seminorm on V. This is basically the same as the argument in the previous paragraph for $\|f\|_p$, at least when $N(v)$ satisfies the positivity condition of a norm. Otherwise, some minor adjustments are needed to deal with $v \in V$ such that $N(v) = 0$ but $v \ne 0$.

70 Hölder's inequality

Let $1 < p, q < \infty$ be conjugate exponents, in the sense that

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{70.1}$$

If E is a nonempty set, $f \in \ell^p(E)$, and $g \in \ell^q(E)$, then *Hölder's inequality* states that $f\,g \in \ell^1(E)$, and

$$\|f\,g\|_1 \le \|f\|_p\,\|g\|_q. \tag{70.2}$$

This also works when $p = 1$ and $q = \infty$, or the other way around, and is much simpler. The $p = q = 2$ case can be reduced to the Cauchy–Schwarz inequality.

Using the convexity of the exponential function, one can check that

$$a\,b \le \frac{a^p}{p} + \frac{b^q}{q} \tag{70.3}$$

for every $a, b \ge 0$. Applying this to $a = |f(x)|$, $b = |g(x)|$, and summing over $x \in E$, we get that

$$\sum_{x \in E} |f(x)|\,|g(x)| \le p^{-1} \sum_{x \in E} |f(x)|^p + q^{-1} \sum_{x \in E} |g(x)|^q. \tag{70.4}$$

In particular, $f\,g \in \ell^1(E)$, and

$$\|f\,g\|_1 \le p^{-1}\,\|f\|_p^p + q^{-1}\,\|g\|_q^q, \tag{70.5}$$

which implies Hölder's inequality in the special case where $\|f\|_p = \|g\|_q = 1$. If f and g are not identically 0 on E, then one can reduce to this case, by considering $f' = f/\|f\|_p$, $g' = g/\|g\|_q$. Otherwise, if f or g is identically 0 on E, then the result is trivial.

If $f \in \ell^p(E)$, $g \in \ell^q(E)$, then put

$$\lambda_g(f) = \sum_{x \in E} f(x)\, g(x). \tag{70.6}$$

Hölder's inequality implies that

$$|\lambda_g(f)| \le \|f\|_p \, \|g\|_q, \tag{70.7}$$

so that $\lambda_g(f)$ defines a continuous linear functional on $\ell^p(E)$ for each $g \in \ell^q(E)$, with dual norm less than or equal to $\|g\|_q$. One can check that the dual norm of λ on $\ell^p(E)$ is actually equal to $\|g\|_q$, by choosing g such that

$$f(x)\, g(x) = |f(x)|^p = |g(x)|^q \tag{70.8}$$

for every $x \in E$. These conditions on g are consistent with each other, because p and q are conjugate exponents.

Conversely, if λ is a continuous linear functional on $\ell^p(E)$, then one can show that $\lambda = \lambda_g$ for some $g \in \ell^q(E)$. As usual, one can start by putting $g(x) = \lambda(\delta_x)$, where δ_x is the function on E equal to 1 at x and to 0 elsewhere. This permits $\lambda_g(f)$ to be defined as in the previous paragraph when f has finite support on E, in which cas it agrees with $\lambda(f)$, by linearity. The next step is to show that

$$\Big(\sum_{x \in A} |g(x)|^q\Big)^{1/q} \tag{70.9}$$

is bounded by the dual norm of λ on $\ell^p(E)$ when A is a finite subset of E. This can be done by choosing f such that (70.8) holds when $x \in A$, and $f(x) = 0$ when $x \in E \backslash A$. This implies that $g \in \ell^q(E)$, and that $\|g\|_q$ is less than or equal to the dual norm of λ on $\ell^p(E)$. The remaining point is that $\lambda(f) = \lambda_g(f)$ for every $f \in \ell^p(E)$. We already know that this holds when f has finite support on E, which implies that it holds for every $f \in \ell^p(E)$, because functions with finite support are dense in $\ell^p(E)$, and because λ and λ_g are continuous on $\ell^p(E)$.

71 $p < 1$

Let E be a nonempty set, and let p be a positive real number strictly less than 1. As in Section 68,

$$d_p(f, g) = \|f - g\|_p^p \tag{71.1}$$

defines a metric on $\ell^p(E)$. It is easy to see that addition and scalar multiplication are continuous with respect to the topology associated to this metric, so that $\ell^p(E)$ becomes a topological vector space. If E has only finitely many elements, then $\ell^p(E)$ can be identified with $\mathbf{R}^n$ or $\mathbf{C}^n$, as appropriate, where n is the number of elements of E, and the topology on $\ell^p(E)$ determined by this metric corresponds exactly to the standard topology on $\mathbf{R}^n$ or $\mathbf{C}^n$.

If $f \in \ell^p(E)$ and $g \in \ell^\infty(E)$, then $f\,g \in \ell^p(E) \subseteq \ell^1(E)$, and we can put

$$\lambda_g(f) = \sum_{x \in E} f(x)\,g(x). \tag{71.2}$$

Moreover,

$$|\lambda_g(f)| \le \|f\|_1\,\|g\|_\infty \le \|f\|_p\,\|g\|_\infty. \tag{71.3}$$

Using this estimate, it is easy to see that λ_g is a continuous linear functional on $\ell^p(E)$ with respect to the topology associated to the metric defined in the previous paragraph.

Conversely, suppose that λ is a continuous linear functional on $\ell^p(E)$. This implies that there is a $\delta > 0$ such that

$$|\lambda(f)| \le 1 \tag{71.4}$$

for all $f \in \ell^p(E)$ such that $d_p(f, 0) = \|f\|_p^p < \delta$. Equivalently, there is a $C \ge 0$ such that

$$|\lambda(f)| \le C\,\|f\|_p \tag{71.5}$$

for every $f \in \ell^p(E)$, because of linearity. Put $g(x) = \lambda(\delta_x)$ for each $x \in E$, where δ_x is the function on E equal to 1 at x and to 0 elsewhere. Thus $|g(x)| \le C$ for every $x \in E$, because $\|\delta_x\|_p = 1$. This permits us to define λ_g as in the preceding paragraph. By construction, $\lambda(f) = \lambda_g(f)$ when f has finite support on E. It is easy to see that functions with finite support on E are dense in $\ell^p(E)$, for basically the same reasons as when $1 \le p < \infty$. Hence $\lambda(f) = \lambda_g(f)$ for every $f \in \ell^p(E)$, since λ, λ_g are both continuous on $\ell^p(E)$.

If E has at least two elements, then the unit ball in $\ell^p(E)$ is not convex, unlike the situation when $p \ge 1$. If E has infinitely many elements, then the convex hull of the unit ball in $\ell^p(E)$ is not even bounded with respect to $\|f\|_p$, since it contains all functions f on E with finite support such that $\|f\|_1 \le 1$, for instance. However, if $f, g \in \ell^p(E)$, $0 \le t \le 1$, and h is another function on E that satisfies

$$|h(x)| \le |f(x)|^t\,|g(x)|^{1-t} \tag{71.6}$$

for every $x \in E$, then $h \in \ell^p(E)$, and

$$\|h\|_p \le \|f\|_p^t\,\|g\|_p^{1-t}. \tag{71.7}$$

This follows from Hölder's inequality, and works for all $p > 0$. In particular, $\|h\|_p \le 1$ when $\|f\|_p, \|g\|_p \le 1$, which is a multiplicative convexity property of the unit ball in $\ell^p(E)$.

72 Bounded linear mappings, revisited

Let V be a real or complex vector space with a norm $\|v\|_V$, and consider the space $\mathcal{BL}(V) = \mathcal{BL}(V, V)$ of bounded linear mappings from V into itself. This is an associative algebra, with composition of linear operators as multiplication, and the identity operator I on V as the multiplicative identity element. Note that $\|I\|_{op} = 1$, except in the trivial case where V consists of only the zero element. If V is complete, then $\mathcal{BL}(V)$ is also complete with respect to the operator norm, as in Section 22. Thus $\mathcal{BL}(V)$ is a Banach algebra when V is a Banach space and $V \neq \{0\}$. If V is finite-dimensional, then $\mathcal{BL}(V)$ is the same as the algebra of all linear transformations on V. In particular, $\mathcal{BL}(V)$ is not commutative when the dimension of V is greater than or equal to 2. This includes the case where V is infinite-dimensional, since the Hahn–Banach theorem may be used to get plenty of bounded linear operators on V with finite rank.

As an example, let V be the space of real or complex-valued continuous functions on $[0, 1]$, equipped with the supremum norm. If f is a continuous function on $[0, 1]$, then let $T(f)$ be the function defined on $[0, 1]$ by

$$T(f)(x) = \int_0^x f(y)\,dy. \tag{72.1}$$

Note that $T(f)$ is continuously-differentiable on $[0, 1]$, with derivative equal to f. In particular, $T(f)$ is continuous on $[0, 1]$. Moreover,

$$|T(f)(x)| \le \int_0^x |f(y)|\,dy \le \int_0^1 |f(y)|\,dy \le \|f\|_{sup} \tag{72.2}$$

for every $f \in C([0, 1])$ and $x \in [0, 1]$, which implies that

$$\|T(f)\|_{sup} \le \int_0^1 |f(y)|\,dy \le \|f\|_{sup}. \tag{72.3}$$

It follows that T is a bounded linear mapping from $C([0, 1])$ into itself, with operator norm less than or equal to 1. It is easy to see that $\|T\|_{op} = 1$, by considering the case where f is the constant function equal to 1 on $[0, 1]$.

Let n be a positive integer, and let $T^n = T \circ \cdots \circ T$ be the n-fold composition of T. This can be expressed by the n-fold integral

$$T^n(f)(x) = \int_0^x \int_0^{y_n} \cdots \int_0^{y_2} f(y_1)\,dy_1 \cdots dy_{n-1}\,dy_n. \tag{72.4}$$

Thus

$$\begin{aligned} |T^n(f)(x)| &\le \int_0^x \int_0^{y_n} \cdots \int_0^{y_2} |f(y_1)|\,dy_1 \cdots dy_{n-1}\,dy_n \\ &\le \int_0^1 \int_0^{y_n} \cdots \int_0^{y_2} |f(y_1)|\,dy_1 \cdots dy_{n-1}\,dy_n. \end{aligned} \tag{72.5}$$

If

$$\sigma(n) = \int_0^1 \int_0^{y_n} \cdots \int_0^{y_2} dy_1 \cdots dy_{n-1}\,dy_n, \tag{72.6}$$

then we get that

$$\|T^n(f)\|_{sup} \le \sigma(n)\,\|f\|_{\mathrm{sup}}. \tag{72.7}$$

This shows that the operator norm of T^n on $C([0,1])$ is less than or equal to $\sigma(n)$, and it is again easy to see that $\|T^n\|_{op} = \sigma(n)$, by considering the case where f is the constant function equal to 1 on $[0,1]$.

In fact, if $\mathbf{1}_{[0,1]}$ denotes the constant function equal to 1 on $[0,1]$, then it is easy to check that

$$T^n(\mathbf{1}_{[0,1]})(x) = \frac{x^n}{n!}, \tag{72.8}$$

using induction on n. In particular,

$$\sigma(n) = T^n(\mathbf{1}_{[0,1]})(1) = \frac{1}{n!}. \tag{72.9}$$

Alternatively, $\sigma(n)$ is the same as the n-dimensional volume of the n-dimensional simplex

$$\Sigma(n) = \{y \in \mathbf{R}^n : 0 \le y_1 \le y_2 \le \cdots \le y_{n-1} \le y_n \le 1\}. \tag{72.10}$$

That the volume of $\Sigma(n)$ is equal to $1/n!$ can also be seen geometrically, by decomposing the unit cube in $\mathbf{R}^n$ into $n!$ copies of $\Sigma(n)$ with disjoint interiors. These copies of $\Sigma(n)$ are obtained by permuting the standard coordinates of $\mathbf{R}^n$, using the $n!$ permutations on the set $\{1, \ldots, n\}$. Each copy of $\Sigma(n)$ has the same n-dimensional volume as $\Sigma(n)$, and the intersection of any two distinct copies has measure 0. Thus the sum of the volumes of all of these copies of $\Sigma(n)$ is equal to $n!$ times the volume of $\Sigma(n)$, and is also equal to the volume of the unit cube, which is equal to 1.

Observe that $n! \ge k^{n-k+1}$ for each positive integer k when $n \ge k$, so that

$$(n!)^{-1/n} \le k^{(k-1)/n-1} \tag{72.11}$$

when $n \ge k$. In particular,

$$(n!)^{-1/n} \le k^{-1/2} \tag{72.12}$$

when $n \ge 2k$, which implies that

$$\lim_{n \to \infty} (n!)^{-1/n} = 0, \tag{72.13}$$

since the previous statement works for every positive integer k. It follows that

$$\lim_{n \to \infty} \|T^n\|_{op}^{1/n} = 0, \tag{72.14}$$

because $\|T^n\|_{op} = \sigma(n) = 1/n!$.

Equivalently, this shows that $r(T) = 0$, in the notation of Section 58. This would be trivial if $T^n = 0$ for some positive integer n, which is clearly not the case in this example.

Let $\mathcal{A}$ be an associative algebra over the real or complex numbers with a multiplicative identity element, such as the algebra of bounded linear operators on a vector space with a norm. If $x \in \mathcal{A}$, then let $\mathcal{A}(x)$ be the subalgebra of $\mathcal{A}(x)$ generated by x, consisting of linear combinations of the multiplicative identity element and positive powers of x. It is easy to see that this is a commutative subalgebra of $\mathcal{A}$, even if $\mathcal{A}$ is not commutative. If $\mathcal{A}$ is a topological algebra, then the closure of a commutative subalgebra of $\mathcal{A}$ is also commutative. If $\mathcal{A}$ is a Banach algebra, then closed subalgebras of $\mathcal{A}$ are Banach algebras too.

73 Involutions

Let $\mathcal{A}$ be an associative algebra over the real or complex numbers. A mapping

$$x \mapsto x^* \tag{73.1}$$

on $\mathcal{A}$ is said to be an *involution* if it satisfies the following three conditions. First, (73.1) should be linear in the real case, and conjugate-linear in the complex case. This means that

$$(x + y)^* = x^* + y^* \tag{73.2}$$

for every $x, y \in \mathcal{A}$ in both cases,

$$(t\, x)^* = t\, x^* \tag{73.3}$$

for every $x \in \mathcal{A}$ and $t \in \mathbf{R}$ in the real case, and

$$(t\, x)^* = \overline{t}\, x^* \tag{73.4}$$

in the complex case. Second, (73.1) should be compatible with multiplication in $\mathcal{A}$, in the sense that

$$(x\, y)^* = y^*\, x^* \tag{73.5}$$

for every $x, y \in \mathcal{A}$. Of course, (73.5) is the same as

$$(x\, y)^* = x^*\, y^* \tag{73.6}$$

when $\mathcal{A}$ is commutative. The third condition is that

$$(x^*)^* = x \tag{73.7}$$

for every $x \in \mathcal{A}$. In particular, this implies that (73.1) is a one-to-one mapping of $\mathcal{A}$ onto itself. If $\mathcal{A}$ has a multiplicative identity element e, then it follows from the multiplicativity condition (73.5) that

$$e^* = e. \tag{73.8}$$

If $\mathcal{A}$ is equipped with a norm, then one normally asks also that the involution be isometric, so that

$$\|x^*\| = \|x\| \tag{73.9}$$

for every $x \in \mathcal{A}$.

If $\mathcal{A}$ is the algebra of continuous complex-valued functions on a topological space, then

$$f(p) \mapsto \overline{f(p)} \tag{73.10}$$

defines an involution on $\mathcal{A}$. This would not work for holomorphic functions, because the complex-conjugate of a holomorphic function f is also holomorphic only when f is constant. If $\mathcal{A}$ is the algebra of $n \times n$ matrices of real numbers with respect to matrix multiplication, then the transpose of a matrix defines an involution on $\mathcal{A}$. If instead $\mathcal{A}$ is the algebra of $n \times n$ matrices of complex numbers with respect to matrix multiplication, then one can get an involution on $\mathcal{A}$ by taking the complex conjugates of the entries of the transpose of a matrix.

If $(V, \langle v, w \rangle)$ is a real or complex Hilbert space and T is a bounded linear operator on V, then it is well known that there is a unique bounded linear operator T^* on V such that

$$\langle T(v), w \rangle = \langle v, T^*(w) \rangle \tag{73.11}$$

for every $v, w \in V$, known as the *adjoint* of T. It is easy to see that this defines an involution on the algebra $\mathcal{BL}(V)$ of bounded linear operators on V. The adjoint of T corresponds exactly to the transpose of a real matrix or the complex conjugate of the transpose of a complex matrix when T is represented by a matrix with respect to an orthonormal basis for V.

Using the definition of the norm associated to an inner product and the Cauchy–Schwarz inequality, one can check that

$$\|T\|_{op} = \sup\{|\langle T(v), w \rangle| : v, w \in V,\ \|v\|, \|w\| \le 1\} \tag{73.12}$$

for every bounded linear operator T on V. This implies that

$$\|T^*\|_{op} = \|T\|_{op} \tag{73.13}$$

for every $T \in \mathcal{BL}(V)$, using the symmetry properties of the inner product and interchanging the roles of v and w in the previous expression for the operator norm of T^*. Moreover,

$$\|T^* \circ T\|_{op} = \|T\|_{op}^2. \tag{73.14}$$

Of course,

$$\|T^* \circ T\|_{op} \le \|T^*\|_{op} \, \|T\|_{op} = \|T\|_{op}^2, \tag{73.15}$$

and so it suffices to show the opposite inequality. Observe that

$$\langle (T^*(T(v)), v \rangle = \langle T(v), T(v) \rangle = \|T(v)\|^2, \tag{73.16}$$

by the definition of the adjoint operator T^*. This implies that

$$\|T(v)\|^2 \le \|(T^*(T(v))\| \, \|v\| \le \|T^* \circ T\|_{op} \, \|v\|^2, \tag{73.17}$$

by the Cauchy–Schwarz inequality and the definition of the operator norm. Thus

$$\|T\|_{op}^2 \le \|T^* \circ T\|_{op}, \tag{73.18}$$

as desired.

A Banach algebra $(\mathcal{A}, \|x\|)$ equipped with an isometric involution $x \mapsto x^*$ is said to be a C^* *algebra* if

$$\|x^* \, x\| = \|x\|^2 \tag{73.19}$$

for every $x \in \mathcal{A}$. This includes the algebras of bounded linear operators on real or complex Hilbert spaces, as in the previous paragraphs. This also includes the algebra of real or complex-valued bounded continuous functions on a topological space X with respect to the supremum norm, where the involution is given by complex conjugation as in (73.10) in the complex case, and by the identity operator in the real case. The same involutions are defined and isometric on the algebras of real and complex-valued continuously-differentiable functions on the unit interval, as in Section 57, but the C^1 norm does not satisfy the C^* condition (73.19).

Suppose that τ is a continuous involution on a topological space X, which is to say a continuous mapping from X into itself such that

$$\tau(\tau(p)) = p \tag{73.20}$$

for every $p \in X$. Equivalently, τ is its own inverse, and hence a homeomorphism from X onto itself. Under these conditions,

$$f(p) \mapsto f(\tau(p)) \tag{73.21}$$

is an involution on the algebra of real-valued continuous functions on X, and

$$f(p) \mapsto \overline{f(\tau(p))} \tag{73.22}$$

is an involution on the algebra of complex-valued continuous functions on X. These involutions also preserve the supremum norms of bounded continuous functions on X. However, the C^* condition (73.19) does not work when τ is not the identity mapping on X, at least when X is sufficiently regular to have enough continuous functions.

As a variant of this, let U be the open disk in the complex plane. If $f(z)$ is a holomorphic function on U, then it is well known that

$$\overline{f(\overline{z})} \tag{73.23}$$

is also holomorphic on U. It is easy to see that this defines an involution on the algebra of holomorphic functions on U, which preserves the supremum norm of bounded holomorphic functions on U. However, if

$$f(z) = z + i, \tag{73.24}$$

then the supremum norm of f on U is equal to 2, and the supremum norm of

$$\overline{f(\overline{z})}\, f(z) = (z - i)(z + i) = z^2 + 1 \tag{73.25}$$

is equal to 2 as well. Thus the C^* condition (73.19) does not work in this case either, when we restrict our attention to bounded holomorphic functions on U, since the supremum norm of (73.25) on U is strictly less than the square of the supremum norm of f.

Let $(\mathcal{A}, \|x\|, x^*)$ be a real or complex C^* algebra, and suppose that $x \in \mathcal{A}$ satisfies

$$x^* = x. \tag{73.26}$$

In this case, the C^* condition (73.19) reduces to

$$\|x^2\| = \|x\|^2. \tag{73.27}$$

If l is a positive integer, then

$$(x^l)^* = (x^*)^l = x^l, \tag{73.28}$$

and so we can apply the previous statement to x^l to get that

$$\|x^{2l}\| = \|x^l\|^2. \tag{73.29}$$

Applying this repeatedly, we get that

$$\|x^{2^n}\| = \|x\|^{2^n} \tag{73.30}$$

for each positive integer n. Of course,

$$\|x^l\| \le \|x\|^l \tag{73.31}$$

for any positive integer n, by the submultiplicative property of the norm. If we choose a positive integer n such that $l \le 2^n$, then we get that

$$\|x\|^{2^n} = \|x^{2^n}\| \le \|x^l\| \, \|x\|^{2^n - l}, \tag{73.32}$$

using the submultiplicative property of the norm again. This implies that

$$\|x\|^l \le \|x^l\|, \tag{73.33}$$

and hence that

$$\|x^l\| = \|x\|^l \tag{73.34}$$

for each positive integer l.

If y is any element of $\mathcal{A}$, then $x = y^* \, y$ satisfies $x^* = x$. Thus we get

$$\|(y^* \, y)^l\| = \|y^* \, y\|^l = \|y\|^{2\,l} \tag{73.35}$$

for each positive integer l. Suppose that y^* commutes with y, so that

$$(y^* \, y)^l = (y^*)^l \, y^l = (y^l)^* \, y^l \tag{73.36}$$

for each l, and hence

$$\|(y^* \, y)^l\| = \|(y^l)^* \, y^l\| = \|y^l\|^2. \tag{73.37}$$

This implies that

$$\|y^l\| = \|y\|^l \tag{73.38}$$

for each positive integer l, as before.

Part III
Several variables

74 Power series

Let n be a positive integer, and let

$$\sum_\alpha a_\alpha \, z^\alpha \tag{74.1}$$

be a power series in n complex variables. More precisely, the sum is taken over all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$, $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ is the corresponding monomial, and the coefficients a_α are complex numbers. Let A be the set of $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$ for which this series converges absolutely, in the sense that $a_\alpha \, z^\alpha$ is a summable function of α on the set of multi-indices. Thus $0 \in A$ trivially, and $w \in A$ whenever there is a $z \in A$ such that $|w_j| \le |z_j|$ for $j = 1, \ldots, n$, by the comparison test.

Let $\sum_\alpha b_\alpha \, z^\alpha$ be another power series, and let B be the set of $z \in \mathbf{C}^n$ on which this series converges absolutely, as before. Note that

$$\sum_\alpha (a_\alpha + b_\alpha) \, z^\alpha \tag{74.2}$$

converges absolutely for every $z \in A \cap B$. The product of these two power series can be expressed formally as

$$\Big(\sum_\alpha a_\alpha \, z^\alpha\Big) \Big(\sum_\beta b_\beta \, z^\beta\Big) = \sum_\gamma c_\gamma \, z^\gamma, \tag{74.3}$$

where

$$c_\gamma = \sum_{\alpha + \beta = \gamma} a_\alpha \, b_\beta. \tag{74.4}$$

More precisely, the sum on the right is taken over all multi-indices α, β such that $\alpha + \beta = \gamma$, of which there are only finitely many. If $z \in A \cap B$, then one can check that $\sum_\gamma c_\gamma \, z^\gamma$ converges absolutely, and that the sum satisfies (74.3). As a first step, one can verify that

$$\sum_\gamma |c_\gamma| \, |z^\gamma| \le \Big(\sum_\alpha |a_\alpha| \, |z^\alpha|\Big) \Big(\sum_\beta |b_\beta| \, |z^\beta|\Big), \tag{74.5}$$

by estimating the sum over finitely many γ's in terms of the product of sums over finitely many α's and β's. This implies that $\sum_\gamma c_\gamma \, z^\gamma$ converges absolutely when $z \in A \cap B$, and one can show that (74.3) holds by approximating infinite sums by sums with only finitely many nonzero terms. It suffices to consider the case where $z = (1, \ldots, 1)$, since otherwise the monomials in z can be absorbed into the coefficients. One can also use linearity to reduce to the case where the coefficients are nonnegative real numbers, and estimate products of sums of finitely many a_α's and b_β's in terms of sums of finitely many c_γ's.

Let us return to a single power series $\sum_\alpha a_\alpha \, z^\alpha$, and suppose that $w, z \in A$ and $u \in \mathbf{C}^n$ satisfy

$$|u_j| \le |w_j|^t \, |z_j|^{1-t} \tag{74.6}$$

for some $t \in \mathbf{R}$, $0 < t < 1$, and each $j = 1, \ldots, n$. Hence

$$|u^\alpha| \le |w^\alpha|^t \, |z^\alpha|^{1-t} \tag{74.7}$$

for each multi-index α. The convexity of the exponential function on the real line implies that

$$k^t \, l^{1-t} \le t \, k + (1-t) \, l \tag{74.8}$$

for every $k, l \ge 0$. Applying this to $k = |w^\alpha|$, $l = |z^\alpha|$ and summing over α, we get that $u \in A$, because

$$\sum_\alpha |a_\alpha| \, |u^\alpha| \le t \sum_\alpha |a_\alpha| \, |w^\alpha|^t + (1-t) \sum_\alpha |a_\alpha| \, |z^\alpha|^{1-t}. \tag{74.9}$$

75 Power series, continued

Let n be a positive integer, and let $\sum_\alpha a_\alpha \, z^\alpha$ be a power series with complex coefficients in $z = (z_1, \ldots, z_n)$. If l is a nonnegative integer, then

$$p_l(z) = \sum_{|\alpha| = l} a_\alpha \, z^\alpha \tag{75.1}$$

is a homogeneous polynomial of degree l in z, where more precisely the sum is taken over the finitely many multi-indices α such that $|\alpha| = l$. Of course,

$$\sum_{l=0}^\infty p_l(z) = \sum_\alpha a_\alpha \, z^\alpha \tag{75.2}$$

formally, which gives another way to look at the convergence of $\sum_\alpha a_\alpha \, z^\alpha$. In particular, if $\sum_\alpha a_\alpha \, z^\alpha$ converges absolutely for some $z \in \mathbf{C}^n$, then $\sum_{l=0}^\infty p_l(z)$ converges absolutely, and the two sums are the same. This uses the fact that

$$|p_l(z)| \le \sum_{|\alpha| = l} |a_\alpha| \, |z^\alpha| \tag{75.3}$$

for each l.

Let $\sum_\alpha b_\alpha \, z^\alpha$ be another power series, with the corresponding polynomials

$$q_l(z) = \sum_{|\alpha| = l} b_\alpha \, z^\alpha. \tag{75.4}$$

Thus $p_l(z) + q_l(z)$ are the polynomials associated to $\sum_\alpha (a_\alpha + b_\alpha)\, z^\alpha$. Suppose that $\sum_\gamma c_\gamma\, z^\gamma$ is the power series obtained by formally multiplying $\sum_\alpha a_\alpha\, z^\alpha$ and $\sum_\beta b_\beta\, z^\beta$, so that

(75.5) $$c_\gamma = \sum_{\alpha+\beta=\gamma} a_\alpha\, b_\beta.$$

It is easy to check that the corresponding polynomials

(75.6) $$r_l = \sum_{|\gamma|=l} c_\gamma\, z^\gamma$$

are also given by

(75.7) $$r_l = \sum_{j=0}^{l} p_j(z)\, q_{l-j}(z).$$

This shows that r_l is the Cauchy product of the p_j's and q_k's.

Note that

(75.8) $$\sum_{l=0}^{\infty} p_l(t\, z) = \sum_{l=0}^{\infty} t^l\, p_l(z)$$

may be considered as an ordinary power series in $t \in \mathbf{C}$ for each $z \in \mathbf{C}^n$. This gives another way to look at the Cauchy product in the preceding paragraph, as the coefficients of the product of two power series in t. If $\sum_{l=0}^\infty p_l(z)$ converges for some $z \in \mathbf{C}^n$, then $\{p_l(z)\}_{l=1}^\infty$ converges to 0, and hence $\{p_l(z)\}_{l=1}^\infty$ is bounded. This implies that (75.8) converges absolutely when $|t| < 1$, by the comparison test.

Consider

(75.9) $$p^*(z) = \limsup_{l\to\infty} |p_l(z)|^{1/l},$$

which takes values in $[0, \infty]$. Observe that

(75.10) $$p^*(t\, z) = |t|\, p^*(z)$$

for each $t \in \mathbf{C}$ and $z \in \mathbf{C}^n$, because $p_l(z)$ is homogeneous of degree l. The right side of (75.10) should be interpreted as being 0 when $t = 0$, even when $p^*(z) = +\infty$, because $p^*(0) = 0$. The *root test* states that $\sum_{l=0}^\infty p_l(z)$ converges absolutely when $p^*(z) < 1$, and diverges when $p^*(z) > 1$. It follows that the radius of convergence of (75.8) as a power series in t is equal to $1/p^*(z)$.

76 Linear transformations

Let n be a positive integer, and let T be a one-to-one linear transformation from $\mathbf{C}^n$ onto itself. Consider the mapping ρ_T acting on complex-valued functions on $\mathbf{C}^n$ defined by

(76.1) $$\rho_T(f)(z) = f(T^{-1}(z)).$$

Thus

(76.2) $$\rho_T(f + g) = \rho_T(f) + \rho_T(g)$$

and

(76.3) $$\rho_T(f\,g) = \rho_T(f)\,\rho_T(g).$$

for any pair of functions f, g on $\mathbf{C}^n$. If f is a polynomial on $\mathbf{C}^n$, then it is easy to see that $\rho_T(f)$ is a polynomial too. If f is a homogeneous polynomial, then $\rho_T(f)$ is a homogeneous polynomial as well, of the same degree.

Of course, $\rho_T(f) = f$ for every function f on $\mathbf{C}^n$ when T is the identity transformation on $\mathbf{C}^n$. If R, T are arbitrary invertible linear transformation on $\mathbf{C}^n$, then

(76.4) $$\begin{aligned} \rho_R(\rho_T(f))(z) &= \rho_T(f)(R^{-1}(z)) = f(T^{-1}(R^{-1}(z))) \\ &= f((R \circ T)^{-1}(z)) = \rho_{R\circ T}(f)(z). \end{aligned}$$

In particular, $\rho_{T^{-1}} = (\rho_T)^{-1}$. Let $GL(\mathbf{C}^n)$ be the group of invertible linear transformations on $\mathbf{C}^n$, with composition of mappings as the group operation. It follows that $T \mapsto \rho_T$ is a homomorphism from $GL(\mathbf{C}^n)$ into the group of invertible linear transformations on the space of functions on $\mathbf{C}^n$, which is to say a representation of $GL(\mathbf{C}^n)$ on the space of functions on $\mathbf{C}^n$.

Let $f(z) = \sum_\alpha a_\alpha\, z^\alpha$ be a formal power series with complex coefficients. This can also be expressed as $\sum_{l=0}^\infty p_l(z)$, where $p_l(z)$ is a homogeneous polynomial of degree l for each $l \ge 0$. If T is an invertible linear transformation on $\mathbf{C}^n$, then we can take $\rho_T(f)$ to be the formal power series that corresponds to $\sum_{l=0}^\infty \rho_T(p_l)$. It is easy to see that this preserves sums and products of power series, just as for ordinary functions. In particular, this defines a representation of $GL(\mathbf{C}^n)$ on the space of formal power series.

If $\sum_{l=0}^\infty p_l(z)$ converges for some $z \in \mathbf{C}^n$, then $\sum_{l=0}^\infty \rho_T(p_l)(T(z))$ converges and has the same sum, because it is the same series of complex numbers. If $\sum_{l=0}^\infty \rho_l(z)$ converges for every $z \in \mathbf{C}^n$, then $\sum_{l=0}^\infty \rho_T(p_l)(T(z))$ converges for every $z \in \mathbf{C}^n$, and has the same sum. Hence the formal and pointwise definitions of $\rho(f)$ are consistent with each other in this case.

77 Abel summability

Let $\sum_{j=0}^\infty a_j$ be an infinite series of complex numbers, and put

(77.1) $$A(r) = \sum_{j=0}^\infty a_j\, r^j$$

when $0 \le r < 1$. More precisely, we suppose that the sum on the right converges for each $r < 1$, which implies that $\{a_j\, r^j\}_{j=0}^\infty$ converges to 0 for each $r < 1$, and hence that $\{a_j\, r^j\}_{j=0}^\infty$ is bounded for each $r < 1$. Conversely, if $\{a_j\, r^j\}_{j=0}^\infty$ is bounded for each $r < 1$, then $\sum_{j=0}^\infty a_j\, t^j$ converges absolutely for each $t < 1$, as one can see by taking $t < r < 1$ and using the comparison test, since $\sum_{j=0}^\infty (t/r)^j$ is a convergent geometric series under these conditions. The expressions $A(r)$

are known as the *Abel sums* associated to $\sum_{j=0}^\infty a_j$, and we say that $\sum_{j=0}^\infty a_j$ is *Abel summable* if

$$\lim_{r \to 1-} A(r) \tag{77.2}$$

exists.

If $\sum_{j=0}^\infty a_j$ converges in the usual sense, then it is Abel summable. To see this, let

$$s_n = \sum_{j=0}^n a_j \tag{77.3}$$

be the nth partial sum of $\sum_{j=0}^\infty a_j$ when $n \ge 0$, and put $s_{-1} = 0$. Thus $a_j = s_j - s_{j-1}$ for each $j \ge 0$, and hence

$$A(r) = \sum_{j=0}^\infty (s_j - s_{j-1})\, r^j = \sum_{j=0}^\infty s_j\, r^j - \sum_{j=0}^\infty s_{j-1}\, r^j \tag{77.4}$$

when $0 \le r < 1$. There is no problem with the convergence of the series on the right, because the convergence of $\sum_{j=0}^\infty a_j$ implies that $\{a_j\}_{j=0}^\infty$ converges to 0 and is therefore bounded, which implies that $s_n = O(n)$. Of course,

$$\sum_{j=0}^\infty s_{j-1}\, r^j = \sum_{j=1}^\infty s_{j-1}\, r^j = \sum_{j=0}^\infty s_j\, r^{j+1}, \tag{77.5}$$

because $s_{-1} = 0$, which implies that

$$A(r) = \sum_{j=0}^\infty s_j (r^j - r^{j+1}) = (1 - r) \sum_{j=0}^\infty s_j\, r^j \tag{77.6}$$

when $r < 1$.

We would like to show that

$$\lim_{r \to 1-} A(r) = \lim_{j \to \infty} s_j \tag{77.7}$$

when the limit on the right side exists. Put $s = \lim_{j \to \infty} s_j$, let $\epsilon > 0$ be given, and choose $L \ge 0$ such that

$$|s_j - s| < \frac{\epsilon}{2} \tag{77.8}$$

for every $j \ge L$. Observe that

$$A(r) - s = (1 - r) \sum_{j=0}^\infty (s_j - s)\, r^j \tag{77.9}$$

when $r < 1$, because $(1 - r) \sum_{j=0}^\infty r^j = 1$. It follows that

$$|A(r) - s| \quad \le \quad (1 - r) \sum_{j=0}^\infty |s_j - s|\, r^j \tag{77.10}$$

$$
\begin{aligned}
&< \quad (1-r)\sum_{j=0}^{L-1}|s_j - s|\,r^j + (1-r)\sum_{j=L}^{\infty}(\epsilon/2)\,r^j \\
&\le \quad (1-r)\sum_{j=0}^{L-1}|s_j - s|\,r^j + \frac{\epsilon}{2}
\end{aligned}
$$

for each $r < 1$. If r is sufficiently close to 1, then

$$
(1-r)\sum_{j=0}^{L-1}|s_j - s|\,r^j \le (1-r)\sum_{j=0}^{L-1}|s_j - s| < \frac{\epsilon}{2} \tag{77.11}
$$

so that $|A(r) - s| < \epsilon/2 + \epsilon/2 = \epsilon$, as desired.

If $a \in \mathbf{C}$ satisfies $|a| = 1$, then

$$
\sum_{j=0}^{\infty} a^j\,r^j = \frac{1}{1 - a\,r} \tag{77.12}
$$

when $0 \le r < 1$. Hence $\sum_{j=0}^{\infty} a^j$ is Abel summable when $a \ne 1$, with the sum equal to $(1-a)^{-1}$. Let $\sum_{j=0}^{\infty} a_j$, $\sum_{j=0}^{\infty} b_j$ be infinite series of complex numbers with Abel sums $A(r)$, $B(r)$, respectively, and note that $\sum_{j=0}^{\infty}(a_j + b_j)$ has Abel sums given by $A(r) + B(r)$. If $\sum_{j=0}^{\infty} a_j$, $\sum_{j=0}^{\infty} b_j$ are Abel summable, then it follows that $\sum_{j=0}^{\infty}(a_j + b_j)$ is Abel summable, with the Abel sum of the latter equal to the sum of the Abel sums of the first two series. Suppose now that $c_n = \sum_{j=0}^{n} a_j\,b_{n-j}$ is the Cauchy product of the a_j's and b_j's, and let $C(r)$ be the corresponding Abel sums. As in Section 63,

$$
C(r) = A(r)\,B(r) \tag{77.13}
$$

when $0 \le r < 1$. More precisely, if the series defining $A(r)$, $B(r)$ converge absolutely, then the series defining $C(r)$ also converges absolutely, and satisfies (77.13). The existence of the Abel sums for $\sum_{j=0}^{\infty} a_j$, $\sum_{j=0}^{\infty} b_j$ for each $r < 1$ implies that this condition holds for every $r < 1$, as discussed at the beginning of this section. If $\sum_{j=0}^{\infty} a_j$, $\sum_{j=0}^{\infty} b_j$ are Abel summable, then it follows that $\sum_{n=0}^{\infty} c_n$ is Abel summable, and that the Abel sum of the latter equal to the product of the Abel sums of the former.

78 Multiple Fourier series

Let n be a positive integer, and let $\mathbf{T}^n$ be the n-dimensional torus, consisting of $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$ such that $|z_j| = 1$ for $j = 1, \ldots, n$. If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of integers, then put

$$
z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}, \tag{78.1}
$$

with the usual convention that $z_j^{\alpha_j} = 1$ when $\alpha_j = 0$. Thus

$$(78.2) \qquad \frac{1}{(2\pi)^n} \int_{\mathbf{T}^n} z^\alpha \, |dz| = \prod_{j=1}^n \frac{1}{2\,\pi} \int_{\mathbf{T}} z_j^{\alpha_j} \, |dz_j|$$

is equal to 0 when $\alpha \neq 0$, and is equal to 1 when $\alpha = 0$. Here $|dz|$ is the n-dimensional element of integration on $\mathbf{T}^n$ corresponding to the element $|dz_j|$ of arc length in each variable.

If f is a continuous complex-valued function on $\mathbf{T}^n$ and $\alpha \in \mathbf{Z}^n$, then we put

$$(78.3) \qquad \widehat{f}(\alpha) = \frac{1}{(2\pi)^n} \int_{\mathbf{T}^n} f(z) \, z^{-\alpha} \, |dz|.$$

The corresponding Fourier series is given by

$$(78.4) \qquad \sum_{\alpha \in \mathbf{Z}^n} \widehat{f}(\alpha) \, z^\alpha.$$

For example, if $f(z) = z^\beta$ for some $\beta \in \mathbf{Z}^n$, then $\widehat{f}(\alpha) = 1$ when $\alpha = \beta$ and is equal to 0 otherwise. Thus (78.4) reduces to f in this case, or when f is a finite linear combination of z^β's. Note that

$$(78.5) \qquad |\widehat{f}(\alpha)| \le \frac{1}{(2\pi)^n} \int_{\mathbf{T}^n} |f(z)| \, |dz|$$

for any continuous function f on $\mathbf{T}^n$ and $\alpha \in \mathbf{Z}^n$.

Let U^n be the n-dimensional open unit polydisk, consisting of $z \in \mathbf{C}^n$ with $|z_j| < 1$ for $j = 1, \ldots, n$. The n-dimensional Poisson kernel $P_n(z, w)$ can be defined for $z \in U^n$ and $w \in \mathbf{T}^n$ by

$$(78.6) \qquad P_n(z, w) = \prod_{j=1}^n P(z_j, w_j),$$

where $P(z_j, w_j)$ is the ordinary Poisson kernel evaluated at z_j, w_j, as in Section 62. If f is a continuous function on $\mathbf{T}^n$, then its Poisson integral is defined on U^n by

$$(78.7) \qquad \phi(z) = \int_{\mathbf{T}^n} P_n(z, w) \, f(w) \, |dw|.$$

As before, one can show that $\phi(z) \to f(z_0)$ as $z \in U^n$ tends to $z_0 \in \mathbf{T}^n$, but one can also do more than this.

Let $\overline{U}^n$ be the n-dimensional closed unit polydisk, consisting of $z \in \mathbf{C}^n$ such that $|z_j| \le 1$ for each j. Of course, this is the same as the closure of U^n in $\mathbf{C}^n$. The boundary ∂U^n of U^n in $\mathbf{C}^n$ consists of $z \in \mathbf{C}^n$ such that $|z_j| \le 1$ for each j and $|z_j| = 1$ for at least one j. In particular, $\mathbf{T}^n \subseteq \partial U^n$, but $\mathbf{T}^n$ is a relatively small subset of ∂U^n when $n > 1$. More precisely, U^n has complex dimension n and hence real dimension $2\,n$, ∂U^n has real dimension $2n - 1$, and $\mathbf{T}^n$ has real dimension n.

One can extend $\phi(z)$ to $z \in \overline{U}^n$ in the following way. If $z \in \mathbf{T}^n$, then we simply put $\phi(z) = f(z)$. If $z \in \partial U^n \backslash \mathbf{T}^n$, then $|z_j| < 1$ for at least one j, and we define $\phi(z)$ by taking the Poisson integral of f in the jth variable when $|z_j| < 1$, and simply evaluating f at z_j in the jth variable when $|z_j| = 1$. It is not too difficult to show that this defines a continuous function on $\overline{U}^n$.

If $z \in \mathbf{C}^n$ and $\alpha \in \mathbf{Z}^n$, then put

$$\widetilde{z}^\alpha = \widetilde{z}_1^{\alpha_1} \cdots \widetilde{z}_n^{\alpha_n}, \tag{78.8}$$

where $\widetilde{z}_j^{\alpha_j} = z_j^{\alpha_j}$ when $\alpha_j \geq 0$ and $\widetilde{z}_j^{\alpha_j} = \overline{z_j}^{\alpha_j}$ when $\alpha_j < 0$. Thus $\widetilde{z}^\alpha = z^\alpha$ when $z \in \mathbf{T}^n$, and

$$|\widetilde{z}^\alpha| = |z_1|^{|\alpha_1|} \cdots |z_n|^{|\alpha_n|} \tag{78.9}$$

for every $z \in \mathbf{C}^n$. If $z \in U^n$, then it is easy to see that

$$\phi(z) = \sum_{\alpha \in \mathbf{Z}^n} \widehat{f}(\alpha)\, \widetilde{z}^\alpha, \tag{78.10}$$

using the analogous expansion for the Poisson kernel in one variable. Note that this series converges absolutely for every $z \in U^n$, since the Fourier coefficients $\widehat{f}(\alpha)$ are bounded, as in (78.5).

If $z \in \mathbf{T}^n$ and $0 \leq r < 1$, then put

$$f_r(z) = \phi(r\, z) = \sum_{\alpha \in \mathbf{Z}^n} \widehat{f}(\alpha)\, r^{|\alpha|}\, z^\alpha, \tag{78.11}$$

where $|\alpha| = |\alpha_1| + \cdots + |\alpha_n|$. One can check that $f_r \to f$ as $r \to 1$ uniformly on $\mathbf{T}^n$, using the fact that continuous functions on compact sets are uniformly continuous. The sum on the right side of (78.11) can be approximated by finite subsums uniformly on $\mathbf{T}^n$ for each $r < 1$, as in Weierstrass' M-test. It follows that every continuous function f on $\mathbf{T}^n$ can be approximated uniformly by finite linear combinations of z^α's, $\alpha \in \mathbf{Z}^n$.

Observe that $\phi(z)$ is "polyharmonic", in the sense that it is harmonic as a function of z_j on the set where $|z_j| < 1$ for each j. This follows from the remarks about harmonic functions of one complex variable in Section 62. In addition,

$$\sup_{z \in \overline{U}^n} |\phi(z)| = \sup_{z \in \mathbf{T}^n} |f(z)|. \tag{78.12}$$

More precisely, the right side of (78.12) is less than or equal to the left side because $\mathbf{T}^n \subseteq \overline{U}^n$ and $\phi = f$ on $\mathbf{T}^n$. To get the opposite inequality, one can use the fact that the Poisson kernel is positive and has integral equal to 1.

If f, g are continuous complex-valued functions on $\mathbf{T}^n$, then put

$$\langle f, g \rangle = \frac{1}{(2\pi)^n} \int_{\mathbf{T}^n} f(z)\, \overline{g(z)}\, |dz|. \tag{78.13}$$

This defines an inner product on the vector space $C(\mathbf{T}^n)$ of continuous complex-valued functions on $\mathbf{T}^n$, for which the corresponding norm is given by

$$\|f\| = \Big(\frac{1}{(2\pi)^n} \int_{\mathbf{T}^n} |f(z)|^2\, |dz| \Big)^{1/2}. \tag{78.14}$$

It is easy to see that the functions z^α, $\alpha \in \mathbf{Z}^n$ are orthonormal with respect to this inner product, and that the Fourier coefficients of a continuous function f on $\mathbf{T}^n$ can be expressed by

$$\widehat{f}(\alpha) = \langle f, z^\alpha \rangle. \tag{78.15}$$

The n-dimensional version of Parseval's formula states that

$$\sum_{\alpha \in \mathbf{Z}^n} |\widehat{f}(\alpha)|^2 = \frac{1}{(2\pi)^n} \int_{\mathbf{T}^n} |f(z)|^2 \, |dz|, \tag{78.16}$$

where the summability of the sum on the left is part of the conclusion. This follows from the orthonormality of the z^α's and the fact that their finite linear combinations are dense in $C(\mathbf{T})$, as in the one-dimensional case.

Suppose that f, g are continuous functions on $\mathbf{T}^n$, and let us check that

$$\widehat{(f\,g)}(\alpha) = \sum_{\beta \in \mathbf{Z}^n} \widehat{f}(\alpha - \beta)\, \widehat{g}(\beta). \tag{78.17}$$

This can be derived formally by multiplying the Fourier series for f, g and collecting terms. To make this rigorous, observe first that $\widehat{f}(\alpha - \beta)\, \widehat{g}(\beta)$ is summable in β, because $\widehat{f}, \widehat{g} \in \ell^2(\mathbf{Z}^n)$, as in the previous paragraph. If $g(z) = z^\gamma$ for some $\gamma \in \mathbf{Z}^n$, then it is easy to see that both sides of (78.17) are equal to $\widehat{f}(\alpha - \gamma)$. It follows that (78.17) holds when g is a finite linear combination of z^γ's, and the same conclusion for an arbitrary continuous function g on $\mathbf{T}^n$ can be obtained by approximation by linear combinations of z^γ's.

If $a(\alpha)$, $b(\alpha)$ are summable functions on $\mathbf{Z}^n$, then their convolution can be defined by

$$(a * b)(\alpha) = \sum_{\beta \in \mathbf{Z}^n} a(\alpha - \beta)\, b(\beta), \tag{78.18}$$

as in the one-dimensional case. More precisely, $a * b$ is also summable on $\mathbf{Z}^n$, and satisfies

$$\|a * b\|_1 \le \|a\|_1 \, \|b\|_1, \tag{78.19}$$

where $\|a\|_1$ is the ℓ^1 norm of a on $\mathbf{Z}^n$. This follows by interchanging the order of summation, as before, and one can also check that $\ell^1(\mathbf{Z}^n)$ is a commutative Banach algebra with respect to convolution. The Fourier transform of a in $\ell^1(\mathbf{Z}^n)$ is defined by

$$\widehat{a}(z) = \sum_{\alpha \in \mathbf{Z}^n} a(\alpha)\, z^\alpha \tag{78.20}$$

for $z \in \mathbf{T}^n$. The sum on the right is absolutely summable for each $z \in \mathbf{T}^n$, because $a(\alpha)$ is summable, and can be approximated by finite subsums uniformly on $\mathbf{T}^n$, as in Weierstrass' M-test. This implies that $\widehat{a}(z)$ is continuous on $\mathbf{T}^n$, and it is easy to see that

$$\widehat{(a * b)}(z) = \widehat{a}(z)\, \widehat{b}(z) \tag{78.21}$$

for every $a, b \in \ell^1(\mathbf{Z}^n)$ and $z \in \mathbf{T}^n$, as before. Conversely, every nonzero multiplicative homomorphism on $\ell^1(\mathbf{Z}^n)$ with respect to convolution can be represented as $a \mapsto \widehat{a}(z)$ for some $z \in \mathbf{T}^n$, as in the one-dimensional situation. Note that the Fourier coefficients of $\widehat{a}$ are given by $a(\alpha)$ for every $a \in \ell^1(\mathbf{Z}^n)$, because of the orthogonality properties of the z^α's. Observe also that

$$\sum_{\alpha \in \mathbf{Z}^n} a(\alpha)\, \widetilde{z}^\alpha \tag{78.22}$$

is absolutely summable for every $z \in \overline{U}^n$, which is the analogue of the function ϕ discussed earlier. As usual, (78.22) can be approximated by finite subsums uniformly on $\overline{U}^n$ under these conditions, which implies more directly that it defines a continuous function on $\overline{U}^n$ than in the earlier discussion.

79 Functions of analytic type

Let $A(\mathbf{T}^n)$ be the collection of continuous functions f on $\mathbf{T}^n$ such that

$$\widehat{f}(\alpha) = 0 \tag{79.1}$$

when $\alpha \in \mathbf{Z}^n$ satisfies $\alpha_j < 0$ for some j. If $f, g \in A(\mathbf{T}^n)$, then it is easy to see from (78.17) that their product $f\,g$ is in $A(\mathbf{T}^n)$ too. Note that the sum on the right side of (78.17) has only finitely many nonzero terms in this situation. It follows that $A(\mathbf{T}^n)$ is a subalgebra of $C(\mathbf{T}^n)$, since the former is clearly a linear subspace of the latter.

If $f \in A(\mathbf{T}^n)$, then (78.10) reduces to

$$\phi(z) = \sum_\alpha \widehat{f}(\alpha)\, z^\alpha, \tag{79.2}$$

where now the sum is taken over all multi-indices α, which is to say $\alpha \in \mathbf{Z}^n$ such that $\alpha_j \geq 0$ for each j. Thus we get an ordinary power series in this case, in the sense that the z^α's are the usual monomials, instead of the modified monomials $\widetilde{z}^\alpha$ that may include complex conjugation. In particular, this implies that f can be approximated uniformly on $\mathbf{T}^n$ by a finite linear combinations of z^α's, where the α's are multi-indices, by the same type of argument as in the previous section. Of course, $z^\alpha \in A(\mathbf{T}^n)$ for every multi-index α, and $A(\mathbf{T}^n)$ is a closed set in $C(\mathbf{T}^n)$ with respect to the supremum norm. It follows that $A(\mathbf{T}^n)$ is the same as the closure in $C(\mathbf{T}^n)$ of the linear span of the z^α's, where α is a multi-index.

Let ϕ_f be the continuous function ϕ on the closed unit polydisk $\overline{U}^n$ associated to $f \in C(\mathbf{T}^n)$ as in the previous section. If $f, g \in A(\mathbf{T}^n)$, then

$$\phi_{f\,g} = \phi_f\, \phi_g. \tag{79.3}$$

This follows by multiplying the series expansions for ϕ_f, ϕ_g in the previous paragraph and collecting terms, as in (74.3) and (74.4). This also uses the formula (78.17) for the Fourier coefficients of the product $f\,g$. The main point is that

$$z^\alpha\, z^\beta = z^{\alpha+\beta} \tag{79.4}$$

while $\widetilde{z}^\alpha\,\widetilde{z}^\beta$ is not necessarily the same as $\widetilde{z}^{\alpha+\beta}$. More precisely, this argument works on the open unit polydisk U^n, where the series expansions for ϕ_f, ϕ_g are absolutely summable. This implies that (79.3) holds on $\overline{U}^n$, by continuity.

Observe that $A(\mathbf{T}^n)$ is a commutative Banach algebra with respect to the supremum norm, since it is a closed subalgebra of $C(\mathbf{T}^n)$ that contains the constant functions, and hence the multiplicative identity element. If $p \in \overline{U}^n$, then $f \mapsto \phi_f(p)$ defines a nonzero homomorphism from $A(\mathbf{T}^n)$ into the complex numbers. Conversely, suppose that h is a nonzero homomorphism on $A(\mathbf{T}^n)$, and let us show that there is a $p \in \overline{U}^n$ such that $h(f) = \phi_f(p)$ for every $f \in A(\mathbf{T}^n)$. As usual, $h(\mathbf{1}_{\mathbf{T}^n}) = 1$, where $\mathbf{1}_{\mathbf{T}^n}$ is the constant function equal to 1 on $\mathbf{T}^n$, and

$$|h(f)| \le \sup_{z \in \mathbf{T}^n} |f(z)| \tag{79.5}$$

for every $f \in A(\mathbf{T}^n)$. Consider $f_j(z) = z_j$, $j = 1, \ldots, n$, as an element of $A(\mathbf{T}^n)$. If $p_j = h(f_j)$, then $|p_j| \le 1$ for each j, by (79.5). Hence $p = (p_1, \ldots, p_n) \in \overline{U}^n$. By construction, $h(f) = \phi_f(p)$ when $f = f_j$ for some j, and it follows that this also holds when f is a polynomial, because h is a homomorphism. Using (79.5) again, we get that $h(f) = \phi_f(p)$ for every $f \in A(\mathbf{T}^n)$, because polynomials are dense in $A(\mathbf{T}^n)$.

Let $\ell^1_A(\mathbf{Z}^n)$ be the set of $a \in \ell^1(\mathbf{Z}^n)$ such that $a(\alpha) = 0$ whenever $\alpha \in \mathbf{Z}^n$ satisfies $\alpha_j < 0$ for some j. It is easy to see that this is a closed subalgebra of $\ell^1(\mathbf{Z}^n)$ with respect to convolution. If $a \in \ell^1_A(\mathbf{Z}^n)$, then (78.22) reduces to an ordinary power series

$$\sum_\alpha a(\alpha)\, z^\alpha, \tag{79.6}$$

where the sum is taken over all multi-indices α. If $b \in \ell^1_A(\mathbf{Z}^n)$ too, then

$$\Big(\sum_\alpha a(\alpha)\, z^\alpha\Big)\Big(\sum_\beta b(\beta)\, z^\beta\Big) = \sum_\gamma (a * b)(\gamma)\, z^\gamma \tag{79.7}$$

for every $z \in \overline{U}^n$, which is basically the same as (74.3) again. Thus the mapping from a to (79.6) defines a homomorphism from $\ell^1(\mathbf{Z}^n)$ into the complex numbers for each $z \in \overline{U}^n$, using convolution as multiplication on $\ell^1_A(\mathbf{Z}^n)$.

Conversely, let us check that any nonzero homomorphism h from $\ell^1_A(\mathbf{Z}^n)$ into the complex numbers is of this form. If $\alpha \in \mathbf{Z}^n$, then let δ_α be the function on $\mathbf{Z}^n$ defined by $\delta_\alpha(\beta) = 1$ when $\alpha = \beta$ and $\delta_\alpha(\beta) = 0$ otherwise. Thus $\delta_\alpha \in \ell^1_A(\mathbf{Z}^n)$ when $\alpha_j \ge 0$ for each j. In particular, $\delta_0 \in \ell^1_A(\mathbf{Z}^n)$, which is the multiplicative identity element for $\ell^1(\mathbf{Z}^n)$, and hence for $\ell^1_A(\mathbf{Z}^n)$. It follows that $h(\delta_0) = 1$, and we also have that

$$|h(a)| \le \|a\|_1 \tag{79.8}$$

for every $a \in \ell^1_A(\mathbf{Z}^n)$, since $\ell^1_A(\mathbf{Z}^n)$ is a Banach algebra. Let $\alpha(l)$ be the element of $\mathbf{Z}^n$ with lth component equal to 1 and other components equal to 0, for $l = 1, \ldots, n$. Put $z_l = h(\delta_{\alpha(l)})$, so that $|z_l| \le 1$, since $\|\delta_{\alpha(l)}\|_1 = 1$. Thus $z = (z_1, \ldots, z_n) \in \overline{U}^n$, and $h(\delta_\alpha) = z^\alpha$ for every $\alpha \in \mathbf{Z}^n$ with $\alpha_j \ge 0$ for each

j, because h is a homomorphism with respect to convolution on $\ell^1_A(\mathbf{Z}^n)$. More precisely, this uses the fact that

$$\delta_\alpha * \delta_\beta = \delta_{\alpha+\beta} \tag{79.9}$$

for every $\alpha, \beta \in \mathbf{Z}^n$. This implies that $h(a)$ is equal to (79.6) for every a in $\ell^1_A(\mathbf{Z}^n)$ and this choice of z, by the linearity and continuity of h.

If h were a homomorphism on all of $\ell^1(\mathbf{Z}^n)$, then we would have (79.8) for every $a \in \ell^1(\mathbf{Z}^n)$, which would imply that $|z_l| = 1$ for each l. This is because $\delta_{\alpha(l)} * \delta_{-\alpha(l)} = \delta_0$, so that

$$z_l \, h(\delta_{-\alpha(l)}) = h(\delta_{\alpha(l)}) \, h(\delta_{-\alpha(l)}) = 1, \tag{79.10}$$

while $|h(\delta_{-\alpha(l)})| \le 1$ by (79.8). In this case, we would get that $h(a)$ is equal to $\widehat{a}(z)$ as in (78.20) for every $a \in \ell^1(\mathbf{Z}^n)$, by essentially the same argument as before. Of course, $a \mapsto \widehat{a}(z)$ defines a homomorphism on $\ell^1(\mathbf{Z}^n)$ for every $z \in \mathbf{T}^n$, as in the previous section.

Similarly, if h is a nonzero homomorphism on all of $C(\mathbf{T}^n)$ and $f_j(z) = z_j$, then $|h(f_j)| = 1$ for each j, because z_j^{-1} is also a continuous function on $\mathbf{T}^n$ with supremum norm equal to 1. Using this, one can show that $h(f) = f(p)$ for every $f \in C(\mathbf{T}^n)$, where $p = (p_1, \ldots, p_n) \in \mathbf{T}^n$ is defined by $p_j = h(f_j)$, in the same way as before. Although this is a special case of the results discussed in Section 34, the present approach has the advantage of making the relationship with $A(\mathbf{T}^n)$ more clear.

80 The maximum principle

Let D be a nonempty bounded connected open set in the complex plane $\mathbf{C}$. If f is a continuous complex-valued function on the closure $\overline{D}$ of D in $\mathbf{C}$, then the extreme value theorem implies that the $|f(z)|$ attains its maximum on $\overline{D}$. If f is also holomorphic on D, then the *maximum modulus principle* implies that the maximum of $|f(z)|$ on $\overline{D}$ is attained on the boundary ∂D of D. More precisely, if $|f(z)|$ has a local maximum on D, then f is constant. This follows from the fact that a nonconstant holomorphic function on a connected open set in $\mathbf{C}$ is an open mapping, in the sense that it maps open sets to open sets.

Alternatively, suppose that $z \in D$, and that the closed disk centered at z with radius $r > 0$ is contained in D. If f is holomorphic on D, then

$$f(z) = \frac{1}{2\pi r} \int_{|w-z|=r} f(w) \, |dw|, \tag{80.1}$$

by the Cauchy integral formula. If $|f|$ has a local maximum at z, then one can use this to show that $f(w) = f(z)$ when $|w - z|$ is sufficiently small, and hence that f is constant on D when D is connected. This identity is known as the "mean value property", since it says that the value of f at z is given by the average of f on the circle $|w - z| = r$. This also works for harmonic functions on D, and the analogous statement for harmonic functions on open subsets of

$\mathbf{R}^n$ holds for every n. In particular, if f is a harmonic function on a connected open set $D \subseteq \mathbf{R}^n$, and if $|f|$ has a local maximum on D, then one can show that f is constant on D. Similarly, if f is a real-valued harmonic function on D with a local maximum on D, then f is constant on D.

A holomorphic function of several complex variables is holomorphic in each variable separately. In particular, such a function is harmonic, but one can get stronger versions of the maximum principle by considering restrictions of the function to complex lines, or even "analytic disks" that do not have to be flat.

Suppose for instance that D is the unit polydisk U^n. If f is a continuous complex-valued function on $\overline{U}^n$ that is holomorphic on U^n, then one can show that the maximum of $|f(z)|$ on $\overline{U}^n$ is actually attained on $\mathbf{T}^n$. This is the same as the boundary of U^n when $n = 1$, but otherwise is significantly smaller, as mentioned previously. This version of the maximum principle was implicitly given already in (78.12), using Poisson integrals. This also works for functions that are polyharmonic instead of holomorphic, which is to say harmonic in z_j for $j = 1, \ldots, n$. This can also be derived from the maximum principle for the unit disk, by looking at restrictions of the function to disks in which all but one variable is constant.

81 Convex hulls

Let A be a nonempty subset of $\mathbf{R}^n$ for some positive integer n. The *convex hull* of A is denoted $\mathrm{Con}(A)$ and is defined to be the set of $x \in \mathbf{R}^n$ for which there are finitely many elements $y_1, \ldots, y_l$ of A and nonnegative real numbers $t_1, \ldots, t_l$ such that $\sum_{j=1}^{l} t_j = 1$ and

$$x = \sum_{j=1}^{l} t_j \, y_j. \tag{81.1}$$

It is easy to see that $\mathrm{Con}(A)$ is a convex set in $\mathbf{R}^n$, and that $\mathrm{Con}(A) \subseteq B$ whenever $A \subseteq B$ and $B \subseteq \mathbf{R}^n$ is convex. Thus $\mathrm{Con}(A)$ is the smallest convex set in $\mathbf{R}^n$ that contains A.

It is well known that every element of $\mathrm{Con}(A)$ can be expressed as a convex combination of less than or equal to $n + 1$ elements of A. This uses the fact that $\mathbf{R}^n$ is an n-dimensional real vector space, while the definition of the convex hull and the other remarks in the previous paragraph would work just as well in any real vector space. Using this, one can show that $\mathrm{Con}(A)$ is compact when $A \subseteq \mathbf{R}^n$ is compact. Otherwise, the *closed convex hull* of A is defined to be the closure of the convex hull of A, and is automatically convex, because the closure of any convex set in $\mathbf{R}^n$ is also convex. This is the smallest closed convex set that contains A, because any closed convex set that contains A also contains $\mathrm{Con}(A)$ and hence $\overline{\mathrm{Con}(A)}$.

If E is a nonempty closed convex set in $\mathbf{R}^n$ and $x \in \mathbf{R}^n \backslash E$, then a well-known separation theorem states that there is a linear function $\lambda(y)$ on $\mathbf{R}^n$

such that

$$\sup_{y \in E} \lambda(y) < \lambda(x). \tag{81.2}$$

To see this, observe first that there is an element u of E that minimizes the distance to x with respect to the standard Euclidean metric, so that

$$\sum_{j=1}^{n} (y_j - x_j)^2 \geq \sum_{j=1}^{n} (u_j - x_j)^2 \tag{81.3}$$

for every $y \in E$. This follows immediately from the extreme value theorem when E is compact, and otherwise one can reduce to that case by considering the intersection of E with a closed ball centered at x with sufficiently large radius. Without loss of generality, we may suppose that $u = 0$, since otherwise we can translate everything by $-u$ to reduce to this case. Thus the previous inequality becomes

$$\sum_{j=1}^{n} (y_j - x_j)^2 \geq \sum_{j=1}^{n} x_j^2, \tag{81.4}$$

which holds for every $y \in E$. Equivalently,

$$\sum_{j=1}^{n} y_j^2 \geq \sum_{j=1}^{n} y_j\, x_j \tag{81.5}$$

for every $y \in E$. Because $u = 0 \in E$ and E is convex, $t\, y \in E$ for every $y \in E$ and $t \in [0, 1]$. Hence

$$\sum_{j=1}^{n} (t\, y_j)^2 \geq \sum_{j=1}^{n} (t\, y_j)\, x_j \tag{81.6}$$

for every $y \in E$ and $0 \leq t \leq 1$. This implies that

$$t \sum_{j=1}^{n} y_j^2 \geq \sum_{j=1}^{n} y_j\, x_j \tag{81.7}$$

when $y \in E$ and $0 < t \leq 1$. Taking the limit as $t \to 0$, we get that

$$\sum_{j=1}^{n} y_j\, x_j \leq 0 \tag{81.8}$$

for every $y \in E$. Put $\lambda(y) = \sum_{j=1}^{n} y_j\, x_j$, so that $\lambda(y) \leq 0$ for every $y \in E$, by the preceding inequality. Note that $x \neq u = 0$, because $x \notin E$ and $u \in E$. Thus we also have that $\lambda(x) = \sum_{j=1}^{n} x_j^2 > 0$, as desired.

Let A be a nonempty set in $\mathbf{R}^n$, and let λ be a linear function on $\mathbf{R}^n$. Suppose that $x \in \mathrm{Con}(A)$, so that there are $y_1, \ldots y_l \in A$ and $t_1, \ldots, t_l \geq 0$ such that $\sum_{j=1}^{j} t_j = 1$ and $x = \sum_{j=1}^{l} t_j\, y_j$. In particular,

$$\lambda(x) = \sum_{j=1}^{l} t_j\, \lambda(y_j) \leq \max_{1 \leq j \leq l} \lambda(y_j). \tag{81.9}$$

This implies that

$$\lambda(x) \le \sup_{y \in A} \lambda(y), \tag{81.10}$$

where the supremum on the right side may be $+\infty$, in which case the inequality is trivial. If $x \in \overline{\mathrm{Con}(A)}$, then it is easy to see that (81.10) also holds, by continuity. However, if $x \in \mathbf{R}^n \backslash \overline{\mathrm{Con}(A)}$, then there is a linear function λ on $\mathbf{R}^n$ for which (81.10) does not hold, as in the previous paragraph. Thus the closed convex hull of A is the same as the set of $x \in \mathbf{R}^n$ such that (81.10) holds for every linear function λ on $\mathbf{R}^n$.

82 Polynomial hulls

Let E be a nonempty subset of $\mathbf{C}^n$ for some positive integer n. The *polynomial hull* of E in $\mathbf{C}^n$ is denoted $\mathrm{Pol}(E)$ and defined to be the set of $z \in \mathbf{C}^n$ such that

$$|p(z)| \le \sup_{w \in E} |p(w)| \tag{82.1}$$

for every polynomial p on $\mathbf{C}^n$. More precisely, to say that p is a polynomial on $\mathbf{C}^n$ means that p can be expressed as

$$p(w) = \sum_{|\alpha| \le N} a_\alpha \, w^\alpha \tag{82.2}$$

for some nonnegative integer N, where the sum is taken over all multi-indices α with $|\alpha| \le N$, and $a_\alpha \in \mathbf{C}$ for each α. If E is unbounded, then p may be unbounded on E, so that the supremum in (82.1) is $+\infty$, and the inequality is trivial.

Of course,

$$E \subseteq \mathrm{Pol}(E) \tag{82.3}$$

by definition. If $E_1 \subseteq E_2 \subseteq \mathbf{C}^n$, then

$$\mathrm{Pol}(E_1) \subseteq \mathrm{Pol}(E_2). \tag{82.4}$$

It is easy to see that $\mathrm{Pol}(E)$ is always a closed set in $\mathbf{C}^n$, because polynomials are continuous. Similarly,

$$\mathrm{Pol}(E) = \mathrm{Pol}(\overline{E}), \tag{82.5}$$

and so we may as well restrict our attention to closed sets $E \subseteq \mathbf{C}^n$.

If E is any nonempty subset of $\mathbf{C}^n$ and p is a polynomial on $\mathbf{C}^n$, then

$$\sup_{z \in \mathrm{Pol}(E)} |p(z)| = \sup_{w \in E} |p(w)|. \tag{82.6}$$

More precisely, the right side is less than or equal to the left side because $E \subseteq \mathrm{Pol}(E)$, while the opposite inequality follows from the definition of $\mathrm{Pol}(E)$. If $\zeta \in \mathrm{Pol}(\mathrm{Pol}(E))$, then we get that

$$|p(\zeta)| \le \sup_{z \in \mathrm{Pol}(E)} |p(z)| = \sup_{w \in E} |p(w)| \tag{82.7}$$

for every polynomial p on $\mathbf{C}^n$, which implies that $\zeta \in \mathrm{Pol}(E)$. Thus $\mathrm{Pol}(\mathrm{Pol}(E))$ is contained in $\mathrm{Pol}(E)$, and hence

$$\mathrm{Pol}(\mathrm{Pol}(E)) = \mathrm{Pol}(E), \tag{82.8}$$

because $\mathrm{Pol}(E) \subseteq \mathrm{Pol}(\mathrm{Pol}(E))$ automatically.

As an example, let us check that

$$\mathrm{Pol}(\mathbf{T}^n) = \overline{U}^n. \tag{82.9}$$

If $z \in \overline{U}^n$, then $z \in \mathrm{Pol}(\mathbf{T}^n)$, as in Section 80, and so $\overline{U}^n \subseteq \mathrm{Pol}(\mathbf{T}^n)$. However, if $z \in \mathbf{C}^n \backslash \overline{U}^n$, then $|z_j| > 1$, and one can check that $z \not\in \mathrm{Pol}(\mathbf{T}^n)$, by taking $p(w) = w_j$. Thus $\mathrm{Pol}(\mathbf{T}^n) \subseteq \overline{U}^n$, as desired.

If E is any nonempty bounded subset of $\mathbf{C}^n$, then

$$\mathrm{Pol}(E) \subseteq \overline{\mathrm{Con}(E)}. \tag{82.10}$$

To see this, we identify $\mathbf{C}^n$ with $\mathbf{R}^{2n}$ as a real vector space, so that the results in the previous section are applicable. If $z \in \mathbf{C}^n \backslash \overline{\mathrm{Con}(E)}$, then there is a real-valued real-linear function λ on $\mathbf{C}^n \cong \mathbf{R}^{2n}$ such that

$$\sup_{w \in E} \lambda(w) < \lambda(z), \tag{82.11}$$

as in the previous section. Equivalently, λ can be expressed as the real part of a complex-linear function μ on $\mathbf{C}^n$, and

$$\sup_{w \in E} \mathrm{Re}\, \mu(w) < \mathrm{Re}\, \mu(z). \tag{82.12}$$

We would like to show that

$$\sup_{w \in E} |1 + t\, \mu(w)| < |1 + t\, \mu(z)| \tag{82.13}$$

when t is a sufficiently small positive real number, so that $z \not\in \mathrm{Pol}(E)$.

Note that

$$\begin{aligned} |1 + t\, \mu(w)|^2 &= (1 + t\ \mathrm{Re}\, \mu(w))^2 + t^2\, (\mathrm{Im}\, \mu(w))^2 \\ &= 1 + 2\, t\ \mathrm{Re}\, \mu(w) + t^2\, (\mathrm{Re}\, \mu(w))^2 + t^2\, (\mathrm{Im}\, \mu(w))^2 \end{aligned} \tag{82.14}$$

for every $t \in \mathbf{R}$ and $w \in \mathbf{C}^n$. Because E is bounded,

$$|\mu(w)|^2 = (\mathrm{Re}\, \mu(w))^2 + (\mathrm{Im}\, \mu(w))^2 \le C \tag{82.15}$$

for some $C \ge 0$ and every $w \in E$. Thus

$$\sup_{w \in E} |1 + t\, \mu(w)|^2 \le 1 + 2\, t\, \sup_{w \in E} \mathrm{Re}\, \mu(w) + C\, t^2 \tag{82.16}$$

for every $t > 0$. Using this and (82.12), it is easy to see that

$$\sup_{w \in E} |1 + t\, \mu(w)|^2 < |1 + t\, \mu(z)|^2 \tag{82.17}$$

when $t > 0$ is sufficiently small, as desired.

If $n = 1$ and $E \subseteq \mathbf{C}$ is unbounded, then every nonconstant polynomial p on $\mathbf{C}$ is unbounded on E. Thus $\mathrm{Pol}(E) = \mathbf{C}$ in this case. In particular, $\mathrm{Pol}(E)$ may not be contained in $\overline{\mathrm{Con}(E)}$ when E is unbounded.

Suppose that E is a nonempty set in $\mathbf{C}^n$ with only finitely many elements. If $z \in \mathbf{C}^n \backslash E$, then it is easy to see that there is a polynomial p on $\mathbf{C}^n$ such that $p(w) = 0$ for each $w \in E$ and $p(z) \neq 0$, by taking a product of affine functions that vanish at the elements of E, one at a time, and are nonzero at z. This implies that $z \notin \mathrm{Pol}(E)$, so that $\mathrm{Pol}(E) \subseteq E$. Hence $\mathrm{Pol}(E) = E$ when E has only finitely many elements, since $E \subseteq \mathrm{Pol}(E)$ automatically. By contrast, the convex hull of a finite set may be much larger.

83 Algebras and homomorphisms

Let E be a nonempty compact set in $\mathbf{C}^n$, and let $C(E)$ be the algebra of continuous complex-valued functions on E. Let $PC(E)$ be the subalgebra of $C(E)$ consisting of the restrictions to E of polynomials on $\mathbf{C}^n$, and let $AC(E)$ be the closure of $PC(E)$ in $C(E)$ with respect to the supremum norm. Thus $AC(E)$ is a closed subalgebra of $C(E)$, and hence a commutative Banach algebra with respect to the supremum norm, since $C(E)$ is. Of course, the constant function equal to 1 on E is the multiplicative identity element in $C(E)$, and is contained in $PC(E) \subseteq AC(E)$.

Suppose that h is a nonzero homomorphism from $AC(E)$ into the complex numbers. Let f_j be the function on E defined by $f_j(w) = w_j$ for $j = 1, \ldots, n$, so that $f_j \in PC(E) \subseteq AC(E)$ for each j. Put

$$z_j = h(f_j) \tag{83.1}$$

for each j, and consider $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$. If p is any polynomial on $\mathbf{C}^n$, and $\widetilde{p}$ is the restriction of p to E, then $\widetilde{p} \in PC(E) \subseteq AC(E)$, and

$$h(\widetilde{p}) = p(z). \tag{83.2}$$

As in Section 49,

$$|h(f)| \leq \sup_{w \in E} |f(w)| \tag{83.3}$$

for every $f \in AC(E)$. It follows that

$$|p(z)| = |h(\widetilde{p})| \leq \sup_{w \in E} |\widetilde{p}(w)| = \sup_{w \in E} |p(w)| \tag{83.4}$$

for every polynomial p on $\mathbf{C}^n$. Thus $z \in \mathrm{Pol}(E)$.

Conversely, suppose that $z \in \mathrm{Pol}(E)$, so that

$$|p(z)| \leq \sup_{w \in E} |p(w)| = \sup_{w \in E} |\widetilde{p}(w)| \tag{83.5}$$

for every polynomial p on $\mathbf{C}^n$. In particular, if $p(w) = 0$ for every $w \in E$, then $p(z) = 0$. This implies that

$$h_z(\widetilde{p}) = p(z) \tag{83.6}$$

is well-defined on $PC(E)$, and in fact it is a homomorphism from $PC(E)$ into the complex numbers. Moreover, (83.5) implies that h_z is a continuous linear functional on $PC(E)$ with respect to the supremum norm, so that h_z has a unique extension to a continuous linear functional on $AC(E)$. It is easy to see that this extension is also a homomorphism with respect to multiplication.

The argument in the preceding paragraph would work just as well if

$$|p(z)| \le C \sup_{w \in E} |p(w)| \tag{83.7}$$

for some $C \ge 0$ and every polynomial p on $\mathbf{C}^n$. Note that p^l is also a polynomial on $\mathbf{C}^n$ for every polynomial p and positive integer l. Applying the previous condition to p^l, we get that

$$|p(z)|^l \le C \sup_{w \in E} |p(w)|^l. \tag{83.8}$$

Equivalently,

$$|p(z)| \le C^{1/l} \sup_{w \in E} |p(w)| \tag{83.9}$$

for each $l \ge 1$ and polynomial p on $\mathbf{C}^n$. Taking the limit as $l \to \infty$, it follows that the initial inequality holds with $C = 1$. Hence this apprently weaker condition implies that $z \in \mathrm{Pol}(E)$. This could also be derived from the earlier discussion, but this approach is more direct.

84 The exponential function

Put

$$E(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!} \tag{84.1}$$

for each $z \in \mathbf{C}$, where $j!$ is "j factorial", the product of $1, \ldots, j$. As usual, this is interpreted as being equal to 1 when $j = 0$. It is easy to see that this series converges absolutely for every $z \in \mathbf{C}$, by the ratio test, for instance.

If $z, w \in \mathbf{C}$, then

$$E(z)\,E(w) = \Big(\sum_{j=0}^{\infty} \frac{z^j}{j!}\Big)\Big(\sum_{l=0}^{\infty} \frac{w^l}{l!}\Big) = \sum_{n=0}^{\infty}\Big(\sum_{j=0}^{n} \frac{z^j\,w^{n-j}}{j!\,(n-j)!}\Big), \tag{84.2}$$

as in Section 63. This uses the absolute convergence of the series defining $E(z)$ and $E(w)$. The binomial theorem states that

$$\sum_{j=0}^{n} \frac{n!}{j!\,(n-j)!}\, z^j\, w^{n-j} = (z+w)^n, \tag{84.3}$$

so that

$$E(z)\,E(w) = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = E(z+w). \tag{84.4}$$

In particular,

$$E(z)\,E(-z) = E(0) = 1 \tag{84.5}$$

for every $z \in \mathbf{C}$. Equivalently, $E(z) \neq 0$ for every $z \in \mathbf{C}$, and $1/E(z) = E(-z)$. If x is a nonnegative real number, then it is clear from the definition of $E(x)$ that $E(x) \in \mathbf{R}$ and $E(x) \geq 1$. It follows that $E(x)$ is a positive real number for every $x \in \mathbf{R}$, and that $E(x) \leq 1$ when $x \leq 0$. Similarly, it is easy to see from the definition that $E(x)$ is strictly increasing when $x \geq 0$, and one can extend this to the whole real line using the fact that $E(-x) = 1/E(x)$.

Observe that

$$\overline{E(z)} = E(\overline{z}) \tag{84.6}$$

for every $z \in \mathbf{C}$, by the definition of $E(z)$. This implies that

$$|E(z)|^2 = E(z)\,\overline{E(z)} = E(z)\,E(\overline{z}) = E(z+\overline{z}) = E(2\,\mathrm{Re}\,z), \tag{84.7}$$

and hence

$$|E(z)| = E(\mathrm{Re}\,z) \tag{84.8}$$

for every $z \in \mathbf{C}$.

If $z = i\,y$ for some $y \in \mathbf{R}$, then (84.8) implies that

$$|E(i\,y)| = 1. \tag{84.9}$$

It is well known that

$$E(i\,y) = \cos y + i \sin y \tag{84.10}$$

for every $y \in \mathbf{R}$. One way to see this is to use the standard power series expansions for the sine and cosine. Alternatively,

$$\frac{d}{dy} E(i\,y) = i\,E(i\,y), \tag{84.11}$$

as one can check using the series expansion for $E(i\,y)$. We already know that $E(i\,y)$ maps the real line into the unit circle $\mathbf{T}$ and sends $y = 0$ to 1. This formula for the derivative of $E(i\,y)$ shows that it goes around the circle at unit speed in the positive orientation. It follows that the real and imaginary parts of $E(i\,y)$ are given by the cosine and sine, respectively, by the geometric definitions of the cosine and sine.

85 Entire functions

Suppose that

$$\sum_{\alpha} a_\alpha\, z^\alpha \tag{85.1}$$

is a power series with complex coefficients that is absolutely summable for every $z \in \mathbf{C}^n$, and let $f(z)$ be the sum of this series. If E is a nonempty bounded set in $\mathbf{C}^n$, then

$$|f(z)| \le \sup_{w \in E} |f(w)| \tag{85.2}$$

for every $z \in \mathrm{Pol}(E)$. This follows from the definition of the polynomial hull, and the fact that f can be approximated uniformly by polynomials corresponding to finite subsums of (85.1) on bounded subsets of $\mathbf{C}^n$.

Let $u = (u_1, \ldots, u_n) \in \mathbf{C}^n$ be given, and put

$$\mu(z) = \sum_{j=1}^n u_j\, z_j. \tag{85.3}$$

Observe that $f_\mu(z) = E(\mu(z))$ can be expressed as

$$\prod_{j=1}^n E(u_j\, z_j) = \sum_\alpha \frac{u^\alpha}{\alpha!}\, z^\alpha, \tag{85.4}$$

where $\alpha! = \alpha_1! \cdots \alpha_n!$. In particular, this power series is absolutely summable for every $z \in \mathbf{C}^n$. Moreover,

$$|f_\mu(z)| = E(\mathrm{Re}\,\mu(z)), \tag{85.5}$$

as in the previous section.

If E is a nonempty subset of $\mathbf{C}^n$ and $z \in \mathbf{C}^n \backslash \overline{\mathrm{Con}(E)}$, then there is a complex-linear function μ on $\mathbf{C}^n$ such that

$$\sup_{w \in E} \mathrm{Re}\,\mu(w) < \mathrm{Re}\,\mu(z), \tag{85.6}$$

as in Section 82. Hence

$$\sup_{w \in E} |f_\mu(w)| < |f_\mu(z)|. \tag{85.7}$$

This has the advantage of working for both bounded and unbounded sets E, in exchange for allowing a larger class of functions than polynomials, as before.

86 The three lines theorem

Let D be the open unit strip in the complex plane,

$$D = \{z \in \mathbf{C} : 0 < \mathrm{Re}\, z < 1\}, \tag{86.1}$$

so that the closure of D is the closed unit strip,

$$\overline{D} = \{z \in \mathbf{C} : 0 \le \mathrm{Re}\, z \le 1\}. \tag{86.2}$$

Also let f be a continuous complex-valued function on $\overline{D}$ which is holomorphic on D, and suppose that A_0, A_1 are positive real numbers such that

$$|f(x + i\, y)| \le A_x \tag{86.3}$$

for $x = 0, 1$ and every $y \in \mathbf{R}$. If f is also bounded on D, then the *three lines theorem* states that

$$|f(x + i\, y)| \le A_0^{1-x}\, A_1^x \tag{86.4}$$

when $0 < x < 1$ and $y \in \mathbf{R}$.

To do this, we would first like to show that f satisfies the maximum principle, so that

$$|f(x + i\, y)| \le \max(A_0, A_1) \tag{86.5}$$

when $0 < x < 1$ and $y \in \mathbf{R}$. However, D is not bounded, $\overline{D}$ is not compact, and so we cannot use the ordinary maximum principle in quite the usual way. Let us begin with the case where f satisfies the additional condition that $f(z) \to 0$ uniformly on $\overline{D}$ as $|\operatorname{Im} z| \to \infty$. Put

$$D_R = \{z \in D : |\operatorname{Im} z| < R\} \tag{86.6}$$

for each $R > 0$. Thus D_R is bounded, and we can apply the maximum principle to f on D_R. If

$$B_R = \sup\{|f(x + i\, y)| : 0 \le x \le 1, y = \pm R\}, \tag{86.7}$$

then we get that

$$|f(x + i\, y)| \le \max(A_0, A_1, B_R) \tag{86.8}$$

when $0 < x < 1$ and $|y| < R$. By hypothesis, $B_R \to 0$ as $R \to \infty$, and so (86.5) follows easily in this case.

If f is bounded but does not necessarily tend to 0 at infinity, then we can approximate it by functions that do. Consider

$$f_\epsilon(z) = f(z)\, E(\epsilon\, z^2) \tag{86.9}$$

for each $\epsilon > 0$. Observe that

$$|f_\epsilon(z)| = |f(z)|\, |E(\epsilon\, z^2)| = |f(z)|\, E(\epsilon\, (x^2 - y^2)), \tag{86.10}$$

where $z = x + i\, y$, and hence $\operatorname{Re} z^2 = x^2 - y^2$. Thus $f_\epsilon(z)$ is continuous on $\overline{D}$, holomorphic on D, and tends to 0 uniformly as $|y| \to \infty$ for each $\epsilon > 0$, because f is bounded on $\overline{D}$ by hypothesis and $E(\epsilon\, y^2) \to +\infty$ as $y \to +\infty$ for each $\epsilon > 0$. We also have that

$$|f_\epsilon(i\, y)| \le A_0, \quad |f_\epsilon(i\, y)| \le A_1\, E(\epsilon) \tag{86.11}$$

for each $y \in \mathbf{R}$, since $E(-\epsilon\, y^2) \le 1$ for every $y \in \mathbf{R}$. This permits us to use the version of the maximum principle in the previous paragraph, to get that

$$|f_\epsilon(x + i\, y)| \le \max(A_0, A_1\, E(\epsilon)) \tag{86.12}$$

when $0 < x < 1$ and $y \in \mathbf{R}$. Of course, $f_\epsilon(z) \to f(z)$ for each $z \in \overline{D}$ as $\epsilon \to 0$, and it follows that f satisfies (86.5), by taking the limit as $\epsilon \to 0$ in (86.12).

Note that $E(t) \ge 1 + t$ for every nonnegative real number t, by the definition of $E(t)$. Thus $E(t) \to +\infty$ as $t \to +\infty$, as in the previous paragraph, and hence $E(-t) = 1/E(t) \to 0$ as $t \to +\infty$. If A is a positive real number, then it follows that there is a unique real number $\log A$ such that $E(\log A) = A$, because $E(t)$ is a strictly increasing continuous function on the real line. Put $A^z = E(z \log A)$, and observe that $|A^z| = A^{\operatorname{Re} z}$, by the properties of the exponential function. In order to get (86.4), consider

$$g(z) = f(z)\, A_0^{z-1}\, A_1^{-z}. \tag{86.13}$$

This is a bounded continuous function on $\overline{D}$ which is holomorphic on D and satisfies

$$|g(x + i\, y)| \le 1 \tag{86.14}$$

when $x = 0, 1$ and $y \in \mathbf{R}$, by the corresponding properties of f. The analogue of (86.5) for g implies that (86.14) holds for every $0 < x < 1$ and $y \in \mathbf{R}$, which is the same as (86.4).

87 Completely circular sets

A set $E \subseteq \mathbf{C}^n$ is said to be *completely circular* if

$$(u_1\, z_1, \ldots, u_n\, z_n) \in E \tag{87.1}$$

for every $z = (z_1, \ldots, z_n) \in E$ and $u = (u_1, \ldots, u_n) \in \mathbf{C}^n$ such that $|u_j| \le 1$ for each j. Equivalently, $w \in E$ whenever $w \in \mathbf{C}^n$ satisfies $|w_j| \le |z_j|$ for some $z \in E$ and each j. In particular, $0 \in E$ when $E \ne \emptyset$.

Suppose that $w, z \in E$, $0 < t < 1$, and $v \in \mathbf{C}^n$ satisfy

$$|v_j| \le |z_j|^t\, |w_j|^{1-t} \tag{87.2}$$

for each j. We would like to show that $v \in \operatorname{Pol}(E)$ when E is completely circular. Thus we would like to show that

$$|p(v)| \le \sup_{\zeta \in E} |p(\zeta)| \tag{87.3}$$

for every polynomial p on $\mathbf{C}^n$. To do this, we shall use the version of the maximum principle discussed in the previous section.

By hypothesis, we can express v as

$$v_j = u_j\, |z_j|^t\, |w_j|^{1-t}, \tag{87.4}$$

where $|u_j| \le 1$ for each j. Put

$$g_j(\tau) = u_j\, |z_j|^\tau\, |w_j|^{1-\tau} \tag{87.5}$$

for each $\tau \in \mathbf{C}$ and $1 \le j \le n$. This uses the definition of A^τ for any positive real number A and complex number τ as $E(\tau \log A)$, as in the previous section, and we put $A^\tau = 0$ for every $\tau \in \mathbf{C}$ when $A = 0$. Note that

$$|g_j(\tau)| \le |z_j|^{\operatorname{Re}\tau}\, |w_j|^{1-\operatorname{Re}\tau} \tag{87.6}$$

for each τ and j, and hence

$$g(\tau) = (g_1(\tau), \ldots, g_n(\tau)) \in E \tag{87.7}$$

when $\operatorname{Re}\tau = 0, 1$. We also have that $g(t) = v$, by construction.

Let p be a polynomial on $\mathbf{C}^n$, and consider

$$f(\tau) = p(g(\tau)). \tag{87.8}$$

This is a holomorphic function on the complex plane $\mathbf{C}$, and in particular it is a holomorphic function on the open unit strip D that extends continuously to the closure $\overline{D}$. Moreover, f is bounded on $\overline{D}$, because g is bounded on $\overline{D}$, and p is bounded on bounded subsets of $\mathbf{C}^n$. It follows that

$$|f(t)| \le \sup\{|f(\tau)| : \tau \in \mathbf{C},\ \operatorname{Re}\tau = 0, 1\}, \tag{87.9}$$

as in the previous section. This is the same as saying that

$$|p(v)| \le \sup\{|p(g(\tau))| : \tau \in \mathbf{C},\ \operatorname{Re}\tau = 0, 1\}, \tag{87.10}$$

which is exactly what we wanted, since $g(\tau) \in E$ when $\operatorname{Re}\tau = 0, 1$.

88 Completely circular sets, continued

Let E be a nonempty bounded completely circular set in $\mathbf{C}^n$, and let $z \in \mathbf{C}^n$ be given. Suppose that $z \ne 0$, and let I be the set of $j = 1, \ldots, n$ such that $z_j \ne 0$. Let $\alpha(I) = (\alpha_1(I), \ldots, \alpha_n(I))$ be the multi-index defined by $\alpha_j(I) = 1$ when $j \in I$, and $\alpha_j(I) = 0$ otherwise. Thus $z^{\alpha(I)} \ne 0$, and if $w^{\alpha(I)} = 0$ for every $w \in E$, then $z \notin \operatorname{Pol}(E)$.

Let E_I be the set of $w \in E$ such that $w_j \ne 0$ when $j \in I$, and suppose from now on in this section that $E_I \ne \emptyset$. Also let $\mathbf{R}^I$ be the set of real-valued functions on I, which is basically the same as $\mathbf{R}^l$, where l is the number of elements of I. Thus $\log|w_j|$, $j \in I$, determines an element of $\mathbf{R}^I$ for each $w \in E_I$, and we let A_I be the subset of $\mathbf{R}^I$ corresponding to elements of E_I in this way. If $r \in \mathbf{R}^I$, $t \in A_I$, and

$$r_j \le t_j \tag{88.1}$$

for each $j \in I$, then $r \in A_I$, because E is completely circular.

Let ζ be the element of $\mathbf{R}^I$ given by $\zeta_j = \log|z_j|$ for $j \in I$, and suppose that ζ is not an element of the closure of the convex hull of A_I in $\mathbf{R}^I$. This implies that there is a linear function λ on $\mathbf{R}^I$ such that

$$\sup_{r \in A_I} \lambda(r) < \lambda(\zeta). \tag{88.2}$$

More precisely, $\lambda(r)$ can be given as

$$\lambda(r) = \sum_{j \in I} \lambda_j \, r_j \tag{88.3}$$

for some $\lambda_j \in \mathbf{R}$, $j \in I$, and every $r \in \mathbf{R}^I$. If $\lambda_j < 0$ for some $j \in I$, then there are $r \in A_I$ for which $\lambda(r)$ is arbitrarily large, because A_I is nonempty and satisfies the condition mentioned at the end of the previous paragraph. Thus

$$\lambda_j \geq 0 \tag{88.4}$$

for each $j \in I$, since $\lambda(r)$ is bounded from above for $r \in A_I$.

If $z \in \mathrm{Pol}(E)$, then

$$|z^\alpha| \leq \sup_{w \in E} |w^\alpha| \tag{88.5}$$

for every multi-index α. Let us restrict our attention to multi-indices α such that $\alpha_j \geq 1$ when $j \in I$ and $\alpha_j = 0$ otherwise, so that $w^\alpha = 0$ when $w \in E \backslash E_I$. In this case, the previous inequality reduces to

$$\sum_{j \in I} \alpha_j \log |z_j| \leq \sup_{w \in E_I} \sum_{j \in I} \alpha_j \, \log |w_j|. \tag{88.6}$$

Equivalently,

$$\sum_{j \in I} \alpha_j \, \zeta_j \leq \sup_{r \in A_I} \sum_{j \in I} \alpha_j \, r_j. \tag{88.7}$$

This inequality holds for arbitrary positive integers α_j, $j \in I$, and hence for arbitrary positive rational numbers α_j, by dividing both sides by a positive integer. It follows that this inequality also holds for arbitrary nonnegative real numbers, by approximation. This uses the hypothesis that E be bounded, so that r_j has an upper bound for each $j \in I$ and $r \in A_I$, and more precisely one should approximate nonnegative real numbers α_j by positive rational numbers α'_j such that $\alpha_j \leq \alpha'_j$ for each j. Combining this with the discussion in the previous paragraph, we get that ζ is in the closure of the convex hull of A_I in $\mathbf{R}^I$ when $z \in \mathrm{Pol}(E)$.

89 The torus action

Let $\mathbf{T}^n$ be the set of $t = (t_1, \ldots, t_n) \in \mathbf{C}^n$ such that $|t_j| = 1$ for each j, as usual. If $t \in \mathbf{T}^n$ and $z \in \mathbf{C}^n$, then put

$$T_t(z) = (t_1 \, z_1, \ldots, t_n \, z_n), \tag{89.1}$$

so that T_t is an invertible linear transformation on $\mathbf{C}^n$ for each $t \in \mathbf{T}^n$. Note that $\mathbf{T}^n$ is a commutative group with respect to coordinatewise multiplication, and that $t \mapsto T_t$ is a homomorphism from $\mathbf{T}^n$ into the group of invertible linear transformations on $\mathbf{C}^n$. Suppose that E is a nonempty subset of $\mathbf{C}^n$ such that

$$T_t(E) \subseteq E \tag{89.2}$$

for every $t \in \mathbf{T}^n$. This implies that

$$T_t(E) = E \tag{89.3}$$

for each $t \in \mathbf{T}^n$, because $T_t^{-1}(E) = T_{t^{-1}}(E) \subseteq E$, where $t^{-1} = (t_1^{-1}. \ldots, t_n^{-1})$.

Suppose that $z \in \mathrm{Pol}(E)$, so that

$$|p(z)| \le \sup_{w \in E} |p(w)| \tag{89.4}$$

for every polynomial p on $\mathbf{C}^n$. If $p_t(w) = p(T_t(w))$, then p_t is also a polynomial on $\mathbf{C}^n$ for each $t \in \mathbf{T}^n$, and hence

$$|p_t(z)| \le \sup_{w \in E} |p_t(w)|. \tag{89.5}$$

We also have that

$$\sup_{w \in E} |p_t(w)| = \sup_{w \in E} |p(w)| \tag{89.6}$$

for every $t \in \mathbf{T}^n$, because of (89.3). Thus

$$|p(T_t(z))| = |p_t(z)| \le \sup_{w \in E} |p_t(w)| = \sup_{w \in E} |p(w)| \tag{89.7}$$

for every polynomial p on $\mathbf{C}^n$ and $t \in \mathbf{T}^n$, which implies that $T_t(z) \in \mathrm{Pol}(E)$ for every $t \in \mathbf{T}^n$. This shows that $T_t(\mathrm{Pol}(E)) \subseteq \mathrm{Pol}(E)$ for every $t \in \mathbf{T}^n$, and hence $T_t(\mathrm{Pol}(E)) = \mathrm{Pol}(E)$ for every $t \in \mathbf{T}^n$, as before.

Let U^n be the open unit polydisk in $\mathbf{C}^n$, consisting of $u \in \mathbf{C}^n$ such that $|u_j| < 1$ for each j. If p is a polynomial on $\mathbf{C}^n$, $u \in U^n$, and $w \in \mathbf{C}^n$, then

$$|p(u_1\, w_1, \ldots, u_n\, w_n)| \le \sup_{t \in \mathbf{T}^n} |p(t_1\, w_1, \ldots, t_n\, w_n)| \tag{89.8}$$

More precisely, we can think of $p(u_1\, w_1, \ldots, u_n\, w_n)$ as a polynomial in u for each $w \in \mathbf{C}^n$, and apply the maximum principle as in Section 80. If $z \in \mathrm{Pol}(E)$ and $u \in U^n$, then we get that

$$|p(u_1\, z_1, \ldots, u_n\, z_n)| \le \sup_{t \in \mathbf{T}^n} |p_t(z)| \le \sup_{w \in E} |p(w)| \tag{89.9}$$

for every polynomial p on $\mathbf{C}^n$, where the second step is as in the previous paragraph. Thus

$$(u_1\, z_1, \ldots, u_n\, z_n) \in \mathrm{Pol}(E) \tag{89.10}$$

for every $z \in \mathrm{Pol}(E)$ and $u \in U^n$, and it follows that $\mathrm{Pol}(E)$ is completely circular in this case.

90 Another condition

Let E be a nonempty completely circular closed set in $\mathbf{C}^n$ such that

$$E^* = \{w \in E : w_j \ne 0 \text{ for each } j\} \tag{90.1}$$

is dense in E. This happens when E is the closure of a nonempty completely circular open set in $\mathbf{C}^n$, for instance. Put

$$A = \{y \in \mathbf{R}^n : y_j = \log |w_j| \text{ for some } w \in E^* \text{ and each } j\}. \tag{90.2}$$

so that

$$E^* = \{w \in \mathbf{C}^n : w_j \neq 0 \text{ for each } j, \text{ and } (\log |w_1|, \ldots, \log |w_n|) \in A\}, \tag{90.3}$$

because E is completely circular. Thus $E = \overline{E^*}$ is uniquely determined by A under these conditions. If $x \in \mathbf{R}^n$, $y \in A$, and

$$x_j \leq y_j \tag{90.4}$$

for each j, then we also have that $x \in A$, since E is completely circular.

Let I be a nonempty subset of $\{1, \ldots, n\}$, and let $\mathbf{R}^I$ be the set of real-valued functions on I, as before. There is a natural projection from $\mathbf{R}^n$ onto $\mathbf{R}^I$, in which one keeps the coordinates corresoponding to $j \in I$ and drops the others. Let E_I be the set of $w \in E$ such that $w_j \neq 0$ when $j \in I$, and let A_I be the subset of $\mathbf{R}^I$ whose elements correspond to $\log |w_j|$, $j \in I$, with $w \in E_I$. Observe that

$$\pi_I(A) \subseteq A_I \subseteq \overline{\pi_I(A)}, \tag{90.5}$$

because $E^* \subseteq E_I$ and E^* is dense in E.

If $z \in \mathrm{Pol}(E)$, then

$$|z^\alpha| \leq \sup_{w \in E} |w^\alpha| \tag{90.6}$$

for every multi-index α. Moreover,

$$\sup_{w \in E^*} |w^\alpha| = \sup_{w \in E} |w^\alpha|, \tag{90.7}$$

since E^* is dense in E. Hence

$$\sup_{w \in E_I} |w^\alpha| = \sup_{w \in E} |w^\alpha| \tag{90.8}$$

for any $I \subseteq \{1, \ldots, n\}$, because $E_I \subseteq E^* \subseteq E$. Of course, (90.8) is trivial when $\alpha_j \geq 1$ for each $j \in I$, so that $w^\alpha = 0$ when $w \in E \backslash E_I$.

Let us consider some examples in $\mathbf{C}^2$ where E^* is not dense in E. If

$$E = (\mathbf{C} \times \{0\}) \cup (\{0\} \times \mathbf{C}), \tag{90.9}$$

then E is closed and completely circular, and $E^* = \emptyset$. Equivalently,

$$E = \{z = (z_1, z_2) \in \mathbf{C}^2 : z_1 \, z_2 = 0\}, \tag{90.10}$$

and it is easy to see that $\mathrm{Pol}(E) = E$ in this case.

Put

$$D(r) = \{\zeta \in \mathbf{C} : |\zeta| \leq r\} \tag{90.11}$$

for each $r > 0$, and consider

$$E = (D(r_1) \times \{0\}) \cup (\{0\} \times D(r_2)) \tag{90.12}$$

for some $r_1, r_2 > 0$. Thus E is closed and completely circular again, $E^* = \emptyset$, and E is also bounded in this case. One can check that $\mathrm{Pol}(E) = E$ as well, using the polynomials $p_1(w) = w_1$, $p_2(w) = w_2$, and $p(w) = w_1 \, w_2$.

If $0 < r < R$ and

$$E(r, R) = (D(r) \times \mathbf{C}) \cup (D(R) \times \{0\}), \tag{90.13}$$

then $E(r, R)$ is closed and completely circular, and

$$\overline{E(r, R)^*} = D(r) \times \mathbf{C}. \tag{90.14}$$

If p is a polynomial on $\mathbf{C}$ that is bounded on $E(r, R)$, then $p(w_1, w_2)$ is bounded as a polynomial in w_2 for each $w_1 \in D(r)$. This implies that $p(w_1, w_2)$ is constant in w_2 for each $w_1 \in D(r)$, and hence for every $w_1 \in \mathbf{C}$. Thus $p(w_1, w_2)$ reduces to a polynomial in w_1, and one can use this to show that the polynomial hull of $E(r, R)$ is equal to $D(R) \times \mathbf{C}$.

Put

$$E(r) = (D(r) \times \mathbf{C}) \cup (\mathbf{C} \times \{0\}) \tag{90.15}$$

for $r > 0$, which is the analogue of $E(r, R)$ with $R = +\infty$. As before, $E(r)$ is closed and completely circular, and

$$\overline{E(r)^*} = D(r) \times \{0\}. \tag{90.16}$$

If p is a polynomial on $\mathbf{C}^2$ that is bounded on $E(r)$, then p is constant, as in the previous paragraph, so that $\mathrm{Pol}(E(r)) = \mathbf{C}^2$.

Of course,

$$E = D(r) \times \mathbf{C} \tag{90.17}$$

is closed and completely circular for each $r > 0$, and satisfies $\overline{E^*} = E$. It is also easy to see that $\mathrm{Pol}(E) = E$ in this case, using the polynomial $p(w) = w_1$.

If $r_1, r_2, R > 0$ and $r_1 < R$, then put

$$E(r_1, r_2, R) = (D(r_1) \times D(r_2)) \cup (D(R) \times \{0\}). \tag{90.18}$$

Thus $E(r_1, r_2, R)$ is closed, bounded, and completely circular, and

$$\overline{E(r_1, r_2, R)^*} = D(r_1) \times D(r_2). \tag{90.19}$$

If $z = (z_1, z_2) \in \mathrm{Pol}(E(r_1, r_2, R))$, then it is easy to see that $|z_1| \le R$ and $|z_2| \le r_2$, using the polynomials $p_1(w) = w_1$ and $p_2(w) = w_2$. If $z_2 \ne 0$, then one can show that $|z_1| \le r_1$, using the polynomials $q_n(w) = w_1^n \, w_2$ for each positive integer n. More precisely,

$$|q_n(z)| \le \sup\{|q_n(w)| : w \in E(r_1, r_2, R)\} \tag{90.20}$$

implies that $|z_1|^n \, |z_2| \le r_1^n \, r_2$ for each n, and hence that

$$|z_1| \, |z_2|^{1/n} \le r_1 \, r_2^{1/n}. \tag{90.21}$$

If $z_2 \neq 0$, then we can take the limit as $n \to \infty$ to get that $|z_1| \le r_1$, as desired. Thus $z \in E(r_1, r_2, R)$, which implies that the polynomial hull of $E(r_1, r_2, R)$ is itself.

If E is a closed bi-disk

$$D(r_1) \times D(r_2) \tag{90.22}$$

for some $r_1, r_2 > 0$, then E is completely circular, $\overline{E^*} = E$, and $\mathrm{Pol}(E) = E$. If E is the union of two closed bi-disks

$$(D(r_1) \times D(r_2)) \cup (D(t_1) \times D(t_2)) \tag{90.23}$$

for some $r_1, r_2, t_1, t_2 > 0$, then E is completely circular and $\overline{E^*} = E$ again. Of course, this reduces to the single bi-disk $D(t_1) \times D(t_2)$ when $r_1 \le t_1$ and $r_2 \le t_2$, and to $D(r_1) \times D(r_2)$ when $t_1 \le r_1$ and $t_2 \le r_2$. Otherwise, $E \neq \mathrm{Pol}(E)$, because E is not multiplicatively convex. Note that all of the other examples mentioned in this section are multiplicatively convex.

91 Multiplicative convexity

Suppose that $E \subseteq \mathbf{C}^n$ has the property that

$$(t_1 \, z_1, \ldots, t_n \, z_n) \in E \tag{91.1}$$

when $z = (z_1, \ldots, z_n) \in E$, $t = (t_1, \ldots, t_n) \in \mathbf{C}^n$, and $|t_j| = 1$ for each j, so that $t \in \mathbf{T}^n$. Let us say that E is *multiplicatively convex* if for each $v, w \in E$ and $a \in (0, 1)$, we have that $u \in E$ whenever $u \in \mathbf{C}^n$ and

$$|u_j| = |v_j|^a \, |w_j|^{1-a} \tag{91.2}$$

for each j. Similarly, if E is completely circular and multiplicatively convex, then $u \in E$ whenever

$$|u_j| \le |v_j|^a \, |w_j|^{1-a} \tag{91.3}$$

for some $v, w \in E$, $0 < a < 1$, and each j. If E is completely circular and convex, then E is multiplicatively convex, because

$$|v_j|^a \, |w_j|^{1-a} \le a \, |v_j| + (1 - a) \, |w_j| \tag{91.4}$$

when $0 < a < 1$, by the convexity of the exponential function. More precisely, if E is invariant under the usual action of $\mathbf{T}^n$, as in (91.1), and E is also nonempty and convex, then it is easy to see that $0 \in E$. This implies that $r \, z \in E$ when $z \in E$ and $0 < r < 1$, and hence that E is completely circular. We have also seen examples of sets that are completely circular and multiplicatively convex, but not convex.

If $E \subseteq \mathbf{C}^n$ is completely circular and $\mathrm{Pol}(E) = E$, then E is multiplicatively convex, as in Section 87. Conversely, if E is closed, bounded, completely circular, and multiplicatively convex, then $\mathrm{Pol}(E) = E$. This basically follows from the discussion in Section 88, with a few extra details. The main point is that the sets A_I considered there are closed and convex in this case. The convexity of the A_I's corresponds exactly to the multiplicative convexity of E. To see that each A_I is closed, one can use the fact that E is closed, and that for each $y \in A_I$ there is a $w = w(y) \in E$ such that $w_j > 0$ and $\log |w_j|$ when $j \in I$, and $w_j = 0$ when $j \notin I$. This also uses the complete circularity of E, and otherwise there is a standard argument based on the compactness of E. Although the boundedness of E is not necessary for this step, it is important for the approximation argument in Section 88.

Note that the polynomial hull of a bounded set $E \subseteq \mathbf{C}^n$ is also bounded. More precisely, if $|w_j| \le r_j$ for some $r_j \ge 0$ and each $w \in E$, then $|z_j| \le r_j$ for each $z \in \mathrm{Pol}(E)$, as one can see by considering the polynomial $p_j(w) = w_j$.

92 Coefficients

Let

$$p(z) = \sum_{|\alpha| \le N} a_\alpha \, z^\alpha \tag{92.1}$$

be a polynomial with complex coefficients on $\mathbf{C}^n$, where the sum is taken over all multi-indices α with $|\alpha| \le N$ for some N. Thus

$$p(t_1 \, z_1, \ldots, t_n \, z_n) = \sum_{|\alpha| \le N} a_\alpha \, t^\alpha \, z^\alpha \tag{92.2}$$

for each $t \in \mathbf{T}^n$, and so

$$a_\beta \, z^\beta = \frac{1}{(2\pi)^n} \int_{\mathbf{T}^n} p(t_1 \, z_1, \ldots, t_n \, z_n) \, t^{-\beta} \, |dt| \tag{92.3}$$

for every multi-index β, as in Section 78. In particular,

$$\begin{aligned} |a_\beta| \, |z^\beta| &\le \frac{1}{(2\pi)^n} \int_{\mathbf{T}^n} |p(t_1 \, z_1, \ldots, t_n \, z_n)| \, |dt| \\ &\le \sup_{t \in \mathbf{T}^n} |p(t_1 \, z_1, \ldots, t_n \, z_n)|. \end{aligned} \tag{92.4}$$

Let E be a nonempty subset of $\mathbf{C}^n$ which is completely circular, or at least invariant under the usual action of $\mathbf{T}^n$. If $p(z)$ is bounded on E, then it follows from the discussion in the previous pargraph that each term $a_\beta \, z^\beta$ in $p(z)$ is bounded on E. Equivalently, the monomial z^β is bounded on E whenever its coefficient a_β in $p(z)$ is not equal to 0. If E is unbounded, then it may be that z^β is not bounded on E for any nonzero multi-index β. This implies that the only polynomials on $\mathbf{C}^n$ that are bounded on E are constant, and hence that $\mathrm{Pol}(E) = \mathbf{C}^n$.

As a nice family of examples in $\mathbf{C}^n$, consider

$$E(b) = \{(z_1, z_2) \in \mathbf{C}^2 : |z_1|^b \, |z_2| \le 1\}, \tag{92.5}$$

where b is a positive real number. Thus $E(b)$ is closed, completely circular, and multiplicatively convex for each $b > 0$, and

$$E(b)^* = \{(z_1, z_2) \in \mathbf{C}^2 : 0 < |z_1|^b \, |z_2| \le 1\} \tag{92.6}$$

is dense in $E(b)$ for each b as well, as in Section 90. If b is rational, so that $b = \beta_1/\beta_2$ for some positive integers β_1, β_2, then

$$E(b) = \{(z_1, z_2) \in \mathbf{C}^2 : |z_1|^{\beta_1} \, |z_2|^{\beta_2} \le 1\} = \{z \in \mathbf{C}^2 : |z^\beta| \le 1\}, \tag{92.7}$$

where $\beta = (\beta_1, \beta_2)$. In this case, it is easy to see that $\mathrm{Pol}(E(b)) = E(b)$, using the polynomial $p(z) = z^\beta$. Otherwise, if b is irrational, then one can check that z^β is unbounded on $E(b)$ for every nonzero multi-index β, which implies that every nonconstant polynomial on $\mathbf{C}^n$ is unbounded on $E(b)$, as before, and hence that $\mathrm{Pol}(E(b)) = \mathbf{C}^2$.

93 Polynomial convexity

A set $E \subseteq \mathbf{C}^n$ is said to be *polynomially convex* if $\mathrm{Pol}(E) = E$. Thus E has to be closed in this case, since the polynomial hull of any set is closed. Of course, $E \subseteq \mathrm{Pol}(E)$ automatically, and so E is polynomially convex when $\mathrm{Pol}(E)$ is contained in E. We have seen before that finite subsets of $\mathbf{C}^n$ are polynomially convex, as are compact convex sets. A closed, bounded, and completely circular set is polynomially convex if and only if it is multiplicatively convex, as in Section 91. The polynomial hull of any set $E \subseteq \mathbf{C}^n$ is polynomially convex, because $\mathrm{Pol}(\mathrm{Pol}(E)) = \mathrm{Pol}(E)$. If p is a polynomial on $\mathbf{C}^n$ and k is a nonnegative real number, then it is easy to see that

$$E(p, k) = \{z \in \mathbf{C}^n : |p(z)| \le k\} \tag{93.1}$$

is polynomially convex. In particular, one can take $k = 0$, so that the zero set of any polynomial is polynomially convex.

If E_α, $\alpha \in A$, is any collection of subsets of $\mathbf{C}^n$, then

$$\mathrm{Pol}\Big(\bigcap_{\alpha \in A} E_\alpha\Big) \subseteq \bigcap_{\alpha \in A} \mathrm{Pol}(E_\alpha), \tag{93.2}$$

because $\bigcap_{\alpha \in A} E_\alpha \subseteq E_\beta$ for each $\beta \in A$, so that $\mathrm{Pol}\Big(\bigcap_{\alpha \in A} E_\alpha\Big) \subseteq \mathrm{Pol}(E_\beta)$ for each $\beta \in A$. If E_α is polynomially convex for each $\alpha \in A$, then we get that

$$\mathrm{Pol}\Big(\bigcap_{\alpha \in A} E_\alpha\Big) \subseteq \bigcap_{\alpha \in A} \mathrm{Pol}(E_\alpha) = \bigcap_{\alpha \in A} E_\alpha. \tag{93.3}$$

This implies that $\bigcap_{\alpha \in A} E_\alpha$ is also polynomially convex, since it is automatically contained in its polynomial hull, as in the previous paragraph. The polynomial hull of any set $E \subseteq \mathbf{C}^n$ may be described as the intersection of all sets $E(p, k)$ such that

$$E \subseteq E(p, k), \tag{93.4}$$

where p is a polynomial on $\mathbf{C}^n$ and k is a nonnegative real number, as before. It follows that E is polynomially convex if and only if it can be expressed as the intersection of some collection of sets of the form $E(p, k)$, since these sets are all polynomially convex, and the intersection of any collection of polynomially convex sets is also polynomially convex.

Alternatively, to avoid technical problems with unbounded sets, one can expand the definition to say that a closed set $E \subseteq \mathbf{C}^n$ is polynomially convex if for every compact set $K \subseteq E$ we have that $\mathrm{Pol}(K) \subseteq E$. Of course, this still implies that $\mathrm{Pol}(E) = E$ when E is compact. With this expanded definition, it is easy to see that every closed convex set in $\mathbf{C}^n$ is polynomially convex, for essentially the same reasons as before. Similarly, a closed completely circular set $E \subseteq \mathbf{C}^n$ is polynomially convex in this expanded sense if and only if it is multiplicatively convex.

94 Entire functions, revisited

Let E be a nonempty subset of $\mathbf{C}^n$, and let $\mathrm{Hol}(E)$ be the set of $z \in \mathbf{C}^n$ such that

$$|f(z)| \le \sup_{w \in E} |f(w)| \tag{94.1}$$

for every complex-valued function f on $\mathbf{C}^n$ that can be expressed as

$$f(w) = \sum_\alpha a_\alpha \, w^\alpha. \tag{94.2}$$

More precisely, the a_α's are supposed to be complex numbers, and the sum is taken over all multi-indices α and is supposed to be absolutely convergent for every $w \in \mathbf{C}^n$. This includes the case of polynomials, for which $a_\alpha = 0$ for all but finitely many α, and so

$$\mathrm{Hol}(E) \subseteq \mathrm{Pol}(E). \tag{94.3}$$

If E is bounded, then f can be approximated uniformly on E by finite subsums of (94.2), which are polynomials, and hence

$$\mathrm{Hol}(E) = \mathrm{Pol}(E). \tag{94.4}$$

If E is not bounded, then f may be unbounded on E, so that the supremum in (94.1) is $+\infty$, and (94.1) holds vacuously.

Of course,

$$E \subseteq \mathrm{Hol}(E) \tag{94.5}$$

automatically. If $E_1 \subseteq E_2$, then

$$\mathrm{Hol}(E_1) \subseteq \mathrm{Hol}(E_2). \tag{94.6}$$

Note that functions on $\mathbf{C}^n$ expressed by absolutely summable power series are continuous, because of uniform convergence on compact sets, and continuity of polynomials. This implies that $\mathrm{Hol}(E)$ is always a closed set in $\mathbf{C}^n$, and that

$$\mathrm{Hol}(\overline{E}) = \mathrm{Hol}(E). \tag{94.7}$$

As in the case of polynomial hulls, one can check that

$$\mathrm{Hol}(\mathrm{Hol}(E)) = \mathrm{Hol}(E). \tag{94.8}$$

Using exponential functions as in Section 85, one also gets that

$$\mathrm{Hol}(E) \subseteq \overline{\mathrm{Con}(E)}. \tag{94.9}$$

More precisely, this works for both bounded and unbounded sets E.

If E is invariant under the torus action, as in Section 89, then it is easy to see that $\mathrm{Hol}(E)$ is too, as before. One can also use the maximum principle to show that $\mathrm{Hol}(E)$ is completely circular in this case, as before. One can use the three lines theorem to show that $\mathrm{Hol}(E)$ is multiplicatively convex in this situation as well. However, there are many examples where E is closed, completely circular, and multiplicatively convex, but $\mathrm{Hol}(E) \neq E$. This uses the same type of arguments as in Sections 90 and 92, and of course it is important that E be unbounded in these examples.

If E_α, $\alpha \in A$, is any collection of subsets of $\mathbf{C}^n$, then

$$\mathrm{Hol}\Big(\bigcap_{\alpha \in A} E_\alpha\Big) \subseteq \bigcap_{\alpha \in A} \mathrm{Hol}(E_\alpha), \tag{94.10}$$

as in the previous section. If $\mathrm{Hol}(E_\alpha) = E_\alpha$ for each $\alpha \in A$, then it follows that

$$\mathrm{Hol}\Big(\bigcap_{\alpha \in A} E_\alpha\Big) = \bigcap_{\alpha \in A} E_\alpha, \tag{94.11}$$

as before. Let $g(w)$ be a complex-valued function on $\mathbf{C}^n$ that can be expressed by a power series that is absolutely summable for each $w \in \mathbf{C}^n$, and put

$$E(g,k) = \{w \in \mathbf{C}^n : |g(w)| \le k\} \tag{94.12}$$

for each nonnegative real number k. As in the previous section, it is easy to see that

$$\mathrm{Hol}(E(g,k)) = E(g,k). \tag{94.13}$$

One can also check that $\mathrm{Hol}(E)$ is the same as the intersection of all sets of the form $E(g,k)$ such that $E \subseteq E(g,k)$ for any $E \subseteq \mathbf{C}^n$, as before.

95 Power series expansions

Let R be a positive real number, and put

$$D(R) = \{w \in \mathbf{C} : |w| < R\}, \tag{95.1}$$

as before. Suppose that $f(w)$ is a holomorphic function on $D(R)$, which one can take to mean that $f(w)$ is continuously-differentiable and satisfies the Cauchy–Riemann equations. Of course, it is well known that one can also start with significantly weaker regularity conditions on f. If $|z| < r < R$, then Cauchy's integral formula implies that

$$f(z) = \frac{1}{2\pi i} \oint_{\partial D(r)} \frac{f(w)}{w - z} \, dw. \tag{95.2}$$

More precisely, this uses an oriented contour integral over the circle centered at 0 with radius r, which is the boundary of the corresponding disk $D(r)$.

Let us briefly review the standard argument for obtaining a power series expansion for $f(z)$ from (95.2). If $|z| < r = |w|$, then

$$\frac{1}{w - z} = \frac{1}{w\,(1 - w^{-1}\,z)} = w^{-1} \sum_{j=0}^{\infty} w^{-j}\, z^j, \tag{95.3}$$

where the series on the right is an absolutely convergent geometric series under these conditions. The partial sums of this series also converge uniformly as a function of w on $\partial D(r)$ for each $z \in D(r)$, by Weierstrass' M-test. This permits us to interchange the order of summation and integration in (95.2), to get that

$$f(z) = \sum_{j=0}^{\infty} a_j\, z^j, \tag{95.4}$$

where $|z| < r < R$ and

$$a_j = \frac{1}{2\pi i} \oint_{\partial D(r)} f(w)\, w^{-j-1} \, dw \tag{95.5}$$

for each $j \geq 0$.

Although this expression for a_j implicitly depends on r, different choices of $r < R$ lead to the same value of a_j. This is an immediate consequence of Cauchy's theorem, and one can also observe that a_j is equal to $1/j!$ times the jth derivative of f at 0, which obviously does not depend on r. Alternatively, once one has this power series expansion for f on $D(r)$, one can use it to evaluate integrals of f over circles of radius less than r. In particular, the coefficients of the power series are given by the corresponding integrals over circles of radius less than r, because of the usual orthogonality properties of the w^j's with respect to integration over the unit circle. This also uses the fact that the partial sums of the power series converge uniformly on compact subsets of $D(r)$, to interchange the order of integration and summation.

Note that

$$|a_j| \le \frac{1}{2\pi r^{j+1}} \int_{\partial D(r)} |f(w)|\,|dw| \tag{95.6}$$

for each j, where the integral is now taken with respect to the element of arc length $|dw|$. In particular,

$$|a_j| \le r^{-j} \Big(\sup_{|w|=r} |f(w)| \Big). \tag{95.7}$$

This works for each $r < R$, since a_j does not depend on r, as in the previous paragraph.

96 Power series expansions, continued

Let n be a positive integer, and let $R = (R_1, \ldots, R_n)$ be an n-tuple of positive real numbers. Also let

$$D_n(R) = D(R_1) \times \cdots \times D(R_n) \tag{96.1}$$

be the corresponding polydisk in $\mathbf{C}^n$. To say that a complex-valued function $f(w)$ on $D(R)$ is holomorphic, we mean that $f(w)$ is continuously-differentiable on $D_n(R)$ and holomorphic as a function of w_j for each j, which is to say that $f(w)$ satisfies the Cauchy–Riemann equations as a function of w_j for each j. As in the one-variable case, one can start with weaker regularity conditions on f, but we shall not pursue this here. One might at least note that it would be sufficient in this section to ask that f be continuous on $D_n(R)$ and holomorphic in each variable separately.

If $z \in D(R)$ and $|z_1| < r_1 < R_1$, then we can apply Cauchy's integral formula to $f(w)$ as a holomorphic function of w_1 to get that

$$f(z) = \frac{1}{2\pi i} \oint_{\partial D(r_1)} \frac{f(w_1, z_2, \ldots, z_n)}{w_1 - z_1}\, dw_1, \tag{96.2}$$

as in the previous section. Repeating the process, if $|z_j| < r_j < R_j$ for each j, then we get that

$$f(z) = \frac{1}{(2\pi i)^n} \oint_{\partial D(r_1)} \cdots \oint_{\partial D(r_n)} f(w) \Big(\prod_{j=1}^{n} (w_j - z_j)^{-1} \Big)\, dw_1 \cdots dw_n, \tag{96.3}$$

which is an n-dimensional version of Cauchy's integral formula.

Let us pause for a moment to consider "multiple geometric series". If $\zeta \in \mathbf{C}^n$ and $|\zeta_j| < 1$ for each j, then

$$\prod_{j=1}^{n} (1 - \zeta_j)^{-1} = \prod_{j=1}^{n} \Big(\sum_{\ell_j = 0}^{\infty} \zeta_j^{\ell_j} \Big) = \sum_{\alpha} \zeta^{\alpha}, \tag{96.4}$$

where the last sum is taken over all multi-indices α, and $\zeta^\alpha = \zeta_1^{\alpha_1} \cdots \zeta_n^{\alpha_n}$ is the usual monomial. All of these sums converge absolutely under these conditions.

If $|z_j| < r_j = |w_j|$ for each j, then

$$\prod_{j=1}^n (w_j - z_j)^{-1} = \prod_{j=1}^n w_j^{-1}\,(1 - w_j^{-1}\,z_j)^{-1} = \sum_\alpha w^{-\alpha-1}\,z^\alpha, \tag{96.5}$$

where $w^{-\alpha-1} = w_1^{-\alpha_1-1} \cdots w_n^{-\alpha_n-1}$. As usual, this sum is absolutely convergent under these conditions, and is uniformly approximated by finite subsums as a function of w on $\partial D(r_1) \times \cdots \times \partial D(r_n)$ for each $z \in D_n(r)$, $r = (r_1, \ldots, r_n)$.

If $r_j < R_j$ for each j, then put

$$a_\alpha = \frac{1}{(2\pi i)^n} \oint_{\partial D(r_1)} \cdots \oint_{\partial D(r_n)} f(w)\,w^{-\alpha-1}\,dw_1 \cdots dw_n \tag{96.6}$$

for each multi-index α. Thus

$$|a_\alpha| \le \frac{r^{-\alpha-1}}{(2\pi)^n} \int_{\partial D(r_1)} \cdots \int_{\partial D(r_n)} |f(w)|\,|dw_1| \cdots |dw_n|, \tag{96.7}$$

where $r^{-\alpha-1}$ is as in the previous paragraph, and hence

$$|a_\alpha| \le r^{-\alpha} \sup\{|f(w)| : |w_j| = r_j \text{ for } j = 1, \ldots, n\} \tag{96.8}$$

for each α.

If $|z_j| < r_j < R_j$ for each j, then we get that

$$f(z) = \sum_\alpha a_\alpha\,z^\alpha. \tag{96.9}$$

More precisely, it is easy to see that the sum on the right converges absolutely under these conditions, by comparison with a convergent multiple geometric series. To get (96.9), one can plug (96.5) into (96.3), and interchange the order of summation and integration. This uses the fact that the sum in (96.5) can be approximated uniformly by finite subsums for $w \in \partial D(r_1) \times \cdots \times \partial D(r_n)$.

As in the previous section, the coefficients a_α do not depend on the choice of $r = (r_1, \ldots, r_n)$, as long as $0 < r_j < R_j$ for each j. Thus (96.9) holds on all of $D_n(R)$, with absolute convergence of the sum for every $z \in D_n(R)$.

97 Holomorphic functions, revisited

Let us say that a complex-valued function $f(z)$ on a nonempty open set U in $\mathbf{C}^n$ is holomorphic if it is continuously-differentiable in the real-variable sense and holomorphic in each variable separately. As in the previous section, this implies that f can be represented by an absolutely convergent power series on a neighborhood of any point in U. In particular, f is automatically continuously-differentiable of all orders on U. This would also work if we only asked that f

be continuous on U and holomorphic in each variable separately, but we shall not try to deal with weaker regularity conditions here.

Let $C(U)$ be the algebra of continuous complex-valued functions on U, and let $\mathcal{H}(U)$ be the subspace of $C(U)$ consisting of holomorphic functions. More precisely, $\mathcal{H}(U)$ is a subalgebra of $C(U)$, because the sum and product of two holomorphic functions on U are also holomorphic. Remember that there is also a natural topology on $C(U)$, defined by the supremum seminorms associated to nonempty compact subsets of U. As in the one-variable case, one can check that $\mathcal{H}(U)$ is closed in $C(U)$ with respect to this topology, using the n-dimensional version of the Cauchy integral formula.

Let f be a holomorhic function on U, and let U_0 be the set of $p \in U$ such that $f = 0$ at every point in a neighborhood of p, so that U_0 is an open set in U, by construction. If Z is the set of $p \in U$ such that f and all of its derivatives are equal to 0 at p, then Z is relatively closed in U, because f and its derivatives are continuous on U. Clearly $U_0 \subseteq Z$, and $Z \subseteq U_0$ because of the local power series representation of f at each point in U. Thus $U_0 = Z$ is both open and relatively closed in U. It follows that $U_0 = U$ when $U_0 \neq \emptyset$ and U is connected.

Suppose that h is a continuous complex-valued function on a closed disk in the complex plane which is holomorphic in the interior and not equal to 0 at any point on the boundary. Let a be the number of points in the interior at which h is equal to 0, counted with their appropriate multiplicity. The argument principle implies that a is the same as the winding number of the boundary values of h around 0 in the range. This winding number is not changed by small perturbations of h on the boundary with respect to the supremum norm, and hence a is not changed by small perturbations of h as a continuous function on the closed disk which is holomorphic in the interior with respect to the supremum norm. This implies that a holomorphic function f in $n \geq 2$ complex variables cannot have isolated zeros, by considering f as a continuous family of holomorphic functions in one variable parameterized by the other $n-1$ variables.

98 Laurent expansions

Let R, T be nonnegative real numbers with $R < T$, and let

$$A(R,T) = \{z \in \mathbf{C} : R < |w| < T\} \tag{98.1}$$

be the open annulus in the complex plane with inner radius R and outer radius T. If $f(w)$ is a holomorphic function on $A(R,T)$ and $R < r < |z| < t < T$, then Cauchy's integral formula implies that

$$f(z) = \frac{1}{2\pi i} \oint_{\partial A(r,t)} \frac{f(w)}{w-z}\, dw. \tag{98.2}$$

The boundary of $A(r,t)$ consists of the circles centered at 0 with radii r, t and opposite orientations, and the integral over $\partial A(r,t)$ may be re-expressed as

$$\oint_{|w|=t} \frac{f(w)}{w-z}\, dw - \oint_{|w|=r} \frac{f(w)}{w-z}\, dw, \tag{98.3}$$

where these circles have their usual positive orientations in both integrals.

As in Section 95,

$$\frac{1}{2\pi i} \oint_{|w|=t} \frac{f(w)}{w-z}\, dw = \sum_{j=0}^{\infty} a_j\, z^j, \tag{98.4}$$

where

$$a_j = \frac{1}{2\pi i} \oint_{|w|=t} f(w)\, w^{-j-1}\, dw. \tag{98.5}$$

Note that

$$|a_j| \le \frac{1}{2\pi t^{j+1}} \oint_{|w|=t} |f(w)|\, |dw| \le t^{-j} \Big(\sup_{|w|=t} |f(w)| \Big) \tag{98.6}$$

for each $j \ge 0$, so that $\sum_{j=0}^{\infty} a_j\, z^j$ converges absolutely when $|z| < t$.

The other term is a bit different, because $|z| > |w| = r$. This time we use

$$\frac{-1}{w-z} = \frac{1}{z\,(1 - z^{-1}\, w)} = z^{-1} \sum_{j=0}^{\infty} z^{-j}\, w^j \tag{98.7}$$

to get that

$$-\frac{1}{2\pi i} \oint_{|w|=r} \frac{f(w)}{w-z}\, dw = \sum_{j=-1}^{-\infty} a_j\, z^j, \tag{98.8}$$

where

$$a_j = \frac{1}{2\pi i} \oint_{|w|=r} f(w)\, w^{-j-1}\, dw \tag{98.9}$$

for $j \le -1$. Thus

$$|a_j| \le \frac{1}{2\pi r^{j+1}} \int_{|w|=r} |f(w)|\, |dw| \le r^{-j} \Big(\sup_{|w|=r} |f(w)| \Big) \tag{98.10}$$

for each $j \le -1$, so that $\sum_{j=-1}^{-\infty} a_j\, z^j$ converges absolutely when $|z| > r$.

Combining these two series, we get that

$$f(z) = \sum_{j=-\infty}^{\infty} a_j\, z^j \tag{98.11}$$

when $r < |z| < t$, where the coefficients a_j are given as above for $j \ge 0$ and $j \le -1$, respectively. As in Section 95, these coefficients do not actually depend on the choices of radii $r, t \in (R, T)$.

99 Laurent expansions, continued

Let R, T be nonnegative real numbers with $R < T$, and let V be a nonempty open set in $\mathbf{C}^{n-1}$ for some $n \geq 2$. If $z = (z_1, z_2, \ldots, z_n) \in \mathbf{C}^n$, then we put $z' = (z_2, \ldots, z_n) \in \mathbf{C}^{n-1}$, and identify z with $(z_1, z') \in \mathbf{C} \times \mathbf{C}^{n-1}$, so that

$$U = A(R, T) \times V \tag{99.1}$$

is identified with an open set in $\mathbf{C}^n$.

Let f be a holomorphic function on U, and let z be an element of U, with $r < |z_1| < t$ for some $r, t \in (R, T)$. Applying the discussion in the previous section to $f(z_1, z')$ as a function of z_1, we get that

$$f(z_1, z') = \sum_{j=-\infty}^{\infty} a_j(z')\, z_1^j, \tag{99.2}$$

where

$$a_j(z') = \frac{1}{2\pi i} \oint_{|w|=t} f(w, z')\, w^{-j-1}\, dw \tag{99.3}$$

when $j \geq 0$, and

$$a_j(z') = \frac{1}{2\pi i} \oint_{|w|=r} f(w, z')\, w^{-j-1}\, dw \tag{99.4}$$

when $j \leq -1$. It follows from these expressions that $a_j(z')$ is holomorphic as a function of z' on V for each j, because f is holomorphic.

Suppose that V_1 is a nonempty open subset of V, and that f is actually a holomorphic function on the open set

$$(A(R, T) \times V) \cup (D(T) \times V_1) \tag{99.5}$$

in $\mathbf{C}^n$. Thus $f(w, z')$ is holomorphic as a function of w on the open disk $D(T)$ for each $z' \in V_1$. This implies that

$$a_j(z') = 0 \tag{99.6}$$

when $z' \in V_1$ and $j \leq -1$. If V is connected, then it follows that the same conclusion holds for every $z' \in V$ and $j \leq -1$, because $a_j(z')$ is holomorphic as a function of z' on V for each j.

Under these conditions, we get that

$$f(z_1, z') = \sum_{j=0}^{\infty} a_j(z')\, z_1^j \tag{99.7}$$

for every $z = (z_1, z')$ in (99.5). This series actually converges absolutely when $|z_1| < T$ and $z' \in V$, as one can see by choosing t such that $|z_1| < t < T$, and applying the estimate for $|a_j|$ in the previous section. Similarly, the partial sums of this series converge uniformly on compact subsets of $D(T) \times V$. The partial sums are also holomorphic in z_1 and z', and it follows that the series defines a holomorphic function on $D(T) \times V$. Thus f extends to a holomorphic function on $D(T) \times V$ in this case.

100 Completely circular domains

Let U be a nonempty complete circular open subset of $\mathbf{C}^n$. If $z \in U$, then there is an n-tuple $R = (R_1, \ldots, R_n)$ of positive real numbers such that

$$z \in D_n(R) \subseteq U, \tag{100.1}$$

where $D_n(R) = D(R_1) \times \cdots \times D(R_n)$ is the polydisk associated to R, as before. Thus U can be expressed as a union of open polydisks.

As in Section 90, let U^* be the set of $w \in U$ such that $w_j \neq 0$ for each j, and let A be the set of $y \in \mathbf{R}^n$ for which there is a $w \in U^*$ such that $y_j = \log |w_j|$ for each j. Note that A is an open set in $\mathbf{R}^n$, and that for each $z \in U$ there is a $w \in U^*$ such that $|z_j| < |w_j|$, because U is an open set in $\mathbf{C}^n$. As before, if $x \in \mathbf{R}^n$, $y \in A$, and $x_j \le y_j$ for each j, then $x \in A$, because U is completely circular. Similarly, if $\zeta \in \mathbf{C}^n$, $x \in A$, and $|z_j| \le \exp x_j$ for each j, then $\zeta \in U$. Conversely, for each $\zeta \in U$ there is an $x \in A$ with this property, so that U is completely determined by A under these conditions.

Let f be a holomorphic function on U. If R is an n-tuple of positive real numbers such that $D_n(R) \subseteq U$, then f can be represented by a power series on $D_n(R)$, as in Section 96. More precisely, there are complex numbers a_α for each multi-index α such that

$$f(z) = \sum_\alpha a_\alpha \, z^\alpha \tag{100.2}$$

for each $z \in D_n(R)$, where the sum converges absolutely. The coefficients a_α can be given by the derivatives of f at 0 in the usual way, since

$$\frac{\partial^{|\alpha|} f}{\partial z^\alpha}(0) = \alpha! \cdot a_\alpha, \tag{100.3}$$

where $\alpha! = \alpha_1! \cdots \alpha_n!$. In particular, the coefficients a_α do not depend on R, and so this power series representation for $f(z)$ holds for every $z \in U$.

Remember that $\mathrm{Con}(A)$ denotes the convex hull of A in $\mathbf{R}^n$, which is an open set in $\mathbf{R}^n$ in this case, because A is open. Similarly, if $x \in \mathbf{R}^n$, $y \in \mathrm{Con}(A)$, and $x_j \le y_j$ for each j, then $x \in \mathrm{Con}(A)$, because of the corresponding property of A. Consider

$$V = \{\zeta \in \mathbf{C}^n : \text{ there is an } x \in \mathrm{Con}(A) \text{ such that } |\zeta_j| \le \exp x_j \text{ for } j = 1, \ldots, n\}. \tag{100.4}$$

It is easy to see that V is open, completely circular, and multiplicatively convex under these conditions. We also have that $U \subseteq V$, with $U = V$ exactly when U is multiplicatively convex. As in Section 74, the set of $z \in \mathbf{C}^n$ for which $\sum_\alpha a_\alpha \, z^\alpha$ is absolutely summable is completely circular and multiplicatively convex. It is not difficult to check that this happens for each $z \in V$, so that f extends to a holomorphic function on V.

101 Convex sets

Let A be a nonempty convex set in $\mathbf{R}^n$. As in Section 81, if $x \in \mathbf{R}^n \backslash \overline{A}$, then there is a linear function λ on $\mathbf{R}^n$ such that

$$\sup_{y \in A} \lambda(y) < \lambda(x). \tag{101.1}$$

More precisely, we can express λ as

$$\lambda(y) = \sum_{j=1}^{n} a_j \, y_j \tag{101.2}$$

for some $a \in \mathbf{R}^n$. Of course, $a \neq 0$, and we can normalize a so that

$$\max_{1 \le j \le n} |a_j| = 1, \tag{101.3}$$

by multiplying a by a positive real number.

Suppose now that $x \in \partial A$, and let us show that there is a nonzero linear functional λ on $\mathbf{R}^n$ such that

$$\lambda(y) \le \lambda(x) \tag{101.4}$$

for every $y \in A$. By hypothesis, there is a sequence $\{x(l)\}_{l=1}^{\infty}$ of elements of $\mathbf{R}^n \backslash \overline{A}$ that converges to x. As in the previous paragraph, for each l there is an $a(l) \in \mathbf{R}^n$ such that

$$\max_{1 \le j \le n} |a_j(l)| = 1 \tag{101.5}$$

and $\lambda_l(y) = \sum_{j=1}^{n} a_j(l) \, y_j$ satisfies

$$\sup_{y \in A} \lambda_l(y) < \lambda_l(x(l)). \tag{101.6}$$

Passing to a subsequence if necessary, we may suppose that $\{a(l)\}_{l=1}^{\infty}$ converges to some $a \in \mathbf{R}^n$, which also satisfies (101.3). If λ is the linear functional on $\mathbf{R}^n$ corresponding to a as before, then it is easy to see that λ satisfies (101.4), as desired.

If in addition A is an open set in $\mathbf{R}^n$, then we get that

$$\lambda(y) < \lambda(x) \tag{101.7}$$

for every $y \in A$. Otherwise, if $\lambda(y) = \lambda(x)$ for some $y \in A$, then one can use the facts that A is open and $\lambda \neq 0$ to get that $\lambda(z) > \lambda(x)$ for some $z \in A$.

As another special case, suppose that A has the property that for each $u \in \mathbf{R}^n$ and $y \in A$ with $u_j \le y_j$ for each j we have that $u \in A$ too. If $\lambda(y) = \sum_{j=1}^{n} a_j \, y_j$ satisfies (101.4), then $a_j \ge 0$ for each j.

102 Completely circular domains, continued

Let U be a nonempty open subset of $\mathbf{C}^n$ that is also completely circular and multiplicatively convex, and let w be an element of the boundary of U. Note that $w \neq 0$, because $0 \in U$. Let I be the set of $j = 1, \ldots, n$ such that $w_j \neq 0$, and let U_I be the set of $z \in U$ such that $z_j \neq 0$ when $j \in I$. Also let $\mathbf{R}^I$ be the set of real-valued functions on I, and let A_I be the set of elements of $\mathbf{R}^I$ of the form $\log |z_j|$, $j \in I$, with $z \in U_I$. If $u \in \mathbf{R}^I$, $v \in A_I$, and $u_j \leq v_j$ for each $j \in I$, then $u \in A_I$ too, because U is completely circular. It is easy to see that A_I is open and convex in $\mathbf{R}^I$, because U is open and multiplicatively convex. One can also check that $\log |w_j|$, $j \in I$, corresponds to an element of the boundary of A_I in $\mathbf{R}^I$ under these conditions.

As in the previous section, there is an $a \in \mathbf{R}^I$ such that $a_j \geq 0$ for each $j \in I$, $\max_{j \in I} a_j = 1$, and

$$\sum_{j \in I} a_j \, v_j < \sum_{j \in I} a_j \, \log |w_j| \tag{102.1}$$

for each $v \in A_I$. If $j \in I$ and l is a positive integer, then let $\alpha_j(l)$ be the smallest positive integer such that

$$a_j \, l \leq \alpha_j(l). \tag{102.2}$$

Put $\alpha_j(l) = 0$ when $j \notin I$, so that $\alpha(l) = (\alpha_1(l), \ldots, \alpha_n(l))$ is a multi-index for each positive integer. By construction, $a_{j_0} = 1$ for some $j_0 \in I$, which implies that $\alpha_{j_0}(l) = l$ for each l. In particular, the multi-indices $\alpha(l)$ are all distinct.

Consider

$$f_w(z) = \sum_{l=1}^{\infty} w^{-\alpha(l)} \, z^{\alpha(l)}. \tag{102.3}$$

This is a power series in z, with coefficients $w^{-\alpha(l)} = \prod_{j \in I} w_j^{-\alpha_j(l)}$, and we would like to show that it converges absolutely when $z \in U$. If $z \in U \backslash U_I$, so that $z_j = 0$ for some $j \in I$, then $z^{\alpha(l)} = 0$ for each l, because $\alpha_j(l) \geq 1$ for every $j \in I$ and $l \geq 1$ by construction. Thus we may as well suppose that $z \in U_I$, so that $\log |z_j|$, $j \in I$, determines an element of A_I, and hence

$$\sum_{j \in I} a_j \, \log |z_j| < \sum_{j \in I} a_j \, \log |w_j|. \tag{102.4}$$

Equivalently,

$$\prod_{j \in I} |z_j|^{a_j} < \prod_{j \in I} |w_j|^{a_j}. \tag{102.5}$$

Observe that

$$0 \leq \alpha_j(l) - a_j \, l \leq 1 \tag{102.6}$$

for each $j \in I$ and $l \geq 1$. Remember that $\alpha_j(l)$ is the smallest positive integer greater than or equal to $a_j \, l$, so that $\alpha_j(l) - a_j \, l \geq 0$ in particular. If $a_j > 0$, then $\alpha_j(l) - a_j < 1$ for each l. Otherwise, if $a_j = 0$, then $\alpha_j(l) = 1$ for each l.

Of course,

$$|w^{-\alpha(l)}|\,|z^{\alpha(l)}| = \prod_{j \in I} \Big(\frac{|z_j|}{|w_j|}\Big)^{\alpha_j(l)}. \tag{102.7}$$

Using the observation in the previous paragraph, we get that

$$\prod_{j \in I} \Big(\frac{|z_j|}{|w_j|}\Big)^{\alpha_j(l) - a_j\, l} \le C \tag{102.8}$$

for some $C \ge 0$, where C depends on w and z but not l. Hence

$$|w^{-\alpha(l)}|\,|z^{\alpha(l)}| \le C \prod_{j \in I} \Big(\frac{|z_j|}{|w_j|}\Big)^{a_j\, l} \tag{102.9}$$

for each l.

Equivalently,

$$|w^{-\alpha(l)}|\,|z^{\alpha(l)}| \le C \Big(\prod_{j \in I} \frac{|z_j|^{a_j}}{|w_j|^{a_j}}\Big)^{l} \tag{102.10}$$

for each l. Note that the quantity in parentheses on the right side is strictly less than 1, by (102.5). It follows that the series in (102.3) converges absolutely when $z \in U_I$, by comparison with a convergent geometric series, as desired.

Thus $f_w(z)$ defines a holomorphic function of z on U. If $z = w$, then the series in (102.3) does not converge, because every term in the series is equal to 1. It is easy to see that $t\,w \in U$ when t is a nonnegative real number strictly less than 1, because $w \in \partial U$ and U is completely circular. In this case,

$$f_w(t\,w) = \sum_{l=1}^{\infty} t^{|\alpha(l)|}, \tag{102.11}$$

which tends to $+\infty$ as $t \to 1$. It follows that $f_w(z)$ does not have a holomorphic extension to a neighborhood of w, since it is not even bounded on U near w.

103 Convex domains

Let U be a nonempty convex open set in $\mathbf{C}^n$, and let w be an element of the boundary of U. As in Section 101, there is a complex-linear function μ on $\mathbf{C}^n$ such that

$$\operatorname{Re} \mu(z) < \operatorname{Re} \mu(w) \tag{103.1}$$

for every $z \in U$. In particular,

$$\mu(z) \ne \mu(w) \tag{103.2}$$

for every $z \in U$. It follows that

$$g_w(z) = \frac{1}{\mu(z) - \mu(w)} \tag{103.3}$$

is a holomorphic function on U that is unbounded on the intersection of U with any neighborhood of w, and hence does not have a holomorphic extension to the union of U with any neighborhood of w.

104 Planar domains

Let U be a nonempty open set in the complex plane, and let w be an element of the boundary of U. Observe that

$$h_w(z) = \frac{1}{z - w} \tag{104.1}$$

is a holomorphic function on U that is unbounded on the intersection of U with any neighborhood of w, and hence cannot be extended to a holomorphic function on the union of U with any neighborhood of w. In particular, holomorphic functions in one complex variable can have isolated zeros, and thus isolated singularities. We have seen before that holomorphic functions in two or more complex variables cannot have isolated zeros, and they also cannot have isolated singularities, by the earlier discussion about Laurent expansions.

Part IV
Convolution

105 Convolution on $\mathbf{T}^n$

Let f, g be continuous complex-valued functions on the n-dimensional torus $\mathbf{T}^n$. The *convolution* $f * g$ is the function defined on $\mathbf{T}^n$ by

$$(f * g)(z) = \frac{1}{(2\pi)^n} \int_{\mathbf{T}^n} f(z \cdot w^{-1})\, g(w)\, |dw|. \tag{105.1}$$

As before, $|dw|$ is the n-dimensional element of integration on $\mathbf{T}^n$ corresponding to the element $|dw_j|$ of arc length in each variable. Alternatively, $|dw|$ represents the appropriate version of Lebesgue measure on $\mathbf{T}^n$. If $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ are elements of $\mathbf{T}^n$, then we put

$$w^{-1} = (w_1^{-1}, \ldots, w_n^{-1}) \tag{105.2}$$

and

$$z \cdot w = (z_1\, w_1, \ldots, z_n\, w_n), \tag{105.3}$$

so that $z \cdot w^{-1}$ is also defined.

It is easy to see that $f * g$ is also a continuous function on $\mathbf{T}^n$ when f, g are continuous, using the fact that continuous functions on $\mathbf{T}^n$ are uniformly continuous, since $\mathbf{T}^n$ is compact. Observe that

$$f * g = g * f, \tag{105.4}$$

as one can see using the change of variables $w \mapsto w^{-1} \cdot z$ in (105.1). More precisely, this also uses the fact that the measure on $\mathbf{T}^n$ is invariant under the mappings $w \mapsto w^{-1}$ and $w \mapsto u \cdot w$ for each $u \in \mathbf{T}^n$. Similarly, one can check that

$$(f * g) * h = f * (g * h) \tag{105.5}$$

for all continuous functions f, g, and h on $\mathbf{T}^n$.

If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of integers, then the corresponding Fourier coefficient of a continuous function f on $\mathbf{T}^n$ is defined as usual by

$$\widehat{f}(\alpha) = \frac{1}{(2\pi)^n} \int_{\mathbf{T}^n} f(z)\, z^{-\alpha}\, |dz|. \tag{105.6}$$

It is easy to check that

$$(\widehat{f * g})(\alpha) = \widehat{f}(\alpha)\, \widehat{g}(\alpha) \tag{105.7}$$

for all continuous functions f, g on $\mathbf{T}^n$ and $\alpha \in \mathbf{Z}^n$. More precisely, if we substitute the definition of $f * g$ into the definition of the Fourier coefficient, then we get a double integral in z and w. This double integral can be evaluated by integrating in z first, using the change of variables $z \mapsto z \cdot w$ and the fact that

$$(z \cdot w)^{-\alpha} = z^{-\alpha}\, w^{-\alpha} \tag{105.8}$$

for all $z, w \in \mathbf{T}^n$ and $\alpha \in \mathbf{Z}^n$. The double integral then splits into a product of integrals over z and w separately, which leads to (105.7).

Note that the convolution $f * g$ can be defined as before when f is continuous on $\mathbf{T}^n$ and g is Lebesgue integrable, and satisfies

$$\sup_{z \in \mathbf{T}^n} |(f * g)(z)| \le \Big(\sup_{z \in \mathbf{T}^n} |f(z)|\Big) \Big(\frac{1}{(2\pi)^n} \int_{\mathbf{T}^n} |g(w)|\, |dw|\Big). \tag{105.9}$$

In this case, it is easy to see that $f * g$ is still continuous, because f is uniformly continuous on $\mathbf{T}^n$. Of course, the analogous statements also hold when the roles of f and g are reversed, because convolution is commutative. If f is bounded and measurable on $\mathbf{T}^n$ and g is integrable, then the convolution $(f * g)(z)$ can be defined in the same way for each $z \in \mathbf{T}^n$, and satisfies (105.9). The convolution $f * g$ is actually continuous in this case as well, as one can show by approximating g by continuous functions with respect to the L^1 norm on $\mathbf{T}^n$, so that $f * g$ is approximated uniformly by continuous functions on $\mathbf{T}^n$ by (105.9) and the previous remarks.

Suppose that f, g are nonnegative real-valued integrable functions on $\mathbf{T}^n$. In this case,

$$\frac{1}{(2\pi)^n} \int_{\mathbf{T}^n} (f * g)(z)\, |dz| = \Big(\frac{1}{(2\pi)^n} \int_{\mathbf{T}^n} f(z)\, |dz|\Big)\Big(\frac{1}{(2\pi)^n} \int_{\mathbf{T}^n} g(z)\, |dw|\Big). \tag{105.10}$$

To see this, one can substitute the definition of $(f * g)(z)$ into the integral on the left, which leads to a double integral in w and z. One can then interchange the order of integration and use the change of variable $z \mapsto z \cdot w$ to split the double integral into a product of integrals in z and w, as before. In particular, it follows that $(f * g)(z)$ is finite for almost every $z \in \mathbf{T}^n$.

Now let f, g be integrable complex-valued functions on $\mathbf{T}^n$. Observe that

$$\int_{\mathbf{T}^n} |f(z \cdot w^{-1})| \, |g(w)| \, |dw| < \infty \tag{105.11}$$

for almost every $z \in \mathbf{T}^n$, by the argument in the previous paragraph applied to $|f|$, $|g|$. Thus $(f * g)(z)$ is defined for almost every $z \in \mathbf{T}^n$, and satisfies

$$|(f * g)(z)| \le \frac{1}{(2\pi)^n} \int_{\mathbf{T}^n} |f(z \cdot w^{-1})| \, |g(w)| \, dw. \tag{105.12}$$

Using Fubini's theorem, one may conclude that $f * g$ is an integrable function on $\mathbf{T}^n$, and that

$$\begin{aligned} \frac{1}{(2\pi)^n} \int_{\mathbf{T}^n} |(f * g)(z)| \, |dz| \le \\ \Big(\frac{1}{(2\pi)^n} \int_{\mathbf{T}^n} |f(z)| \, |dz|\Big)\Big(\frac{1}{(2\pi)^n} \int_{\mathbf{T}^n} |g(w)| \, |dw|\Big). \end{aligned} \tag{105.13}$$

One can also check that convolution is commutative and associative on $L^1(\mathbf{T}^n)$, as before.

If f is an integrable function on $\mathbf{T}^n$, then the Fourier coefficients $\widehat{f}(\alpha)$ can be defined in the usual way, and satisfy

$$|\widehat{f}(\alpha)| \le \frac{1}{(2\pi)^n} \int_{\mathbf{T}^n} |f(z)| \, |dz| \tag{105.14}$$

for each $\alpha \in \mathbf{Z}^n$. If f and g are integrable functions on $\mathbf{T}^n$, so that their convolution $f * g$ is also integrable, as in the preceding paragraph, then the Fourier coefficients of $f * g$ are equal to the product of the Fourier coefficients of f and g, as in (105.7). This follows from Fubini's theorem, as before.

106 Convolution on $\mathbf{R}^n$

Let f and g be nonnegative real-valued integrable functions on $\mathbf{R}^n$, and put

$$(f * g)(x) = \int_{\mathbf{R}^n} f(x - y) \, g(y) \, dy, \tag{106.1}$$

where dy denotes Lebesgue measure on $\mathbf{R}^n$, as usual. It is easy to see that

$$\int_{\mathbf{R}^n} (f * g)(x) \, dx = \Big(\int_{\mathbf{R}^n} f(x) \, dx\Big)\Big(\int_{\mathbf{R}^n} g(y) \, dy\Big), \tag{106.2}$$

by interchanging the order of integration and using the change of variables $x \mapsto x + y$, as in the previous section. Thus $f * g$ is integrable on $\mathbf{R}^n$ under these conditions, and finite almost everywhere on $\mathbf{R}^n$ in particular.

If f and g are arbitrary real or complex-valued integrable functions on $\mathbf{R}^n$, then it follows that

$$\int_{\mathbf{R}^n} |f(x-y)|\,|g(y)|\,dy < \infty \tag{106.3}$$

for almost every $x \in \mathbf{R}^n$, by applying the preceding argument to $|f|$ and $|g|$. This shows that the definition (106.1) of $(f * g)(x)$ also makes sense in this case for almost every $x \in \mathbf{R}^n$, and satisfies

$$|(f * g)(x)| \le \int_{\mathbf{R}^n} |f(x-y)|\,|g(y)|\,dy. \tag{106.4}$$

One can also check that $f * g$ is measurable, using Fubini's theorem. Integrating in x as before, we get that

$$\int_{\mathbf{R}^n} |(f * g)(x)|\,dx \le \Big(\int_{\mathbf{R}^n} |f(x)|\,dx\Big)\Big(\int_{\mathbf{R}^n} |g(y)|\,dy\Big), \tag{106.5}$$

and that $f * g$ is integrable in particular.

As in the previous section, it is easy to see that

$$f * g = g * f, \tag{106.6}$$

using the change of variables $y \mapsto x - y$. Similarly, one can verify that

$$(f * g) * h = f * (g * h) \tag{106.7}$$

for any integrable functions f, g, and h on $\mathbf{R}^n$.

If f is an integrable function on $\mathbf{R}^n$ and g is bounded and measurable, then the convolution $f * g$ can be defined using (106.1) as before, and satisfies

$$\sup_{x \in \mathbf{R}^n} |(f * g)(x)| \le \Big(\int_{\mathbf{R}^n} |f(x)|\,dx\Big)\Big(\sup_{y \in \mathbf{R}^n} |g(y)|\Big). \tag{106.8}$$

One can also check that $f * g$ is uniformly continuous under these conditions, as follows. If f is a continuous function on $\mathbf{R}^n$ with compact support, then f is uniformly continuous, and it is easy to see that $f * g$ is uniformly continuous directly from the definitions. Otherwise, if f is any integrable function on $\mathbf{R}^n$, then it is well known that f can be approximated by continuous functions on $\mathbf{R}^n$ with compact support in the L^1 norm. This implies that $f * g$ can be approximated by uniformly continuous functions on $\mathbf{R}^n$ with respect to the supremum norm, and hence that $f * g$ is uniformly continuous as well.

107 The Fourier transform

If f is an integrable complex-valued function on $\mathbf{R}^n$, then the *Fourier transform* $\widehat{f}$ of f is defined by

$$\widehat{f}(\xi) = \int_{\mathbf{R}^n} f(x)\,\exp(-i\xi \cdot x)\,dx. \tag{107.1}$$

Here $\xi \in \mathbf{R}^n$, and $\xi \cdot x$ is the usual dot product, given by

$$\xi \cdot x = \sum_{j=1}^{n} \xi_j \, x_j. \tag{107.2}$$

Also, $\exp(-i\xi \cdot x)$ refers to the complex exponential function, which satisifies $|\exp(it)| = 1$ for every $t \in \mathbf{R}$. Thus the integrand in (107.1) is an integrable function, and

$$|\widehat{f}(\xi)| \le \int_{\mathbf{R}^n} |f(x)| \, dx \tag{107.3}$$

for every $\xi \in \mathbf{R}^n$.

Let R be a positive real number, and put $f_R(x) = f(x)$ when $|x| \le R$ and $f_R(x) = 0$ when $|x| > R$. Thus

$$\widehat{f}_R(\xi) = \int_{|x| \le R} f(x) \, \exp(-i\xi \cdot x) \, dx, \tag{107.4}$$

and

$$|\widehat{f}(\xi) - \widehat{f}_R(\xi)| \le \int_{|x| > R} |f(x)| \, dx \tag{107.5}$$

for every $\xi \in \mathbf{R}^n$ and $R > 0$. In particular, $\widehat{f}_R \to \widehat{f}$ uniformly on $\mathbf{R}^n$ as $R \to \infty$. It is easy to see that $\widehat{f}_R(\xi)$ is uniformly continuous on $\mathbf{R}^n$ for each $R > 0$, using the fact that $\exp(it)$ is uniformly continuous on the real line. It follows that $\widehat{f}(\xi)$ is also uniformly continuous on $\mathbf{R}^n$, since it is the uniform limit of uniformly continuous functions on $\mathbf{R}^n$.

Now let f, g be integrable functions on the real line, so that their convolution $f * g$ is also integrable, as in the previous section. The Fourier transform of $f * g$ is given by

$$\begin{aligned} (\widehat{f * g})(\xi) &= \int_{\mathbf{R}^n} (f * g)(x) \, \exp(-i\xi \cdot x) \, dx \\ &= \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} f(x - y) \, g(y) \, \exp(-i\xi \cdot x) \, dy \, dx. \end{aligned} \tag{107.6}$$

This is the same as

$$\int_{\mathbf{R}^n} \int_{\mathbf{R}^n} f(x) \, g(y) \, \exp(-i\xi \cdot (x + y)) \, dx \, dy, \tag{107.7}$$

by interchanging the order of integration and using the change of variables $x \mapsto x + y$. Because $\exp(i(r + t)) = \exp(ir) \exp(it)$ for every $r, t \in \mathbf{R}$, this double integral reduces to

$$\Big(\int_{\mathbf{R}^n} f(x) \, \exp(-i\xi \cdot x) \, dx \Big) \Big(\int_{\mathbf{R}^n} g(y) \, \exp(-i\xi \cdot y) \, dy \Big). \tag{107.8}$$

Thus

$$(\widehat{f * g})(\xi) = \widehat{f}(\xi) \, \widehat{g}(\xi) \tag{107.9}$$

for every $\xi \in \mathbf{R}^n$.

108 Holomorphic extensions

Let $L^1(\mathbf{R}^n)$ be the space of Lebesgue integrable functions on $\mathbf{R}^n$ equipped with the L^1 norm

$$\|f\|_1 = \int_{\mathbf{R}^n} |f(x)|\,dx, \tag{108.1}$$

as usual. Let us say that $f \in L^1(\mathbf{R}^n)$ has support contained in a closed set $E \subseteq \mathbf{R}^n$ if $f(x) = 0$ almost everywhere on $\mathbf{R}^n\backslash E$. The space $L^1_{com}(\mathbf{R}^n)$ of $f \in L^1(\mathbf{R}^n)$ with compact support is a dense linear subspace of $L^1(\mathbf{R}^n)$ which is closed under convolution, in the sense that $f * g \in L^1_{com}(\mathbf{R}^n)$ for every $f, g \in L^1_{com}(\mathbf{R}^n)$. If $f \in L^1_{com}(\mathbf{R}^n)$ is supported in a compact set K, then the Fourier transform $\widehat{f}(\xi)$ extends to a holomorphic function $\widehat{f}(\zeta)$ on $\mathbf{C}^n$, given by

$$\widehat{f}(\zeta) = \int_K f(x)\,\exp(-i\zeta\cdot x)\,dx. \tag{108.2}$$

Here $\zeta \in \mathbf{C}^n$ may be expressed as $\xi + i\eta$, with $\xi, \eta \in \mathbf{R}^n$, and

$$\zeta \cdot x = \sum_{j=1}^n \zeta_j\,x_j, \tag{108.3}$$

as before. Thus (108.2) reduces to (107.1) when $\zeta = \xi \in \mathbf{R}^n$, and otherwise it is easy to check that $\widehat{f}(\zeta)$ is a holomorphic function on $\mathbf{C}^n$, since the exponential function is holomorphic. In addition,

$$(\widehat{f * g})(\zeta) = \widehat{f}(\zeta)\,\widehat{g}(\zeta) \tag{108.4}$$

for every $f, g \in L^1_{com}(\mathbf{R}^n)$ and $\zeta \in \mathbf{C}^n$, for the same reasons as in the previous section.

Let $L^1_+(\mathbf{R})$ be the space of $f \in L^1(\mathbf{R})$ that are supported in $[0, \infty)$, and let $L^1_-(\mathbf{R})$ be the space of $f \in L^1(\mathbf{R})$ that are supported in $(-\infty, 0]$. These are closed linear subspaces of $L^1(\mathbf{R})$ that are closed under convolution, in the sense that $f * g \in L^1_+(\mathbf{R})$ when $f, g \in L^1_+(\mathbf{R})$, and similarly for $L^1_-(\mathbf{R})$. Let H_+, H_- be the upper and lower open half-planes in the complex plane, consisting of complex numbers with positive and negative imaginary parts, respectively. Thus their closures $\overline{H}_+$, $\overline{H}_-$ are the upper and lower closed half-planes in $\mathbf{C}$, consisting of complex numbers with imaginary part greater than or equal to 0 and less than or equal to 0, respectively. If $f \in L^1_+(\mathbf{R})$, then

$$\widehat{f}(\zeta) = \int_0^\infty f(x)\,\exp(-i\zeta\,x)\,dx = \int_0^\infty f(x)\,\exp(-i\xi\,x + \eta\,x)\,dx \tag{108.5}$$

is defined for all $\zeta = \xi + i\,\eta \in \overline{H}_-$. In this case, $\eta \leq 0$, so that

$$|\exp(-i\xi\,x + \eta\,x)| = \exp(\eta\,x) \leq 1 \tag{108.6}$$

for every $x \geq 0$, and hence

$$|\widehat{f}(\zeta)| \leq \|f\|_1 \tag{108.7}$$

for every $\zeta \in \overline{H}_-$. As in the previous section, one can check that $\widehat{f}(\zeta)$ is uniformly continuous on $\overline{H}_-$. This uses the fact that $\exp(-i\zeta\, x)$ is uniformly continuous as a function of ζ on $\overline{H}_-$ for each $x \ge 0$, and it is easier to first consider the case where f has compact support in $[0, \infty)$, and then get the same conclusion for any $f \in L_+(\mathbf{R})$ by approximation. One can also check that $\widehat{f}(\zeta)$ is holomorphic on H_-, using the holomorphicity of the exponential function and the integrability of the expressions in (108.5). If $f, g \in L^1_+(\mathbf{R})$, then

$$(108.8) \qquad \widehat{(f * g)}(\zeta) = \widehat{f}(\zeta)\, \widehat{g}(\zeta)$$

for every $\zeta \in \overline{H}_-$, for the same reasons as before. In the same way, the Fourier transform of a function in $L^1_-(\mathbf{R})$ has a natural extension to a bounded uniformly continuous function on $\overline{H}_+$ that is holomorphic on H_+, and with the analogous property for convolutions.

Let $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ be an n-tuple with $\epsilon_j \in \{1, -1\}$ for each j, which is to say an element of $\{1, -1\}^n$. Put

$$(108.9) \qquad Q_{n,\epsilon} = \{x \in \mathbf{R}^n : \epsilon_j\, x_j \ge 0 \text{ for } j = 1, \ldots, n\},$$

which is the closed "quadrant" in $\mathbf{R}^n$ associated to ϵ. Let $L^1_\epsilon(\mathbf{R}^n)$ be the set of $f \in L^1(\mathbf{R}^n)$ which are supported in $Q_{n,\epsilon}$. It is easy to see that $L^1_\epsilon(\mathbf{R}^n)$ is a closed linear subspace of $L^1(\mathbf{R}^n)$ that is closed with respect to convolution, as before. Consider

$$(108.10) \qquad H_{n,\epsilon} = \{\zeta = \xi + i\,\eta \in \mathbf{C}^n : \epsilon_j\, \eta_j > 0 \text{ for } j = 1, \ldots, n\},$$

so that the closure $\overline{H}_{n,\epsilon}$ of $H_{n,\epsilon}$ consists of the $\zeta = \xi + \eta \in \mathbf{C}^n$ with $\eta \in Q_{n,\epsilon}$. If $f \in L^1_\epsilon(\mathbf{R}^n)$, then

$$(108.11) \qquad \begin{aligned} \widehat{f}(\zeta) &= \int_{Q_{n,\epsilon}} f(x)\, \exp(-i\zeta \cdot x)\, dx \\ &= \int_{Q_{n,\epsilon}} f(x)\, \exp(-i\xi \cdot x + \eta \cdot x)\, dx \end{aligned}$$

is defined for every $\zeta = \xi + \eta \in H_{n,-\epsilon}$, where $-\epsilon = (-\epsilon_1, \ldots, -\epsilon_n)$. In this case, $\eta \cdot x \le 0$ for every $x \in Q_{n,\epsilon}$, so that

$$(108.12) \qquad |\exp(-i\xi \cdot x + \eta \cdot x)| = \exp(\eta \cdot x) \le 1,$$

and hence $|\widehat{f}(\zeta)| \le \|f\|_1$ for every $\zeta \in \overline{H}_{n,-\epsilon}$. As before, one can check that $\widehat{f}(\zeta)$ is uniformly continuous on $\overline{H}_{n,-\epsilon}$, and holomorphic on $H_{n,-\epsilon}$. If $f, g \in L^1_\epsilon(\mathbf{R}^n)$, then the extension of the Fourier transform of $f * g$ to $H_{n,-\epsilon}$ is equal to the product of the extensions of the Fourier transforms of f and g to $H_{n,-\epsilon}$, as usual.

109 The Riemann–Lebesgue lemma

If a, b are real numbers with $a < b$, then the Fourier transform of the indicator function $\mathbf{1}_{[a,b]}$ of the interval $[a, b]$ in the real line is equal to

$$\int_a^b \exp(-i\xi\, x)\, dx = i\xi^{-1} \left(\exp(-i\,\xi\, b) - \exp(-i\xi\, a)\right) \tag{109.1}$$

when $\xi \neq 0$, and to $b - a$ when $\xi = 0$. In particular, this tends to 0 as $|\xi| \to \infty$.

If $f \in L^1(\mathbf{R})$, then the Riemann–Lebesgue lemma states that

$$\lim_{|\xi| \to \infty} \widehat{f}(\xi) = 0. \tag{109.2}$$

This follows immediately from the remarks in the previous paragraph when f is a step function, which is to say a finite linear combination of indicator functions of intervals in the real line. Otherwise, any integrable function f on the real line can be approximated by step functions in the L^1 norm, which leads to an approximation of the Fourier transform $\widehat{f}$ of f by Fourier transforms of step functions in the supremum norm, by (107.3). This permits one to derive (109.2) for f from the corresponding statement for step functions.

This also works for integrable functions on $\mathbf{R}^n$. In this case, we can start with a rectangular box B in $\mathbf{R}^n$, which is to say the Cartesian product of n intervals in the real line. The indicator function of B on $\mathbf{R}^n$ is the same as the product of the n indicator functions of the corresponding intervals in $\mathbf{R}$, as functions of $x_1, \ldots, x_n$. Thus the Fourier transform of the indicator function of B is the same as the product of the n one-dimensional Fourier transforms of these indicator functions of intervals in $\mathbf{R}$, as functions of $\xi_1, \ldots, \xi_n$. This implies that the Fourier transform of the indicator function of B tends to 0 at infinity, as before. Hence the Fourier transform of any finite linear combination of indicator functions of rectangular boxes in $\mathbf{R}^n$ also tends to 0 at infinity. Any integrable function f on $\mathbf{R}^n$ can be approximated by a finite linear combination of indicator functions of rectangular boxes in the L^1 norm, which implies (109.2) as in the one-dimensional case.

As in the previous section, the Fourier transform of the indicator function $\mathbf{1}_{[a,b]}$ of an interval $[a, b]$ in the real line extends to a holomorphic function on the complex plane, given by

$$\int_a^b \exp(-i\zeta\, x)\, dx = i\,\zeta^{-1} \left(\exp(-i\zeta\, b) - \exp(-i\zeta\, a)\right) \tag{109.3}$$

when $\zeta \neq 0$, and equal to $b - a$ when $\zeta = 0$. If $a, b \geq 0$, then it is easy to see that this tends to 0 as $|\zeta| \to \infty$ when ζ is in the closed lower half-plane $\overline{H}_-$. If $f \in L^1_+(\mathbf{R})$, so that the Fourier transform of f has a natural extension $\widehat{f}(\zeta)$ to $\zeta \in \overline{H}_-$, as in the preceding section, then one can use this to show that $\widehat{f}(\zeta) \to 0$ as $|\zeta| \to \infty$ in $\overline{H}_-$, by approximating f by step functions as before. Of course, there is an analogous statement for the extension to the closed upper half-plane $\overline{H}_+$ of the Fourier transform of a function in $L^1_-(\mathbf{R}^n)$. There is also an analogous statement for the extension to $\overline{H}_{n,-\epsilon}$ of the Fourier transform of a function in $L^1_\epsilon(\mathbf{R}^n)$, as in the previous section.

110 Translation and multiplication

If $f \in L^1(\mathbf{R}^n)$ and $t \in \mathbf{R}^n$, then let $T_t(f)$ be the function on $\mathbf{R}^n$ obtained by translating f by t, so that

$$T_t(f)(x) = f(x - t). \tag{110.1}$$

Thus $T_t(f) \in L^1(\mathbf{R}^n)$ too, and $\|T_t(f)\|_1 = \|f\|_1$. It is easy to see that

$$(\widehat{T_t(f)})(\xi) = \exp(-i\xi \cdot t)\, \widehat{f}(\xi), \tag{110.2}$$

for each $\xi \in \mathbf{R}^n$, using the change of variable $x \mapsto x + t$ in the definition of $\widehat{T_t(f)}$. Similarly, if f has compact support in $\mathbf{R}^n$, then $T_t(f)$ does too, and the natural extension of the Fourier transform of $T_t(f)$ to a holomorphic function on $\mathbf{C}^n$ satisfies

$$(\widehat{T_t(f)})(\zeta) = \exp(-i\zeta \cdot t)\, \widehat{f}(\zeta) \tag{110.3}$$

for each $\zeta \in \mathbf{C}^n$.

Suppose now that $\epsilon \in \{1, -1\}^n$, and that $f \in L^1_\epsilon(\mathbf{R}^n)$, as in Section 108. Thus f is supported in the "quadrant" $Q_{n,\epsilon}$ defined in (108.9). If $t \in Q_{n,\epsilon}$, then it is easy to see that $T_t(f)$ is supported in $Q_{n,\epsilon}$ as well, so that $T_t(f) \in L^1_\epsilon(\mathbf{R}^n)$. As in Section 108, the Fourier transform of f and $T_t(f)$ have natural extensions to $\overline{H}_{n,-\epsilon}$, which are related by the same expression (110.3) as in the previous paragraph. Note that

$$|\exp(-i\zeta \cdot t)| \le 1 \tag{110.4}$$

for each $\zeta \in H_{n,-\epsilon}$ and $t \in Q_{n,\epsilon}$, as in Section 108.

If $w \in \mathbf{R}^n$ and $f \in L^1(\mathbf{R}^n)$, then let $M_w(f)$ be the function on $\mathbf{R}^n$ defined by multiplying f by $\exp(iw \cdot x)$, so that

$$(M_w(f))(x) = \exp(iw \cdot x)\, f(x). \tag{110.5}$$

Thus $M_w(f) \in L^1(\mathbf{R}^n)$ and $\|M_w(f)\|_1 = \|f\|_1$, since $|\exp(iw \cdot x)| = 1$ for every $x, w \in \mathbf{R}^n$. It is easy to see that

$$(\widehat{M_w(f)})(\xi) = \widehat{f}(\xi - w) \tag{110.6}$$

for every $\xi, w \in \mathbf{R}^n$, directly from the definition of the Fourier transform. If $w \in \mathbf{C}^n$, then we can still define $M_w(f)$ for $f \in L^1(\mathbf{R}^n)$ by (110.5), and $M_w(f)$ will be locally integrable on $\mathbf{R}^n$, but it may not be integrable on $\mathbf{R}^n$. However, if f has compact support in $\mathbf{R}^n$, then $M_w(f)$ also has compact support in $\mathbf{R}^n$ for every $w \in \mathbf{C}^n$, and $M_w(f)$ is integrable on $\mathbf{R}^n$ for every $w \in \mathbf{C}^n$. In this case, the Fourier transform of f extends to a holomorphic function on $\mathbf{C}^n$, as in Section 108, and the Fourier transform of $M_w(f)$ is defined and extends to a holomorphic function on $\mathbf{C}^n$ for each $w \in \mathbf{C}^n$. As before, we have that

$$(110.7) \qquad (\widehat{M_w(f)})(\zeta) = \widehat{f}(\zeta - w)$$

for every $\zeta, w \in \mathbf{C}^n$ when $f \in L^1_{com}(\mathbf{R}^n)$.

Let ϵ be an element of $\{1, -1\}^n$ again, and suppose that $f \in L^1_\epsilon(\mathbf{R}^n)$. As before, $M_w(f)$ is a locally integrable function on $\mathbf{R}^n$ with support contained in $Q_{n,\epsilon}$ for every $w \in \mathbf{C}^n$. If $w \in \overline{H}_{n,\epsilon}$, then $|\exp(iw \cdot x)| \le 1$ for every $x \in Q_{n,\epsilon}$, and hence $M_w(f) \in L^1_\epsilon(\mathbf{R}^n)$, with $\|M_w(f)\|_1 \le \|f\|_1$. As in Section 108, the Fourier transforms of f and $M_w(f)$ have natural extensions to $\overline{H}_{n,-\epsilon}$ under these conditions, and one can check that they are related as in (110.7) for each $\zeta \in \overline{H}_{n,-\epsilon}$. Note that $\widehat{f}(\zeta - w)$ is defined in this case, because $-w$ and hence $\zeta - w$ is in $\overline{H}_{n,-\epsilon}$.

111 Some examples

Let a be a positive real number, and put

$$(111.1) \qquad q_{a,+}(x) = \exp(-a\, x)$$

when $x \ge 0$, and $q_{a,+}(x) = 0$ when $x < 0$. Thus $q_{a,+} \in L^1_+(\mathbf{R})$, and so the Fourier transform of $q_{a,+}$ should have a natural extension to the closed lower half-plane in $\mathbf{C}$, as in Section 108. More precisely,

$$(111.2) \qquad \widehat{q_{a,+}}(\zeta) = \int_0^\infty \exp(-a\, x - i\zeta\, x)\, dx = \frac{-1}{-a - i\zeta} = \frac{1}{a + i\zeta}$$

for every $\zeta \in \overline{H}_-$. Note that $\mathrm{Re}(a + i\zeta) \ge a > 0$ when $\zeta \in \overline{H}_-$ and $a > 0$.

Similarly, put

$$(111.3) \qquad q_{a,-}(x) = \exp(a\, x) = \exp(-a\, |x|)$$

when $x \le 0$, and $q_{a,-}(x) = 0$ when $x > 0$. In this case, $q_{a,-} \in L^1_-(\mathbf{R}^n)$, so that the Fourier transform of $q_{a,-}$ should have a natural extension to the closed upper half-plane in $\mathbf{C}$, as in Section 108. Indeed,

$$(111.4) \qquad \widehat{q_{a,-}}(\zeta) = \int_{-\infty}^0 \exp(a\, x - i\zeta\, x)\, dx = \frac{1}{a - i\zeta}$$

for every $\zeta \in \overline{H}_+$. As before, $\mathrm{Re}(a - i\zeta) \ge a > 0$ when $\zeta \in \overline{H}_+$ and $a > 0$.

Now let $a = (a_1, \ldots, a_n)$ be an n-tuple of positive real numbers, and let ϵ be an element of $\{1, -1\}^n$. Put

$$(111.5) \qquad q_{n,a,\epsilon}(x) = \exp\Big(-\sum_{j=1}^n a_j\, \epsilon_j\, x_j\Big) = \exp\Big(-\sum_{j=1}^n a_j |x_j|\Big)$$

when $x \in Q_{n,\epsilon}$, and $q_{n,a,\epsilon}(x) = 0$ when $x \in \mathbf{R}^n \backslash Q_{n,\epsilon}$. Equivalently,

$$q_{n,a,\epsilon}(x) = \prod_{j=1}^{n} q_{a_j,\epsilon_j}(x_j), \tag{111.6}$$

where the subscript ϵ_j on the right should be interpreted as $+$ when $\epsilon_j = 1$ and as $-$ when $\epsilon_j = -1$. Of course, $q_{n,a,\epsilon} \in L^1_\epsilon(\mathbf{R}^n)$, and so its Fourier transform should have a natural extension to $\overline{H}_{n,-\epsilon}$, as usual. In fact, the Fourier transform of $q_{n,a,\epsilon}$ can be given as the product of the one-dimensional Fourier transforms of the factors q_{a_j,ϵ_j}, so that

$$\widehat{q_{n,a,\epsilon}}(\zeta) = \prod_{j=1}^{n} \widehat{q_{a_j,\epsilon_j}}(\zeta_j) = \prod_{j=1}^{n} \frac{1}{(a_j + i\epsilon_j\, \zeta_j)} \tag{111.7}$$

for each $\zeta \in \overline{H}_{n,-\epsilon}$.

112 Some examples, continued

Let a be a positive real number, and put

$$p_a(x) = \exp(-a\,|x|) = q_{a,+}(x) + q_{a,-}(x). \tag{112.1}$$

This defines an integrable function on the real line, whose Fourier transform is given by

$$\widehat{p_a}(\xi) = \widehat{q_{a,+}}(\xi) + \widehat{q_{a,-}}(\xi) = \frac{1}{a+i\xi} + \frac{1}{a-i\xi} = 2\,\mathrm{Re}\left(\frac{1}{a+i\xi}\right) \tag{112.2}$$

for each $\xi \in \mathbf{R}$. Of course,

$$\frac{1}{a+i\xi} = \frac{a-i\xi}{(a+i\xi)(a-i\xi)} = \frac{a-i\xi}{a^2+\xi^2}, \tag{112.3}$$

and so (112.2) is the same as

$$\widehat{p_a}(\xi) = \frac{2a}{a^2+\xi^2}. \tag{112.4}$$

It follows easily from (112.4) that $\widehat{p_a}(\xi)$ is an integrable function of ξ on the real line. In order to compute its integral, observe that

$$\int_{\mathbf{R}} \widehat{p_a}(\xi)\,d\xi = \lim_{R\to\infty} \int_{-R}^{R} \widehat{p_a}(\xi)\,d\xi = \lim_{R\to\infty} 2\,\mathrm{Re} \int_{-R}^{R} \frac{1}{a+i\xi}\,d\xi. \tag{112.5}$$

Using the change of variables $\xi \mapsto R\,\xi$, we get that

$$\int_{-R}^{R} \frac{1}{a+i\xi}\,d\xi = \int_{-1}^{1} \frac{1}{a+R\,\xi}\,R\,d\xi = \int_{-1}^{1} \frac{1}{a\,R^{-1}+i\xi}\,d\xi \tag{112.6}$$

for each $R > 0$. Hence

$$\int_{\mathbf{R}} \widehat{p_a}(\xi)\, d\xi = \lim_{r \to 0+} 2\,\mathrm{Re} \int_{-1}^{1} \frac{1}{r + i\xi}\, d\xi. \tag{112.7}$$

Let $\log z$ be the principal branch of the logarithm. Remember that this is a holomorphic function defined on the set of $z \in \mathbf{C}$ such that z is not a real number less than or equal to 0, which agrees with the ordinary natural logarithm of z when z is a positive real number, and whose derivative is equal to $1/z$. Thus

$$\int_{-1}^{1} \frac{1}{r + i\xi}\, i d\xi = \log(r + i) - \log(r - i) \tag{112.8}$$

for each $r > 0$, which implies that

$$\begin{aligned} 2\,\mathrm{Re} \int_{-1}^{1} \frac{1}{r + i\xi}\, d\xi &= 2\,\mathrm{Im} \int_{-1}^{1} \frac{1}{r + i\xi}\, i d\xi \\ &= 2\,\mathrm{Im}(\log(r + i) - \log(r - i)). \end{aligned} \tag{112.9}$$

Taking the limit as $r \to 0+$, we get that

$$\int_{\mathbf{R}} \widehat{p_a}(\xi)\, d\xi = 2\,\mathrm{Im}(\log i - \log(-i)) = 2\pi, \tag{112.10}$$

since $\log i = (\pi/2)i$ and $\log(-i) = -(\pi/2)i$.

Similarly, if $a = (a_1, \ldots, a_n)$ is an n-tuple of positive real numbers, then

$$p_{n,a}(x) = \prod_{j=1}^{n} p_{a_j}(x_j) = \exp\Big(- \sum_{j=1}^{n} a_j\, |x_j| \Big) \tag{112.11}$$

is an integrable function on $\mathbf{R}^n$. The Fourier transform of $p_{n,a}$ is the product of the one-dimensional Fourier transforms of its factors, given by

$$\widehat{p_{n,a}}(\xi) = \prod_{j=1}^{n} \widehat{p_{a_j}}(\xi_j) = \prod_{j=1}^{n} \frac{2\, a_j}{(a_j^2 + \xi_j^2)}. \tag{112.12}$$

The integral of this is equal to the product of the one-dimensional integrals of its factors, so that

$$\int_{\mathbf{R}^n} \widehat{p_{n,a}}(\xi)\, d\xi = (2\pi)^n. \tag{112.13}$$

113 The multiplication formula

Let f, g be integrable functions on $\mathbf{R}^n$. The *multiplication formula* states that

$$\int_{\mathbf{R}^n} \widehat{f}(\xi)\, g(\xi)\, d\xi = \int_{\mathbf{R}^n} f(x)\, \widehat{g}(x)\, dx. \tag{113.1}$$

Note that both sides of this equation make sense, because the Fourier transforms of f and g are bounded. Equivalently, (113.1) states that

$$\int_{\mathbf{R}^n} \Big(\int_{\mathbf{R}^n} f(x)\, \exp(-i\xi \cdot x)\, dx \Big)\, g(\xi)\, d\xi = \int_{\mathbf{R}^n} \Big(\int_{\mathbf{R}^n} g(\xi)\, \exp(-ix \cdot \xi)\, d\xi \Big) f(x)\, dx, \tag{113.2}$$

which follows from Fubini's theorem.

Let h be integrable function on $\mathbf{R}^n$, and let w be an element of $\mathbf{R}^n$. If

$$g(\xi) = \exp(i\xi \cdot w)\, h(\xi), \tag{113.3}$$

then

$$\widehat{g}(x) = \widehat{h}(x - w), \tag{113.4}$$

as in Section 110. If $f \in L^1(\mathbf{R}^n)$, then the multiplication formula implies that

$$\int_{\mathbf{R}^n} \widehat{f}(\xi)\, \exp(i\xi \cdot w)\, h(\xi)\, d\xi = \int_{\mathbf{R}^n} f(x)\, \widehat{h}(x - w)\, dx. \tag{113.5}$$

The right side is similar to $(f * \widehat{h})(w)$, but not quite the same.

If $h_1(\xi) = h(-\xi)$, then

$$\widehat{h_1}(x) = \int_{\mathbf{R}^n} h(-\xi)\, \exp(-i\xi \cdot x)\, d\xi = \int_{\mathbf{R}^n} h(\xi)\, \exp(i\xi \cdot x)\, dx = \widehat{h}(-x), \tag{113.6}$$

using the change of variable $x \mapsto -x$. Hence

$$\int_{\mathbf{R}^n} f(x)\, \widehat{h}(x - w)\, dx = \int_{\mathbf{R}^n} f(x)\, \widehat{h_1}(w - x)\, dx = (f * \widehat{h_1})(w). \tag{113.7}$$

Suppose now that h is an even function on $\mathbf{R}^n$, so that $h_1 = h$. Thus $\widehat{h}$ is even too, by (113.6). In this case, (113.5) reduces to

$$\int_{\mathbf{R}^n} \widehat{f}(\xi)\, \exp(i\xi \cdot w)\, h(\xi)\, d\xi = (f * \widehat{h})(w). \tag{113.8}$$

114 Convergence

Let $a = (a_1, \ldots, a_n)$ be an n-tuple of positive real numbers, and put

$$P_{n,a}(x) = \pi^{-n} \prod_{j=1}^{n} \frac{a_j}{(a_j^2 + x_j^2)}. \tag{114.1}$$

Thus $P_{n,a}$ is a nonnegative integrable function on $\mathbf{R}^n$ that satisfies

$$\int_{\mathbf{R}^n} P_{n,a}(x)\, dx = 1 \tag{114.2}$$

for each a, as in Section 112.

Let f be a bounded continuous function on $\mathbf{R}^n$. By standard arguments,

$$\lim_{a \to 0} (P_{n,a} * f)(x) = f(x) \tag{114.3}$$

for each $x \in \mathbf{R}^n$. Because f is uniformly continuous on compact subsets of $\mathbf{R}^n$, one also gets uniform convergence on compact subsets of $\mathbf{R}^n$ in (114.3). If f is uniformly continuous on $\mathbf{R}^n$, then one gets uniform convergence on $\mathbf{R}^n$.

If f is a continuous function on $\mathbf{R}^n$ with compact support, then f is bounded and uniformly continuous in particular, so that $P_{n,a} * f \to f$ uniformly on $\mathbf{R}^n$ as $a \to 0$, as in the previous paragraph. In this case, it is easy to check that $P_{n,a} * f \to f$ as $a \to 0$ in the L^1 norm on $\mathbf{R}^n$ too.

If f is any integrable function on $\mathbf{R}^n$, then

$$\|P_{n,a} * f\|_1 \le \|P_{n,a}\|_1 \, \|f\|_1 = \|f\|_1 \tag{114.4}$$

for each a, as in Section 106. One can also check that $P_{n,a} * f \to f$ as $a \to 0$ in the L^1 norm on $\mathbf{R}^n$, since this holds on a dense subset of $L^1(\mathbf{R}^n)$, as in the preceding paragraph.

115 Inversion

If f is an integrable function on $\mathbf{R}^n$, then

$$\int_{\mathbf{R}^n} \widehat{f}(\xi) \exp(i\xi \cdot w) \exp\Big(- \sum_{j=1}^n a_j \, |\xi_j| \Big) \, d\xi = (2\pi)^n \, (P_{n,a} * f)(w) \tag{115.1}$$

for every n-tuple $a = (a_1, \ldots, a_n)$ of positive real numbers and $w \in \mathbf{R}^n$. This follows from (113.8), with h equal to $p_{n,a}$ from Section 112. This also uses the fact that $p_{n,a}$ is even and satisfies

$$\widehat{p}_{n,a} = (2\pi)^n \, P_{n,a}, \tag{115.2}$$

where $P_{n,a}$ is as in the previous section.

If $\widehat{f}$ is also integrable on $\mathbf{R}^n$, then it is easy to see that

$$\lim_{a \to 0} \int_{\mathbf{R}^n} \widehat{f}(\xi) \exp(i\xi \cdot w) \exp\Big(- \sum_{j=1}^n a_j \, |\xi_j| \Big) \, d\xi = \int_{\mathbf{R}^n} \widehat{f}(\xi) \exp(i\xi \cdot w) \, d\xi \tag{115.3}$$

for every $w \in \mathbf{R}^n$. More precisely,

$$\widehat{f}(\xi) \exp\Big(- \sum_{j=1}^n a_j \, |\xi_j| \Big) \to \widehat{f}(\xi) \tag{115.4}$$

as $a \to 0$ in the L^1 norm on $\mathbf{R}^n$, so that one has uniform convergence in w in the previous statement. This can be derived from the dominated convergence theorem, but one can also use the same type of argument a bit more directly. The main points are that

$$\exp\Big(-\sum_{j=1}^{n} a_j\,|\xi_j|\Big) \le 1 \tag{115.5}$$

for every a and ξ, and that

$$\exp\Big(-\sum_{j=1}^{n} a_j\,|\xi_j|\Big) \to 1 \tag{115.6}$$

as $a \to 0$ uniformly on compact subsets of $\mathbf{R}^n$.

It follows that

$$\int_{\mathbf{R}^n} \widehat{f}(\xi)\,\exp(i\xi\cdot w)\,d\xi = (2\pi)^n\,f(w) \tag{115.7}$$

for almost every $w \in \mathbf{R}^n$ when f and $\widehat{f}$ are integrable functions on $\mathbf{R}^n$, since $P_{n,a} * f \to f$ in $L^1(\mathbf{R}^n)$ as $a \to 0$, as in the preceding section. In particular, $f = 0$ almost everywhere on $\mathbf{R}^n$ when $\widehat{f} = 0$.

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