

L-SPACE SURGERIES, GENUS BOUNDS, AND THE CABLING CONJECTURE

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ABSTRACT. We prove that if positive integer p -surgery along a knot $K \subset S^3$ produces an L-space and it bounds a sharp 4-manifold, then the knot genus obeys the bound

$$2g(K) - 1 \leq p - \sqrt{3p + 1}.$$

Moreover, there exists an infinite family of pairs (K_n, p_n) attaining this bound, where K_n denotes an n -fold iterated cable of the unknot and $p_n \rightarrow \infty$. In particular, the stated bound applies when the knot surgery produces a lens space or a connected sum thereof. Combined with work of Gordon-Luecke, Hoffman, and Matignon-Sayari, it follows that if surgery along a knot produces a connected sum of lens spaces, then the knot is either a torus knot or a cable thereof, confirming the cabling conjecture in this case.

1. INTRODUCTION

1.1. Lens space surgeries. Denote by K a knot in S^3 , p a positive integer, and q a non-zero integer. For a knot K and slope $r = p/q$, let K_r denote the result of r -framed Dehn surgery along K . By definition, a lens space takes the form U_r , where U denotes the unknot and $r \neq 1/n$.

When can surgery along a non-trivial knot K produce a lens space? This question remains unanswered forty years since Moser first raised it [21], although work by several researchers has led to significant progress on it. For example, the cyclic surgery theorem of Culler-Gordon-Luecke-Shalen asserts that either K is a torus knot or the surgery slope is an integer [4], and a conjecturally complete construction due to Berge accounts for all the known examples [3].

On the basis of Berge's construction, Goda-Teragaito conjectured an inequality relating the surgery slope that produces a lens space and the knot genus $g(K)$ [8]. Reflect K if necessary in order to assume that the slope is positive; then their conjecture asserts that for a *hyperbolic* knot K ,

$$(1) \quad \frac{p-1}{2} \leq 2g(K) - 1 \leq p - 9.$$

The case of a non-hyperbolic knot is well-understood. Note that $2g(K) - 1$ equals *minus* the maximum Euler characteristic of a Seifert surface for K .

Both bounds in (1) are now close to settled. Rasmussen established the inequality

$$\frac{p-5}{2} \leq 2g(K) - 1$$

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for *any* knot K for which K_p is a lens space, noting that it is attained for $p = 4k + 3$ and K the $(2, 2k + 1)$ -torus knot [29, Theorem 1]. Kronheimer-Mrowka-Ozsváth-Szabó established the bound

$$(2) \quad 2g(K) - 1 \leq p$$

by an application of monopole Floer homology [19, Corollary 8.5]. Their argument utilizes the fact that the Floer homology of a lens space is as simple as possible: $\text{rk } \widehat{HF}(Y) = |H_1(Y; \mathbb{Z})|$. A space with this property is called an *L-space*, and a knot with a positive L-space surgery is called an *L-space knot*. The proof of (2) extends to show that the set of positive rational slopes for which surgery along K results in an L-space is either empty or consists of all rational values $\geq 2g(K) - 1$. This fact holds in the setting of Heegaard Floer homology as well [23], the framework in place for the remainder of this paper.

As remarked in [19, pp. 537-8], the bound (2) can often be improved for the case of a lens space surgery. Indeed, a closer examination of the Berge knots suggests the bound

$$(3) \quad 2g(K) - 1 \leq p - 2\sqrt{(4p + 1)/5}$$

whenever K_p is a lens space, with the exception of K the right-hand trefoil and $p = 5$ (cf. [31]). This bound is attained by an infinite sequence of type VIII Berge knots K and slopes $p \rightarrow \infty$. Indeed, work of Rasmussen, including extensive computer calculations, implies that the bound (3) holds for all $p \leq 100,000$ [30].

The current work addresses an improvement on the bound (2) in the direction of (3). We begin with the method introduced and carried out in [19, 23], which uses a version of Theorem 2.4 below. That theorem uses the *correction terms* of a lens space $U_{p/q}$ to place a restriction on the genus of a knot K with $K_p = U_{p/q}$. However, the formulae for these correction terms often prove unwieldy towards the end of extracting explicit bounds on the knot genus. The key advance presented here stems from the observation that a lens space bounds a *sharp* four-manifold (Definition 2.1), whose existence enables us to distill the desired information. In this more general set-up, we obtain the following result.

Theorem 1.1. *Let K denote an L-space knot and suppose that K_p bounds a smooth, negative definite 4-manifold X with $H_1(X; \mathbb{Z})$ torsion-free. Then the knot genus is bounded above by*

$$(4) \quad 2g(K) - 1 \leq p - \sqrt{p} - 1.$$

If X can be chosen sharp, then we obtain the improved bound

$$(5) \quad 2g(K) - 1 \leq p - \sqrt{3p + 1}.$$

Furthermore, there exists an infinite family of pairs (K_n, p_n) that attain equality in (5), where K_n denotes an n -fold iterated cable of the unknot, and $p_n \rightarrow \infty$.

We do not know as much concerning the tightness of inequality (4). It does, however, lead to an improvement over [22, Proposition 1.3] for $p \geq 9$, which under the same assumptions establishes that $2g(K) - 1 \leq p - 4$ for the specific case of a torus knot K .

For the case of a lens space surgery, we establish the bound (3) in forthcoming work [14].

Theorem 1.2. *Suppose that $K \subset S^3$, p is a positive integer, and K_p is a lens space. Then*

$$2g(K) - 1 \leq p - 2\sqrt{(4p+1)/5},$$

unless K is the right-hand trefoil and $p = 5$. Moreover, this bound is attained by an infinite family of distinct type VIII Berge knots K and slopes $p \rightarrow \infty$. \square

For comparison between Theorems 1.1 and 1.2, note that $2\sqrt{4/5} \approx 1.79$ and $\sqrt{3} \approx 1.73$. We touch on Theorem 1.2 again in Section 5.3.

Thus, both (3) and (5) dramatically improve on Goda-Teragaito's second conjectured bound (1) for $p \gg 0$. In Section 5.2 we indicate how that bound follows for all except two values $p \in \{14, 19\}$. In addition, Baker-Grigsby-Hedden [2] and Rasmussen [30] have proposed programs to prove the completeness of Berge's construction using Floer homology. One step involved in both approaches is to argue the non-existence of a non-trivial knot K for which $K_{2g(K)-1}$ is a lens space. This fact follows immediately from Theorem 1.1.

1.2. Reducible surgeries. When can surgery along a knot K produce a reducible 3-manifold? The cabling conjecture of Gonzalez-Acuña – Short asserts that this can only occur when the knot is a cable knot, with the surgery slope provided by the cabling annulus [9, Conjecture A], [18, Problem 1.79]. From this it would follow that the surgery slope is an integer, and the reducible manifold has two prime summands, one of which is a lens space.

Analogous to the cyclic surgery theorem in this context, Gordon-Luecke proved that the surgery slope of a reducible surgery is an integer p , which we can again take to be positive upon reflecting the knot [11]. They also proved that K_p has a lens space summand [12, Theorem 3]. In this vein, further work of Howie, Sayari, and Valdez Sánchez implies that K_p has at most three prime summands, and if it has three, then two are lens spaces of coprime orders and the third is a homology sphere [17, 32, 33].

Apparently unknown to practitioners of Floer homology, a bound strikingly opposite to (2) holds in this setting. Building on work of Hoffman, Matignon-Sayari showed that if K_p is reducible, then either K and p satisfy the conclusions of the cabling conjecture, or else

$$(6) \quad p \leq 2g(K) - 1$$

[16, 20]. Note that if K is a cable knot with cabling slope p , then there is no relation in general between p and $g(K)$. On the other hand, assuming the surgered manifold is an L-space, we have the following easy result.

Proposition 1.3. *If K is a cable knot with cabling slope p , and K_p is an L-space, then $2g(K) - 1 < p$.*

Proof. Let $K = C_{q,r}(\kappa)$ denote the cable knot, where $|q| \geq 2$. Thus, $p = qr$ and $K_p \cong \kappa_{r/q} \# L(q, r)$. In order for K_p to be an L-space, $\kappa_{r/q}$ must be as well, so (2) implies that

$$2g(\kappa) - 1 < r/q;$$

the inequality is strict since the left side is an integer while the right side is not. On the other hand, an elementary calculation shows that

$$2g(K) - 1 = qr + q(2g(\kappa) - 1) - r.$$

Thus, $2g(K) - 1 < qr = p$, as desired. □

Thus, in light of the Matignon-Sayari bound (6) and Proposition 1.3, in order to establish the cabling conjecture under the assumption that the surgered manifold is an L-space, it suffices to show that $K_{2g(K)-1}$ is never a reducible L-space. Theorem 1.1 shows that this is the case if we further assume that the surgered manifold bounds a negative definite 4-manifold with torsion-free H_1 .

Corollary 1.4. *Suppose that K_p is a reducible L-space and it bounds a smooth, negative definite 4-manifold X with $H_1(X; \mathbb{Z})$ torsion-free. Then K is a cable knot with cabling slope p .* □

In particular, Corollary 1.4 applies to a connected sum of lens spaces, a natural case of interest in view of the fact that any reducible surgery has a lens space summand. Accordingly, the cabling conjecture follows in this case. A quick appeal to [11, §3] fills in the details of the following result.

Theorem 1.5. *Suppose that surgery along a knot $K \subset S^3$ produces a connected sum of lens spaces. Then K is either a (p, q) -torus knot or a (p, q) -cable of an (r, s) -torus knot with $p = qrs \pm 1$, and the surgery slope is pq . The surgered manifold is $L(p, q) \# L(q, p)$ or $L(p, qs^2) \# L(q, \pm 1)$, respectively.* □

1.3. Conventions. We use homology groups with integer coefficients throughout. All 4-manifolds are assumed smooth. For a compact 4-manifold X , regard $H_2(X)$ as an inner product space equipped with the intersection pairing on X . Let $W_{\pm p}(K)$ denote the 4-manifold obtained by attaching a $\pm p$ -framed 2-handle to D^4 along the knot $K \subset S^3 = \partial D^4$.

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2. INPUT FROM FLOER HOMOLOGY

In [23], Ozsváth-Szabó associated a numerical invariant $d(Y, \mathfrak{t}) \in \mathbb{Q}$ called a *correction term* to an oriented rational homology sphere Y equipped with a spin^c structure \mathfrak{t} . It is analogous to Frøyshov's h -invariant in monopole Floer homology [7]. They proved that this invariant obeys the relation $d(-Y, \mathfrak{t}) = -d(Y, \mathfrak{t})$, and that if Y is the boundary of a negative definite 4-manifold X , then

$$(7) \quad c_1(\mathfrak{s})^2 + b_2(X) \leq 4d(Y, \mathfrak{t})$$

for every $\mathfrak{s} \in \text{Spin}^c(X)$ which extends $\mathfrak{t} \in \text{Spin}^c(Y)$ [23, Theorem 9.6].

Definition 2.1. *A negative definite 4-manifold X is sharp if, for every $\mathfrak{t} \in \text{Spin}^c(Y)$, there exists some extension $\mathfrak{s} \in \text{Spin}^c(X)$ that attains equality in the bound (7).*

The following result provides the examples of L-spaces and sharp 4-manifolds that we will need.

Proposition 2.2 (Proposition 3.3 and Theorem 3.4, [28]). *Let L denote a non-split alternating link. Then the branched double-cover $\Sigma(L)$ is an L-space¹, and there exists a sharp 4-manifold X with $\partial X = \Sigma(L)$ and $H_1(X) = 0$.*

Every lens space $U_{p/q}$ arises as the branched double-cover of a 2-bridge link. In this case, the 4-manifold $X_{p/q}$ implied by Proposition 2.2 admits the following description. Assume $0 < q < p$, and write $p/(p-q) = [a_1, \dots, a_n]^-$ as a Hirzebruch-Jung continued fraction, with each $a_i \geq 2$. Then $X_{p/q}$ denotes plumbing along a linear chain of disk bundles over S^2 with Euler numbers $-a_1, \dots, -a_n$, in that order. In particular, $W_{-p}(U)$ is sharp, since it is diffeomorphic to the disk bundle of Euler number $-p$ over S^2 .

In order to make use of (7), we must understand spin^c structures on K_p . Given $\mathfrak{t} \in \text{Spin}^c(K_p)$, it extends to some $\mathfrak{s} \in \text{Spin}^c(W_p(K))$ since $H_1(W_p(K)) = 0$. The group $H_2(W_p(K))$ is generated by the class of a surface Σ obtained by smoothly gluing the core of the handle attachment to a copy of a Seifert surface for K with its interior pushed into $\text{int}(D^4)$. The quantity $\langle c_1(\mathfrak{s}), [\Sigma] \rangle + p$ is an even value $2i$ whose residue class (mod $2p$) does not depend on the choice of extension \mathfrak{s} . The assignment $\mathfrak{t} \mapsto i$ sets up a 1-1 correspondence

$$(8) \quad \text{Spin}^c(K_p) \xrightarrow{\sim} \mathbb{Z}/p\mathbb{Z}.$$

Next, suppose that K_p bounds a smooth, negative definite 4-manifold X . The manifold $W := -W_p(K) \cong W_{-p}(\overline{K})$ is negative definite and has boundary $\overline{K}_{-p} = -K_p$, where \overline{K} denotes the mirror image of K . Form the closed, smooth, oriented 4-manifold $Z := X \cup W$. Since $b_1(K_p) = 0$, it follows that $n := b_2(Z) = b_2(X) + b_2(W)$; and since $H_2(X) \oplus H_2(W) \hookrightarrow H_2(Z)$, it follows that Z is negative definite. In particular, the square of a class in $H^2(Z)$ equals the sum of the squares of its restrictions to $H^2(X)$ and $H^2(W)$.

Lemma 2.3. *Suppose that K_p bounds a smooth, negative definite 4-manifold X with $H_1(X)$ torsion-free, and form $Z = X \cup W$ as above. Then every $i \in \text{Spin}^c(K_p)$ extends to some $\mathfrak{s} \in \text{Spin}^c(Z)$, and*

$$(9) \quad c_1(\mathfrak{s})^2 + n \leq 4d(K_p, i) - 4d(U_p, i).$$

Furthermore, if X is sharp, then for every i there exists some extension \mathfrak{s} that attains equality in (9).

Proof. The fact that every spin^c structure on K_p extends across Z follows from the fact that $H_1(X)$ and $H_1(W)$ are torsion-free. Now fix some $i \in \text{Spin}^c(K_p)$ and an extension $\mathfrak{s} \in \text{Spin}^c(Z)$. From (7) we obtain

$$c_1(\mathfrak{s}|X)^2 + b_2(X) \leq 4d(K_p, i).$$

Observe that the maximum value of $c_1(\mathfrak{s}|W)^2 + 1$ does not depend on the knot K . Since $W_{-p}(U)$ is sharp, it follows that this value equals $4d(U_{-p}, i)$. Therefore,

$$c_1(\mathfrak{s}|W)^2 + 1 \leq -4d(U_p, 1).$$

¹No relation between the ‘‘L’’s!

Summing these two inequalities results in (9).

We obtain equality in (9) under the assumption that X is sharp by taking an extension of i to some $\mathfrak{s}_X \in \text{Spin}^c(X)$ that attains equality in (7) and gluing it to an extension $\mathfrak{s}_W \in \text{Spin}^c(W)$ that attains the value $-4d(U_p, i)$.

□

Let K denote an L-space knot. We aim to use (9) to obtain information about the knot genus. Consider the Alexander polynomial of K ,

$$\Delta_K(T) = \sum_{j=-g}^g a_j \cdot T^j, \quad g := \deg(\Delta_K),$$

and define the *torsion coefficient*

$$t_i(K) = \sum_{j \geq 1} j \cdot a_{|i|+j}.$$

Since K is an L-space knot, [27, Theorem 1.2] implies that the knot Floer homology group $\widehat{HFK}(K)$ is uniquely determined by the Alexander polynomial Δ_K . In particular, the maximum Alexander grading in which this group is supported is equal to the degree g of Δ_K . On the other hand, [24, Theorem 1.2] implies that this grading equals the knot genus:

$$g = g(K).$$

Furthermore, [27, Theorem 1.2] implies that the non-zero coefficients of the Alexander polynomial take values ± 1 and alternate in sign, beginning with $a_g = 1$. It follows that for all $i \geq 0$, the quantity

$$t_i(K) - t_{i+1}(K) = \sum_{j \geq 1} a_{i+j}$$

is always 0 or 1, so the $t_i(K)$ form a sequence of monotonically decreasing, non-negative integers for $i \geq 0$. Therefore, we obtain

$$(10) \quad t_i(K) = 0 \text{ if and only if } |i| \geq g(K).$$

Owens-Strle state the following result explicitly [22, Theorem 6.1]; it slightly extends the case $q = 1$ of [25, Theorem 1.2] (see also [19, Theorem 8.5] or the identical [23, Corollary 7.5]).

Theorem 2.4. *Let K denote an L-space knot and p a positive integer. Then the torsion coefficients and correction terms satisfy*

$$(11) \quad -2t_i(K) = d(K_p, i) - d(U_p, i), \text{ for all } |i| \leq p/2.$$

In [19, p. 538], the stated version of Theorem 2.4 is used in conjunction with (10) to enumerate the lens spaces obtained by surgery on a knot K with genus $g(K) \leq 5$. By using this approach in tandem with Lemma 2.3, we will obtain the estimates presented in Theorem 1.1. To that end, we focus our attention to the left-hand side of (9). Donaldson's theorem implies that $H_2(Z) \cong -\mathbb{Z}^n$, where $-\mathbb{Z}^n$ denotes the integer lattice equipped with *minus* the

standard Euclidean inner product [5]. Choose an orthonormal basis $\{e_1, \dots, e_n\}$ for $-\mathbb{Z}^n$: $\langle e_i, e_j \rangle = -\delta_{ij}$ for all i, j . The first Chern class map

$$c_1 : \text{Spin}^c(Z) \rightarrow H^2(Z)$$

has image the set of *characteristic covectors* for the inner product space $H_2(Z)$. Identify $H_2(Z) \cong H^2(Z)$ by Poincaré duality; then Donaldson's theorem implies that this set corresponds to

$$\text{Char}(-\mathbb{Z}^n) = \left\{ \mathbf{c} = \sum_{i=1}^n \mathbf{c}_i e_i \mid \mathbf{c}_i \text{ odd for all } i \right\}.$$

Write

$$\sigma = \sum_{i=1}^n \sigma_i e_i$$

for the image of the class $[\Sigma]$ under the inclusion $H_2(W) \hookrightarrow H_2(Z) \cong -\mathbb{Z}^n$.

With the preceding notation in place, the following Lemma follows on combination of Lemma 2.3 with Theorem 2.4.

Lemma 2.5. *Let K denote an L-space knot, and suppose that K_p bounds a smooth, negative definite 4-manifold X with $H_1(X)$ torsion-free. Then*

$$(12) \quad \mathbf{c}^2 + n \leq -8t_i(K)$$

for all $|i| \leq p/2$ and $\mathbf{c} \in \text{Char}(-\mathbb{Z}^n)$ such that $\langle \mathbf{c}, \sigma \rangle + p \equiv 2i \pmod{2p}$. Furthermore, if X is sharp, then for every $|i| \leq p/2$ there exists \mathbf{c} that attains equality in (12). \square

3. THE GENUS BOUNDS

We proceed to establish the bounds appearing in Theorem 1.1. Both bounds stem from the following result, whose proof and application are elementary.

Proposition 3.1. *Under the hypotheses of Lemma 2.5,*

$$(13) \quad 2g(K) \leq p - |\sigma|_1,$$

with equality if X is sharp. Here $|\sigma|_1$ denotes the L^1 norm $\sum_{i=1}^n |\sigma_i|$.

Proof. Select a value $0 \leq i \leq p/2$ for which the left-hand side of (12) vanishes. Hence there exists $\mathbf{c} \in \{\pm 1\}^n \subset -\mathbb{Z}^n$ for which $\langle \mathbf{c}, \sigma \rangle + p \equiv 2i \pmod{2p}$. On the other hand, we have $|\langle \mathbf{c}, \sigma \rangle| \leq |\langle \sigma, \sigma \rangle| = p$, so that in fact $\langle \mathbf{c}, \sigma \rangle + p = 2i$. Now, the assumption on i and the non-negativity of the torsion coefficients together imply that $t_i(K) = 0$, so that in fact $i \geq g(K)$ by (10). It follows that for all $\mathbf{c} \in \{\pm 1\}^n$, we have

$$(14) \quad 2g(K) \leq \langle \mathbf{c}, \sigma \rangle + p.$$

The minimum value of the right-hand side of (14) is attained by the *sign vector* $s(\sigma)$, defined by

$$s(\sigma)_j := \begin{cases} +1, & \text{if } \sigma_j \geq 0; \\ -1, & \text{otherwise.} \end{cases}$$

For it, (14) produces the desired bound (13). The equality under the assumption that X is sharp follows immediately. \square

The bound (4) in Theorem 1.1 follows at once from Proposition 3.1 and the trivial inequality $p = |\langle \sigma, \sigma \rangle| \leq |\sigma|_1^2$. Now suppose that X is sharp. Then $2g(K) = p - |\sigma|_1$ by Proposition 3.1, and its proof extends to show that for all $p - |\sigma|_1 \leq 2i \leq p$, there exists $\mathbf{c} \in \{\pm 1\}^n$ with $\langle \mathbf{c}, \sigma \rangle + p = 2i$. Replacing any such \mathbf{c} by its negative, we obtain this fact for all $p - |\sigma|_1 \leq 2i \leq p + |\sigma|_1$. In other words, for all $-|\sigma|_1 \leq j \leq |\sigma|_1$ with $j \equiv p \equiv |\sigma|_1 \pmod{2}$, there exists $\mathbf{c} \in \{\pm 1\}^n$ for which $\langle \mathbf{c}, \sigma \rangle = j$. By a change of basis of $-\mathbb{Z}^n$, we may assume that the vector σ has the property that

$$0 \leq \sigma_1 \leq \cdots \leq \sigma_n.$$

Write a vector $\mathbf{c} \in \{\pm 1\}^n$ in the form $(-1, \dots, -1) + 2\chi$, where $\chi \in \{0, 1\}^n$. Then we obtain that for every $0 \leq k \leq |\sigma|_1$, there exists $\chi \in \{0, 1\}^n$ for which $|\langle \chi, \sigma \rangle| = -\langle \chi, \sigma \rangle = k$. In other words, for every such k , there exists a subset $S \subset \{1, \dots, n\}$ for which $\sum_{i \in S} \sigma_i = k$.

Lemma 3.2. *Consider a sequence of integers $0 = \sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_n$. For every value $0 \leq k \leq \sigma_1 + \cdots + \sigma_n$, there exists a subset $S \subset \{1, \dots, n\}$ such that $\sum_{i \in S} \sigma_i = k$ if and only*

$$(15) \quad \sigma_i \leq \sigma_0 + \cdots + \sigma_{i-1} + 1 \text{ for all } 1 \leq i \leq n.$$

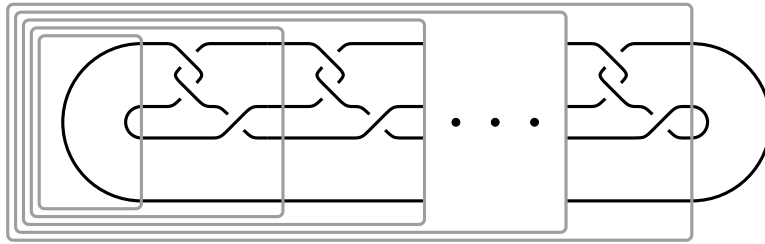
If we imagine the σ_i as values of coins, then Lemma 3.2 provides a necessary and sufficient condition under which one can make exact change from them in any amount up to their total value. We call such a vector $\sigma = (\sigma_0, \dots, \sigma_n)$ a *changemaker* (cf. [13]).

Proof. (\implies) We proceed by induction on n . The statement is obvious when $n = 1$. For the induction step, select any value $1 \leq k \leq \sigma_1 + \cdots + \sigma_n$, and pick the largest value j for which $k \geq \sigma_0 + \cdots + \sigma_{j-1} + 1$. By (15), $k - \sigma_j \geq 0$, and by the choice of j , we have $k - \sigma_j \leq \sigma_0 + \cdots + \sigma_{j-1}$. By induction on n , there exists $S' \subset \{1, \dots, j-1\}$ (possibly the empty set) for which $\sum_{i \in S'} \sigma_i = k - \sigma_j$; now $S = S' \cup \{j\}$ provides the desired subset with $\sum_{i \in S} \sigma_i = k$.

(\impliedby) We establish the contrapositive statement. If the inequality (15) failed for some i , then let k denote the value $\sigma_0 + \cdots + \sigma_{i-1} + 1$. For any $S \subset \{1, \dots, n\}$, either $j < i$ for all $j \in S$, in which case $\sum_{j \in S} \sigma_j < k$, or there exists some $j \in S$ with $j \geq i$, in which case $\sum_{j \in S} \sigma_j \geq \sigma_i > k$. Therefore, there does not exist a subset S such that $\sum_{j \in S} \sigma_j = k$. \square

Returning to the case at hand, we appeal to Lemma 3.2 and invoke the inequality (15) for each $1 \leq i \leq n$ to obtain the estimate

$$\begin{aligned} (\sigma_1 + \cdots + \sigma_n + 1)^2 &= 1 + \sum_{i=1}^n \sigma_i^2 + 2\sigma_i(\sigma_0 + \cdots + \sigma_{i-1} + 1) \\ &\geq 1 + \sum_{i=1}^n 3\sigma_i^2 = 3p + 1. \end{aligned}$$

FIGURE 1. The knot κ_n .

It follows that $|\sigma|_1 \geq \sqrt{3p+1} - 1$, and on combination with the equality in (13) we obtain the desired bound (5).

4. ITERATED CABLES

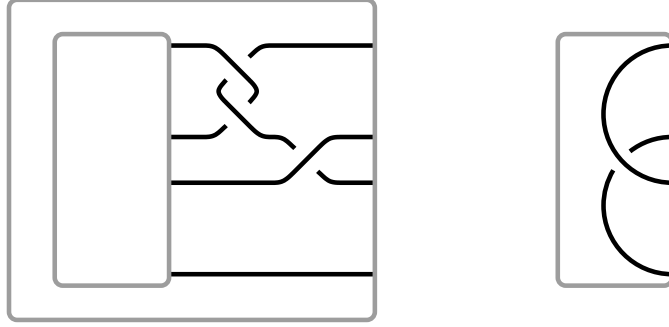
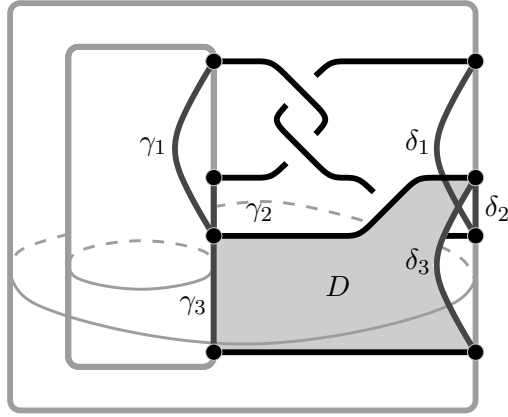
In this section, we prove the final assertion of Theorem 1.1.

Proposition 4.1. *Let $p_0 = 0$, and for $n \geq 0$, inductively define $a_{n+1} = 2p_n + 1$ and $p_n = 2a_n - 1$. Let K_n denote the $(2, a_n)$ -cable of the $(2, a_{n-1})$ -cable of the \cdots $(2, a_1)$ -cable of the unknot. Then for all $n \geq 1$, p_n -surgery along K_n is an L-space which bounds a sharp 4-manifold, and the bound in Equation (5) is attained in this case.*

We proceed by constructing a sharp 4-manifold X_n for which $\partial X_n = \Sigma(\kappa_n)$ for a particular alternating knot κ_n . It follows quickly from the presentation of κ_n that $\Sigma(\kappa_n) = (K_n)_{p_n}$ for some knot K_n and slope $p_n \in \mathbb{Q}$. The bulk of the argument consists in identifying the pair (K_n, p_n) with the one stated in the Proposition.

Proof. Let κ_n denote the alternating knot depicted in Figure 1. It contains n copies of the tangle \mathcal{T} displayed in Figure 2. According to Proposition 2.2, $\Sigma(\kappa_n)$ is an L-space for all $n \geq 1$, and there exists a sharp 4-manifold X_n with $\partial X_n = \Sigma(\kappa_n)$. This space admits an explicit Kirby calculus description by attaching 2-handles along a framed link $\mathbb{L}_n \subset S^3 = \partial D^4$. Here \mathbb{L}_n denotes a linear chain of $n - 1$ unknots, with each component framed by -5 and oriented clockwise, and with each consecutive pair in the chain linked twice positively. This Kirby description begins from the one described on [26, p. 719], replacing each 1-handle by a 0-framed unknot, and blowing down the (-1) -curves.

The space $\Sigma(\mathcal{T})$ is the (unique) Seifert-fibered space over the annulus with a single exceptional fiber of multiplicity 2. Equivalently, it is homeomorphic to a $C_{2,q}$ cable space (here q can denote any odd number). In Figure 3 we redraw \mathcal{T} with emphasis on a collection of arcs drawn on its boundary. Filling along the preimage $\tilde{\gamma}_1$ in $\Sigma(\mathcal{T})$ produces a solid torus with meridian given by $\tilde{\delta}_1$. Observe that by filling \mathcal{T} with the other tangle \mathcal{T}' in Figure 2, we obtain a tangle isotopic as a marked tangle to \mathcal{T}' itself. Let \mathcal{T}_n denote the complement to the inner-most tangle in the picture for κ_n . By construction, one rational filling of \mathcal{T}_n produces κ_n , while filling with \mathcal{T}' produces the unknot. It follows that the space $\Sigma(\mathcal{T}_n)$ is the complement of some knot $K_n \subset S^3$ for which p_n -surgery produces $\Sigma(\kappa_n)$ for some $p_n \in \mathbb{Q}$. Identify the picture of \mathcal{T} in Figure 3 with the inner-most copy appearing in the diagram for κ_n .

FIGURE 2. A pair of tangles \mathcal{T} and \mathcal{T}' .FIGURE 3. Arcs on the boundary of \mathcal{T} .

We claim that for all $n \geq 0$,

- (1) the pair (K_n, p_n) agrees with the pair stated in the Proposition;
- (2) the curve $\tilde{\gamma}_1$ represents a meridian μ for K_n ; and
- (3) the curve $\tilde{\gamma}_2$ represents $p_n \cdot \mu + \lambda$, where λ denotes the Seifert-framed longitude of K_n , and μ and λ are oriented so that $\langle \mu, \lambda \rangle = +1$.

We proceed by induction on n . When $n = 0$, κ_0 is a two-component unlink, and $\Sigma(\kappa_0) = S^1 \times S^2$. Assertions (1)-(3) follow easily by direct inspection. Now assume that $n > 0$. The space $\Sigma(\mathcal{T}_n)$ consists of filling $\Sigma(\mathcal{T}_{n-1})$ with the cable space $\Sigma(\mathcal{T})$, where a meridian $\tilde{\delta}_1$ of $\Sigma(\mathcal{T})$ gets identified with the meridian μ' of K_{n-1} . It follows at once that $\Sigma(\mathcal{T}_n)$ is the complement of some 2-cable of K_{n-1} ; it stands to determine which precisely. Observe that $\tilde{\gamma}_1$ is a meridian μ for K_n since filling along it produces S^3 . Also, $\tilde{\gamma}_3$ is a longitude for K_n since it meets μ in a single point. Furthermore, the annulus $\Sigma(D)$ connects $\tilde{\gamma}_3$ with $\tilde{\delta}_3$, which is a cable of K_{n-1} . Let λ' denote the Seifert-framed longitude of K_{n-1} , oriented so that $\langle \mu', \lambda' \rangle = +1$. Then for

one of the orientations on $\tilde{\delta}_3$, we have

$$\langle \mu', \tilde{\delta}_3 \rangle = \langle \tilde{\delta}_1, \tilde{\delta}_3 \rangle = 2$$

and

$$\langle \tilde{\delta}_3, \lambda' \rangle = \langle \tilde{\delta}_3, \tilde{\delta}_2 - p_{n-1} \cdot \mu' \rangle = 1 + 2p_{n-1} = a_n.$$

Thus, $\tilde{\delta}_3$ represents the class $a_n \cdot \mu' + 2\lambda'$. It follows that K_n is isotopic to the $(2, a_n)$ -cable of K_{n-1} . To complete the induction step, we use the fact that $\tilde{\gamma}_3$ represents the class $2a_n \cdot \mu + \lambda$ (cf. [10, p. 32]). Orienting $\tilde{\gamma}_2$ appropriately, we have

$$\langle \mu, \tilde{\gamma}_2 \rangle = \langle \tilde{\gamma}_1, \tilde{\gamma}_2 \rangle = 1$$

and

$$\langle \tilde{\gamma}_2, \lambda \rangle = \langle \tilde{\gamma}_2, \tilde{\gamma}_3 - 2a_n \cdot \mu \rangle = -1 + 2a_n = p_n.$$

It follows that $\tilde{\gamma}_2$ represents the stated class.

It stands to verify that $2g(K_n) - 1 = p_n - \sqrt{3p_n + 1}$. This follows easily from the behavior of the knot genus under cabling. An alternative argument runs as follows. Since $b_2(X_n) = n - 1$, the vector σ belongs to $-\mathbb{Z}^n$. Furthermore, $|\langle \sigma, \sigma \rangle| = p_n$. In light of (15), it follows that $\sigma = \sum_{i=1}^n 2^{i-1} e_i$. The formula for $g(K_n)$ now follows on application of (13). \square

Fintushel-Stern have given a construction for a Kirby diagram of an iterated cable [6]. It would be illuminating to identify the spaces $\Sigma(\kappa_n)$ and $(K_n)_{p_n}$ using their technique.

5. CONCLUDING REMARKS

5.1. Iterated cables. We discovered the construction in Proposition 4.1 in the following indirect way. Suppose that (K, p) attains equality in (5), where K_p bounds a sharp 4-manifold X . It follows that the vector σ representing the class $[\Sigma]$ must attain equality in (15) for all i . Thus, σ takes the form $\sum_{i=1}^n 2^{i-1} e_i$ for some $n \geq 1$, and $p = |\langle \sigma, \sigma \rangle| = p_n$. Now, $H_2(X)$ embeds in $-\mathbb{Z}^n$ as the orthogonal complement $(\sigma)^\perp$. This subspace is spanned by the vectors $2e_i - e_{i+1}$, for $i = 1, \dots, n-1$. With respect to this basis, the intersection pairing on X equals the linking matrix for \mathbb{L}_n . Thus, the simplest choice for X is the result of attaching 2-handles to D^4 along the framed link \mathbb{L}_n . The knot κ_n results from reverse-engineering the process for producing a sharp 4-manifold from the branched double-cover of a non-split alternating link [26, p. 719]. The family of knots K_n follows in turn.

It appears difficult to address whether the family of knots K_n attaining equality in (5) is unique. Any other candidate knot must have the same torsion coefficients, and hence knot Floer homology groups, as some K_n . Examples of distinct L-space knots with identical knot Floer homology groups do exist, but not in great abundance (cf. [15, §1.1.3]).

5.2. The Goda-Teragaito conjecture. Theorem 1.2 implies the second bound in (1) for all $p \geq 20$. Furthermore, a quick analysis of changemakers of norm 18 and 20, coupled with an application of Proposition 3.1, settles (1) for these two values of p as well. The values $p \leq 17$, with the exception of $p = 14$, fall to a theorem of Baker [1, Theorem 1.6]. Combining these results, the second bound in (1) follows for all except the two values $p \in \{14, 19\}$. Part of the difficulty in handling these remaining cases owes to the fact that 14-surgery on the $(3, 5)$ -torus knot and 19-surgery on the $(4, 5)$ -torus knot both produce lens spaces, while neither of these knots is hyperbolic. The best that our methods establish is that any putative counterexample to (1) must have the same knot Floer homology groups as one of these two knots.

5.3. The realization problem. If $K_p = U_{p/q}$ for some knot K , then we have seen that $H_2(X_{p/q}) \oplus H_2(-W_p(K))$ embeds as a full-rank sublattice of $-\mathbb{Z}^n$, and the vector σ corresponding to the generator of $H_2(-W_p(K))$ is a changemaker. This fact places a restriction on the intersection pairing of the plumbing manifold $X_{p/q}$. In fact, this necessary condition turns out to be sufficient as well. This is the main thrust of [14], which answers the *realization problem*: which lens spaces arise by positive integer surgery along a knot $K \subset S^3$? The refined techniques of that paper also lead to Theorem 1.2.

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