ON QUASICONFORMAL SELFMAPPINGS OF THE UNIT DISK AND ELLIPTIC PDE IN THE PLANE

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ABSTRACT. In this paper we prove the following theorem: if w is a quasiconformal mapping of the unit disk onto itself satisfying elliptic partial differential inequality $|L[w]| \leq \mathcal{B}|\nabla w|^2 + \Gamma$, then w is Lipschitz continuous. This results extends some recent results, where instead of elliptic differential operator is considered Laplace operator only. By using this result, we show that a quasiconformal selfmapping of the unit disk is Lipschitz continuous provided that the Beltrami coefficient is Lipschitz continuous.

1. INTRODUCTION AND NOTATION

1.1. **Quasiconformal mappings.** Let $A = \begin{pmatrix} a^{11} & a^{12} \\ a^{21} & a^{22} \end{pmatrix}$. We will consider the matrix norm:

$$|A| = \max\{|Az| : z \in \mathbf{R}^2, |z| = 1\}$$

and the matrix function

$$l(A) = \min\{|Az| : z \in \mathbf{R}^2, |z| = 1\}.$$

Let D and Ω be subdomains of the complex plane $\mathbf{C}$, and $w = u + iv : D \to \Omega$ be a function that has both partial derivatives at a point $z \in D$. By $\nabla w(z)$ we denote the matrix $\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$. For the matrix ∇w we have

$$|\nabla w| = |\partial w| + |\bar{\partial} w| \tag{1.1}$$

and

$$l(\nabla w) = ||\partial w| - |\bar{\partial} w||, \tag{1.2}$$

where

$$\partial w = w_z := \frac{1}{2}\left(w_x + \frac{1}{i}w_y\right) \text{ and } \bar{\partial} w = w_{\bar{z}} := \frac{1}{2}\left(w_x - \frac{1}{i}w_y\right).$$

We say that a function $u : D \to \mathbf{R}$ is ACL (absolutely continuous on lines) in the region D, if for every closed rectangle $R \subset D$ with sides parallel to the x and y-axes, u is absolutely continuous on a.e. horizontal and a.e. vertical line in R. Such a function has of course, partial derivatives u_x, u_y a.e. in D.

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A sense-preserving homeomorphism $w\colon D \to \Omega$, where D and Ω are subdomains of the complex plane $\mathbf{C}$, is said to be K-quasiconformal (K-q.c), $K \geq 1$, if w is ACL in D in the sense that the real and imaginary part are ACL in D, and

$$|\nabla w| \leq Kl(\nabla w) \quad \text{a.e. on } D, \tag{1.3}$$

(cf. [1], pp. 23–24). Notice that the condition (1.3) can be written as

$$|w_{\bar z}| \leq k|w_z| \quad \text{a.e. on } D \text{ where } k = \frac{K-1}{K+1} \text{ i.e. } K = \frac{1+k}{1-k}\,.$$

If in the previous definition replace the condition "w is a sense-preserving homeomorphism" by the condition "w is continuous" we obtain the definition of a quasiregular mapping.

1.2. **Elliptic operator.** Let $A(z) = \{a^{ij}(z)\}_{i,j=1}^2$ be a symmetric matrix function defined in a domain $D \subset \mathbf{C}$ ($a^{ij} = a^{ji}$). Assume that

$$\Lambda^{-1} \leq \langle A(z)h, h\rangle \leq \Lambda \quad \text{for} \quad |h| = 1, \tag{1.4}$$

where Λ is a constant ≥ 1 or in coordinates

$$\Lambda^{-1} \leq \sum_{i,j=1}^{2} a^{ij}(z)h_i h_j \leq \Lambda \text{ for} \sum_{i=1}^{2} h_i^2 = 1. \tag{1.5}$$

In addition we assume that

$$|A(z) - A(\zeta)| \leq \mathfrak{L}|\zeta - z| \quad \text{for any} \quad z, \zeta \in D. \tag{1.6}$$

For

$$L[u] := \sum_{i,j=1}^{2} a^{ij}(z)D_{ij}u(z), \tag{1.7}$$

subjected to conditions (1.5) and (1.6) we consider the following differential inequality

$$|L[u]| \leq \mathcal{B}|\nabla u|^2 + \Gamma, \tag{1.8}$$

or, by using Einstein convention

$$|a^{ij}(z)D_{ij}u| \leq \mathcal{B}|\nabla u|^2 + \Gamma, \tag{1.9}$$

and call it *elliptic partial differential inequality.* Observe that, if A is the identity matrix, then L is the Laplace operator Δ. A C^2 solutions $u : D \to \mathbf{R}(\mathbf{C})$ of the equation $\Delta u = 0$ is called a harmonic function (mappings) and the corresponding inequality (1.7) is called *Poisson differential inequality*. The class of harmonic quasiconformal mappings (HQC) has been one of recent main topics of investigation of some authors. See the subsection below. For the connection between quasiconformal mappings and PDE we refer to the book [2]. See also [9, Chapter 12], [6], [40] and [46].

1.3. **Background and statement of the main results.** Let γ be a Jordan curve. By the Riemann mapping theorem there exists a Riemann conformal mapping of the unit disk onto a Jordan domain $\Omega = \operatorname{int} \gamma$. By Caratheodory's theorem it has a continuous extension to the boundary. Moreover if $\gamma \in C^{n,\alpha}$, $n \in \mathbb{N}$, $0 \le \alpha < 1$, then the Riemann conformal mapping has $C^{n,\alpha}$ extension to the boundary (this result is known as Kellogg's theorem), see [41], [42], [45], [49] and [50] for related results. Conformal mappings are quasiconformal and harmonic. Hence quasiconformal harmonic (shortly HQC) mappings are natural generalization of conformal mappings. O. Martio [35] was the first who considered harmonic quasiconformal mappings on the complex plane. Hengartner and Schober have shown that, for a given second dilatation ($a = \overline{f_{\bar z}}/f_z$, with $\|a\| < 1$) there exist a q.c. harmonic mapping f between two Jordan domains with analytic boundary ([13, Theorem 4.1]).

Recently there has been a number of authors who are working on the topic. The situation in which the image domain is different from the unit disk firstly has been considered by the author in [22]. There it is observed that if f is harmonic K-quasiconformal mapping of the upper half-plane onto itself normlized such that $f(\infty) = \infty$, then $\mathrm{Im} f(z) = cy$, where $c > 0$; hence f is bi-Lipschitz. In [22] and [25] also characterization of HQC automorphisms of the upper half-plane by means of integral representation of analytic functions is given.

Using the result of Heinz ([12]): If w is a harmonic diffeomorphism of the unit disk onto itself with $w(0) = 0$, then $|w_z|^2 + |w_{\bar z}|^2 \ge \frac{1}{\pi^2}$, it can be shown that, every quasiconformal harmonic mapping of the unit disk onto itself is co-Lipschitz. Further, Pavlović [44], proved that every quasiconformal selfmapping of the unit disk $\mathbf{U} := \{z \in \mathbf{C} : |z| < 1\}$ is Lipschitz continuous, using the Mori's theorem on the theory of quasiconformal mappings. Partyka and Sakan ([43]) yield explicit Lipschitz and co-Lipschitz constants depending on a constant of quasiconformality. Using the Hilbert transforms of the derivative of boundary function, the first characterizations of HQC automorphisms of the upper half-plane and of the unit disk have been given in [44, 25]; for further result cf. [36]. Among the other things Knežević and Mateljević in [16] showed that a q.c. harmonic mapping of the unit disk onto itself is a $(1/K, K)$ quasi-isometry with respect to Poincaré distance. See also the paper of Chen and Fang [3] for a generalization of the previous result to convex domains.

Since the composition of a harmonic mapping and of a conformal mapping is itself harmonic, using the case of the unit disk and Kellogg's theorem, these theorems can be generalized to the class of mappings from arbitrary Jordan domain with $C^{1,\alpha}$ boundary onto the unit disk. However the composition of a conformal and a harmonic mapping is not, in general, a harmonic mapping. This means in particular that the results of this kind for arbitrary image domain do not follow from the case of the unit disk or the upper half-plane and Kellogg's theorem.

Using some new methods the results concerning the unit disk and the half-plane have been extended properly in the papers [24]–[20], [33] and [36]. In particular, in [26] (and in subsequent paper [28]) it was shown how to apply Kellogg's theorem and that simple proof in the case of the upper half-plane has analogy for C^2 domains; namely, by using a Heinz-Berensetin theorem [11, Theorem 4] it was proved a version of "inner estimate" for quasi-conformal diffeomorphisms, which satisfies a certain Poisson differential inequality. As an application of this estimate, it was shown that quasi-conformal harmonic mappings between smooth domains (with respect to the approximately analytic metric), have bounded partial derivatives; in particular, these mappings are Lipschitz.

For related result about quasiconformal harmonic mappings with respect to the hyperbolic metric we refer to the paper of Wan [47] and of Marković [34].

Very recently, Iwaniec, Kovalev and Onninen in [14] have shown that, the class of quasiconformal harmonic mappings is also interesting concerning the modulus of annuli in complex plane.

In this paper we study Lipschitz continuity of the class of K-q.c. self-mappings of the unit disk satisfying elliptic differential inequality. This class contains conformal mappings and quasiconformal harmonic mappings.

The main result of this paper is the following theorem which can be considered as an extension of Kellogg theorem and results of Martio, Pavlović, Partyka, Sakan, Mateljević and the author.

Note that, we replace the laplace operator Δ by a strictly elliptic operator L.

Theorem 1.1. *If $w : \mathbf{U} \to \mathbf{U}$, $w(a) = 0$ is a K q.c. solution of the elliptic partial differential inequality*

$$|L[w]| \leq \mathcal{B}|\nabla w|^2 + \Gamma,$$

then ∇w is bounded by a constant depending only on $\mathcal{B}$, Γ K, Λ, $\mathfrak{L}$ and a and w is lipschitz continuous.

By using Theorem 1.1, Riemann measure mapping theorem and the fact that Beltrami equation (under certain smoothness of Beltrami coefficient) reduces to an elliptic partial differential inequality, we obtain the following result which we believe could be of interest for the experts in the quasiconformal mappings.

Theorem 1.2. *Let w be a q.c. mapping of the unit disk onto itself such that the Beltrami coefficient $\mu = \frac{w_{\bar{z}}}{w_z}$ is Lipschitz continuous in $\mathbf{U}$. Then w is lipschitz continuous in $\mathbf{U}$.*

The proof of Theorem 1.1 is given in the Section 3. The methods of the proof differ from the methods of the proof of corresponding results for the class HQC. In Section 2 we make some estimates concerning the Green function of the disk, and some estimates concerning the gradient of a solution to elliptic partial differential inequality, satisfying certain boundary condition similar to those of the excellent paper of Nagumo [39]. We first prove interior estimates of the gradient of a solution u of elliptic PDE in terms of constants of elliptic operator, and modulus of continuity of u (Theorem 2.5). After that we recall a theorem of Nagumo ([39]), which shows that, if u is a solution of elliptic PDE, with vanishing boundary condition defined in a domain D whose boundary has bounded curvature from above by a constant κ, then $|\nabla u(z)| \leq \gamma, \ \ z \in D$, where γ is a constant depending not depending on u providing that $16\mathcal{B}\Gamma\|u\|_\infty < 1$ (Theorem 2.8). In order to prove Theorem 1.1, we previously show that the function $u = |w|$ satisfies a certain elliptic differential inequality near the boundary of the unit disk. In order to show a priory bound, we make use of Mori's theorem which implies that the modulus of continuity of a K-q.c. self-mapping of the unit disk depends only on K. By using Theorem 2.5 we show that the gradient is a priory bounded in compacts of the unit disk, while Theorem 2.8 serve to obtain the a priory bound of gradient of u in some "neighborhood" of the boundary of the unit disk. By using the quasiconformality, we prove that ∇w is a priory bounded as well.

2. Auxiliary results

2.1. **Green function.** If $h(z, w)$ is a real function, then by $\nabla_z h$ we denote the gradient (h_x, h_y).

Lemma 2.1. *Let*

$$h(z, w) = \log \frac{|1 - z\bar{w}|}{|z - w|},$$

then

$$\nabla_z h(z, w) = \frac{1 - |w|^2}{(\bar{z} - \bar{w})(w\bar{z} - 1)} \tag{2.1}$$

and

$$\partial_w \nabla_z h(z, w) = -\frac{1}{(1 - w\bar{z})^2}, \quad \partial_{\bar{w}} \nabla_z h(z, w) = -\frac{1}{(\bar{w} - \bar{z})^2}. \tag{2.2}$$

Proof. First of all

$$\nabla h = (h_x, h_y) = h_x + ih_y.$$

Since

$$h_{\bar{z}} = \frac{1}{2}(h_x + ih_y),$$

it follows that

$$\nabla h = 2h_{\bar{z}}.$$

Since

$$2h(z) = \log \left(\frac{1 - z\bar{w}}{z - w} \frac{1 - \bar{z}w}{\bar{z} - \bar{w}} \right).$$

Differentiating we obtain

$$2h_{\bar{z}}(z) = \log \left(\frac{1 - \bar{z}w}{\bar{z} - \bar{w}} \right)_{\bar{z}} = \frac{|w|^2 - 1}{(\bar{z} - \bar{w})^2} \frac{\bar{z} - \bar{w}}{1 - \bar{z}w}.$$

This implies (2.1). From

$$\frac{1 - |w|^2}{(\bar{z} - \bar{w})(w\bar{z} - 1)} = \frac{w}{w\bar{z} - 1} + \frac{1}{\bar{w} - \bar{z}}$$

it follows (2.2). □

Corollary 2.2. *Let $G(\zeta, \omega)$ be the Green's function of the disk $\{\zeta : |\zeta - \zeta_0| \le R\}$ defined by*

$$G(\zeta, \omega) := \log \frac{|\varphi(\zeta) - \varphi(\omega)|}{|1 - \varphi(\zeta)\overline{\varphi(\omega)}|},$$

where

$$\varphi(\zeta) = \frac{1}{R}(\zeta - \zeta_0).$$

Then

$$|\nabla_\zeta G(\zeta, \omega)| \le \frac{2}{|\zeta - \omega|} \tag{2.3}$$

and

$$|\partial_{\omega_j} \nabla_\zeta G(\zeta, \omega)| \le \frac{2}{|\zeta - \omega|^2}, j = 1, 2. \tag{2.4}$$

Proof. Let

$$\varphi(\zeta) = \frac{1}{R}(\zeta - z_0).$$

Then

$$\varphi'(\zeta) = \frac{1}{R}.$$

Take $z = \varphi(\zeta)$ and $w = \varphi(\omega)$ and define $h(z, w) = G(\zeta, \omega)$. It follows that

$$\nabla_\zeta G(\zeta, \omega) = \nabla_z h(z, w) \cdot \varphi'(\zeta) = \frac{1}{R}\nabla_z h(z, w). \tag{2.5}$$

Thus

$$|\nabla_\zeta G(\zeta, \omega)| = \frac{1}{R}|\nabla_z h(z, w)| \tag{2.6}$$

Since

$$\frac{1 - |w|^2}{|1 - \bar{z}w|} \le \frac{1 - |w|^2}{1 - |w|} \le 2.$$

Combining with (2.6) and (2.1), we obtain (2.3). To get (2.4), observe first that, for $\omega = \omega_1 + i\omega_2$

$$\partial_{\omega_1} = \partial_\omega + \partial_{\bar{\omega}} \tag{2.7}$$

and

$$\partial_{\omega_2} = i(\partial_\omega - \partial_{\bar{\omega}}). \tag{2.8}$$

On the other hand for $|z| \le 1$ and $|w| \le 1$ we have

$$\left|\frac{1}{(1 - w\bar{z})^2}\right| \le \left|\frac{1}{(w - z)^2}\right|.$$

From (2.7), (2.8), (2.2), (2.5) we deduce (2.4). □

2.2. **Interior estimates of gradient.**

Lemma 2.3. *Let $u:\overline{\mathbf{U}}\to\mathbb{C}$ be a continuous mapping. Then there exists a positive function $\varpi=\varpi_u(t)$, $t\in(0,2)$, such that $\lim_{t\to 0}\varpi_u(t)=0$ and*

$$|u(z)-u(w)|\le\varpi(|z-w|),\quad z,w\in\mathbf{U}.$$

The function ϖ is called the modulus of continuity of u

Lemma 2.4. *Let $Y:D\to\mathbf{U}$ be a C^2 mapping of a domain $D\subset\mathbf{U}$. Let $\mathbf{U}(z_0,\rho)\subset D$ and let $Z\in\mathbf{C}$ be any constant number. Then we have the estimate:*

$$|\nabla h(z_0)|\le\frac{2}{\rho^2}\int_{|y-z_0|=\rho}|Y(y)-Z|d\mathcal{H}^1(y) \tag{2.9}$$

where $h(z)$, $|z-z_0|\le\rho$ is the Poisson integral of $Y|_{z_0+\rho\mathbf{T}}$ and $d\mathcal{H}^1$ is the Hausdorff probability measure.

Proof. Assume that $v\in C^2(\overline{\mathbf{U}})$ and define

$$H(z)=\int_{\mathbf{T}}P(z,\eta)v(\eta)d\mathcal{H}^1(\eta), \tag{2.10}$$

where

$$P(z,\eta)=\frac{1-|z|^2}{|z-\eta|^2},\quad |\eta|=1,\ \ |z|<1. \tag{2.11}$$

Then H is a harmonic function. It follows that

$$\langle\nabla H(z),e\rangle=\int_{\mathbf{T}}\langle\nabla_z P(z,\eta),e\rangle\, v(\eta)d\mathcal{H}^1(\eta),\quad e\in\mathbf{R}^2. \tag{2.12}$$

By differentiating (2.11), we obtain

$$\nabla_z P(z,\eta)=\frac{-2z}{|z-\eta|^2}-\frac{2(1-|z|^2)(z-\eta)}{|z-\eta|^{2+2}}.$$

Hence

$$\nabla_z P(0,\eta)=\frac{2\eta}{|\eta|^4}=2\eta.$$

Therefore

$$|\langle\nabla_z P(0,\eta),e\rangle|\le|\nabla_z P(0,\eta)||e|=2|e|. \tag{2.13}$$

Using (2.12), (2.13), we obtain

$$|\langle\nabla H(0),e\rangle|\le\int_{\mathbf{T}}|\nabla_z P(0,\eta)||e||v(\eta)|d\mathcal{H}^1(\eta).$$

Hence we have

$$|\nabla H(0)|\le 2\int_{\mathbf{T}}|v(\eta)|d\mathcal{H}^1(\eta) \tag{2.14}$$

Let $v(z) = Y(z_0+\rho z) - Z$ and $H(z) = P[v|_{\mathbf{T}}](z)$. Then $H(z) = h(z_0+\rho z) - Z$ and $\nabla H(0) = \rho \nabla h(z_0)$. Inserting this into (2.14) we obtain

$$\rho|\nabla h(z_0)| = |\nabla H(0)| \le 2\int_{\mathbf{T}} |Y(z_0+\rho\eta) - Z| d\mathcal{H}^1(\eta). \tag{2.15}$$

Introducing the change of variables $\zeta = z_0 + \rho\eta$ in the integral (2.15) we obtain

$$|\nabla Y(z_0)| \le \frac{2}{\rho^2}\int_{|\zeta - z_0| = \rho} |Y(\zeta) - Z| d\mathcal{H}^1(\zeta) \tag{2.16}$$

which is identical with (2.9). □

Theorem 2.5. *Let D be a bounded domain, whose diameter is d. Let $A(z) = \{a^{ij}(z)\}_{i,j=1}^2$ be a symmetric matrix function defined in a domain $\Omega \subset \mathbf{C}$ ($a^{ij} = a^{ji}$) satisfying the condition* (1.5) *and* (1.6)*. Let $u(z)$ be any C^2 solution of elliptic partial differential inequality* (1.8) *such that*

$$|u(z)| \le M \text{ in } D. \tag{2.17}$$

Then there exist constants $C^{(0)}$ and $C^{(1)}$, depending on modulus of continuity of u, Λ, $\mathfrak{L}$, B, Γ, M and d such that

$$|\nabla u(z)| < C^{(0)} \rho(z)^{-1} \max_{|\zeta - z| \le \rho(z)} \{|u(\zeta)|\} + C^{(1)} \tag{2.18}$$

where $\rho(z) = \operatorname{dist}(z, \partial D)$.

Proof. Fix a point $a \in D$ and let B_p, $0 < p < 1$, be a closed disk defined by

$$B_p = \{z; |z - a| \le p \operatorname{dist}(a, \partial D)\}.$$

Its radius is

$$R_p = p \operatorname{dist}(a, \partial D).$$

Define the function μ_p:

$$\mu_p = \max_{z \in B_p} \{|\nabla u| r_p(z)\} \tag{2.19}$$

where $r_p(z) = \operatorname{dist}\,(z, \partial B_p) = R_p - |z - a|$. Then there exists a point $z_p \in B_p$ such that

$$|\nabla u(z_p)| r_p(z_p) = \mu_p \quad (z_p \in B_p). \tag{2.20}$$

We need the following result in the sequel.

Lemma 2.6. *The function μ_p is continuous on $(0,1)$ and has continuous extension at 0: $\mu_0 = 0$.*

Proof of Lemma 2.6. Let p_n be a sequence converging to a number p, let $\mu_{p_n} = |\nabla u(z_n)| r_{p_n}(z_n)$ and assume it converges to μ'_p. Prove that $\mu'_p = \mu_p$. Passing to a subsequence, we can assume that $z_n \to z'_p$. Then $z'_p \in B_p$. Thus $\mu'_p \le \mu_p$. On the other hand $\mu_{p_n} \ge |\nabla u((1-\varepsilon_n)z_p)| r_{p_n}((1-\varepsilon_n)z_p)$, where ε_n is a positive sequence converging to zero. It follows that $\mu'_p \ge \lim_{n\to\infty} |\nabla u((1-$

$\varepsilon_n)z_p)|r_{p_n}((1-\varepsilon_n)z_p) = \mu_p$. Furthermore, since $r_p \le R_p = p\,\mathrm{dist}(a, \partial D)$, we obtain that

$$\lim_{p\to 0^+} \mu_p \le |\nabla u(0)| \lim_{p\to 0^+} R_p = 0.$$

□

Now let $Tz = \zeta$ be a linear transformation of coordinates such that

$$\sum_{i,j=1}^{2} a^{ij}(z_p) D_{ij} u = \Delta v, \tag{2.21}$$

where $v(\zeta) = u(z)$. By [29, Lemma 11.2.1] the transformation T can be chosen to so that

$$T = \begin{pmatrix} {\lambda_1}^{-\frac{1}{2}} & 0 \\ 0 & {\lambda_2}^{-\frac{1}{2}} \end{pmatrix} \cdot R, \tag{2.22}$$

where λ_1 and λ_2 are eigenvalues of the matrix $A(z_p)$ and R is some orthogonal matrix. Then

$$\frac{1}{\Lambda} \le \lambda_1, \lambda_2 \le \Lambda.$$

Let $\nabla^2 u$ denotes the Hessian matrix of u:

$$\nabla^2 u = \begin{pmatrix} D_{11}u & D_{12}u \\ D_{21}u & D_{22}u \end{pmatrix}.$$

Since

$$\nabla^2 u = T^t \nabla^2 v T,$$

we obtain:

$$\begin{aligned} Tr(A^t \nabla^2 u) &= Tr(A^t T^t \nabla^2 v T) \\ &= Tr((TA)^t \nabla^2 v T) \\ &= Tr(\nabla^2 v T (TA)^t) \\ &= Tr(\nabla^2 v T A^t T^t) \\ &= Tr(B^t \nabla^2 v), \end{aligned}$$

where

$$B(\zeta) = TA(z)T^t. \tag{2.23}$$

Then

$$B(\zeta_p) = I,$$

$$b^{ij}(\zeta) D_{ij} v(\zeta) \;= a^{ij}(z) D_{ij} u(z) \tag{2.24}$$

and

$$\Delta v = (\delta_{ij} - b^{ij}(\zeta)) D_{ij} v + b^{ij}(\zeta) D_{ij} v. \tag{2.25}$$

Further $T(D(z_p, r_p)) \subset T(B_p) \subset T(D) =: D'$. From (2.22) we see that $T(D(z_p, r_p))$ is an ellipse with axes equal to $\lambda_1^{-1/2} \cdot r_p$ and $\lambda_2^{-1/2} \cdot r_p$ and with the center at $\zeta_p = T(z_p)$. Then $D_\lambda := \{\zeta : |\zeta - \zeta_p| \le \lambda r_p\}$ is a closed disk in $T(B_p)$ provided that

$$0 < \lambda < \frac{1}{2\sqrt{\Lambda}}. \tag{2.26}$$

Let $G(\zeta,\omega)$ be the Green's function of the disk D_λ. So that from (2.25)

$$\begin{aligned} v = &-\frac{1}{\pi}\int_{D_\lambda} G(\zeta,\omega)(\delta_{ij} - b^{ij}(\omega))D_{ij}v(\omega)d\mathcal{L}^2(\omega) \\ &- \frac{1}{\pi}\int_{D_\lambda} G(\zeta,\omega)b^{ij}(\omega)D_{ij}v(\omega)d\mathcal{L}^2(\omega) + h(\zeta), \end{aligned}$$

where $d\mathcal{L}^2(z) = dxdy$ is the Lebesgue two-dimensional measure in the complex plane and $h(\zeta)$ is the harmonic function which takes the same value as $v(\zeta)$ for $\zeta \in \partial D_\lambda$. Then

$$|\nabla v(\zeta_p)| \le \mathcal{P} + \mathcal{Q} + \mathcal{R}, \tag{2.27}$$

where

$$\begin{aligned} \mathcal{P} &= |\frac{1}{\pi}\int_{D_\lambda} \nabla_\zeta G(\zeta_p,\omega)b^{ij}(\omega)D_{ij}v(\omega)d\mathcal{L}^2(\omega)| \\ \mathcal{Q} &= |\frac{1}{\pi}\int_{D_\lambda} \nabla_\zeta G(\zeta_p,\omega)(\delta_{ij} - b^{ij}(\omega))D_{ij}v(\omega)d\mathcal{L}^2(\omega)| \\ \mathcal{R} &= |\nabla_\zeta h(\zeta_p)|. \end{aligned}$$

Further, it follows by (1.5) and (2.23) that

$$\nabla B(\zeta)\cdot T = T\cdot\nabla A(z)\cdot T^t.$$

Since $\Lambda^{-1/2}|z| \le |Tz| \le \Lambda^{1/2}|z|$, we obtain

$$|\nabla B(\zeta)| \le |T|^3|\nabla A| \le \Lambda^{3/2}\mathfrak{L}. \tag{2.28}$$

Thus

$$\|B(\zeta) - B(\zeta_p)\| \;=\; \|B(\zeta) - \mathrm{I}\| \le \Lambda^{3/2}\mathfrak{L}|\zeta - \zeta_p| \tag{2.29}$$

As

$$d(T(z), T(z_p)) \le \lambda r_p(z_p),$$

by using the inequalities

$$r_p(z_p) \le d(z, z_p) + r_p(z),$$

$$d(z, z_p) \le \Lambda^{1/2} d(T(z), T(z_p))$$

and

$$|\nabla u(z)| r_p(z) \le \mu_p,$$

we obtain

$$|\nabla u(z)| \le (1 - \lambda\Lambda^{1/2})^{-1} r_p(z_p)^{-1}\mu_p \text{ for } z \in T^{-1}(D_\lambda)(\subset B_p).$$

From (2.26) we obtain that

$$(1-\lambda\Lambda^{1/2})^{-2} < 4. \tag{2.30}$$

Having in mind the formula $\nabla u(z) = \nabla v(\zeta)\cdot T$ we obtain

$$|\nabla v(\zeta)| \le 2\Lambda^{1/2} r_p(z_p)^{-1}\mu_p \tag{2.31}$$

for $\omega \in D_\lambda$.
Since

$$|a^{ij}(z)D_{ij}u| \le \mathcal{B}|\nabla u|^2 + \Gamma,$$

$$|b^{ij}(\zeta)D_{ij}v(\zeta)| = |a^{ij}(z)D_{ij}u(z)|,$$

it follows that

$$|b^{ij}(\zeta)D_{ij}v(\zeta)| \le \mathcal{B}|T|^2|\nabla v|^2 + \Gamma = \mathcal{B}\Lambda|\nabla v|^2 + \Gamma \tag{2.32}$$

and

$$|b^{ij}(\zeta)D_{ij}v(\zeta)| \le 2\mathcal{B}r_p(z_p)^{-2}\mu_p^2 + \Gamma. \tag{2.33}$$

From now on we divide the proof into four steps:
Step 1: Estimation of $\mathcal{P}$. From (2.3) and (2.32) we first have.

$$\begin{aligned}
&|\frac{1}{\pi}\int_{D_\lambda} \nabla_\zeta G(\zeta_p,\omega) b^{ij}(\omega)D_{ij}v(\omega)d\mathcal{L}^2(\omega)| \\
&\quad \le \frac{2}{\pi}\int_{|\omega-\zeta_p|\le\lambda r_p(z_p)} \frac{1}{|\omega-\zeta_p|}|b^{ij}(\omega)D_{ij}v(\omega)|d\mathcal{L}^2(\omega) \\
&\quad \le \frac{2}{\pi}\int_{|\omega-\zeta_p|\le\lambda r_p(z_p)} \frac{1}{|\omega-\zeta_p|}(\mathcal{B}\Lambda|\nabla v|^2+\Gamma)d\mathcal{L}^2(\omega)
\end{aligned}$$

Proceeding as in the proof of [39, Theorem 2] we obtain that

$$\mathcal{P} \le \frac{4\mathcal{B}\lambda\mu_p^2}{r_p} + 2\Gamma r_p\lambda. \tag{2.34}$$

Step 2: Estimation of $\mathcal{Q}$. Let $\mathbf{n}_\omega = (\cos\alpha_1, \cos\alpha_2)$ be the unit inner vector of ∂D_λ at ω. Then from Green formula

$$\int_{\partial D_\lambda}\sum_{i=1}^2 u_i(\omega)\cos\alpha_i d\mathcal{H}^1(\omega) = \int_{D_\lambda}(\partial_{\omega_1}u_1 + \partial_{\omega_2}u_2)d\mathcal{L}^2(\omega),$$

proceeding as in [39, Theorem 2], we obtain

$$\begin{aligned}
\mathcal{Q} \le\ & |\frac{1}{\pi}\int_{|\omega-\zeta_p|=\lambda r_p(z_p)} \nabla_\zeta G(\zeta_p,\omega)(\delta_{ij}-b^{ij}(\omega))\partial_i v(\omega)\cos\alpha_j d\mathcal{H}^1(\omega)| \\
&+ |\frac{1}{\pi}\int_{|\omega-\zeta_p|\le\lambda r_p(z_p)} \nabla_\zeta G(\zeta_p,\omega)\partial_{\omega_j} b^{ij}(\omega)\partial_i v(\omega)d\mathcal{L}^2(\omega)| \\
&+ |\frac{1}{\pi}\int_{|\omega-\zeta_p|\le\lambda r_p(z_p)} \partial_{\omega_j}\nabla_\zeta G(\zeta_p,\omega)(\delta_{ij}-b^{ij}(\omega))\partial_i v(\omega)d\mathcal{L}^2(\omega)|.
\end{aligned} \tag{2.35}$$

By using the Cauchy-Schwarz inequality, (2.3), (2.4), (2.31) and the fact that $|\zeta_p - \omega| = \lambda r_p(z_p)$ we obtain

$$\mathcal{Q} \leq 16\Lambda\mathfrak{L}\lambda\mu_p. \tag{2.36}$$

Step 3: Estimation of $\mathcal{R}$.
Let $\varpi(t) = \varpi_v(t)$ be the modulus of continuity of v as in Lemma 2.3. From (2.9), for $Z = v(\zeta_p)$ ($Z = 0$), $Y(\zeta) = v(\zeta)$ and $\rho = \lambda r_p(z_p)$, by using Lemma 2.4 and 2.3 we obtain

$$\begin{aligned}\mathcal{R} \leq |\nabla h(z_p)| &\leq \frac{2}{\lambda r_p(z_p)^2}\int_{|\omega-\zeta_p|=\lambda r_p(z_p)} |v(\omega) - Z| d\mathcal{H}^1(\omega)\\ &\leq \frac{2}{\lambda r_p(z_p)} \max\{|v(\zeta) - Z| : |\zeta - \zeta_p| = \lambda r_p(z_p)\}\\ &\leq \frac{\min\{2\varpi(\lambda r_p(z_p)), 2K\}}{\lambda r_p(z_p)},\end{aligned} \tag{2.37}$$

where

$$K = \sup_{|z-a|\leq\rho(a)} |u(z)|. \tag{2.38}$$

Step 4: The finish of the proof. As

$$|\nabla v(\zeta) \geq \Lambda^{-1/2}|\nabla u(z_p)| = \Lambda^{-1/2} r_p(z_p)^{-1}\mu_p$$

and $r_p(z_p) < 2\rho(a) \leq d$, we get from (2.27), (2.34), (2.36) and (2.37),

$$A_0\mu_p^2 + B_0\mu_p + C_0 \geq 0, \tag{2.39}$$

where

$$A_0 = 4\mathcal{B}\lambda,$$
$$B_0 = 16\Lambda\mathfrak{L}\lambda r_p(z_p) - \Lambda^{-1/2}$$

and

$$C_0 = 2\Gamma {r_p}^2(z_p)\lambda + \frac{2\min\{\varpi(\lambda r_p(z_p)), K\}}{\lambda}.$$

We can take $\lambda > 0$ depending on ϖ, Λ, $\mathfrak{L}$, B, Γ and d so small that

$$B_0^2 > 4A_0C_0 \tag{2.40}$$

and

$$16AL\lambda \leq 1/2\Lambda^{-1/2}. \tag{2.41}$$

Let μ_1 and μ_2 ($\mu_1 < \mu_2$) be the distinct real roots of the equation

$$A_0\mu^2 + B_0\mu + C_0 = 0. \tag{2.42}$$

Then we have from (2.39)

$$\mu_p \leq \mu_1 \text{ or } \mu_p \geq \mu_2(\mu_1 < \mu_2).$$

Lemma 2.6 asserts that μ_p depends on p continuously for $0 < p < 1$ and $\lim_{p\to 0}\mu_p = 0$. Then we have only $\mu_p \le \mu_1$. And, letting p tend to 1, by the definition of μ_p

$$|\nabla u(a)| \le \mu_1 \rho(a)^{-1}. \tag{2.43}$$

As μ_1 is the smaller root of (2.42),

$$\begin{aligned}\mu_1 &= \frac{-B_0 - \sqrt{B_0^2 - 4A_0C_0}}{2A_0}\\ &= \frac{2C_0}{-B_0 + \sqrt{B_0^2 - 4A_0C_0}}\\ &\le -\frac{2C_0}{B_0}.\end{aligned}$$

From (2.43) and (2.38) we get

$$|\nabla u(a)| \le C^{(1)}\rho(a)^{-1} \sup_{|z-a|\le \rho(a)} |u(z)| + C^{(2)} \tag{2.44}$$

where $C^{(1)}$ and $C^{(2)}$ depend on Λ, $\mathfrak{L}$, B, M, Γ, d and on modulus of continuity of u.

□

2.3. Boundedness of gradient.

Definition 2.7. We say that a domain D satisfies the *exterior sphere condition* for some $\kappa > 0$, if: To any point p of ∂D there corresponds a ball $B_p \subset \mathbf{C}$ with radius κ such that $\overline{D} \cap B_p = \{p\}$.

Theorem 2.8 (A priory bound)**.** [39, Lemma 2] *Let D be a complex domain with diameter d satisfying exterior sphere condition for some $\kappa > 0$. Let $u(z)$ be a twice differentiable mapping satisfying the elliptic differential inequality* (1.8) *in D satisfying the boundary condition $u = 0$ ($z \in G$). Assume in addition that $|u(z)| \le M$, $z \in D$,*

$$\frac{4}{\pi}\cdot 16\mathcal{B}\Gamma M < 1 \tag{2.45}$$

and $u \in C(\overline{D})$. Then

$$|\nabla u| \le \gamma, \;\; z \in D, \tag{2.46}$$

where γ is a constant depending on κ, M, $\mathcal{B}$, Γ, $\mathfrak{L}$, Λ and d only.

Remark 2.9. See [9, Theorem 15.9] for a related result. In the statement of [39, Lemma 2] instead of condition (2.45) appears

$$16\mathcal{B}\Gamma M < 1,$$

however its proof lays on [39, Theorem 2], whose proof, it seems that, works only under the condition (2.45). Indeed the right hand side of the inequality in the first line on [39, p. 214] should be multiplied by

$$\frac{2\Gamma(1+m/2)}{\sqrt{\pi}\Gamma((m+1)/2)},$$

where m is the dimension of the space (in our case $m = 2$) and

$$\frac{2\Gamma(1+2/2)}{\sqrt{\pi}\Gamma((2+1)/2)} = \frac{4}{\pi}.$$

3. Proof of the main theorems

We need the following lemmas.

Lemma 3.1. [23] *Every $K-$q.r. mapping $w(z) = \rho(z)S(z) : D \to \Omega$, $D, \Omega, \subset \mathbb{C}$, $\rho = |w|$, $S(z) = e^{is(z)}$, $s(z) \in [0, 2\pi)$, satisfies the inequalities*

$$\rho|\nabla S| \le K|\nabla\rho| \tag{3.1}$$

and

$$|\nabla\rho| \le K\rho|\nabla S| \tag{3.2}$$

almost everywhere on D. Inequalities (3.1) *and* (3.2) *are sharp: the equality*

$$\rho|\nabla S| = |\nabla\rho| \tag{3.3}$$

holds if w is a 1-quasiregular mapping. We also have

$$K^{-1}|\nabla w| \le |\nabla\rho| \le |\nabla w|. \tag{3.4}$$

Lemma 3.2. *If $w = \rho S : \mathbf{U} \to \mathbf{U}$, $\rho = |w|$, is twice differentiable, then*

$$L[\rho] = \rho(a^{11}|p|^2 + 2a^{12}\langle p, q\rangle + a^{22}|q|^2) + \frac{1}{2}\langle L[w], S\rangle, \tag{3.5}$$

where $p = D_1S$ and $q = D_2S$.

If in addition w is $K - q.c.$ and satisfies

$$|L[w]| = |\sum_{i,j=1}^{2} a^{ij}(z)D_{ij}w| \le \mathcal{B}|\nabla w|^2 + \Gamma, \tag{3.6}$$

then there exists constants Θ and Π depending on K, $\mathcal{B}$ and Γ such that

$$|L[\rho]| \le \frac{\Theta}{\rho}|\nabla\rho|^2 + \Pi. \tag{3.7}$$

Proof. Let $w = (w_1, w_2)$ (here w_i are real), let $S = (S_1, S_2)$ and let $f = (f_1, f_2)$. For real differentiable functions a and b define the bi-linear operator

$$D[a, b] = \sum_{k,l=1}^{2} a^{kl}(z)D_k a(z)D_l b(z).$$

Since $w_i = \rho S_i$, $i \in \{1, 2\}$ and

$$\rho = \sum_{i=1}^{2} S_i w_i,$$

we obtain:

$$L[w_i] = S_i L[\rho] + \rho L[S_i] + 2D[\rho, S_i],\ i \in \{1, 2\} \tag{3.8}$$

and

$$L[\rho] = \sum_{i=1}^{2} w_i L[S_i] + \sum_{i=1}^{2} S_i L[w_i] + 2\sum_{i=1}^{2} D[S_i, w_i]. \tag{3.9}$$

From (3.8) we obtain

$$\begin{aligned} L[\rho] &= L[\rho]|S|^2 \\ &= \sum_{i=1}^{n} S_i \cdot S_i L[\rho] \\ &= \sum_{i=1}^{2} S_i L[w_i] - \rho\sum_{i=1}^{2} S_i L[S_i] - 2\sum_{i=1}^{2} S_i D[\rho, S_i]. \end{aligned} \tag{3.10}$$

By adding (3.9) and (3.10) we obtain

$$L[\rho] = \sum_{i=1}^{2}(D[S_i, w_i] - S_i D[\rho, S_i]) + \frac{1}{2}\langle L[w], S\rangle .$$

On the other hand

$$\begin{aligned} D[S_i, w_i] - S_i D[S_i, \rho] &= \sum_{k,l=1}^{2} a^{kl}(z) D_k S_i D_l w_i - S_i \sum_{k,l=1}^{2} a^{kl}(z) D_k S_i D_l \rho \\ &= \sum_{k,l=1}^{2} a^{kl}(z) D_k S_i(\rho D_l S_i + S_i D_l \rho) - S_i \sum_{k,l=1}^{2} a^{kl}(z) D_k S_i D_l \rho \\ &= \rho \sum_{k,l=1}^{2} a^{kl}(z) D_k S_i D_l S_i, \quad i = 1, 2. \end{aligned}$$

Thus

$$\begin{aligned} L[\rho] &= \rho \sum_{i,k,l=1}^{2} a^{kl}(z) D_k S_i D_l S_i + \frac{1}{2}\langle L[w], S\rangle \\ &= \rho(a^{11}|p|^2 + 2a^{12}\langle p, q\rangle + a^{22}|q|^2) + \frac{1}{2}\langle L[w], S\rangle , \end{aligned}$$

where $p = (D_1S_1, D_1S_2)$ and $q = (D_2S_1, D_2S_2)$. Therefore

$$\begin{aligned} |L[\rho]| &\le \Lambda\rho(|p|^2 + |q|^2) + \frac{1}{2}(\mathcal{B}|\nabla w|^2 + \Gamma) \\ &= \Lambda\rho\|\nabla S\|^2 + \frac{1}{2}(\mathcal{B}|\nabla w|^2 + \Gamma). \end{aligned}$$

Here $\|\cdot\|$ is Hilbert-Schmidt norm which satisfies the inequality $\|P\| \le \sqrt{2}|P|$. If w is $K-$q.c., then according to (3.1) and (3.3) we have

$$|L[\rho]| \le \frac{\sqrt{2}K\Lambda}{\rho}|\nabla\rho|^2 + \frac{1}{2}(\mathcal{B}K|\nabla\rho|^2 + \Gamma).$$

Taking $\Theta = \sqrt{2}K\Lambda + \mathcal{B}K/2$ and $\Pi = \Gamma/2$ we obtain (3.7). □

Lemma 3.3. *If $f = u + iv$ is a K q.c. mapping satisfying elliptic differential inequality, then u and v satisfy the elliptic differential inequality.*

Proof. Let

$$A := |\nabla u|^2 = 2(|u_z|^2 + |u_{\bar z}|^2) = \frac{1}{2}(|f_z + \overline{f_{\bar z}}|^2 + |f_{\bar z} + \overline{f_z}|^2)$$

and

$$B := |\nabla v|^2 = 2(|v_z|^2 + |v_{\bar z}|^2) = \frac{1}{2}(|f_z - \overline{f_{\bar z}}|^2 + |f_{\bar z} - \overline{f_z}|^2).$$

Then

$$\frac{A}{B} = \frac{|1+\mu|^2}{|1-\mu|^2}$$

where $\mu = \overline{f_{\bar z}}/f_z$. Since $|\mu| \le k = \frac{K-1}{K+1}$

$$\frac{(1-k)^2}{(1+k)^2} \le \frac{A}{B} \le \frac{(1+k)^2}{(1-k)^2}. \tag{3.11}$$

As

$$|L[f]| = |L[u] + iL[v]| \le \mathcal{B}|\nabla f|^2 + \Gamma \le \mathcal{B}(|\nabla u|^2 + |\nabla v|^2) + \Gamma,$$

the relation (3.11) yields

$$|L[u]| \le \mathcal{B}\left(1 + \frac{(1+k)^2}{(1-k)^2}\right)|\nabla u|^2 + \Gamma$$

and

$$|L[v]| \le \mathcal{B}\left(1 + \frac{(1+k)^2}{(1-k)^2}\right)|\nabla v|^2 + \Gamma.$$

□

Before proving the main theorems of this paper let us recall one of the most fundamental results concerning quasiconformal mappings

Proposition 3.4 (Mori). *If $w : \mathbf{U} \to \mathbf{U}$, $w(0) = 0$, is a K quasiconformal harmonic mapping of the unit disk onto itself, then*

$$|w(z_1) - w(z_2)| \le 16|z_1 - z_2|^{1/K}, \;\; z_1, z_2 \in \mathbf{U}.$$

Theorem 3.5. *If $w : \mathbf{U} \to \mathbf{U}$, $w(a) = 0$, is a K q.c. solution of the elliptic differential inequality*

$$|L[w]| \le \mathcal{B}|\nabla w|^2 + \Gamma,$$

then ∇w is bounded by a constant $C(K, \mathcal{B}, \Gamma, \Lambda, \mathfrak{L}, a)$ and w is lipschitz continuous.

Proof. The idea of the proof is to estimate the gradient of w in some "neighborhood" of the boundary together with some interior estimate in the rest of the unit disk. Let $\frac{1+|a|}{2} \le \alpha < 1$ and $\beta = \frac{\alpha+1}{2}$. Define $D_\alpha = \{z : |z| \le \beta\}$ and $A_\alpha = \{z : \alpha \le |z| < 1\}$.

Let $w = (w_1, w_2)$. According to Theorem 2.5 and Lemma 3.3, there exists a constant C_i depending only on modulus of continuity of w_i, $\mathcal{B}$, Γ, K, Λ, $\mathfrak{L}$ and α such that

$$|\nabla w_i(z)| \le C_i, \;\; z \in D_\alpha, i = 1, 2. \tag{3.12}$$

By Mori's theorem, the modulus of continuity of w_i depends only on K and a. Thus

$$|\nabla w(z)| \le |\nabla w_1| + |\nabla w_2| \le C_1 + C_2 = C_3(K, \mathcal{B}, \Gamma, \Lambda, \mathfrak{L}, \alpha), \;\; z \in D_\alpha. \tag{3.13}$$

As w is K quasiconformal selfmapping of the unit disk, by Mori's theorem ([48]) it satisfies the inequality:

$$4^{1-K}\left|\frac{a-z}{1-z\bar{a}}\right|^K \le |w(z)|, \;\; |z| < 1, \tag{3.14}$$

where $a = w^{-1}(0)$. Let $u = |w|$. From Lemma 3.2 and (3.14)

$$|L[u]| \le 2^{3K-2}\left(\frac{1+|a|}{1-|a|}\right)^K \Theta|\nabla u|^2 + \Pi, (1+|a|)/2 < |z| < 1. \tag{3.15}$$

Let g be a function

$$g : D \to \mathbf{R}$$

defined such that

$$g(z) = \begin{cases} 1, & \text{if } \beta < |z| \le 1; \\ 1 + (u(z)-1)\dfrac{\exp\frac{1}{|z|^2-\beta^2}}{\exp\frac{1}{\alpha^2-\beta^2}}, & \text{if } \alpha \le |z| \le \beta. \end{cases}$$

Let

$$\phi(z) := \frac{\exp\frac{1}{|z|^2-\beta^2}}{\exp\frac{1}{\alpha^2-\beta^2}}.$$

Then

$$L[g] = \begin{cases} 0, & \text{if } \beta < |z| \le 1; \\ (u(z)-1)L[\phi] + \phi L[u] + D[u,\phi], & \text{if } \alpha \le |z| \le \beta. \end{cases}$$

Therefore

$$|L[g]| \le \begin{cases} 0, & \text{if } \beta < |z| \le 1; \\ \mathcal{B}_1|\nabla u|^2 + \Gamma_1, & \text{if } \alpha \le |z| \le \beta, \end{cases} \tag{3.16}$$

where

$$\mathcal{B}_1 = 2^{3K-2}\left(\frac{1+|a|}{1-|a|}\right)^K\left(\sqrt{2}K\Lambda + \frac{\mathcal{B}K}{2}\right)$$

and Γ_1 is a constant depending only on K, $\mathcal{B}$, Γ, Λ, $\mathfrak{L}$ and α. By (3.4), (3.13) and (3.16) we have

$$|L[g]| \le C_4(K, \mathcal{B}, \Gamma, \Lambda, \mathfrak{L}, \alpha), \;\; z \in A_\alpha \tag{3.17}$$

and

$$|\nabla g| \le C_5(K, \mathcal{B}, \Gamma, \Lambda, \mathfrak{L}, \alpha), \;\; z \in A_\alpha. \tag{3.18}$$

Furthermore, by using the inequalities (3.15), (3.17) and (3.18) and $|a+b|^2 \leq 2(|a|^2+|b|^2)$ we have

$$\begin{aligned}|L[u-g]| &\leq |L[u]| + |L[g]| \\ &\leq \mathcal{B}_1|\nabla u|^2 + C_7(K,\mathcal{B},\Gamma,\Lambda,\mathfrak{L},\alpha) \\ &\leq 2\mathcal{B}_1|\nabla u - \nabla g|^2 + C_8(K,\mathcal{B},\Gamma,\Lambda,\mathfrak{L},\alpha), \quad z \in A_\alpha.\end{aligned}$$

By Mori's theorem, there exists a constant $\alpha = \alpha(K,a) < 1$ such that if

$$M = \max\{|u(z)-g(z)| : z \in A_\alpha\}$$

then there holds

$$\frac{64}{\pi} \cdot 2\mathcal{B}_1 M\Lambda < 1. \tag{3.19}$$

Thus $\tilde{u} = u - g$ satisfies the conditions of Theorem 2.8 in the domain $D = A_\alpha$. The conclusion is that ∇u is bounded in $\beta < |z| < 1$ by a constant depending only on K, $\mathcal{B}$, Γ, Λ, $\mathfrak{L}$ and a and on modulus of continuity of $\tilde{u}$. From Mori's theorem, the modulus of continuity of u depends only on K and a. From (3.18), the modulus of continuity of g do not depends on u. Combining the last fact and the relation (3.4) we obtain

$$|\nabla w| \leq C_0(K,\mathcal{B},\Gamma,\Lambda,\mathfrak{L},a), \quad \beta < |z| < 1. \tag{3.20}$$

From (3.13) and (3.20) we obtain the desired conclusion.

□

Mori's theorem for q.c. selfmappings of the unit disk has been generalized in various directions in the plane and in the space. See for example the papers [15], [8] and [7].

In the following theorem we show that, a q.c. mapping is Lipschitz continuous under certain regularity condition on Beltrami coefficient.

Theorem 3.6. *Let w be a K-q.c. mapping of the unit disk* $\mathbf{U}$ *onto itself such that the Beltrami coefficient $\mu = \frac{w_{\bar{z}}}{w_z}$ is Lipschitz continuous in* $\mathbf{U}$. *Then w is Lipschitz continuous in* $\mathbf{U}$.

Remark 3.7. Under the condition of Theorem 3.6, the function w is $C^{1,\alpha}(\mathbf{U})$, $\alpha < 1$, ([2, Theorem 15.0.7]) but the last fact of course do not implies that w is Lipschitz in $\mathbf{U}$.

Proof. Let

$$\Phi(z) = \begin{cases} C\exp\left(\frac{1}{|z|^2-1}\right), & \text{if } |z|<1 \\ 0, & \text{otherwise.}\end{cases}$$

Here the constant C is chosen so that $\int_{\mathbf{C}} \Phi(z) = 1$. Let $\varepsilon > 0$,

$$\phi_\varepsilon = \varepsilon^2 \Phi\left(\frac{z}{\varepsilon}\right),$$

$\mu(z) := 0, z \in \mathbf{C}\setminus\mathbf{U}$, and define

$$\mu_\varepsilon(z) = \phi_\varepsilon * \mu(z) = \int_{\mathbf{C}} \phi_\varepsilon(z-\tau)\mu(\tau)d\mathcal{L}^2(\tau), \quad z \in \mathbf{U}.$$

Then μ_ε is $C^\infty(\mathbf{U})$ and satisfies the inequalities

$$\|\mu_\varepsilon\|_\infty \le \|\mu\|_\infty \tag{3.21}$$

and

$$\|\nabla\mu_\varepsilon\|_\infty \le \|\nabla\mu\|_\infty = \Upsilon. \tag{3.22}$$

According to Riemann measure mapping theorem there exists a homeomorphic solution $w_\varepsilon : \mathbf{U} \to \mathbf{U}$ of Beltrami equation $w_{\bar z} = \mu_\varepsilon w_z$ normalized by $w_\varepsilon(a) = w(a) = 0$ and $w_\varepsilon(1) = w(1)$ (see e.g. [2, Theorem 9.0.3]). The solution w_ε is $C^\infty(\mathbf{U})$ because $\mu_\varepsilon \in C^\infty$ (see e.g. [2, Theorem 15.0.7]). Then w_ε converges uniformly on compact subsets of $\mathbf{U}$ to the mapping w as $\varepsilon \to 0$.

Let $\mu_\varepsilon = \alpha + i\beta$ and $w_\varepsilon = u + iv$. Then $w_{\varepsilon\bar z} = \mu_\varepsilon w_{\varepsilon z}$ is equivalent to the system:

$$\begin{aligned}(1-\alpha)u_x - (1+\alpha)v_y &= \beta(u_y - v_x)\\ (1+\alpha)u_y + (1-\alpha)v_x &= \beta(u_x + v_y).\end{aligned}$$

Hence

$$\begin{aligned}v_x &= -a^{22}u_y - a^{12}u_x\\ v_y &= a^{11}u_x + a^{12}u_y,\end{aligned}$$

where

$$a^{11} = \frac{|\mu_\varepsilon|^2 + 1 - 2\alpha}{1 - |\mu_\varepsilon|^2},$$

$$a^{22} = \frac{|\mu_\varepsilon|^2 + 1 + 2\alpha}{1 - |\mu_\varepsilon|^2}$$

and

$$a^{21} = a^{12} = \frac{-2\beta}{1 - |\mu_\varepsilon|^2}.$$

Since $v_{xy} = v_{yx}$ it follows

$$\mathcal{L}[u] := L[u] + (a^{11}_x + a^{12}_y)u_x + (a^{22}_y + a^{12}_x)u_y = 0, \tag{3.23}$$

where

$$L[u] := a^{11}u_{xx} + 2a^{12}u_{xy} + a^{22}u_{yy}.$$

It is easily to see that, the matrix $A_\varepsilon = \{a^{ij}\}_{i,j=1}^2$ satisfies the elliptic condition

$$\frac{1}{K(z)} \le \langle A_\varepsilon(z)h, h\rangle \le K(z), \ \ |h| = 1,$$

where, because of (3.21)

$$K(z) = \frac{1 + |\mu_\varepsilon(z)|}{1 - |\mu_\varepsilon(z)|} \le K.$$

Furthermore, since μ_ε is lipschitz with Lipschitz constant $\Upsilon_\varepsilon \le \Upsilon$, it follows that

$$|A_\varepsilon(z) - A_\varepsilon(w)| \le \mathfrak{L}|z - w|,$$

where $\mathfrak{L}$ depends only on Υ and on K.

Similarly we obtain that v satisfies the same PDE

$$\mathcal{L}[v] = 0. \tag{3.24}$$

Thus

$$\mathcal{L}[w_\varepsilon] = \mathcal{L}[u] + i\mathcal{L}[v] = 0. \tag{3.25}$$

It follows that w_ε is a solution of elliptic partial differential inequality

$$|L[w_\varepsilon]| \le \mathcal{B}|\nabla w_\varepsilon|^2 + \Gamma,$$

for some $\mathcal{B} > 0$ and $\Gamma > 0$ depending on Υ and K but not depending on ε (because of (3.22)). From Theorem 3.5 it follows that

$$|w_\varepsilon(z_1) - w_\varepsilon(z_2)| \le C(K, \Upsilon, a)|z_1 - z_2| \quad z_1, z_2 \in \mathbf{U}.$$

As w_ε converges to w we obtain the desired conclusion. □

An immediate consequence of Theorem 3.6 is the following corollary:

Corollary 3.8. *If w is a q.c. selfmapping of the unit disk with constant Beltrami coefficient, then w is bi-Lipschitz continuous.*

Remark 3.9. Under the conditions of Corollary 3.8 we can say much more. Namely, $w = \varphi(az + b\overline{z})$, where a, b are two complex constants and φ is a conformal mapping of an elipse to the unit disk.

By using Theorem 3.6, Kellogg theorem and the formula

$$\mu_{\varphi\circ f} = \mu_f,$$

where φ is a conformal mapping we obtain

Corollary 3.10. *Let w be a K-q.c. mapping of the unit disk $\mathbf{U}$ onto a Jordan domain Ω with $C^{1,\alpha}$ boundary such that the Beltrami coefficient μ_w is Lipschitz continuous in $\mathbf{U}$. Then w is lipschitz continuous in $\mathbf{U}$.*

Example 3.11. [2, p. 391]. Let $g(z) = -z\log|z|^2$, where $|z| \le r = e^{-2}$, then $g : r\mathbf{U} \to 4r\mathbf{U}$ is a homeomorphism and

$$\frac{\partial g}{\partial \bar{z}} = -\frac{z}{\bar{z}}, \quad \frac{\partial g}{\partial z} = -1 - \log|z|^2, \quad \mu_g(z) = \frac{z}{\bar{z}(1 + \log|z|^2)}.$$

Thus g is quasiconformal with continuous Beltrami coefficient, and yet g is not Lipschitz. The mapping $f(z) = \frac{1}{4r}g(rz)$ is a q.c. mapping of the unit disk onto itself with continuous Beltrami coefficient, but g is not Lipschitz neither locally Lipschitz. Thus the condition μ_f is Lipschitz continuous in Theorem 3.6 is important even for local Lipschitz behavior of a solution to Beltrami equation $w_{\bar{z}} = \mu(z)w_z$.

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