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Weight Distributions of Multi-Edge type LDPC Codes

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SUMMARY The multi-edge type LDPC codes, introduced by Richardson and Urbanke, present the general class of structured LDPC codes. In this paper, we derive the average weight distributions of the multi-edge type LDPC code ensembles. Furthermore, we investigate the asymptotic exponential growth rate of the average weight distributions and investigate the connection to the stability condition of the density evolution.
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1. Introduction

In 1963, Gallager invented low-density parity-check (LDPC) codes [1]. Due to the sparseness of the representation of the codes, LDPC codes are efficiently decoded by the sum-product (SP) decoders [2] or Log-SP decoders [3]. The Log-SP decoding is also known as the belief propagation. By the powerful method *density evolution* [3], invented by Richardson and Urbanke, the messages of the Log-SP decoding are statistically evaluated. The optimized LDPC codes can realize the reliable transmissions at rate close to the Shannon limit [4].

Recently, many *structured* LDPC codes have been proposed: accumulate repeat accumulate codes [5], irregular repeat accumulate codes [6], MacKay-Neal codes [7], protograph codes [8], raptor codes [9], low-density generator-matrix codes [10] and so on. These structured codes are usually designed for exploiting the structure to realize an excellent decoding performance, efficient encoding, a fast decoding algorithm, a parallel implementation and so on. Above all, the multi-edge type LDPC (MET-LDPC) codes [11] give a general framework that unifies all those structured LDPC codes.

The average weight distribution of codewords, which is simply referred to as *the average weight distribution*, helps the analysis of the average performance of the maximum likelihood (ML) decoding [12] and the typical minimum distance [1]. The decoding errors for the high SNR regions are mainly brought by codewords of small weight. Constructing LDPC codes without small weight codewords helps to lower the error floors. For the average weight distribution of the standard irregular LDPC codes are studied in [12]–[16] and those of structured codes are studied in [17]–[19]. Specifically for the standard irregular LDPC codes, the average weight distribution is derived in [15] as the coefficients of some polynomial. And many useful properties [13], [16] are derived from the coefficient expression.

In [18], we have already derived the average weight distributions for the MET-LDPC code ensembles. However, the derived equation [18] is not as simple as that of the standard irregular LDPC codes [15] and is hard to investigate further properties. Indeed the derived equation in [18] is not written in a closed form but given as a recursive form which concatenates the weight distributions of constituent codes of the MET code ensembles. In this paper, we derive the average weight distributions of the MET-LDPC code ensembles in a simple closed form. Furthermore, we investigate the asymptotic exponential growth rate of the average weight distributions and investigate the connection to the stability condition of the density evolution.

The rest of this paper is organized as follows. Section 2 gives the definition of the MET-LDPC code ensemble. Section 3 gives the simple expression for the average weight distributions of the MET-LDPC code ensemble. In Section 4 we derive the asymptotic exponential growth rate of the average weight distributions and investigate it for the codeword of small weight in Section 5. Furthermore, in Section 6, we show that the connection between the the asymptotic exponential growth rate of the codeword of small weight and the stability condition of the density evolution [4].

2. Multi-Edge type LDPC Codes

Before we give the general definition of the ME-LDPC codes, we show a specific example of MET-LDPC code for a better understanding. The Tanner graph of an example of MET-LDPC code is shown in Fig. 1. We

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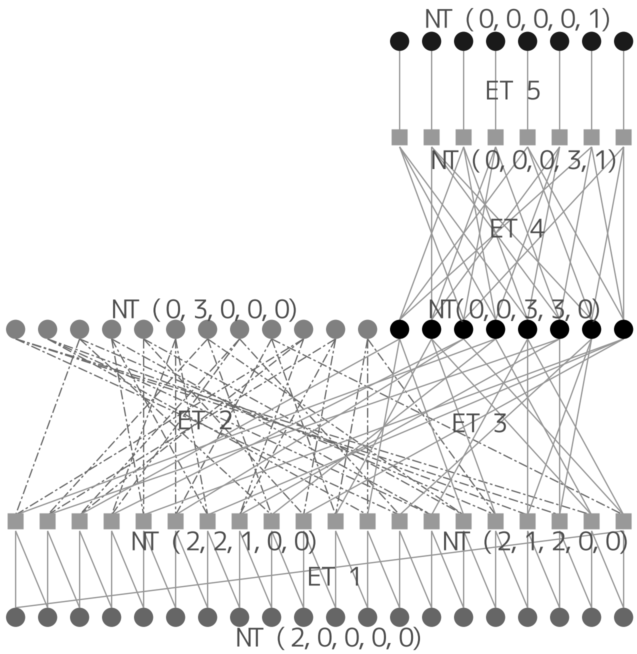

Fig. 1 The Tanner graph of a Multi-Edge type LDPC code. NT and ET stand for variable node-type, check node-type and edge-type, respectively.

use the terms, a "code" and its "Tanner graph" interchangeably. The edges in the graphs are divided into 5 types of edges labeled from "ET 1" to "ET 5". Note that there are two types of edges in the third row of edges. With this classification, each edge is said to have *edge-type* i for $i = 1, \dots, 5$. Furthermore variable and check nodes are classified into types according to the number of each edge-type they have. The types are called variable and check *node-types*, respectively. For example, the check nodes labeled "NT(2,2,1,0,0)" are said to have node-type (2,2,1,0,0) since these check nodes have two edges of edge-type 1, two edges of edge-type 2 and 1 edge of edge-type 3. And the variable nodes labeled "NT(0,0,0,0,1)" are said to have node-type (0,0,0,0,1) since these variable nodes have 1 edge of edge-type 5.

The original definition of MET-LDPC [11] codes involves the transmissions over the $n_{\mathfrak{r}}$ types of parallel channels. Since our interest in this paper is limited to the average weight distributions, we restrict ourselves to the transmissions over a single channel, i.e. $n_{\mathfrak{r}} = 1$.

For simplicity of notation, we define $\mathbf{1} = (1, \dots, 1)$ and $\mathbf{0} = (0, \dots, 0)$. And define that $\mathbf{x} \geq \mathbf{0}$ means that $x_i \geq 0$ for $i = 1, \dots, n$. Moreover, we use the notation

$$\mathbf{x}^{\mathbf{y}} := \prod_{i=1}^{n} x_i^{y_i}$$

for two vectors $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n)$ of size n.

Now, we give the definition of an MET-LDPC code ensemble. Analogously to the degree distribution pair $(\lambda(x), \rho(x))$ [16] for the standard irregular LDPC code ensemble, an MET-LDPC code ensemble is specified by a multivariate polynomial pair $(\nu(\mathbf{r}, \mathbf{x}), \mu(\mathbf{x}))$ which is also referred to as *the degree distribution pair*.

$$\begin{aligned}
&\nu(\mathbf{r}, \mathbf{x}) := \sum_{\mathbf{b},\mathbf{d} \geq \mathbf{0}} \nu_{\mathbf{b},\mathbf{d}} \mathbf{r}^{\mathbf{b}} \mathbf{x}^{\mathbf{d}}, \\
&\mu(\mathbf{x}) := \sum_{\mathbf{d} \geq \mathbf{0}} \mu_{\mathbf{d}} \mathbf{x}^{\mathbf{d}}, \\
&\mathbf{b} := (b_0, b_1, \dots, b_{n_{\mathfrak{r}}}), \mathbf{d} := (d_1, \dots, d_{n_{\mathfrak{e}}}), \\
&\mathbf{r} := (r_0, r_1, \dots, r_{n_{\mathfrak{r}}}), \mathbf{x} := (x_1, \dots, x_{n_{\mathfrak{e}}})
\end{aligned}$$

where $n_{\mathfrak{r}}$ is the number of channel-types and $n_{\mathfrak{e}}$ is the number of edge-types.

For a given degree distribution pair $(\nu(\mathbf{r}, \mathbf{x}), \mu(\mathbf{x}))$ and code length n, we define an equi-probable ensemble of LDPC codes with graphs G that satisfy the followings.

1. G has n variable nodes of channel-type $\boldsymbol{b} = (0, 1)$, i.e. G has n un-punctured transmitted bits.
2. G has $n\nu_{\mathbf{b},\mathbf{d}}$ variable nodes of channel-type $\mathbf{b}$ and node-type $\mathbf{d}$.
3. G has $n\mu_{\mathbf{d}}$ check nodes of node-type $\mathbf{d}$.

We denote this code ensemble by $\mathcal{C}(n, \nu(\mathbf{r}, \mathbf{x}), \mu(\mathbf{x}))$.

It is easy to see that the number of edges of edge-type i incident to variable and check nodes of node-type $\mathbf{d}$ are respectively given as

$$\begin{aligned}
&\sum_{\mathbf{b} \geq \mathbf{0}} d_i n \nu_{\mathbf{b},\mathbf{d}} = d_i n \nu_{(1,0),\mathbf{d}} + d_i n \nu_{(0,1),\mathbf{d}}, \\
&\sum_{\mathbf{d} \geq \mathbf{0}} d_i n \mu_{\mathbf{d}}.
\end{aligned}$$

It follows that the number of edges of edge-type i incident to variable nodes and check nodes are respectively given as

$$\begin{aligned}
n\nu_i(\mathbf{1}, \mathbf{1}) &:= n \frac{\partial}{\partial x_i} \nu(\mathbf{r}, \mathbf{x}) \Big|_{\substack{\mathbf{r}=\mathbf{1} \\ \mathbf{x}=\mathbf{1}}} = n \sum_{\mathbf{b},\mathbf{d} \geq \mathbf{0}} d_i \nu_{\mathbf{b},\mathbf{d}}, \\
n\mu_i(\mathbf{1}) &:= n \frac{\partial}{\partial x_i} \mu(\mathbf{x}) \Big|_{\mathbf{x}=\mathbf{1}} = n \sum_{\mathbf{d} \geq \mathbf{0}} d_i \mu_{\mathbf{d}},
\end{aligned}$$

for $i = 1, \dots, n_{\mathfrak{e}}$, where $\mathbf{1} := (1, \dots, 1)$. They are constrained to be identical, and we denote this number by E_i, i.e. for $i = 1, \dots, n_{\mathfrak{e}}$

$$E_i := n\nu_i(\mathbf{1}, \mathbf{1}) = n\mu_i(\mathbf{1}).$$

Being permuted the connection among E_i edges of edge-type i in a graph, the resulting Tanner graph has the same degree distribution pair. The number of graphs in the MET-LDPC ensemble is given as follows.

$$\#\mathcal{C}(n, \nu(\mathbf{r}, \mathbf{x}), \mu(\mathbf{x})) = \prod_{i=1}^{n_{\mathfrak{e}}} E_i!. \tag{1}$$

In this setting, we can see that the graph shown in Fig. 1 is an MET-LDPC code in the MET-LDPC code ensemble $\mathcal{C}(n=40,\nu(\mathbf{r},\mathbf{x}),\mu(\mathbf{x}))$, where

$$\begin{aligned}\nu(\mathbf{r},\mathbf{x}) &= 0.5r_1x_1^2+0.3r_1x_2^3+0.2r_0x_3^3x_4^3+0.2r_1x_5,\\ \mu(\mathbf{x}) &= 0.4x_1^2x_2^2x_3+0.1x_1^2x_2x_3^2+0.2x_4^3x_5.\end{aligned}$$

3. Weight Distribution of Multi-Edge type Codes

In this section, we derive the average weight distribution of the MET-LDPC code ensemble $\mathcal{C}(n,\nu(\mathbf{r},\mathbf{x}),\mu(\mathbf{x}))$. For readers who are unfamiliar with the enumeration technique of the weight distributions in LDPC code ensembles, we refer the readers to [1], [12].

We consider all the 2^N maps from each variable node to $\{0,1\}$,

$$x: v\mapsto x_v\in\{0,1\}.$$

We say a map x is a codeword of a code G if $\sum_{v\in V_c}x_v$ is an even number for every check node c in G, where V_c is the set of variable nodes adjacent to the check node c in G. The weight $w(x)$ of a map x is defined as the number of un-punctured variable nodes v such that $x_v=1$. Let $A_G(\ell)$ be the number of codewords of weight ℓ in a code G. Let $A(\ell)$ be the average number of codewords of weight ℓ for the MET-LDPC code ensemble $\mathcal{C}(n,\nu(\mathbf{r},\mathbf{x}),\mu(\mathbf{x}))$ defined as follows.

$$A(\ell)=\sum_{G\in\mathcal{C}(n,\nu(\mathbf{r},\mathbf{x}),\mu(\mathbf{x}))}A_G(\ell)/\#\mathcal{C}(n,\nu(\mathbf{r},\mathbf{x}),\mu(\mathbf{x})).$$

Theorem 1: For a given MET-LDPC code ensemble $\mathcal{C}(n,\nu(\mathbf{r},\mathbf{x}),\mu(\mathbf{x}))$, the average number of codewords of weight ℓ is given as follows.

$$A(\ell)=\sum_{\mathbf{e}\ge\mathbf{0}}\frac{\mathrm{coef}\left((Q(t,\mathbf{s})P(\mathbf{u}))^n, t^\ell\mathbf{s}^{\mathbf{e}}\mathbf{u}^{\mathbf{e}}\right)}{\prod_i\binom{E_i}{e_i}}, \tag{2}$$

$$Q(t,\mathbf{s})=\prod_{\mathbf{b},\mathbf{d}\ge\mathbf{0}}(1+t^{b_1}\mathbf{s}^{\mathbf{d}})^{\nu_{\mathbf{b},\mathbf{d}}},$$

$$P(\mathbf{u})=\prod_{\mathbf{d}\ge\mathbf{0}}\left(\frac{(\mathbf{1}+\mathbf{u})^{\mathbf{d}}+(\mathbf{1}-\mathbf{u})^{\mathbf{d}}}{2}\right)^{\mu_{\mathbf{d}}},$$

where

$$\begin{aligned}\mathbf{e}&=(e_1,\dots,e_{n_\mathfrak{e}}),\\ \mathbf{u}&=(u_1,\dots,u_{n_\mathfrak{e}}),\\ \mathbf{s}&=(s_1,\dots,s_{n_\mathfrak{e}}).\end{aligned}$$

And $\mathrm{coef}(g(\mathbf{x}),\mathbf{x}^{\mathbf{d}})$ is the coefficient of a term $\mathbf{x}^{\mathbf{d}}$ in a multivariate polynomial $g(\mathbf{x})$.

Proof : An edge is said to be *active*, if the edge is incident to a variable node v such that $x_v=1$. We will count all the codewords of weight ℓ in all graphs in the ensemble $\mathcal{C}(n,\nu(\mathbf{r},\mathbf{x}),\mu(\mathbf{x}))$ with e_i active edge of edge-type i for $i=1,\dots,n_\mathfrak{e}$, and sum them up for all $\mathbf{e}=(e_1,\dots,e_{n_\mathfrak{e}})\ge\mathbf{0}$. Counting all the codewords involves the following 3 parts:

1. Count the active edge constellations satisfying all the parity-check constraints.
2. Count the active edge constellations which stem from maps of weight ℓ.
3. Count the edge permutations among active edges and non-active edges.

Before we start counting the active edge constellations satisfying all the parity-check constraints, first, let us count the active edge constellations satisfying a single parity-check constraint. Consider a check node c of node-type $\mathbf{d}$. In other words, the check node c has d_i edges of edge-type i for $i=1,\dots,n_\mathfrak{e}$. The check node c is satisfied if the total number of active edges is even, i.e. $\sum_{i=1}^{n_\mathfrak{e}}e_i=$ even. Let $a_\mathrm{c}(\mathbf{e})$ be the number of active edge constellations which satisfy the check node c with given e_i active incident edges of edge-type i for $i=1,\dots,n_\mathfrak{e}$. It is easily checked that

$$a_\mathrm{c}(\mathbf{e})=\begin{cases}\prod_{i=1}^{n_\mathfrak{e}}\binom{d_i}{e_i} & \sum_{i=1}^{n_\mathfrak{e}}e_i=\text{even}\\ 0 & \sum_{i=1}^{n_\mathfrak{e}}e_i=\text{odd}\end{cases}$$

Let $f_\mathbf{d}(\mathbf{u})$ the generating function of $a_\mathrm{c}(\mathbf{e})$ defined as

$$f_\mathbf{d}(\mathbf{u}):=\sum_{\mathbf{e}\ge\mathbf{0}}a_\mathrm{c}(\mathbf{e})\mathbf{u}^{\mathbf{e}}.$$

We can simply describe $f_\mathbf{d}(\mathbf{u})$ as

$$\begin{aligned}f_\mathbf{d}(\mathbf{u})&=\frac{\prod_{i=1}^{n_\mathfrak{e}}(1+u_i)^{d_i}+\prod_{i=1}^{n_\mathfrak{e}}(1-u_i)^{d_i}}{2}\\ &=\frac{(\mathbf{1}+\mathbf{u})^{\mathbf{d}}+(\mathbf{1}-\mathbf{u})^{\mathbf{d}}}{2}.\end{aligned}$$

Next, count the active edge constellations satisfying all the $n\mu(\mathbf{1})$ parity-check constraints with given e_i active edges of edge-type i for $i=1,\dots,n_\mathfrak{e}$. Since there are $n\mu_\mathbf{d}$ check nodes of node-type $\mathbf{d}$ for $\mathbf{d}\ge\mathbf{0}$, the number of active edge constellations to satisfy all the parity-check constraints is given by

$$\mathrm{coef}\Big(\prod_{\mathbf{d}\ge\mathbf{0}}f_\mathbf{d}(\mathbf{u})^{n\mu_\mathbf{d}},\mathbf{u}^{\mathbf{e}}\Big). \tag{3}$$

Secondly, we will count the active edge constellations which stem from maps of weight ℓ. Count the active edge constellations which stem from a single variable node of weight $\ell=0,1$ at first. This may be somewhat confusing since it is too trivial. Consider a variable node v of channel-type $\mathbf{b}$ and node-type $\mathbf{d}$. Assume v is given e_i active edges of edge-type i for $i=1,\dots,n_\mathfrak{e}$. Let $a_\mathrm{v}(\ell,\mathbf{b},\mathbf{e})$ be the number of constellations which stem from the maps of weight $\ell\in\{0,1\}$. From the definition of the active edges, it is easily checked that

$$a_{\mathrm{v}}(\ell, \mathbf{b}, \mathbf{e}) = \begin{cases} 1 & (\ell = 0, \mathbf{e} = \mathbf{0}), \\ 1 & (\ell = b_1, \mathbf{e} = \mathbf{d}), \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the generating function of $a_{\mathrm{v}}(\ell, \mathbf{b}, \mathbf{e})$ is simply written as

$$\sum_{\ell \in \{0,1\}, \mathbf{b} \geq \mathbf{0}, \mathbf{e} \geq \mathbf{0}} a_{\mathrm{v}}(\ell, \mathbf{b}, \mathbf{e}) t^{\ell} \mathbf{s}^{\mathbf{e}} = 1 + t^{b_1} \mathbf{s}^{\mathbf{d}}.$$

Next, consider all n variable nodes. There are $n\nu_{\mathbf{b},\mathbf{d}}$ variable nodes of channel-type $\mathbf{b}$ and of node-type $\mathbf{d}$ for $\mathbf{b} \geq \mathbf{0}$ and $\mathbf{d} \geq \mathbf{0}$. It is consequent that for given e_i active edges of edge-type i, the number of active edge constellations which stem from maps $x : v \mapsto x_v \in \{0, 1\}$ of weight ℓ is given by

$$\mathrm{coef}\Big(\prod_{\mathbf{b},\mathbf{d} \geq \mathbf{0}} (1 + t^{b_1} \mathbf{s}^{\mathbf{d}})^{n\nu_{\mathbf{b},\mathbf{d}}}, t^{\ell} \mathbf{s}^{\mathbf{e}}\Big). \tag{4}$$

In the third place, consider that we are given e_i active edges of edge-type i and hence there are $E_i - e_i$ non-active edges of type i for $i = 1, \ldots, n_{\mathfrak{e}}$. For these active edge and non-active edges, the number of possible ways of permuting active and non-active edges is given as

$$\prod_{i=1}^{n_{\mathfrak{e}}} e_i!(E_i - e_i)!. \tag{5}$$

Let $A_{\mathbf{e}}(\ell)$ be the average number of graphs which have codewords of weight ℓ for given e_i active edges of type i for $i = 1, \ldots, n_{\mathfrak{e}}$. By multiplying Eq. (3), Eq. (4) and Eq. (5), and dividing by the number of codes in the ensemble given in Eq. (1), we obtain

$$\begin{aligned} A_{\mathbf{e}}(\ell) =& \mathrm{coef}\Big(\prod_{\mathbf{d} \geq \mathbf{0}} f_{\mathbf{d}}(\mathbf{u})^{n\mu_{\mathbf{d}}}, \mathbf{u}^{\mathbf{e}}\Big) \\ & \cdot \mathrm{coef}\Big(\prod_{\mathbf{b},\mathbf{d} \geq \mathbf{0}} (1 + t^{b_1} \mathbf{s}^{\mathbf{d}})^{n\nu_{b,\mathbf{d}}}, t^{\ell} \mathbf{s}^{\mathbf{e}}\Big) \Big/ \prod_{i=1}^{n_{\mathfrak{e}}} \binom{E_i}{e_i}. \end{aligned}$$

The average number of codewords of weight ℓ for the ensemble is obtained by summing up $A_{\mathbf{e}}(\ell)$ over the all possible active edge numbers.

$$A(\ell) = \sum_{\mathbf{e} \geq \mathbf{0}} A_{\mathbf{e}}(\ell) \tag{6}$$

□

4. Asymptotic Analysis

LDPC codes are usually used with large code length. We are interested in the asymptotic average weight distributions in the limit of large code length. The average number of codewords of weight ωn is usually increases or decays exponentially in n. We focus our interest in the asymptotic exponential growth rate of the $A(\ell)$ which is simply referred to as the *growth rate* $\gamma(\omega)$ defined as follows.

$$\gamma(\omega) := \lim_{n \to \infty} \frac{1}{n} \log A(\omega n),$$

where ω called is the normalized weight of codewords.

In this section, we derive the growth rate for the MET-LDPC code ensemble. To this end, first introduce the following lemma.

Lemma 1 ([12], III.2): For an m-variable polynomial $g(x_1, \ldots, x_m)$ with non-negative coefficients, it holds that

$$\lim_{n \to \infty} \frac{1}{n} \log \mathrm{coef}(g(\mathbf{x})^n, \mathbf{x}^{\boldsymbol{\alpha} n}) = \inf_{\mathbf{x} > \mathbf{0}} \log \frac{g(\mathbf{x})}{\mathbf{x}^{\boldsymbol{\alpha}}},$$

where $\mathbf{x} > \mathbf{0}$ means $x_i > 0$ for all $i = 1, \ldots, m$. The point $\mathbf{x}$ that takes the minimum of $\frac{g(\mathbf{x})}{\mathbf{x}^{\boldsymbol{\alpha}}}$ is given by a solution of the following equations.

$$\frac{x_i}{g(\mathbf{x})} \frac{\partial g(\mathbf{x})}{\partial x_i} = \alpha_i \quad (i = 1, 2, \ldots, m)$$

The number of terms in Eq. (2) is upper-bounded by $\prod_{i=1}^{n_{\mathfrak{e}}} E_i$. Therefore the largest term alone contributes the growth rate of $A(\ell)$. Therefore, from Eq. (6) we have

$$\max_{\mathbf{e} \geq \mathbf{0}} A_{\mathbf{e}}(\ell) \leq A(\ell) \leq \Big(\prod_{i=1}^{n_{\mathfrak{e}}} E_i\Big) \max_{\mathbf{e} \geq \mathbf{0}} A_{\mathbf{e}}(\ell) \tag{7}$$

$$\frac{1}{n} \log A(\ell) = \frac{1}{n} \log \max_{\mathbf{e} \geq \mathbf{0}} A_{\mathbf{e}}(\ell) + o(1). \tag{8}$$

Rewriting $A_{\mathbf{e}}(\ell)$ as

$$\begin{aligned} A_{n\boldsymbol{\beta}}(\omega n) &= \frac{\mathrm{coef}\big((Q(t, \mathbf{s})P(\mathbf{u}))^n, (t^{\omega} \mathbf{s}^{\boldsymbol{\beta} \geq \mathbf{0}} \mathbf{u}^{\boldsymbol{\beta}})^n\big)}{\prod_{i=1}^{n_{\mathfrak{e}}} \binom{\mu_i(\mathbf{1})n}{\beta_i n}}, \\ \boldsymbol{\beta} &= (\beta_1, \ldots, \beta_{n_{\mathfrak{e}}}), \end{aligned}$$

where $\boldsymbol{e} = n\boldsymbol{\beta}$ and using Lemma 1, we obtain that

$$\begin{aligned} \lim_{n \to \infty} \frac{1}{n} \log A(\ell) &= \sup_{\boldsymbol{\beta} \geq \mathbf{0}} \inf_{t > 0, \mathbf{s} > \mathbf{0}, \mathbf{u} > \mathbf{0}} \Big[\\ & \log Q(\mathbf{s}, t) + \log P(\mathbf{u}) - \sum_{i=1}^{n_{\mathfrak{e}}} \beta_i \log(u_i) \\ & - \sum_{i=1}^{n_{\mathfrak{e}}} \beta_i \log(s_i) - \omega \log(t) - \sum_{i=1}^{n_{\mathfrak{e}}} \mu_i(\mathbf{1}) h\left(\frac{\beta_i}{\mu_i(\mathbf{1})}\right) \Big] \\ &=: \sup_{\boldsymbol{\beta} \geq \mathbf{0}} \gamma(\boldsymbol{\beta}) \end{aligned} \tag{9}$$

A point $(\mathbf{u}, \mathbf{s}, t)$ that takes $\inf_{t,\mathbf{s},\mathbf{u}}$ is given as a solution of the following equations.

$$\omega = \frac{t \frac{\partial Q}{\partial t}}{Q} = \sum_{\mathbf{b},\mathbf{d} \geq \mathbf{0}} \frac{\nu_{\mathbf{b},\mathbf{d}} b_1 t \mathbf{s}^{\mathbf{d}}}{1 + t^{b_1} \mathbf{s}^{\mathbf{d}}}, \tag{10}$$

$$\beta_i = u_i \frac{\frac{\partial P}{\partial u_i}}{P} = u_i \sum_{\mathbf{d}\ge\mathbf{0}} \mu_{\mathbf{d}} d_i \frac{\frac{(\mathbf{1}+\mathbf{u})^{\mathbf{d}}}{1+u_i} - \frac{(\mathbf{1}-\mathbf{u})^{\mathbf{d}}}{1-u_i}}{(\mathbf{1}+\mathbf{u})^{\mathbf{d}}+(\mathbf{1}-\mathbf{u})^{\mathbf{d}}}, \qquad (11)$$

$$\beta_i = s_i \frac{\frac{\partial Q}{\partial s_i}}{Q} = \sum_{\mathbf{b},\mathbf{d}\ge\mathbf{0}} \frac{\nu_{\mathbf{b},\mathbf{d}} d_i t^{b_1} \mathbf{s}^{\mathbf{d}}}{1+t^{b_1}\mathbf{s}^{\mathbf{d}}}, \text{ for } i=1,\dots,n_{\mathfrak{e}}. \qquad (12)$$

A point $\boldsymbol{\beta} = (\beta_1, \dots, \beta_{n_\mathfrak{e}})$ which gives $\sup_{\boldsymbol{\beta}}$ needs to satisfy the stationary condition

$$\frac{\beta_i}{\mu_i(\mathbf{1}) - \beta_i} = u_i s_i. \qquad (13)$$

Thus, we obtain the following theorem.

Theorem 2: For a given MET-LDPC code ensemble $\mathcal{C}(n, \nu(\mathbf{r}, \mathbf{x}), \mu(\mathbf{x}))$, the growth rate of the normalized weight ω is given by

$$\gamma(\omega) := \lim_{n\to\infty} \frac{1}{n} A(\omega n) = \max_{\boldsymbol{\beta}\in\mathfrak{B}(\omega)} \gamma(\boldsymbol{\beta}),$$

where $\mathfrak{B}(\omega)$ is a set of $\boldsymbol{\beta}$ such that (10), (11), (12) and (13) hold.

The derivative of $\gamma(\boldsymbol{\beta})$ in terms of ω can be expressed in the following simple expression.

Lemma 2: For $\boldsymbol{\beta}$ and t such that $t \neq 0$ and equations (10), (11) and (12) hold, we have the following.

$$\frac{d}{d\omega}\gamma(\boldsymbol{\beta}) = -\log(t(\omega))$$

Proof : Let x' denote the derivation of x with respect to ω. Differentiating $\gamma(\omega)$ defined in (9), we have

$$\frac{d}{d\omega}\gamma(\boldsymbol{\beta}) = \frac{Q'}{Q} + \frac{P'}{P} - w\frac{t'}{t} - \sum_{i=1}^{n_\mathfrak{e}} \log\frac{\mu_i(\mathbf{1}) - \beta_i}{\beta_i}\beta_i'$$
$$- \log t - \sum_{i=1}^{n_\mathfrak{e}} (\beta_i' \log u_i + \beta_i \frac{u_i'}{u_i} + \beta_i' \log s_i + \beta_i \frac{s_i'}{s_i}),$$

where $\mathbf{s}$ is given by equations (10), (11) and (12). From (13), we see

$$-\beta_i' \log u_i - \beta_i' \log s_i - \beta_i' \log \frac{\mu_i(\mathbf{1}) - \beta_i}{\beta_i} = 0.$$

Combining (11) and $P' = \sum_{i=1}^{n_\mathfrak{e}} \frac{\partial P}{\partial u_i} u_i'$, we have

$$\frac{P'}{P} - \sum_{i=1}^{n_\mathfrak{e}} \beta_i \frac{u_i'}{u_i} = 0 \qquad (14)$$

From (10), (12) and $Q' = \frac{\partial Q}{\partial t} t' + \sum_{i=1}^{n_\mathfrak{e}} \frac{\partial Q}{\partial s_i} s_i'$, we have

$$\frac{Q'}{Q} - w\frac{t'}{t} + \sum_{i=1}^{n_\mathfrak{e}} \beta_i \frac{s_i'}{s_i} = 0$$

Thus, we can conclude the proof since the remaining term in the right hand side of (14) is $-\log t$. □

5. Analysis of Small Weight Codeword

In this section we restrict ourselves to considering unpunctured MET-LDPC codes, i.e.

$$\mathbf{b} = (b_0, b_1) = (0, 1) \text{ for } \nu_{\mathbf{b},\mathbf{d}} \neq 0. \qquad (15)$$

Furthermore, we assume that for every edge-type i there exists a check node which has at least 2 edges of edge-type i. In precise, for $i = 1, \dots, n_\mathfrak{e}$,

$$\exists\mathbf{d} \text{ such that } d_i \ge 2 \text{ and } \mu_{\mathbf{d}} \neq 0. \qquad (16)$$

For the standard irregular LDPC codes [20] with a degree distribution pair $(\lambda(x), \rho(x))$, this assumption reduces to the condition of the non-existence of check nodes of degree 1, i.e. $\rho'(1) > 0$.

We investigate how the growth rate behaves for codewords of small weight, i.e. for small normalized weight ω. From the linearity of MET-LDPC codes, $A(0) = 1$ and $\gamma(0) = 0$, then from (9) and Lemma 2, it follows that for $\omega \to 0$,

$$\gamma(\omega) = \gamma'(0)\omega + o(\omega) \qquad (17)$$
$$= \sup_{t\in\mathfrak{T}} -\log(t)\omega + o(\omega), \qquad (18)$$

where $\mathfrak{T}$ is a set of t such that (10), (11), (12) and (13) hold for $\omega \to 0$. From the assumption of non-puncturing (15) and (10), for $\omega \to 0$, it holds that $t\mathbf{s}^{\mathbf{d}} \to 0$ for $\mathbf{d}$ with $\nu_{\mathbf{b},\mathbf{d}} \neq 0$. Using this, it follows that $\beta_i \to 0$ for $i = 1, \dots, n_\mathfrak{e}$ from (12). Using the assumption of check node-types Eq. (16) and Eq. (11), it is consequent that $u_i \to 0$ for $i = 1, \dots, n_\mathfrak{e}$. Moreover, from (11) it follows that as $\mathbf{u} \to \mathbf{0}$,

$$\beta_i = \sum_{\mathbf{d}\ge\mathbf{0}} \mu_{\mathbf{d}} u_i d_i ((d_i - 1)u_i + \sum_{j\neq i} d_j u_j) + o((\textstyle\sum_{i=1}^{n_\mathfrak{e}} u_i)^2).$$

Substituting this to (13), we have

$$s_i = \frac{\mu_{i,i}(\mathbf{1})}{\mu_i(\mathbf{1})} u_i + \sum_{j\neq i} \frac{\mu_{i,j}(\mathbf{1})}{\mu_i(\mathbf{1})} u_j + o(\textstyle\sum_{i=1}^{n_\mathfrak{e}} u_i), \qquad (19)$$

$$\mu_{i,j}(\mathbf{x}) = \frac{\partial^2}{\partial x_i \partial x_j} \mu(\mathbf{x}).$$

As $\mathbf{s} \to \mathbf{0}$, from (12) we have

$$\beta_i = t s_i (\nu_{i,i}(\mathbf{1}, \mathbf{0}) s_i + \sum_{j\neq i} \nu_{i,j}(\mathbf{1}, \mathbf{0}) s_j) + o((\textstyle\sum_{i=1}^{n_\mathfrak{e}} s_i)^2)$$

Substituting this to (13), we obtain the following.

$$u_i = t\Big(\frac{\nu_{i,i}(\mathbf{1}, \mathbf{0})}{\nu_i(\mathbf{1}, \mathbf{1})} s_i + \sum_{j\neq i} \frac{\nu_{i,j}(\mathbf{1}, \mathbf{0})}{\nu_i(\mathbf{1}, \mathbf{1})} s_j\Big) + o(\textstyle\sum_{i=1}^{n_\mathfrak{e}} s_i) \qquad (20)$$

$$\nu_{i,j}(\mathbf{r},\mathbf{x}) = \frac{\partial^2}{\partial x_i \partial x_j}\nu(\mathbf{r},\mathbf{x})$$

We can represent (19) and (20) by matrices as $\mathbf{s} = \mathrm{P}\mathbf{u}$ and $\mathbf{u} = t\Lambda(\mathbf{1})\mathbf{s}$, respectively, where

$$\Lambda_{i,j}(\mathbf{r}) := \frac{\left.\frac{\partial^2 \nu(\mathbf{r},\mathbf{x})}{\partial x_i \partial x_j}\right|_{\mathbf{x}=\mathbf{0}}}{\nu_i(\mathbf{1},\mathbf{1})},$$
$$\mathrm{P}_{i,j} := \frac{\left.\frac{\partial^2 \mu(\mathbf{x})}{\partial x_i \partial x_j}\right|_{\mathbf{x}=\mathbf{1}}}{\mu_i(\mathbf{1})}.$$

In summary, for $t \neq 0$ we obtain

$$\frac{1}{t}\mathbf{u} = \Lambda(\mathbf{1})\mathrm{P}\mathbf{u} + o(\textstyle\sum_{i=1}^{n_\mathfrak{e}} u_i). \tag{21}$$

This implies that $\frac{1}{t}$ is an eigenvalue of $\Lambda(\mathbf{1})\mathrm{P}$. Therefore $\sup_{t\in\mathfrak{T}}$ of (17) is achieved by the largest eigenvalue $\frac{1}{t}$ of $\Lambda(\mathbf{1})\mathrm{P}$. Then we have the following theorem.

Theorem 3: For an MET-LDPC code ensemble $\mathcal{C}(n,\nu(\mathbf{r},\mathbf{x}),\mu(\mathbf{x}))$, assume the largest eigenvalue $\frac{1}{t}$ of $\Lambda(\mathbf{1})\mathrm{P}$ is not zero. The growth rate $\gamma(\omega) := \lim_{n\to\infty}\frac{1}{n}\log A(\omega n)$ of the average number $A(\omega n)$ of codewords of weight ωn, in the limit of code length, is given by

$$\gamma(\omega) = \log\left(\frac{1}{t}\right)\omega + O(\omega^2).$$

Furthermore, there exists $\delta > 0$ such that if $\frac{1}{t} < 1$, there are exponentially few codewords of weight ωn for $\omega < \delta$.

For a standard irregular LDPC code ensemble [16] with a given degree distribution pair $(\lambda(x),\rho(x))$, can be viewed as an MET-LDPC code ensemble

$$\mathcal{C}\Big(n, \nu(r_1,x) = r_1\frac{\sum_i \lambda_i x^i/i}{\sum_i \lambda_i/i}, \mu(x) = \frac{\sum_i \rho_i x^i/i}{\sum_i \lambda_i/i}\Big).$$

The eigenvalue is given by $\lambda'(0)\rho'(1)$ which is zero if there are no variable nodes of degree 2. The condition $\frac{1}{t} < 1$ in Theorem 3 reduces to $\lambda'(0)\rho'(1) < 1$, which coincides with the known result [13].

6. Relation with Stability Condition

In this section, we investigate the connection between the growth rate and the stability condition [4]. For simplicity, we assume the transmission takes place over the binary erasure channels (BEC) with the erasure probability ϵ.

For the standard irregular LDPC code ensemble [16] with degree distribution pair $(\lambda(x),\rho(x))$, in the limit of the code length, we denote the average decoding erasure probability of messages sent from variable nodes to check nodes at the ℓ-th iteration round by $p^{(\ell)}$. From density evolution [4], $p^{(\ell)}$ is given by

$$p^{(0)} = \varepsilon,$$
$$p^{(\ell)} = \varepsilon\lambda(1-\rho(1-p^{(\ell-1)})).$$

The following is shown in [4], if $\varepsilon\lambda'(0)\rho'(1) > 1$, there exists $\gamma > 0$ such that $\lim_{\ell\to\infty} p^\ell > \gamma$. The inequality

$$\varepsilon\lambda'(0)\rho'(1) < 1 \tag{22}$$

is called *the stability condition of density evolution*. Furthermore, it is shown in [21], the capacity-achieving LDPC code ensemble have the degree distribution pair $(\lambda(x),\rho(x))$ with $\varepsilon\lambda'(0)\rho'(1) = 1$. Meanwhile, it is known that the growth rate of the average number of codewords of small linear weight ωn is given by

$$\lim_{n\to\infty}\frac{1}{n}\log A(\omega n) = \log(\lambda'(0)\rho'(1))\omega + o(\omega). \tag{23}$$

Interestingly, the same parameter $\lambda'(0)\rho'(1)$ appears in both the stability condition Eq. (22) and the growth rate Eq. (23). Does this correspondence also hold for the MET-LDPC code ensembles?

For the MET-LDPC code ensemble with the degree distribution pair $(\nu(\mathbf{r},\mathbf{x}),\mu(\mathbf{x}))$, in the limit of large code length, let $p_i^{(\ell)}$ denote the erasure probability of the message sent along the edges of edge-type i from variable nodes to check nodes at the ℓ-th interation round.

From the density evolution developed for the MET-LDPC codes [11, Eq. (8)], $p_i^{(\ell)}$ is recursively given by

$$\mathbf{p}^{(\ell)} = \boldsymbol{\lambda}((1,\varepsilon), \mathbf{1} - \boldsymbol{\rho}(\mathbf{1} - \mathbf{p}^{(\ell-1)})),$$
$$\boldsymbol{\lambda}(\mathbf{r},\mathbf{x}) := (\lambda_1(\mathbf{r},\mathbf{x}),\dots,\lambda_{n_\mathfrak{e}}(\mathbf{r},\mathbf{x})),$$
$$\boldsymbol{\rho}(\mathbf{x}) := (\rho_1(\mathbf{x}),\dots,\rho_{n_\mathfrak{e}}(\mathbf{x})),$$
$$\lambda_i(\mathbf{r},\mathbf{x}) = \frac{\nu_i(\mathbf{r},\mathbf{x})}{\nu_i(\mathbf{1},\mathbf{1})},$$
$$\rho_i(\mathbf{x}) = \frac{\mu_i(\mathbf{x})}{\mu_i(\mathbf{1})},$$

where $p_i^{(0)} = \varepsilon$ for $i = 1,\dots,n_\mathfrak{e}$. And it follows that [11, Theorem 7] is given as follows. If the spectral radius of $\Lambda(1,\varepsilon)\mathrm{P}$ is less than 1, there exists $\gamma > 0$ such that

$$\lim_{\ell\to\infty}\sum_i p_i^{(\ell)} > \gamma.$$

In short, the stability condition for the MET-LDPC code is given as follows.

$$1 > \text{the spectral radius of } \Lambda(1,\varepsilon)\mathrm{P}.$$

Since $\Lambda(1,\varepsilon)\mathrm{P}$ is a non-negative matrix, the spectral radius of $\Lambda(1,\varepsilon)\mathrm{P}$ is an eigenvalue of $\Lambda(1,\varepsilon)\mathrm{P}$, it follows $\Lambda(1,1)\mathrm{P}$ coincides with the parameter which appears in Theorem 3.

7. Conclusion

We present a simple expression of the average weight distributions of MET-LDPC code ensembles which gives us a general framework of LDPC codes. We showed that the correspondence between the growth rate of the weight distributions and the stability condition is also the case with the MET-LDPC codes.

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