

EXOTIC HEAT PDE'S.II

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This paper is dedicated to Stephen Smale in occasion of his 80th birthday

ABSTRACT. Exotic heat equations that allow to prove the Poincaré conjecture and its generalizations to any dimension are considered. The methodology used is the PDE's algebraic topology, introduced by A. Prástaro in the geometry of PDE's, in order to characterize global solutions. In particular it is shown that this theory allows us to identify n -dimensional *exotic spheres*, i.e., homotopy spheres that are homeomorphic, but not diffeomorphic to the standard S^n .

AMS Subject Classification: 55N22, 58J32, 57R20; 58C50; 58J42; 20H15; 32Q55; 32S20.

Keywords: Integral bordisms in PDE's; Existence of local and global solutions in PDE's; Conservation laws; Crystallographic groups; Singular PDE's; PDE's on complex manifolds category; PDE's on quantum supermanifolds category; Exotic spheres.

1. INTRODUCTION

"How exotic are exotic spheres ?"

The term "exotic sphere" was used by J. Milnor to characterize smooth manifolds that are homotopy equivalent and homeomorphic to S^n , but not diffeomorphic to S^n .¹ This strange mathematical phenomenon, never foreseen before the introduction just by J. Milnor of the famous 7-dimensional exotic sphere [28], has stimulated a lot of mathematical research in algebraic topology. The starting points, were, other than the cited paper by J. W. Milnor, also a joint paper with M. A. Kervaire [25] and some papers by S. Smale [55], Freedman [13] and J. Cerf [6] on generalizations of the Poincaré conjecture in dimension $n \geq 4$. There the principal mathematical tools utilized were Morse theory (Milnor), h-cobordism theory (Smale), surgery techniques and Hirzebruch signature formula. Surprising, from this beautiful mathematical architecture was remained excluded just the famous

¹In this paper we will use the following notation: \approx homeomorphism; \cong diffeomorphism; \cong homotopy equivalence; \simeq homotopy.

Poincaré conjecture for 3-dimensional manifolds. In fact, the surgery techniques do not give enough tools in low dimension ($n < 5$), where surgery obstructions disappear. Really, it was necessary to recast the Poincaré problem as a problem to find solutions in a suitable PDE equation (*Ricci flow equation*), to be able to obtain more informations just on dimension three. (See works by R.S. Hamilton [17, 18, 19, 20, 21], G. Perelman [33, 34] and A. Prástaro [50, 1].) The idea by R. S. Hamilton to recast the problem in the study of the Ricci flow equation has been the real angular stone that has allowed to look to the solution of the Poincaré conjecture from a completely new point of view. In fact, with this new perspective it was possible to G. Perelman to obtain his results and to A. Prástaro to give a new proof of this conjecture, by using his PDE's algebraic topologic theory. To this respect, let us emphasize that the usual geometric methods for PDE's (Spencer, Cartan), were able to formulate for nonlinear PDE's, local existence theorems only, until the introduction, by A. Prástaro, of the algebraic topologic methods in the PDE's geometric theory. These give suitable tools to calculate integral bordism groups in PDE's, and to characterize global solutions. Then, on the ground of integral bordism groups, a new geometric theory of stability for PDE's and solutions of PDE's has been built. These general methodologies allowed to A. Prástaro to solve fundamental mathematical problems too, other than the Poincaré conjecture and some of its generalizations, like characterization of global smooth solutions for the Navier-Stokes equation and global smooth solutions with mass-gap for the quantum Yang-Mills superequation. (See [39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51].²) The main purpose of this paper is to show how, by using the PDE's algebraic topology, introduced by A. Prástaro, one can prove the Poincaré conjecture in any dimension for the category of smooth manifolds, but also to identify exotic spheres. In the part I [52] we have just emphasized as in dimension 3, the method followed by A. Prástaro allows us to prove the Poincaré conjecture and to state also that 3-dimensional homotopy spheres are diffeomorphic to S^3 . (Related problems are considered there too.) In the framework of the PDE's algebraic topology, the identification of exotic spheres is possible thanks to an interaction between integral bordism groups of PDE's, conservation laws, surgery and geometric topology of manifolds. With this respect we shall enter in some details on these subjects, in order to well understand and explain the meaning of such interactions. So the paper splits in three sections other this Introduction. 2. Integral bordism groups in Ricci flow PDE's. 3. Morse theory in Ricci flow PDE's. 4. h-Cobordism in Ricci flow PDE's. The main result is contained just in this last section and it is Theorem 4.59 that by utilizing the previously results considered states (and proves) the following.³

Theorem 4.59. *The generalized Poincaré conjecture, for any dimension $n \geq 1$ is true, i.e., any n -dimensional homotopy sphere M is homeomorphic to S^n : $M \approx S^n$. For $1 \leq n \leq 6$ one has also that M is diffeomorphic to S^n : $M \cong S^n$. But for $n \geq 7$, it does not necessitate that M is diffeomorphic to S^n . This happens when the Ricci flow equation, under the homotopy equivalence full admissibility hypothesis, (see below for definition), becomes a 0-crystal.*

²See also Refs. [1, 2, 53], where interesting related applications of the PDE's Algebraic Topology are given.

³In order to allow a more easy understanding, this paper has been written in a large expository style.

Moreover, under the sphere full admissibility hypothesis, the Ricci flow equation becomes a 0-crystal in any dimension $n \geq 1$.

2. INTEGRAL BORDISM GROUPS IN RICCI FLOW PDE's

In this section we shall characterize the local and global solutions of the Ricci flow equation, following the geometric approach of some our previous works on this equation [40, 50, 48, 1]. Let M be a n -dimensional smooth manifold and let us consider the following fiber bundle $\bar{\pi} : E \equiv \mathbb{R} \times \widetilde{S_2^0 M} \rightarrow \mathbb{R} \times M$, $(t, x^i, y_{ij})_{1 \leq i, j \leq n} \mapsto (t, x^i) \equiv (x^\alpha)_{0 \leq \alpha \leq n}$, where $\widetilde{S_2^0 M} \subset S_2^0 M$ is the open subbundle of non-degenerate Riemannian metrics on M . Then the Ricci flow equation is the closed second order partial differential relation, (in the sense of Gromov [16]), on the fiber bundle $\bar{\pi} : E \rightarrow \mathbb{R} \times M$, $(RF) \subset JD^2(E)$, defined by the differential polynomials on $JD^2(E)$ given in (1):

$$(1) \quad \begin{cases} F_{jl} & \equiv |y|[y_{ik}](y_{il,jk} + y_{jk,il} - y_{jl,ik} - y_{ik,jl}) \\ & + [y_{ik}][y_{rs}]([jk, r][il, s] - [jl, r][ik, s]) + \frac{|y|^2}{2} y_{jl,t} \\ & \equiv S_{jl}(y_{rs}, y_{rs,\alpha}, y_{rs,pq}) + \frac{|y|^2}{2} y_{jl,t} = 0, \end{cases}$$

where $[ij, r]$ are the usual Christoffels symbols, given by means of the coordinates $y_{rs,i}$, $|y| = \det(y_{ik})$, and $[y_{ik}]$ is the algebraic complement of y_{ik} . The ideal $\mathfrak{p} \equiv \langle F_{jl} \rangle$ is not prime in $\mathbb{R}[y_{rs}, y_{rs,\alpha}, y_{rs,ij}]$. However, an irreducible component is described by the system in solved form: $y_{rs,t} = -\frac{2}{|y|^2} S_{jl}$. This is formally integrable and also completely integrable.⁴ In fact,

$$(2) \quad \begin{cases} \dim JD^{2+s}(E) = n + 1 + \sum_{0 \leq r \leq 2+s} \frac{(n+1)n}{2} \frac{(n+r)!}{r!n!} \\ \dim (RF)_{+s} = n + 1 + \frac{(n+1)n}{2} \left[\sum_{0 \leq r \leq 2+s} \frac{(n+r)!}{r!n!} - \sum_{0 \leq r' \leq s} \frac{(n+r')!}{r'!n!} \right] \\ \dim g_{2+s} = \frac{(n+1)n}{2} \frac{(n+2+s)!}{(2+s)!n!} - \frac{(n+1)n}{2} \frac{(n+s)!}{s!n!}. \end{cases}$$

Therefore, one has: $\dim(RF)_{+s} = \dim(RF)_{+(s-1)} + \dim g_{2+s}$. This assures that one has the exact sequences in (3).

$$(3) \quad (RF)_{+s} \longrightarrow (RF)_{+(s-1)} \longrightarrow 0, \quad s \geq 1.$$

One can also see that the symbol g_2 is not involutive. By the way a general theorem of the formal geometric theory of PDE's assures that after a finite number of prolongations, say s , the corresponding symbol g_{2+s} becomes involutive. (See refs.[15, 39].) Then, taking into account the surjectivity of the mappings (3), we get that (RF) is formally integrable. Furthermore, from the algebraic character of this equation, we get also that is completely integrable. Therefore, in the neighborhood of any of its points $q \in (RF)$ we can find solutions. (These can be analytic ones, but also smooth if we consider to work on the infinity prolongation $(RF)_{+\infty}$, where the Cartan distribution is "involutive" and of dimension $(n+1)$.) Finally, taking into account that $\dim(RF) > 2(n+1) + 1 = 2n + 3$, we can use Theorem 2.15 in [40] to calculate the n -dimensional singular integral bordism group, $\Omega_{n,s}^{(RF)}$, for n -dimensional closed smooth admissible integral manifolds bording by means of (singular) solutions. (Note that the symbols of (RF) and its prolongations are

⁴We shall denote with the same symbol (RF) the corresponding algebraic manifold. For a geometric algebraic theory of PDE's see the monograph [39], and references quoted there.

non-zero.) This group classifies the structure of the global singular solutions of the Ricci-flow equation. One has:

$$(4) \quad \Omega_{n,s}^{(RF)} \cong \bigoplus_{r+s=n} H_r(M; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} \Omega_s,$$

where Ω_s is the bordism group for s -dimensional closed smooth manifolds.⁵

It is important to underline that with the term "n-dimensional closed smooth admissible integral manifolds" we mean smooth integral manifolds, $N \subset (RF) \subset JD^2(E)$, that diffeomorphically project on their image on E , via the canonical projection $\pi_{2,0} : JD^2(E) \rightarrow E$. In [50] we have proved, that any smooth section $g : M \rightarrow \widetilde{S}_2^0 M$, identifies a space-like n -dimensional smooth integral manifold $N \subset (RF)$, and that for such a Cauchy manifold pass local smooth solutions, contained in a tubular neighbourhood $N \times [0, \epsilon) \subset (RF)$, for suitable $\epsilon > 0$. Therefore, we can represent any n -dimensional smooth compact Riemannian manifold (M, γ) as a space-like Cauchy manifold $N_0 \subset (RF)_{t_0}$, for some initial time t_0 , and ask if there are solutions that bord N_0 with (S^n, γ') , where γ' is the canonical metric of S^n , identified with another space-like Cauchy manifold $N_1 \subset (RF)_{t_1}$, with $t_0 < t_1$. The answer depends on the class of solution that we are interested to have. For weak-singular solutions the corresponding integral bordism group $\Omega_{n,s}^{(RF)}$ is given in (4). The relation with the integral bordism group $\Omega_n^{(RF)}$, for smooth solutions of (RF) is given by the exact commutative diagram (5) where is reported the relation with the bordism group Ω_n for smooth manifolds.

$$(5) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \searrow & & \downarrow & & \\ & & \overline{K}_n^{(RF)} & \xrightarrow{\quad} & K_{n,s;n}^{(RF)} & \longrightarrow & 0 \\ & & \uparrow & \searrow & \downarrow & & \\ 0 & \longrightarrow & K_{n,s}^{(RF)} & \longrightarrow & \Omega_n^{(RF)} & \xrightarrow{a} & \Omega_{n,s}^{(RF)} \longrightarrow 0 \\ & & \uparrow & & \searrow & \downarrow & \\ & & 0 & \longrightarrow & \Omega_n & \xrightarrow{c} & \Omega_n^{(RF)} \longrightarrow 0 \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

Theorem 2.1. *Let M in the Ricci flow equation $(RF) \subset JD^2(E) \subset J_n^2(W)$ be a smooth compact n -dimensional manifold homotopy equivalent to S^n . Then the n -dimensional singular integral bordism group of (RF) is given in (6).*

$$(6) \quad \Omega_{n,s}^{(RF)} = \Omega_n \bigoplus \mathbb{Z}_2.$$

⁵We used the fact that the fiber of $E \rightarrow M$ is contractible.

3. MORSE THEORY IN RICCI FLOW PDE's

Let us now give the fundamental theorem that describe quantum tunnel effects in solutions of PDEs, i.e., the change of sectional topology in the (singular) solutions of (RF) .

Theorem 3.1 (Topology transitions as quantum tunnel effects in Ricci flow equation). *Let $N_0, N_1 \subset (RF) \subset JD^2(E)$ be space-like Cauchy manifolds of (RF) , at two different times $t_0 \neq t_1$. Let $V \subset (RF)$ be a (singular) solution such that $\partial V = N_0 \sqcup N_1$. Then there exists an admissible Morse function $f : V \rightarrow [a, b] \subset \mathbb{R}$ such that:*

(A) (Simple quantum tunnel effect). *If f has a critical point q of index k then there exists a k -cell $e^k \subset V - N_1$ and an $(n - k + 1)$ -cell $e_*^{n-k+1} \subset V - N_0$ such that:*

- (i) $e^k \cap N_0 = \partial e^k$;
- (ii) $e_*^{n-k+1} \cap N_1 = \partial e_*^{n-k+1}$;
- (iii) *there is a deformation retraction of V onto $N_0 \cup e^k$;*
- (iv) *there is a deformation retraction of V onto $N_1 \cup e_*^{n-k+1}$;*
- (v) $e_*^{n-k+1} \cap e^k = q$; $e_*^{n-k+1} \pitchfork e^k$.

(B) (Multi quantum tunnel effect). *If f is of type $(\nu_0, \dots, \nu_{n+1})$ where ν_k denotes the number of critical points with index k such that f has only one critical values c , $a < c < b$, then there are disjoint k -cells $e_i^k \subset V \setminus N_1$ and disjoint $(n - k + 1)$ -cells $(e_*)_i^{n-k+1} \subset V \setminus N_0$, $1 \leq i \leq \nu_k$, $k = 0, \dots, n + 1$, such that:*

- (i) $e_i^k \cap N_0 = \partial e_i^k$;
- (ii) $(e_*)_i^{n-k+1} \cap N_1 = \partial (e_*)_i^{n-k+1}$;
- (iii) *there is a deformation retraction of V onto $N_0 \cup \{\cup_{i,k} (e_*)_i^k\}$;*
- (iv) *there is a deformation retraction of V onto $N_1 \cup \{\cup_{i,k} (e_*)_i^{n-k}\}$;*
- (v) $(e_*)_i^{n-k} \cap e_i^k = q_i$; $(e_*)_i^{n-k+1} \pitchfork e_i^k$.

(C) (No topology transition). *If f has no critical point then $V \cong N_0 \times I$ where $I \equiv [0, 1]$.*

Proof. The proof can be conducted by adapting to the Ricci flow equation (RF) Theorem 23 in [35]. Let us emphasize here some important lemmas only.

Lemma 3.2 (Morse-Smale functions). 1) *On a closed connected compact smooth manifold M , there exists a Morse function $f : M \rightarrow \mathbb{R}$ such that the critical values are ordered with respect to the indexes, i.e., $f(x_\lambda) = f(x_\mu)$, if $\lambda = \mu$, and $f(x_\lambda) > f(x_\mu)$, if $\lambda > \mu$, where x_λ , (resp. x_μ), is the critical point of f with index λ (resp. μ). Such functions are called regular functions, or Morse-Smale functions, and are not dense in $C^\infty(M, \mathbb{R})$, as Morse functions instead are. Furthermore, such functions can be chosen in such a way that they have an unique maximum point (with index $\lambda = n = \dim M$), and an unique minimum point (with index $\lambda = 0$).*

2) *To such functions are associated vector fields $\zeta = \text{grad } f : M \rightarrow TM$ such that $\zeta(x_\lambda) = 0$ iff x_λ is a critical point. Then in a neighborhood of a x_λ , the integral curves of ζ are of two types: ingoing in x_λ , and outgoing from x_λ . These fit in two different disks D^λ and $D^{n-\lambda}$ contained in M called separatrix diagram. (See Fig. 1.)*

Lemma 3.3 (Morse functions and CW complexes). 1) *Let M be a compact n -dimension manifold and $f : M \rightarrow [a, b]$ an admissible Morse function of type*

(ν_0, \dots, ν_n) such that $\partial M = f^{-1}(b)$. Then M has the homotopy type of a finite CW complex having ν_k cells at each dimension $k = 0, \dots, n$. Furthermore $(M, \partial M)$ has the homotopy type of a CW-pair of dimension n .

Furthermore $(M, f^{-1}(a))$ has the homotopy of relative CW complex having ν_k cells of dimension k , for each $k = 0, \dots, n$.⁶

2) An n -dimension manifold M has the homotopy type of a CW complex of dimension $\leq n$.

3) Let M be a compact manifold and $N \subset M$ a compact submanifold with $\partial N = \partial M = \emptyset$. Then (M, N) has the homotopy type of CW pair.

4) The cell decomposition of a closed connected compact smooth manifold M , related to a Morse-Smale function, is obtained attaching step-by-step a cell of higher dimension to the previous ones.

Lemma 3.4 (Homological Euler characteristic). 1) If the compact solution V of (RF) is characterized by an admissible Morse function $f : V \rightarrow [a, b]$ of type $(\nu_0, \dots, \nu_{n+1})$, then its homological Euler characteristic $\chi_{hom}(V)$ is given by the formula (10).

$$(10) \quad \chi_{hom}(V) = \sum_{0 \leq k \leq (n+1)} (-1)^k \beta_k, \quad \beta_k = \dim_F H_k(V, f^{-1}(a); F)$$

where F is any field.⁷ Furthermore, if $f^{-1}(a) = \emptyset$, then β_k in (10) is given by $\beta_k = \dim_F H_k(V; F)$, and $\chi_{hom}(V) = \chi(V)$.

2) If M is a compact odd dimensional manifold with $\partial M = \emptyset$, then $\chi_{hom}(M) = 0$.

3) Let M be a compact manifold such that its boundary can be divided in two components: $\partial M = \partial_- M \cup \partial_+ M$, then $\chi_{hom}(M, \partial_+ M) = \chi_{hom}(M, \partial_- M)$.

4) Let M be a compact manifold such that $\partial M = N_0 \sqcup N_1$, with N_i , $i = 0, 1$, disjoint closed sets. Let $f : M \rightarrow \mathbb{R}$ be a C^2 map without critical points, such that $f(N_0) = 0$, $f(N_1) = 1$. Then one has the diffeomorphisms: $M \cong N_0 \times I$, $M \cong N_1 \times I$.

5) Let M be a compact n -dimensional manifold with $\partial M = \emptyset$, such that has a Morse function $f : M \rightarrow \mathbb{R}$ with only two critical points. Then M is homeomorphic to S^n .

⁶A relative CW complex (Y, X) is a space Y and a closed subspace X such that $Y = \bigcup_{r=-1}^{\infty} Y_r$,

such that $X = Y_{-1} \subset Y_0 \subset \dots$, and Y_r is obtained from Y_{r-1} by attaching r -cells.

⁷Let $H_k(Y, X; F)$ denote the singular homology group of the pair (Y, X) with coefficients in the field F . $\beta_k = \dim_F H_k(Y, X; F)$ are called the F -Betti numbers of (Y, X) . If these numbers are finite and only finitely are nonzero, then the homological Euler characteristic of (Y, X) is defined by the formula: $\chi_{hom}(Y, X) = \sum_{0 \leq k \leq \infty} (-1)^k \beta_k$. When Y is a compact manifold and X is a compact submanifold, then $\chi_{hom}(Y, X)$ is defined. The evaluation of homological Euler characteristic for topological spaces coincides with that of Euler characteristic for CW complexes X given by $\chi(X) = \sum_{i \geq 0} (-1)^i k_i$, where k_i is the number of cells of dimension i . For closed smooth manifolds M , $\chi(M)$ coincides with the Euler number, that is the Euler class of the tangent bundle TM , evaluated on the fundamental class of M . For closed Riemannian manifolds, $\chi(M)$ is given as an integral on the curvature, by the generalized Gauss-Bonnet theorem: $\chi(V) = \frac{1}{(2\pi)^n} \int_V Pf(\Omega)$, where $\partial V = \emptyset$, $\dim V = 2n$, Ω is the curvature of the Levi-Civita connection and $Pf(\Omega) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \epsilon(\sigma) \prod_{i=1}^n \Omega_{\sigma(2i-1)\sigma(2i)}$, where (Ω_{rs}) is the skew-symmetric $(2n) \times (2n)$ matrix representing $\Omega : V \rightarrow \mathfrak{so}(2n) \otimes \Lambda^2_0(V)$, hence $Pf(\Omega) : V \rightarrow \Lambda^0_{2n}(V)$. In Tab. 2 are reported some important properties of Euler characteristic, that are utilized in this paper.

TABLE 2. Euler characteristic χ : properties and examples.

Definition	χ	Remarks	Examples
$\partial V = \emptyset, \dim V = 2n + 1, n \geq 0$	$\chi(V) = 0$	(from Poincaré duality)	
$M \cong N$ (homotopy equivalence)	$\chi(M) = \chi(N)$	from $H^*(M) \cong H^*(N)$	$\chi(pt) = 1$ $\chi(S^n) = 1 + (-1)^n = 0$ ($n = \text{odd}$), 2 ($n = \text{even}$) $\chi(D^3) = \chi(P^3) = \chi(\mathbb{R}^3) = 1$, $P^3 = \text{convex polyhedron}$ $\chi(S^2) = \chi(\partial P^3) = 2$, $\partial P^3 = \text{surface convex polyhedron}$
$V = M \sqcup N$	$\chi(V) = \chi(M) + \chi(N)$	(from homology additivity)	$\chi(S^2 \sqcup \dots \sqcup S^2) = 2n$
excision couple (M, N)	$\chi(M \cup N) = \chi(M) + \chi(N) - \chi(M \cap N)$		$\chi(S^2) = \chi(D^2) + \chi(D^2) - \chi(S^1) = 1 + 1 - 0 = 2$ $\chi(K_{ib}) = \chi(M_{ob}) + \chi(M_{ob}) - \chi(S^1) = 0 + 0 - 0 = 0$ $\chi(C_{rc} \cup_{S^1} D^2 = \mathbb{R}P^2) = \chi(M_{ob}) + \chi(D^2) - \chi(S^1) = 0 + 1 - 0 = 1$
$V = M \times N$	$\chi(V) = \chi(M) \cdot \chi(N)$		$\chi(T^n) = \chi(\underbrace{S^1 \times \dots \times S^1}_n) = 0$
$p: V \rightarrow M$ orientable fibration over field with fibre F M path-connected	$\chi(V) = \chi(F) \cdot \chi(M)$	(from Serre-spectral sequence) (also from transfer map) ($\tau: H_*(M) \rightarrow H_*(V)$) ($p_* \circ \tau = \chi(F) \cdot 1_{H_*(M)}$)	$S^n \rightarrow \mathbb{R}P^n: \chi(S^n) = \chi(\{1, -1\}) \cdot \chi(\mathbb{R}P^n) = 2 \cdot \chi(\mathbb{R}P^n)$ $\chi(\mathbb{R}P^n) = 0$ ($n = \text{odd}$), 1 ($n = \text{even}$).
$p: V \rightarrow M$ k -sheeted covering	$\chi(V) = k \cdot \chi(M)$		$p: M_{ob} \rightarrow S^1: M_{ob} = \text{Möbius strip}: \chi(M_{ob}) = 2 \cdot \chi(S^1) = 0$
$V = \partial M, \dim M = 2m, m \geq 0$	$\chi(V) = 2m, m \geq 0$	(from excision couple)	$\mathbb{R}P^{2n} \neq \partial M$, since $\chi(\mathbb{R}P^{2n}) = 1$

$\chi: \mathcal{O}_{2n}^0 \rightarrow \mathbb{Z}$ is a surjective mapping for $i \geq 1$ and an isomorphism for $i = 1$.
 $\chi(\partial P^3) = V - E + F$, $V = \text{vertex-number}$, $E = \text{edge-number}$, $F = \text{face-number}$.
 Closed oriented surfaces: $\chi = 2 - 2g$, $g = \text{genus}$, (number of handles).
 Closed nonorientable surfaces: $\chi = 2 - \kappa$, $\kappa = \text{nonorientable genus}$, (number of real projective planes in a connected decomposition).
 Examples of nonorientable surfaces: $K_{ib} = \text{Klein bottle}$ ($\partial K_{ib} = \emptyset$); $M_{ob} = \text{Möbius strip}$ ($\partial M_{ob} = S^1$); $\mathbb{R}P^2 = \text{Projective plane}$ ($\partial \mathbb{R}P^2 = \emptyset$).
 Examples of nonorientable surfaces: $C_{rc} \cong M_{ob} = \text{cross-cap}$: surface homotopy equivalent to Möbius strip.

Lemma 3.5. *Let X and Y be closed compact differentiable manifolds without boundaries, then there exists a compact manifold V , such that $\partial V = X \sqcup Y$ iff Y is obtained from X by a sequence of surgeries. (For details see below Theorem 4.12.)*

□

Theorem 3.6 (Smooth solutions and characteristic vector fields). *The characteristic vector field ξ , propagating a space-like n -dimensional smooth, compact, Cauchy manifold $N \subset V$, where V is a smooth solution of (RF) , hence a time-like, $(n + 1)$ -dimensional smooth integral manifold of (RF) , cannot have zero points.*

Proof. In fact, the characteristic vector field ξ coincides with the time-like $\zeta_0 \equiv \partial x_0 + \sum_{|\beta| \geq 0} y_{\alpha\beta}^j \partial y_j^\beta$, where $y_{\alpha\beta}^j$ are determined by the infinity prolongation $(RF)_{+\infty}$ of (RF) . Therefore such a vector field cannot have zero points on a compact smooth solution V , of (RF) , such that $\partial V = N_0 \sqcup N_1$. On the other hand, if $f: V \rightarrow \mathbb{R}$ is the Morse function whose gradient gives just the vector field ξ , then f cannot have critical points.⁸ □

Corollary 3.7. *A $(n + 1)$ -dimensional smooth, compact, manifold $V \subset (RF)$, smooth solution of (RF) , such that $\partial V = N_0 \sqcup N_1$, where $N_i, i = 0, 1$, are smooth Cauchy manifolds, cannot produce a change of topology from N_0 to N_1 , hence these manifolds must necessarily be homeomorphic.*

The following theorem emphasizes the difference between homeomorphic manifolds and diffeomorphic ones.

Theorem 3.8 (Exotic differentiable structures on compact smooth manifolds). *Let M and N be n -dimensional homeomorphic compact smooth manifolds. Then it does not necessitate that M is diffeomorphic to N .*

⁸Let us recall that a compact connected manifold M with boundary $\partial M \neq \emptyset$, admits a nonvanishing vector field. Furthermore, a compact, oriented n -dimensional submanifold $M \subset \mathbb{R}^{2n}$ has a nonvanishing normal vector field. Therefore, above statements about smooth solutions of (RF) agree with well known results of differential topology. (See, e.g. [23].)

Proof. Since M is considered homeomorphic to N , there exist continuous mappings $f : M \rightarrow N$ and $g : N \rightarrow M$, such that $g \circ f = id_M$, and $f \circ g = id_N$. Let us consider, now, the following lemma.

Lemma 3.9. *Let M and N be C^s manifolds, $1 \leq s \leq \infty$, without boundary. Then $C^s(M, N)$ is dense in $C_r^s(M, N)$, (in the strong topology), $0 \leq r < s$.*

Proof. See, e.g., [23]. □

From Lemma 3.9 we can state that the above continuous mappings f and g can be approximated with differentiable mapping, but these do not necessitate to be diffeomorphisms. In fact we have the following lemma.

Lemma 3.10. *Let $G^k(M, N) \subset C^k(M, N)$, $k \geq 1$, denote any one of the following subsets: diffeomorphisms, embeddings, closed embeddings, immersions, submersions, proper maps. Let M and N be compact C^s manifolds, $1 \leq s \leq \infty$, without boundary. Then $G^s(M, N)$ is dense in $G^r(M, N)$ in the strong topology, $1 \leq r < s$. In particular, M and N are C^s diffeomorphic iff they are C^r diffeomorphic with $r \geq 1$.*

Proof. See, e.g., [23]. □

Above lemma can be generalized also to compact manifolds with boundary. In fact, we have the following lemma.

Lemma 3.11. *Let us consider compact manifolds with boundary. Then the following propositions hold.*

- (i) *Every C^r manifold M , $1 \leq r < \infty$, is C^∞ diffeomorphic to a C^∞ manifold and the latter is unique up to C^∞ diffeomorphisms.*
- (ii) *Let $(M, \partial M)$ and $(N, \partial N)$, be C^s manifold pairs, $1 \leq s \leq \infty$. Then, the inclusion $C^s(M, \partial M; N, \partial N) \hookrightarrow C^r(M, \partial M; N, \partial N)$, $0 \leq r < s$, is dense in the strong topology. If, $1 \leq r < s$ and $(M, \partial M)$ and $(N, \partial N)$ are C^r diffeomorphic, they are also C^s diffeomorphic.*

Proof. See, e.g., [23]. □

Therefore, it is not enough to assume that compact smooth manifolds should be homeomorphic in order to state that they are also diffeomorphic, hence the proof of Theorem 3.8 is complete. (To complement Theorem 3.8 see also Lemma 4.28 and Lemma 4.29 below.) □

From Theorem 3.8 we are justified to give the following definition.

Definition 3.12. *Let M and N be two n -dimensional smooth manifolds that are homeomorphic but not diffeomorphic. Then we say that N is an exotic substitute of M .*

Example 3.13. *The sphere S^7 has 28 exotic substitutes, just called exotic 7-dimensional spheres. (See [28, 25].) These are particular 7-dimensional manifolds, built starting from oriented fiber bundle pairs over S^4 . More precisely let us consider $(D^4, S^3) \rightarrow (W, V) \rightarrow S^4$. The 4-plane bundle $D^4 \rightarrow S^4$ is classified by the isomorphism $[S^4, BSO(4)] \cong \mathbb{Z} \oplus \mathbb{Z}$, given by $\omega \mapsto (\frac{1}{4}(2\chi(\omega) + p_1(\omega)), \frac{1}{4}(2\chi(\omega) - p_1(\omega)))$, where $\chi(\omega), p_1(\omega) \in H^4(S^4) = \mathbb{Z}$ are respectively the Euler number and the Pontrjagin class of ω , (related by the congruence $p_1(\omega) = 2\chi(\omega) \pmod{4}$). Let us denote*

by $(W(\omega), V(\omega))$ the above fiber bundle pair identified by ω . The homology groups of $V(\omega) \rightarrow S^4$, are given in (11).⁹

$$(11) \quad H_p(V(\omega)) = \begin{cases} \mathbb{Z} & \text{if } p = 0, 7 \\ \text{coker}(\chi(\omega) : \mathbb{Z} \rightarrow \mathbb{Z}) & \text{if } p = 3 \\ \text{ker}(\chi(\omega) : \mathbb{Z} \rightarrow \mathbb{Z}) & \text{if } p = 4 \\ 0 & \text{otherwise.} \end{cases}$$

The Euler number $\chi(\omega)$ is the Hopf invariant of $J(\omega) \in \pi_7(S^4)$, i.e., $\chi(\omega) = \text{Hopf}(J(\omega)) \in \mathbb{Z}$. If $\chi(\omega) = 1 \in \mathbb{Z}$, then $V(\omega)$ is a homotopy 7-sphere which is boundary of an oriented 8-dimensional manifold $W(\omega)$. In fact ${}^+ \Omega_7 = 0$ and for $\chi(\omega) = 1 \in \mathbb{Z}$ one has $H_p(V(\omega)) = H_p(S^7)$. Let k be an odd integer and let $\omega_k : S^4 \rightarrow BSO(4)$ be the classifying map for orientable 4-plane bundle over S^4 with $p_1(\omega_k) = 2k$, $\chi(\omega_k) = 1 \in \mathbb{Z}$. There exists a Morse function $V(\omega_k) \rightarrow \mathbb{R}$ with two critical points, such that $V(\omega_k) \setminus \{\text{pt}\} \cong \mathbb{R}^7$, hence $V(\omega_k)$ is homeomorphic to S^7 . Let us investigate under which conditions $V(\omega_k)$ is diffeomorphic to S^7 too. So let us assume that such diffeomorphism $f : V(\omega_k) \cong S^7$ exists. Then let us consider the closed oriented 8-dimensional manifold $M = W(\omega_k) \cup_f D^8$. For such a manifold we report in (12) its intersection form and signature.

$$(12) \quad \begin{cases} (H^4(M), \lambda) = (\mathbb{Z}, 1) \\ \sigma(M) = \sigma(H^4(M), \lambda) = 1. \end{cases}$$

By the Hirzebruch signature theorem one has $\sigma(M) = \langle \mathcal{L}_2(p_1, p_2), [M] \rangle = 1 \in \mathbb{Z}$, with $\langle \mathcal{L}_2(p_1, p_2), [M] \rangle = \frac{1}{45}(7p_2(M) - p_1(M)^2) = 1 \in H^8(M) = \mathbb{Z}$, $p_1(M) = 2k$, $p_2(M) = \frac{1}{7}(45 + 4k^2) = \frac{4}{7}(k^2 - 1) + 7 \in H^4(M) = \mathbb{Z}$. Since $p_2(M)$ is an integer, it follows that must be $k^2 \equiv 1 \pmod{7}$. This condition on k , comes from the assumption that $V(\omega_k)$ is diffeomorphic to S^7 , therefore, it follows that under the condition $k^2 \not\equiv 1 \pmod{7}$, $V(\omega_k)$ can be only homeomorphic to S^7 , but not diffeomorphic, hence it is an exotic sphere, and M is only a 8-dimensional topological manifold, to which the Hirzebruch signature theorem does not apply.

Example 3.14. The 4-dimensional affine space \mathbb{R}^4 has infinity exotic substitutes, just called exotic \mathbb{R}^4 . (See [11, 13].)

The surgery theory is a general algebraic topological framework to decide if a homotopy equivalence between n -dimensional manifolds is a diffeomorphism. (See, e.g. [63].) We shall resume here some definitions and results about this theory. In the following section we will enter in some complementary informations and we will continue to develop such approach in connection with other algebraic topological aspects.

Definition 3.15. An n -dimensional geometric Poincaré complex is a finite CW complex such that one has the isomorphism $H^p(X; \Lambda) \cong H_{n-p}(X; \Lambda)$, induced by the cap product, i.e. $[\omega] \mapsto [X] \cap [\omega]$, for every $\mathbb{Z}[\pi_1(X)]$ -module Λ .

Theorem 3.16 (Geometric Poincaré complex properties). 1) An n -dimensional manifold is an n -dimensional geometric Poincaré complex.
2) Let X be a geometric Poincaré complex and Y another CW complex homotopy related to X . Then also Y is a geometric Poincaré complex.

⁹Recall that an odd dimensional oriented compact manifold M , with $\partial M = \emptyset$ has $\chi(M) = 0$. In particular $\chi(S^{2k+1}) = 0$, instead $\chi(S^{2k}) = 2$. Furthermore, if M and N are compact oriented manifolds with $\partial M = \partial N = \emptyset$, then $\chi(M \times N) = \chi(M)\chi(N)$.

3) Any CW complex homotopy equivalent to a manifold is a geometric Poincaré complex.

4) Geometric Poincaré complexes that are not homotopy equivalent to a manifold may be obtained by gluing together n -dimensional manifolds with boundary, $(M, \partial M)$, $(N, \partial N)$, having an homotopic equivalence on the boundaries, $f : \partial M \cong \partial N$, which is not homotopic to a diffeomorphism.

5) (Transfer or Umkehr map). Let $f : N \rightarrow M$ be a mapping between oriented, compact, closed manifolds of arbitrary dimensions. Then the Poincaré duality identifies an homomorphism $\tau : H^\bullet(N; \mathbb{Z}) \rightarrow H^{\bullet-d}(M; \mathbb{Z})$, where $d = \dim N - \dim M$. More precisely one has the commutative diagram (13) that defines τ .

$$(13) \quad \begin{array}{ccc} H^\bullet(N; \mathbb{Z}) & \xrightarrow{\tau} & H^{\bullet-d}(M; \mathbb{Z}) \\ D_N \downarrow & & \uparrow D_M^{-1} \\ H_{\dim N - \bullet}(N; \mathbb{Z}) & \xrightarrow{f_*} & H_{\dim M - \bullet}(M; \mathbb{Z}) \end{array}$$

where $d = \dim N - \dim M$. D_N and D_M are the Poincaré isomorphisms on N and M respectively. One has $\tau(f^*(x) \cup y) = x \cup \tau(y)$, $\forall x \in H^\bullet(M; \mathbb{Z})$ and $y \in H^\bullet(N; \mathbb{Z})$.¹⁰

In particular when $f : \widetilde{M} \rightarrow M$ is a covering map, then one can write $\tau(x)(\sigma) = x(\sum_{f(\bar{\sigma})=\sigma} \bar{\sigma})$, $\forall x \in C^\bullet(\widetilde{M})$ and $\sigma \in C_\bullet(M)$.

Definition 3.17. Let X be a closed n -dimensional geometric Poincaré complex. A manifold structure (M, f) on X is a closed n -dimensional manifold M together with a homotopy equivalence $f : M \cong X$. We say that such two manifold structures (M, f) , (N, g) on X are equivalent if there exists a bordism $(F; f, g) : (V; M, N) \rightarrow X \times (I; \{0\}, \{1\})$, with F a homotopy equivalence. (This means that $(V; M, N)$ is an h -cobordism, (see the next section).) Let $\mathfrak{S}(X)$ denote the set of such equivalence classes. We call $\mathfrak{S}(X)$ the manifold structure set of X . $\mathfrak{S}(X) = \emptyset$ means that X is without manifold structures.

Theorem 3.18 (Manifold structure set properties). 1) $\mathfrak{S}(X)$ is homotopy invariant of X , i.e., a homotopy equivalence $f : X \cong Y$ induces a bijection $\mathfrak{S}(X) \rightarrow \mathfrak{S}(Y)$.

2) A homotopy equivalence $f : M \cong N$ of n -dimensional manifolds determines an element $(M, f) \in \mathfrak{S}(N)$, such that f is h -cobordant to $1 : N \rightarrow N$ iff $(M, f) \in [(N, 1)] \in \mathfrak{S}(N)$.

3) Let M be a n -dimensional closed differentiable manifold. If $\mathfrak{S}(X) = \{pt\}$ then M does not admit exotic substitutes.

4) (Differential structures by gluing manifolds together). Let M and N be n -dimensional manifolds such that their boundary are diffeomorphic: $\partial M \cong \partial N$. Let α and β be two differential structures on $M \cup_f N$ that agree with the differential structures on M and N respectively. Then there exists a diffeomorphism $h : W_\alpha \cong W_\beta$ such that $h|_M = 1_M$.

Definition 3.19. A degree 1 normal map from an n -dimensional manifold M to an n -dimensional geometric Poincaré complex X is given by a couple (f, b) , where $f : M \rightarrow X$ is a mapping such that $f_*[M] = [X] \in H_n(X)$, and $b : \nu_M \rightarrow \eta$ is

¹⁰If $f : N \rightarrow M$ is an orientable fiber bundle with compact, orientable fiber F , integration over the fiber provides another definition of the transfer map: $\tau : H_{de-Rham}^\bullet(N) \rightarrow H_{de-Rham}^\bullet(M)^{\bullet-r}$, where $r = \dim F$.

a stable bundle map over f , from the stable normal bundle $\nu_M : M \rightarrow BO$ to the stable bundle $\eta : X \rightarrow BO$. We write also $(f, b) : M \rightarrow X$.

Theorem 3.20 (Obstructions for manifold structures on geometric Poincaré complex). 1) Let X be an n -dimensional geometric Poincaré complex. Then the criterion to decide if X is homotopy equivalent to an n -dimensional manifold M , is to verify that are satisfied the following two conditions.

(i) X admits a degree 1 normal map $(f, b) : M \rightarrow X$. This is the case when the map $t(\nu_X) : X \rightarrow B(G/O)$, given in (14), is null-homotopic.

$$(14) \quad \begin{array}{ccccc} X & \longrightarrow & BG & \longrightarrow & B(G/O) \\ & & \searrow & \nearrow & \\ & & & & t(\nu_X) \end{array}$$

Then there exists a null-homotopy $t(\nu_X) \simeq \{*\}$ iff the Spivak normal fibration $\nu_X : X \rightarrow BG = \varinjlim_{\overline{k}} BG(k)$ admits a vector bundle reduction $\widehat{\nu}_X : X \rightarrow BO$.

(ii) $(f, b) : M \rightarrow X$ is bordant to a homotopy equivalence $(g, h) : N \cong X$.

2) (J. H. C. Whitehead's theorem). $f : M \rightarrow X$ is a homotopy equivalence iff $\pi_*(f) = 0$. Let $n = 2k$, or $n = 2k + 1$. It is always possible to kill $\pi_i(f)$, for $i \leq k$, i.e., there is a bordant degree 1 normal map $(h, b) : N \rightarrow X$, with $\pi_i(h) = 0$ for $i \leq k$. There exists a normal bordism of (f, b) to a homotopy equivalence iff it is also possible to kill $\pi_{k+1}(h)$. In general there exists an obstruction to killing $\pi_{k+1}(h)$, which for $n \geq 5$ is of algebraic nature.

3) (C. T. C. Wall's surgery obstruction theorem).[63] For any group π there are defined algebraic L -groups $L_n(\mathbb{Z}[\pi])$ depending only on $n \pmod{4}$ as group of stable isomorphism classes of $(-1)^k$ -quadratic forms over $\mathbb{Z}[\pi]$ for $n = 2k$, or as group of stable automorphisms of such forms for $n = 2k + 1$. An n -dimensional degree 1 normal map $(f, b) : N \rightarrow X$ has a surgery obstruction $\sigma_*(f, b) \in L_n(\mathbb{Z}[\pi_1(X)])$, such that $\sigma_*(f, b) = 0$ if (and for $n \geq 5$ only if) (f, b) is bordant to a homotopy equivalence.

Example 3.21. The simply-connected surgery obstruction groups are given in Tab. 3. In particular, we have the following.

TABLE 3. Simply connected surgery obstruction groups

$n \pmod{4}$	0	1	2	3
$L_n(\mathbb{Z})$	\mathbb{Z}	0	\mathbb{Z}_2	0

- The surgery obstruction of a $4k$ -dimensional normal map $(f, b) : M \rightarrow X$ with $\pi_1(X) = \{1\}$ is $\sigma_*(f, b) = \frac{1}{8}\sigma(K_{2k}(M), \lambda) \in L_{4k}(\mathbb{Z}) = \mathbb{Z}$, with λ the nonsingular symmetric form on the middle-dimensional homology kernel \mathbb{Z} -module

$$K_{2k}(M) = \ker(f_* : H_{2k}(M) \rightarrow H_{2k}(X)).$$

- The surgery obstruction of a $(4k + 2)$ -dimensional normal map $(f, b) : M \rightarrow X$ with $\pi_1(X) = \{1\}$ is $\sigma_*(f, b) = \text{Arf}(K_{2k+1}(M; \mathbb{Z}_2), \lambda, \mu) \in L_{4k+2}(\mathbb{Z}) = \mathbb{Z}_2$, with λ, μ the nonsingular quadratic form on the middle-dimensional homology \mathbb{Z}_2 -coefficient homology kernel \mathbb{Z}_2 -module

$$K_{2k+1}(M; \mathbb{Z}_2) = \ker(f_* : H_{2k+1}(M; \mathbb{Z}_2) \rightarrow H_{2k+1}(X; \mathbb{Z}_2)).$$

TABLE 4. Calculated groups Θ_n for $1 \leq n \leq 20$ and some related groups.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Θ_n	0	0	0	0	0	0	\mathbb{Z}_{28}	\mathbb{Z}_2	\mathbb{Z}_8	\mathbb{Z}_6	\mathbb{Z}_{992}	0	\mathbb{Z}_3	\mathbb{Z}_2	\mathbb{Z}_{16256}	\mathbb{Z}_2	\mathbb{Z}_{16}	\mathbb{Z}_{16}	\mathbb{Z}_{523264}	\mathbb{Z}_{24}
bP_{n+1}	0	0	0	0	0	0	\mathbb{Z}_{28}	0	\mathbb{Z}_2	0	\mathbb{Z}_{992}	0	0	0	\mathbb{Z}_{8128}	0	\mathbb{Z}_2	0	\mathbb{Z}_{261632}	0
Θ_n/bP_{n+1}	0	0	0	0	0	0	0	\mathbb{Z}_2	\mathbb{Z}_4	\mathbb{Z}_6	0	0	\mathbb{Z}_3	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_8	\mathbb{Z}_{16}	\mathbb{Z}_2	\mathbb{Z}_{24}
π_n^s/J	0	\mathbb{Z}_2	0	0	0	\mathbb{Z}_2	0	\mathbb{Z}_2	\mathbb{Z}_4	\mathbb{Z}_6	0	0	\mathbb{Z}_3	\mathbb{Z}_4	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_8	\mathbb{Z}_{16}	\mathbb{Z}_2	\mathbb{Z}_{24}
π_n^s	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	0	\mathbb{Z}_2	\mathbb{Z}_{240}	\mathbb{Z}_4	\mathbb{Z}_8	\mathbb{Z}_6	\mathbb{Z}_{504}	0	\mathbb{Z}_3	\mathbb{Z}_4	\mathbb{Z}_{960}	\mathbb{Z}_4	\mathbb{Z}_{16}	—	—	—
J	\mathbb{Z}_2	0	\mathbb{Z}_{24}	0	0	0	\mathbb{Z}_{240}	\mathbb{Z}_2	\mathbb{Z}_2	0	\mathbb{Z}_{504}	0	0	0	\mathbb{Z}_{480}	\mathbb{Z}_2	\mathbb{Z}_2	—	—	—

$\dim_{\mathbb{Z}} \Theta_n$ = number of differential structures on the n -dimensional homotopy sphere.

$bP_{n+1} \triangleleft \Theta_n$: subgroup of n -dimensional homotopy spheres bounding parallelizable manifolds.

bP_{n+1} is a finite cyclic group that vanishes if n is even.

Kervaire-Milnor formula: $\dim_{\mathbb{Z}} bP_{4n} = \frac{3-(-1)^n}{2} 2^{2n-2} (2^{2n-1} - 1) \text{ Numerator}(\frac{B_{4n}}{4n})$, $n \geq 2$, with B_{4n} Bernoulli numbers.

$\dim_{\mathbb{Z}} bP_{4n+2} = 0$, $n = 0, 1, 3, 7, 15$; $\dim_{\mathbb{Z}} bP_{4n+2} = 0$, or \mathbb{Z}_2 , $n = 31$; $\dim_{\mathbb{Z}} bP_{4n+2} = \mathbb{Z}_2$ otherwise.

$\pi_n^s \equiv \varinjlim_k \pi_{n+k}(S^k)$: stable homotopy groups or n -stems. (Serre's theorem. The groups π_n^s are finite.)

There is an injective map $\Theta_n/bP_{n+1} \rightarrow \pi_n^s/J$ where J is the image of Whitehead's J -homomorphisms $\pi_n(SO) \rightarrow \pi_n^s$.

For $n = 0$ one has $\pi_n^s = \mathbb{Z}$ and $J = 0$.

Theorem 3.22 (Browder-Novikov-Sullivan-Wall's surgery exact sequence). *One has the following propositions.*

(i) *Let X be an n -dimensional geometric Poincaré complex with $n \geq 5$. The manifold structure set $\mathfrak{S}(X) \neq \emptyset$ iff there exists a normal map $(f, b) : M \rightarrow X$ with surgery obstruction $\sigma_*(f, b) = 0 \in L_n(\mathbb{Z}[\pi_1(X)])$.*

(ii) *Let M be an n -dimensional manifold. Then $\mathfrak{S}(M)$ fits into the surgery exact sequence of pointed sets reported in (15).*

$$(15) \quad \cdots L_{n+1}(\mathbb{Z}[\pi_1(M)]) \longrightarrow \mathfrak{S}(M) \longrightarrow [M, G/O] \rightarrow L_n(\mathbb{Z}[\pi_1(M)]) .$$

4. h-COBORDISM IN RICCI FLOW PDE'S

In this section we shall relate the h-cobordism with the geometric properties of the Ricci flow equation considered in the previous two sections. With this respect let us recall first some definitions and properties about surgery on manifolds.

Definition 4.1. 1) *The n -dimensional handle, of index p , is $h^p \equiv D^p \times D^{n-p}$. Its core is $D^p \times \{0\}$. The boundary of the core is $S^{p-1} \times \{0\}$. Its cocore is $\{0\} \times D^{n-p}$ and its transverse sphere is $\{0\} \times S^{n-p-1}$.*

2) *Given a topological space Y , the images of continuous maps $D^n \rightarrow Y$ are called the n -cells of Y .*

3) *Given a topological space X and a continuous map $\alpha : S^{n-1} \rightarrow X$, we call $Y \equiv X \bigcup_{\alpha} D^n$ obtained from X by attaching a n -dimensional cell to X .*

4) *We call CW-complex a topological space X obtained from \emptyset by successively attaching cells of non-decreasing dimension:*

$$(16) \quad X \equiv (\cup D^0) \cup D^1 \cup D^2) \cup \dots$$

We call $X^n \equiv \bigcup_{1 \leq i \leq n} D^i$, $n \geq 0$, the (n) -skeleta.

Definition 4.2. (Homotopy groups.) 1) *We define homotopy groups of manifold, (resp. CW-complex), M , the groups*

$$(17) \quad \pi_p(M) = [S^p, M], p \geq 0.$$

2) *Let X be a manifold over a CW-complex and an element $x \in \pi_n(X)$, $n \geq 1$. Let $Y = X \bigcup_{\phi^n} D^{n+1}$ be the CW-complex obtained from X by attaching an $(n+1)$ -cell with map $\phi^n : S^n \rightarrow X$, with $x = [\phi^n] \in \pi_n(X)$. The operation of attaching the $(n+1)$ -cell is said to kill x .*

Theorem 4.3. (CW-substitute) 1) For any manifold, M , we can construct a CW-complex X and a weak homotopy equivalence $f : X \rightarrow M$, (i.e., the induced maps $f_* : \pi_r(X') \rightarrow \pi_r(X)$ on the Hurewicz homotopy groups are bijective for $r \geq 0$).¹¹ Then X' is called the CW-substitute of X . This is unique up to homotopy. 2) Furthermore if $h : X \rightarrow Y$ is a continuous map between manifolds, and (X', f) , (Y', g) are the corresponding CW-substitutes, then we can find a cellular map $h : X' \rightarrow Y'$ so that the following diagram is commutative:

$$(18) \quad \begin{array}{ccc} X' & \xrightarrow{f} & X \\ h' \downarrow & & \downarrow h \\ Y' & \xrightarrow{g} & Y \end{array}$$

h' is unique up to homotopy.

Definition 4.4. An n -dimensional bordism $(W; M_0, f_0; M_1, f_1)$ consists of a compact manifold W of dimension n , and closed $(n-1)$ -dimensional manifolds M_0, M_1 , such that $\partial W = N_0 \sqcup N_1$, and diffeomorphisms $f_i : M_i \cong N_i$, $i = 0, 1$. An n -dimensional h-bordism (resp. s-bordism) is a n -dimensional bordism as above, such that the inclusions $N_i \hookrightarrow W$, $i = 0, 1$, are homotopy equivalences (resp. simply homotopy equivalences).¹² W is a trivial h-bordism if $W \cong M_0 \times [0, 1]$. In such a case M_0 is diffeomorphic to M_1 : $M_0 \cong M_1$

We will simply denote also by $(W; M_0, M_1)$ a n -dimensional bordism. If $\phi^p : S^{p+1} \times D^{n-p-1} \rightarrow M_1$ is an embedding, then

$$(19) \quad W + (\phi^p) \equiv W \bigcup_{\phi^p} D^p \times D^{n-p} \equiv W \bigcup h^p$$

¹¹Note that an homotopy equivalence is an weak homotopy equivalence, but the vice versa is not true. Recall that two pointed topological spaces (X, x_0) and (Y, y_0) have the same homotopy type if $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$, and $\pi_n(X, x_0)$ and $\pi_n(Y, y_0)$ are isomorphic as modules over $\mathbb{Z}[\pi_1(X, x_0)]$ for $n \geq 2$. A *simply homotopy equivalence* between m -dimensional manifolds, (or finite CW complexes), is a homotopy equivalence $f : M \cong N$ such that the Whitehead torsion $\tau(f) \in Wh(\pi_1(M))$, where $Wh(\pi_1(M))$ is the Whitehead group of $\pi_1(M)$. With this respect, let us recall that if A is an associative ring with unity, such that A^m is isomorphic to A^n iff $m = n$, put $GL(A) \equiv \bigcup_{n=1}^{\infty} GL_n(A)$, the *infinite general linear group* of A and $E(A) \equiv [GL(A), GL(A)] \triangleleft GL(A)$. $E(A)$ is the normal subgroup generated by the elementary matrices $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$. The *torsion group* $K_1(A)$ is the abelian group $K_1(A) = GL(A)/E(A)$. Let A^\bullet denote the multiplicative group of units in the ring A . For a commutative ring A , the inclusion $A^\bullet \hookrightarrow K_1(A)$ splits by the determinant map $\det : K_1(A) \rightarrow A^\bullet$, $\tau(\phi) \mapsto \det(\phi)$ and one has the splitting $K_1(A) = A^\bullet \oplus SK_1(A)$, where $SK_1(A) = \ker(\det : K_1(A) \rightarrow A^\bullet)$. If A is a field, then $K_1(A) \cong A^\bullet$ and $SK_1(A) = 0$. The *torsion* $\tau(f)$ of an isomorphism $f : L \cong K$ of finite generated free A -modules of rank n , is the torsion of the corresponding invertible matrix $(f_j^i) \in GL_n(A)$, i.e., $\tau(f) = \tau(f_j^i) \in K_1(A)$. The isomorphism is *simple* if $\tau(f) = 0 \in K_1(A)$. The Whitehead group of a group G is the abelian group $Wh(G) \equiv K_1(\mathbb{Z}[G])/\{\tau(\mp g) | g \in G\}$. $Wh(G) = 0$ in the following cases: (a) $G = \{1\}$; (b) $G = \pi_1(M)$, with M a surface; (c) $G = \mathbb{Z}^m$, $m \geq 1$. There is a conjecture, (Novikov) that extends the case (b) also to m -dimensional compact manifolds M with universal cover $\widetilde{M} = \mathbb{R}^m$. This conjecture has been verified in many cases [12].

¹²Let us emphasize that to state that the inclusions $M_i \hookrightarrow W$, $i = 0, 1$, are homotopy equivalences is equivalent to state the M_i are deformation retracts of W .

is said obtained from W by attaching a handle, $h^p \equiv D^p \times D^{n-p}$, of index p by ϕ^p .¹³ Put $\partial(W + \phi^p)_0 = M_0$, $\partial(W + \phi^p)_1 = \partial(W + \phi^p) - M_0$.

Theorem 4.5 (CW-substitute of manifold and Hurewicz morphisms). *For any manifold M we can construct a CW-complex M' and a weak homotopy equivalence $f : M' \rightarrow M$. Then M' is called the CW-substitute of M . M' is unique up to homotopy. Then the homotopy groups of M and M' are isomorphic, i.e., one has the top horizontal exact short sequence reported in the commutative diagram (20). There the vertical lines represent the Hurewicz morphisms relating homotopy groups and homology groups.*

$$(20) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \pi_p(M) & \longrightarrow & \pi_p(M') & \longrightarrow & 0 \\ & & \downarrow a & & \downarrow a' & & \\ 0 & \longrightarrow & H_p(M) & \longrightarrow & H_p(M') & \longrightarrow & 0 \end{array}$$

If M is $(n-1)$ -connected, $n \geq 2$, then the morphisms a , a' , become isomorphisms for $p \leq n$ and epimorphisms for $p = n+1$.

We call the morphisms a and a' the Hurewicz morphisms of the manifold M and M' respectively.

Definition 4.6. *A p -surgery on a manifold M of dimension n is the procedure of construction a new n -dimensional manifold:¹⁴*

$$(21) \quad N \equiv \overline{(M \setminus S^p \times D^{n-p})} \bigcup_{S^p \times S^{n-p-1}} D^{p+1} \times S^{n-p-1}.$$

Example 4.7. *Since for the n -dimensional sphere S^n we can write*

$$(22) \quad \begin{aligned} S^n &= \partial D^{n+1} = \partial(D^{p+1} \times D^{n-p}) \\ &= S^p \times D^{n-p} \cup D^{p+1} \times S^{n-p-1} \end{aligned}$$

it follows that the surgery removing $S^p \times D^{n-p} \subset S^n$ converts S^n into the product of two spheres

$$(23) \quad D^{p+1} \times S^{n-p-1} \bigcup_{S^p \times S^{n-p-1}} D^{p+1} \times S^{n-p-1} = S^{p+1} \times S^{n-p-1}.$$

Theorem 4.8 (Surgery and Euler characteristic). 1) *Let M be a $2n$ -dimensional smooth manifold and let apply to N obtained by M with a p -surgery as defined in (21). Then the Euler characteristic of N is related to the M one, by the relation reported in (24).*

$$(24) \quad \chi(N) = \begin{cases} \chi(M) + 2 & p = \text{odd} \\ \chi(M) - 2 & p = \text{even}. \end{cases}$$

2) *Let $M = 2n+1$, $n \geq 0$. If $M = \partial V$, then V can be chosen a manifold with $\chi(V) = 0$, i.e., having the same Euler characteristic of M .*

3) *Let $M = 2n$, $n \geq 0$. If $M = \partial V$, then $\chi(M) = 2\chi(V)$.*

¹³In general $W \cup h^p$ is not a manifold but a CW-complex.

¹⁴We say also that a p -surgery removes a framed p -embedding $g : S^p \times D^{n-p} \hookrightarrow M$. Then it kills the homotopy class $[g] \in \pi_p(M)$ of the core $g = g| : S^p \times \{0\} \hookrightarrow M$.

Proof. 1) Let us first note that we can write $M = \overline{(M \setminus S^p \times D^{n-p})} \cup (S^p \times D^{2n-p})$, hence we get

$$(25) \quad \begin{cases} \chi(M) &= \chi(\overline{(M \setminus S^p \times D^{n-p})}) + \chi(S^p \times D^{2n-p}) \\ &= \chi(\overline{(M \setminus S^p \times D^{n-p})}) + (1 + (-1)^p). \end{cases}$$

From (25) we get

$$(26) \quad \chi(\overline{(M \setminus S^p \times D^{n-p})}) = \chi(M) - (1 + (-1)^p) = \begin{cases} \chi(M) & p = \text{odd} \\ \chi(M) - 2 & p = \text{even}. \end{cases}$$

On the other hand one has

$$(27) \quad \begin{cases} \chi(N) &= \chi(\overline{(M \setminus S^p \times D^{n-p})}) + \chi(D^{p+1} \times S^{2n-p-1}) - \chi(S^p \times S^{2n-p-1}) \\ &= \chi(\overline{(M \setminus S^p \times D^{n-p})}) + \begin{cases} 2 & p = \text{odd} \\ 0 & p = \text{even} \end{cases}. \end{cases}$$

Then from (26) and (27) we get

$$(28) \quad \chi(N) = \begin{cases} \chi(M) + 2 & p = \text{odd} \\ \chi(M) - 2 & p = \text{even}. \end{cases}$$

2) In fact, if $n = 1$ then M can be considered the boundary of a Möbius strip M_{ob} , that has just $\chi(M_{ob}) = 0$. If $n \geq 3$, and $\chi(V) = 2q$, we can add to V q times p -surgeries with p even in order to obtain a manifold V' that has the same dimension and boundary of V but with Euler characteristic zero. Furthermore, if $\chi(V) = 2q + 1$, we consider the manifold $V'' = V \sqcup \mathbb{R}P^{2n+1}$ that has the same dimension and boundary of V , but $\chi(V'')$ is even. Then we can proceed as before on V'' .

3) Let us consider $V' = V \cup_M V$. Then one has $\chi(V') = 0 = 2\chi(V) - \chi(M)$. \square

Example 4.9. A connected sum of connected n -dimensional manifolds M and N is the connected n -dimensional manifold

$$(29) \quad M \sharp N = (M \setminus D^n) \cup (S^{n-1} \times D^1) \cup (N \setminus D^n).$$

$M \sharp N$ is the effect of the 0-surgery on the disjoint union $M \sqcup N$ which removes the framed 0-embedding $S^0 \times D^n \hookrightarrow M \times N$ defined by the disjoint union of the embeddings $D^n \hookrightarrow M$, $D^n \hookrightarrow N$.

Example 4.10. Given a $(n+1)$ -dimensional manifold with boundary $(M, \partial M)$ and an embedding $S^{i-1} \times D^{n-i+1} \hookrightarrow \partial M$, $0 \leq i \leq n+1$, we define the $(n+1)$ -dimensional manifold $(W, \partial W)$ obtained from M by attaching a i -handle:

$$(30) \quad W = M \bigcup_{S^{i-1} \times D^{n-i+1}} D^i \times D^{n-i+1} = M \cup h^i.$$

Then ∂W is obtained from ∂M by an $(i-1)$ -surgery:

$$(31) \quad \partial W = (\partial M \setminus S^{i-1} \times D^{n-i+1}) \bigcup_{S^{i-1} \times S^{n-i}} D^i \times S^{n-1}.$$

Definition 4.11. An elementary $(n+1)$ -dimensional bordism of index i is the bordism $(W; M, N)$ obtained from $M \times D^1$ by attaching a i -handle at $S^{i-1} \times D^{n-i+1} \hookrightarrow M \times \{1\}$. The dual of an elementary $(n+1)$ -dimensional bordism $(W; M, N)$ of index i is the elementary $(n+1)$ -dimensional bordism $(W; N, M)$ of index $(n-i+1)$,

obtained by reversing the ends and regarding the i -handle attached to $M \times D^1$ as a $(n - i + 1)$ -handle attached to $N \times D^1$.

Theorem 4.12 (Handle decomposition of bordisms in the category \mathfrak{M}_∞). 1) Every bordism $(W; M, N)$, $\dim W = n + 1$, $\dim M = \dim N = n$, has a handle decomposition of the union of a finite sequence

$$(32) \quad (W; M, N) = (W_1; M_0, M_1) \bigcup (W_2; M_1, M_2) \bigcup \cdots \bigcup (W_k; M_{k-1}, M_k)$$

of adjoining elementary bordisms $(W_s; M_{s-1}, M_s)$ with index (i_s) such that $0 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n + 1$.

2) Closed n -dimensional manifolds M, N are bordant iff N can be obtained from M by a sequence of surgeries.

3) Every closed n -dimensional manifold M can be obtained from \emptyset by attaching handles:

$$(33) \quad M = h^{i_0} \bigcup h^{i_1} \bigcup \cdots \bigcup h^{i_k}.$$

Furthermore, M has a Morse function $f : M \rightarrow \mathbb{R}$ with critical points $\{x_{i_0}, x_{i_1}, \dots, x_{i_k}\}$, where x_λ is a critical point with index λ , and the corresponding vector field $\zeta = \text{grad } f : M \rightarrow TM$ has zero-value only at such critical points. (See Fig. 1.)

Proof. In fact any n -dimensional manifold can be characterized by means of its corresponding CW-substitute. \square

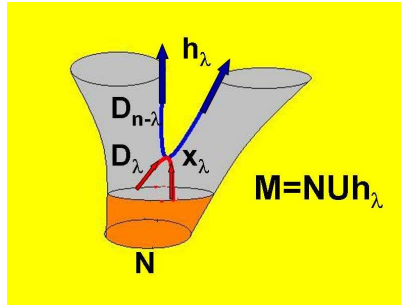


FIGURE 1. Passing through critical point $x_\lambda \in M$, (index λ), of Morse function $f : M \rightarrow \mathbb{R}$, identified by attaching handle to a manifold N . (Separatrix diagram.)

Example 4.13 (Sphere S^2). In this case one has the following handle decomposition: $S^2 = h^0 \bigcup h^2$, with $h^0 = D^0 \times D^2 = \{0\} \times D^2$, the south hemisphere, and $h^2 = D^2 \times D^0 = D^2 \times \{1\}$, the north hemisphere.

Example 4.14 (Torus $T^2 = S^1 \times S^1$). This 2-dimensional manifold has the following CW-complex structure: $T^2 = h^0 \bigcup h^1 \bigcup h^1 \bigcup h^2$, with $h^0 = \{0\} \times D^2$, $h^1 = D^1 \times D^1$, $h^2 = D^2 \times D^0 = D^2 \times \{1\}$.

Remark 4.15. One way to prove whether two manifolds are diffeomorphic is just to suitably use bordism and surgery techniques. (See, e.g. Refs.[62, 63].) In fact we should first prove that they are bordant and then see if some bordism can be modified by successive surgeries on the interior to become an s -bordism.

Theorem 4.16 (Homology properties in \mathfrak{M}_∞). *Let M be a n -dimensional manifold. One has the following homology structures.*

Let $\{C_\bullet(M; A) \cong A \otimes_{\mathbb{R}} C_\bullet(M; \mathbb{R}), \partial\}$ be the chain complex extension of the singular chain complex of M . Then one has the following exact commutative diagram:

(34)

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & B_\bullet(M; A) & \longrightarrow & Z_\bullet(M; A) & \longrightarrow & H_\bullet(M; A) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & C_\bullet(M; A) & \xlongequal{\quad} & C_\bullet(M; A) & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & {}^A\underline{\Omega}_{\bullet,s}(M) & \longrightarrow & Bor_\bullet(M; A) & \longrightarrow & Cyc_\bullet(M; A) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

where:

$$\left\{ \begin{array}{l}
 B_\bullet(M; A) = \ker(\partial|_{C_\bullet(M; A)}); \quad Z_\bullet(M; A) = \text{im}(\partial|_{C_\bullet(M; A)}); \\
 H_\bullet(M; A) = Z_\bullet(M; A)/B_\bullet(M; A), \\
 b \in [a] \in Bor_\bullet(M; A) \Rightarrow a - b = \partial c, \quad c \in C_\bullet(M; A); \\
 b \in [a] \in Cyc_\bullet(M; A) \Rightarrow \partial(a - b) = 0; \\
 b \in [a] \in {}^A\underline{\Omega}_{\bullet,s}(M) \Rightarrow \left\{ \begin{array}{l} \partial a = \partial b = 0 \\ a - b = \partial c, \quad c \in C_\bullet(M; A) \end{array} \right\}.
 \end{array} \right.$$

Furthermore, one has the following canonical isomorphism: ${}^A\underline{\Omega}_{\bullet,s}(M) \cong H_\bullet(M; A)$. As $C_\bullet(M; A)$ is a free two-sided projective A -module, one has the unnatural isomorphism: $Bor_\bullet(M; A) \cong {}^A\underline{\Omega}_{\bullet,s}(M) \oplus Cyc_\bullet(M; A)$.

Proof. It follows from standard results in homological algebra and homology in topological spaces. (For more details about see also [37].) \square

Theorem 4.17 (Cohomology properties in \mathfrak{M}_∞). *Let M be a n -dimensional manifold. One has the following cohomology structures.*

Let $\{C^\bullet(M; A) \equiv \text{Hom}_A(C_\bullet(M; A); A) \cong \text{Hom}_{\mathbb{R}}(C_\bullet(M; \mathbb{R}); A), \delta\}$ be the dual of the chain complex $C_\bullet(M; A)$ considered in above theorem. Then one has the following exact commutative diagram:

(35)

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B^\bullet(M; A) & \longrightarrow & Z^\bullet(M; A) & \longrightarrow & H^\bullet(M; A) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & C^\bullet(M; A) & \xlongequal{\quad} & C^\bullet(M; A) & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & {}^A\underline{\Omega}_s^\bullet(M) & \longrightarrow & Bor^\bullet(M; A) & \longrightarrow & Cyc^\bullet(M; A) \longrightarrow 0 \\
& & & & \downarrow & & \downarrow \\
& & & & 0 & & 0
\end{array}$$

where:

$$\left\{ \begin{array}{l}
B^\bullet(M; A) = \ker(\delta|_{C^\bullet(M; A)}); \quad Z^\bullet(M; A) = \text{im}(\delta|_{C^\bullet(M; A)}); \\
H^\bullet(M; A) = Z^\bullet(M; A)/B^\bullet(M; A); \\
b \in [a] \in Bor^\bullet(M; A) \Rightarrow a - b = \delta c; \quad c \in C^\bullet(M; A); \\
b \in [a] \in Cyc^\bullet(M; A) \Rightarrow \delta(a - b) = 0; \\
b \in [a] \in {}^A\underline{\Omega}_s^\bullet(M) \Rightarrow \left\{ \begin{array}{l} \delta a = \delta b = 0 \\ a - b = \delta c, \quad c \in C^\bullet(M; A) \end{array} \right\}.
\end{array} \right.$$

Furthermore, one has the following canonical isomorphism: ${}^A\underline{\Omega}_s^\bullet(M) \cong H^\bullet(M; A)$. As $C^\bullet(M; A)$ is a free two-sided projective A -module, one has the unnatural isomorphism: $Bor_\bullet(M; A) \cong {}^A\underline{\Omega}_s^\bullet(M) \oplus Cyc^\bullet(M; A)$.

Proof. It follows from standard results in cohomological algebra and cohomology in topological spaces. (See also [37].) \square

Definition 4.18. We say that a manifold M is cohomologically trivial if all the cohomology groups $H^r(M; A)$ vanish for $r \geq 1$.

Theorem 4.19. Let M be a n -dimensional manifold. The following propositions are equivalent.

- (i) M is cohomologically trivial.
- (ii) $H^r(M; \mathbb{K}) = 0, \forall r \geq 1$.
- (iii) $H_r(M; A) \cong H_r(M; \mathbb{K}) = 0, \forall r \geq 1$.
- (iv) The complex $\{C^\bullet(M; A), \delta\}$ is acyclic, i.e., the sequence

(36)

$$0 \longrightarrow Z^0 \xrightarrow{\epsilon} C^0(M; A) \xrightarrow{\delta} C^1(M; A) \xrightarrow{\delta} \cdots \xrightarrow{\delta} C^n(M; A) \xrightarrow{\delta} 0$$

is exact.

Proof. (i) \Leftrightarrow (ii) since $H^r(M; A) \cong H^r(M; \mathbb{K}) \otimes_{\mathbb{K}} A$.

(i) \Leftrightarrow (iii) since $H^r(M; A) \cong Hom_A(H_r(M; A), A)$ and considering the following isomorphism $H_r(M; A) \cong H_r(M; \mathbb{K}) \otimes_{\mathbb{K}} A$.

(i) \Leftrightarrow (iv) since the exactness of the sequence (36) is equivalent to $H^r(M; A) = 0$, for $r \geq 1$. \square

Theorem 4.20. *Let M be a n -dimensional manifold modeled on the algebra A . The following propositions are equivalent.*

- (i) M is cohomologically trivial.
- (ii) $H_r(M; A) = 0$, $r \geq 1$.
- (iii) $H_r(M; \mathbb{K}) = H^r(M; \mathbb{K}) = 0$, $r \geq 1$.

Example 4.21. *A manifold contractible to a point is cohomologically trivial.*

Theorem 4.22 (h-Cobordism groups). 1) *The h-cobordism classes of n -dimensional smooth manifolds, form an abelian group ${}^h\Omega_n$ that has a natural epimorphism on the bordism group Ω_n , i.e., one has the short exact sequence in (37).*

$$(37) \quad {}^h\Omega_n \longrightarrow \Omega_n \longrightarrow 0.$$

2) *If $(W; X_0, X_1)$, $\partial W = X_0 \sqcup X_1$, is a h-cobordant and X_i , $i = 0, 1$, are simply connected, then W is a trivial h-cobordism.*

3) *A simply connected manifold M is h-cobordant to the sphere S^n iff M bounds a contractible manifold.*

4) *If M is a homotopy sphere, then $M \sharp (-M)$ bounds a contractible manifold.*

5) *If a homotopy sphere of dimension $2k$ bounds an S -parallelizable manifold M then it bounds a contractible manifold M_1 .*

6) *Let Θ_n denote the collection of all h-cobordism classes of homotopy n -spheres. Θ_n is an additive group with respect the connected sum,¹⁵ where the sphere S^n serves as zero element. The opposite of an element X is the same manifold with reversed orientation, denoted by $-X$. In Tab. 4 are reported the expressions of some calculated groups Θ_n .¹⁶*

Proof. There are topological manifolds that have not smooth structure. Furthermore, there are examples of topological manifolds that have smooth structure everywhere except a single point. If a neighborhood of that point is removed, the smooth boundary is a homotopy sphere. Any smooth manifold may be *triangulated*, i.e. admits a PL structure, and the underlying PL manifold is unique up to a PL isomorphism.¹⁷ The vice versa is false. No all topological or PL manifolds have at least one smooth structure.

The h-cobordism classes Θ_n of n -dimensional homotopy spheres are trivial for $1 \leq n \leq 6$, i.e. $\Theta_n \cong 0$, and $[S^n] = 0$. For $n = 1, 2$ this follows from the fact that each of such topological manifolds have a unique smooth structure uniquely determined by its homology. For $n = 3$ this follows from the proof of the Poincaré conjecture (see [50]). Furthermore, Each topological 3-manifold has an unique differential

¹⁵The *connected sum* of two connected n -dimensional manifolds X and Y is the n -dimensional manifold $X \sharp Y$ obtained by excising the interior of embedded discs $D^n \subset X$, $D^n \subset Y$, and joining the boundary components $S^{n-1} \subset X \setminus D^n$, $S^{n-1} \subset Y \setminus D^n$, by $S^{n-1} \times I$.

¹⁶It is interesting to add that another related notion of cobordism is the *H-cobordism* of n -dimensional manifold, $(V; M, N)$, $\partial V = M \sqcup N$, with $H_\bullet(M) \cong H_\bullet(N) \cong H_\bullet(V)$. An n -dimensional manifold Σ is a *homology sphere* if $H_\bullet(\Sigma) = H_\bullet(S^n)$. Let Θ_n^H be the abelian group of *H-cobordism classes* of n -dimensional homology spheres, with addition by connected sum. (Kervaire's theorem.) For $n \geq 4$ every n -dimensional homology sphere Σ is H-cobordant to a homology sphere and the forgetful map $\Theta_n \rightarrow \Theta_n^H$ is an isomorphism.

¹⁷A topological manifold M is piecewise linear, i.e., admits a PL structure, if there exists an atlas $\{U_\alpha, \varphi_\alpha\}$ such that the composites $\varphi_\alpha \circ \varphi_{\alpha'}^{-1}$, are piecewise linear. Then there is a polyhedron $P \subset \mathbb{R}^s$, for some s and a homeomorphism $\phi : P \approx M$, (*triangulation*), such that each composite $\varphi_\alpha \circ \phi$ is piecewise linear.

structure [29, 31, 64]. Therefore, since from the proof of the Poincaré conjecture it follows that all 3-homotopy spheres are homeomorphic to S^3 , it necessarily follows that all 3-homotopy spheres are diffeomorphic to S^3 too. Furthermore, for $n = 4$, the triviality of Θ_4 follows from the works by Freedman [13]. (See also J. Cerf [6].)

Lemma 4.23 (Freedman's theorem). [13] *Two closed simply connected 4-manifolds are homeomorphic iff they have the same symmetric bilinear form $\sigma : H^2(M; \mathbb{Z}) \otimes H^2(M; \mathbb{Z}) \rightarrow H^4(M; \mathbb{Z}) \cong \mathbb{Z}$, (with determinant ± 1 , induced by the cup product), and the same Kirby-Siebermann invariant κ .¹⁸ Any σ can be realized by such a manifold. If $\sigma(x \otimes x)$ is odd for some $x \in H^2(M; \mathbb{Z})$, then either value of κ can be realized also. However, if $\sigma(x \otimes x)$ is always even, then κ is determined by σ , being congruent to $\frac{1}{8}\sigma$.*

In particular, if M is homotopy sphere, then $H^2(M, \mathbb{Z}) = 0$ and $\kappa \equiv 0$, so M is homeomorphic to S^4 .¹⁹

The cases $n = 5, 6$, can be proved by using surgery theory and depend by the Smale's h-cobordism theorem.²⁰

Lemma 4.24 (Smale's h-cobordism theorem). [55] *Any n -dimensional simply connected h-cobordism W , $n > 5$, with $\partial W = M \sqcup (-N)$, is diffeomorphic to $M \times [1, 0]$. (All manifolds are considered smooth and oriented. $-N$ denotes the manifold N with reversed orientation.)²¹*

If $n \geq 5$ any two homotopy n -sphere are PL homeomorphic, and diffeomorphic too except perhaps at a single point. (If $n = 5$, (resp. $n = 6$), then any homotopy n -sphere Σ bounds a contractible 6-manifold, (resp. 7-manifold), and is diffeomorphic to S^5 , (resp. S^6). Every smooth manifold M of dimension $n > 4$, having the homotopy of a sphere is a twisted sphere, i.e., M can be obtained by taking two disks D^n and gluing them together by a diffeomorphism $f : S^{n-1} \cong S^{n-1}$ of their boundaries. More precisely one has the isomorphism $\pi_0(\text{Diff}_+(S^n)) \cong \Theta_{n+1}$, $[f] \mapsto \Sigma_f \equiv D^{n+1} \cup_f (-D^{n+1})$, where $\pi_0(\text{Diff}_+(S^n))$ denotes the group of isotopy classes of oriented preserving diffeomorphisms of S^n .

See the paper by M. A. Kervaire and J. W. Milnor [25] and the following ones by S. Smale [55, 56]. In the following we shall give a short summary of this proof for $n \geq 5$. This is really an application of the Browder-Novikov theorem.

¹⁸ κ is \mathbb{Z}_2 -valued and vanishes iff the product manifold $M \times \mathbb{R}$ can be given a differentiable structure.

¹⁹It is not known which 4-manifolds with $\kappa = 0$ actually possess differentiable structure, and it is not known when this structure is essentially unique.

²⁰There exists also a s-cobordism version of such a theorem for non-simply connected manifolds. More precisely, an $(n + 1)$ -dimensional h-cobordism $(V; N, M)$ with $n \geq 5$, is trivial iff it is an s-cobordism. This means that for $n \geq 5$ h-cobordant n -dimensional manifolds are diffeomorphic iff they are s-cobordant. Since the Whitehead group of the trivial group is trivial, i.e., $Wh(\{1\}) = 0$, it follows that h-cobordism theorem is the simply-connected special case of the s-cobordism.

²¹The proof utilizes Morse theory and the fact that for an h-cobordism $H_\bullet(W, M) \cong H_\bullet(W, N) \cong 0$, gives $W \cong M \times [0, 1]$. The motivation to work with dimensions $n \geq 5$ is in the fact that it is used the *Whitney embedding theorem* that states that a map $f : N \rightarrow M$, between manifolds of dimension n and m respectively, such that either $2n + 1 \leq m$ or $m = 2n \geq 6$ and $\pi_1(M) = \{1\}$, is homotopic to an embedding.

Lemma 4.25. *Let Ξ_n denote the set of smooth n -dimensional manifolds homeomorphic to S^n . Let \sim_d denote the equivalence relation in Ξ_n induced by diffeomorphic manifolds. Put $\Gamma_n \equiv \Xi_n / \sim_d$.²² Then the operation of connected sum makes Γ_n an abelian group for $n \geq 5$.*

Proof. Let us first remark that since we are working in Ξ_n , the operation of connected sum there must be considered in smooth sense. Then it is easy to see that for $M_1, M_2, M_3 \in \Xi_n$, one has $M_1 \# M_2 \cong M_2 \# M_1$ and that $(M_1 \# M_2) \# M_3 \cong M_1 \# (M_2 \# M_3)$. Therefore, it is well defined the commutative and associative composition map $+$: $\Gamma_n \times \Gamma_n \rightarrow \Gamma_n$, $[M_1] + [M_2] = [M_1 \# M_2]$. The zero of this composition is the equivalence class $[S^n] \in \Gamma_n$. In fact, since $\overline{S^n \setminus D^n} \cup_{S^{n-1}} (S^{n-1} \times I) \cong D^n$, we get $M \# S^n \cong \overline{M \setminus D^n} \cup_{S^{n-1}} (\overline{S^n \setminus D^n} \cup_{S^{n-1}} (S^{n-1} \times I)) \cong \overline{M \setminus D^n} \cup_{S^{n-1}} D^n \cong M$. Therefore, $[S^n] = 0 \in \Gamma_n$. Furthermore, each element $M \in \Xi_n$ admits, up to diffeomorphisms, a unique opposite $M' \in \Xi_n$. In fact, since $\overline{M \setminus D^n} \cong D^n$, it follows that $M \cong D^n \cup_{\lambda} D^n \cong D^n \cup_{S^{n-1}} D^n$, where $\lambda : S^{n-1} \rightarrow S^{n-1}$ is a given diffeomorphism that identifies the two copies of S^{n-1} . Then M' is defined by $M' = D^n \cup_{\lambda^{-1}} D^n$. In fact one has $M \# M' \cong (D^n \cup_{\lambda} D^n) \# (D^n \cup_{\lambda^{-1}} D^n) \cong D^n \cup_1 D^n \cong S^n$, where $1 = \lambda \circ \lambda^{-1} : S^{n-1} \rightarrow S^{n-1}$. (See Fig. 2.) \square

Lemma 4.26. *One has the group isomorphisms $\Gamma_n \cong \Theta_n$ for any $n \geq 1$. So these groups classify all possible differentiable structures on S^n , up to orientation preserving diffeomorphisms.*

\square

Remark 4.27 (Strange phenomena on dimension four). *It is well known that on \mathbb{R}^4 there are uncountably many inequivalent differentiable structures, i.e., one has exotic \mathbb{R}^4 , say $\widetilde{\mathbb{R}^4}$. (This is a result by M. H. Freedman [13], starting from some results by S. K. Donaldson [11].) On the other hand by the fact that $\Gamma_4 \cong \Theta_4 = 0$ it follows for any 4-dimensional homotopy sphere $\Sigma \cong S^4$. So taking in to account that $S^4 \setminus \{pt\} \cong \mathbb{R}^4$, it natural arises the question: Do exotic 4-sphere $\widetilde{\Sigma}$ exist such that $\widetilde{\Sigma} \setminus \{pt\} \cong \mathbb{R}^4$? The answer to this question, taking seriously the isomorphism $\Gamma_4 \cong \Theta_4 = 0$, should be negative. This means that all exotic \mathbb{R}^4 collapse on the unique one S^4 by the process of one point compactification !*

Lemma 4.28 (M. Hirsch and J. Munkres). [22, 31] *The obstructions to the existence of a smooth structure on a n -dimensional combinatorial (or PL) manifold lie in the groups $H^{k+1}(M; \Gamma_k)$; while the obstruction to the uniqueness of such a smooth structures, when it exists, are elements of $H^k(M; \Gamma_k)$.*

Lemma 4.29 (Kirby and Siebenman). [26] *For a topological manifold M of dimension $n \geq 5$, there is only one obstruction to existence of a PL-structure, living in $H^4(M; \mathbb{Z}_2)$, and only one obstruction to the uniqueness of this structure (when it exists), living in $H^3(M; \mathbb{Z}_2)$.²³*

²² Γ_n is just the set of twisted n -spheres up to orientation-preserving diffeomorphisms. One has the exact sequences given in (38).

$$(38) \quad \pi_0(Diff^+(D^n)) \longrightarrow \pi_0(Diff^+(S^{n-1})) \longrightarrow \Gamma_n \longrightarrow 0$$

²³This result by Kirby and Siebenman does not exclude that every manifold of dimension $n > 4$ can possess some triangulation, even if it cannot be PL-homeomorphic to Euclidean space.

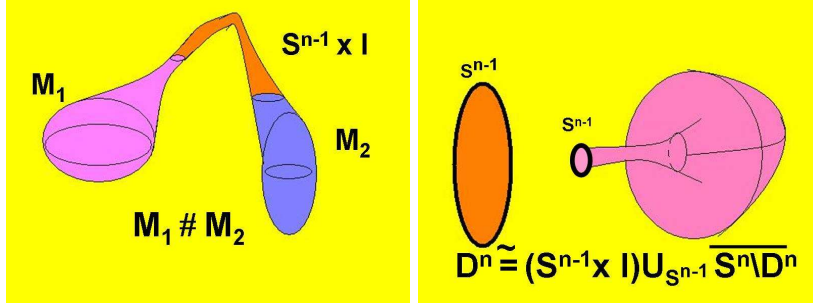


FIGURE 2. Connected sum properties.

Definition 4.30 (Intersection form and signature of manifold). 1) *The intersection form of a $2n$ -dimensional topological manifold with boundary $(M, \partial M)$ is the $(-1)^n$ -symmetric form λ over the \mathbb{Z} -module $H \equiv H^n(M, \partial)/\text{torsion}$, $\lambda(x, y) = \langle x \cup y, [M] \rangle \in \mathbb{Z}$.²⁴*

2) (Milnor's plumbing theorem.) *For $n \geq 3$ every $(-1)^n$ -quadratic form (H, λ, μ) over \mathbb{Z} is realized by an $(n-1)$ -connected $2n$ -dimensional framed manifold with boundary $(V, \partial V)$ with $H_n(V) = H$. The form (H, λ, μ) is nonsingular iff $H_\bullet(\partial V) = H_\bullet(S^{2n-1})$. Let (H, λ) be a nonsingular (-1) -symmetric form over \mathbb{Z} and $\{b_j, c_j\}_{1 \leq j \leq p}$ a basis for the \mathbb{Z} -module H , such that $\lambda(b_r, b_s) = 0$, $\lambda(c_r, c_s) = 0$, $\lambda(b_r, c_s) = 0$, for $r \neq s$, and $\lambda(b_r, c_r) = 1$. Let $\mu : H \rightarrow Q_{-1}(\mathbb{Z}) = \mathbb{Z}_2$ be a (-1) -quadratic function associated to (H, λ) . Then the Arf invariant of a nonsingular (-1) -quadratic form (H, λ, μ) over \mathbb{Z} is $\text{Arf}(H, \lambda, \mu) = \sum_{1 \leq j \leq p} \mu(b_j)\mu(c_j) \in \mathbb{Z}_2 \equiv \{0, 1\}$. If $\partial M = \emptyset$, or $H_\bullet(\partial M) = H_\bullet(S^{2n-1})$, then λ is nonsingular.*

3) *The signature $\sigma(M)$ of a $4k$ -dimensional manifold $(M, \partial M)$ is $\sigma(M) = \sigma(\lambda) \in \mathbb{Z}$, where λ is the symmetric form over the \mathbb{Z} -module $H_{2k}(M)/\text{torsion}$.*

4) *Let M be an oriented manifold with empty boundary, $\partial M = \emptyset$, $\dim M = n = 4k$. Then $H^i(M; \mathbb{Z})$ is finitely generated for each i and $H^{2k}(M; \mathbb{Z}) \cong \mathbb{Z}^s \oplus \text{Tor}$, where Tor is the torsion subgroup. Let $[M] \in H_{4k}(M; \mathbb{Z})$ be the orientation class of M . Let $\langle, \rangle : H^{2k}(M; \mathbb{Z}) \times H^{2k}(M; \mathbb{Z}) \rightarrow \mathbb{Z}$, be the symmetric bilinear form given by $(a, b) \mapsto \langle a, b \rangle = \langle a \cup b, [M] \rangle \in \mathbb{Z}$. This form vanishes on the torsion subgroup, hence it factors on $H^{2k}(M; \mathbb{Z})/\text{Tor} \times H^{2k}(M; \mathbb{Z})/\text{Tor} \cong \mathbb{Z}^s \times \mathbb{Z}^s \rightarrow \mathbb{Z}$. This means that the adjoint map $\phi : H^{2k}(M; \mathbb{Z})/\text{Tor} \rightarrow \text{Hom}_{\mathbb{Z}}(H^{2k}(M; \mathbb{Z})/\text{Tor}; \mathbb{Z}) = (H^{2k}(M; \mathbb{Z})/\text{Tor})^*$, $a \mapsto \phi(a)(b) = \langle a, b \rangle$, is an isomorphism. This is a just a*

²⁴Let R be a commutative ring and H a finite generated free R -module. A ϵ -symmetric form over H is a bilinear mapping $\lambda : H \times H \rightarrow R$, such that $\lambda(x, y) = \epsilon\lambda(y, x)$, with $\epsilon \in \{+1, -1\}$. The form λ is *nonsingular* if the R -module morphism $H \rightarrow H^* \equiv \text{Hom}_R(H; R)$, $x \mapsto (y \mapsto \lambda(x, y))$ is an isomorphism. A ϵ -quadratic form associated to a ϵ -symmetric form λ over H , is a function $\mu : H \rightarrow Q_\epsilon(R) \equiv \text{coker}(1 - \epsilon : R \rightarrow R)$, such that: (i) $\lambda(x, y) = \mu(x + y) - \mu(x) - \mu(y)$; (ii) $\lambda(x, x) = (1 + \epsilon)\mu(x) \in \text{im}(1 + \epsilon : R \rightarrow R) \subseteq \ker(1 - \epsilon : R \rightarrow R)$, $\forall x, y \in H$, $a \in R$. If $R = \mathbb{Z}$ and $\epsilon = 1$, we say *signature* of λ , $\sigma(\lambda) = p - q \in \mathbb{Z}$, where p and q are respectively the number of positive and negative eigenvalues of the extended form on $\mathbb{R} \otimes_{\mathbb{Z}} H$. Then λ has a 1-quadratic function $\mu : H \rightarrow Q_{+1}(\mathbb{Z})$ iff λ has even diagonal entries, i.e., $\lambda(x, x) \equiv 0 \pmod{2}$, with $\mu(x) = \lambda(x, x)/2$, $\forall x \in H$. If λ is nonsingular then $\sigma(\lambda) \equiv 0 \pmod{8}$. Examples. 1) $R = H = \mathbb{Z}$, $\lambda = 1$, $\sigma(\lambda) = 1$.

2) $R = \mathbb{Z}$, $H = \mathbb{Z}^8$, $\lambda = E^8$ -form, given by $(\lambda_{ij}) = \begin{pmatrix} a_{rs} & b_{rs} \\ c_{rs} & d_{rs} \end{pmatrix}$ with $(a_{rs}) = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$, $(b_{rs}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, $(c_{rs}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $(d_{rs}) = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix}$.

TABLE 5. Polynomials $s_I(\sigma_1, \dots, \sigma_n)$, for $0 \leq n \leq 4$.

n	$s_I(\sigma_1, \dots, \sigma_n)$
0	$s = 1$
1	$s_{(1)}(\sigma_1) = \sigma_1$
2	$s_{(2)}(\sigma_1, \sigma_2) = \sigma_1^2 - 2\sigma_2$ $s_{(1,1)}(\sigma_1, \sigma_2) = \sigma_2$
3	$s_{(3)}(\sigma_1, \sigma_2, \sigma_3) = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3$ $s_{(1,2)}(\sigma_1, \sigma_2, \sigma_3) = \sigma_1\sigma_2 - 3\sigma_3$ $s_{(1,1,1)}(\sigma_1, \sigma_2, \sigma_3) = \sigma_3$
4	$s_{(4)}(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \sigma_1^4 - 4\sigma_1^2\sigma_2 + 2\sigma_2^2 + 4\sigma_1\sigma_3 - 4\sigma_4$ $s_{(1,3)}(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \sigma_1^2\sigma_2 - 2\sigma_2^2 - \sigma_1\sigma_3 + 4\sigma_4$ $s_{(2,2)}(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \sigma_2^2 - 2\sigma_1\sigma_3 + 2\sigma_4$ $s_{(1,1,2)}(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \sigma_1\sigma_3 - 4\sigma_4$ $s_{(1,1,1,1)}(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \sigma_4$

direct consequence of Poincaré duality: $H_{2k}(M; \mathbb{Z})/\text{Tor} = (H^{2k}(M; \mathbb{Z})/\text{Tor})^*$ and $a(b \cap [M]) = \langle a \cup b, [M] \rangle$. Then the signature of this bilinear form is the usual signature of the form after tensoring with the rationals \mathbb{Q} , i.e., of the symmetric matrix associated to the form, after choosing a basis for \mathbb{Q}^s . Hence the signature is the difference between the number of $+1$ eigenvalues with the number of -1 eigenvalues of such a matrix. Let us denote by $\sigma(M)$ the signature of the above nondegenerate symmetric bilinear form, and call it signature of M .

Theorem 4.31 (R. Thom's properties of signature). 1) If M has $\dim M = 4k$, and it is a boundary, then $\sigma(M) = 0$.

2) $\sigma(-M) = -\sigma(M)$.

3) Let M and L be two $4k$ -dimensional closed, compact, oriented manifolds without boundary. Then we have $\sigma(M \sqcup L) = \sigma(M) + \sigma(L)$ and $\sigma(M \times L) = \sigma(M) \cdot \sigma(L)$, where the orientation on $M \times L$ is $[M \times L] = [M] \otimes [L]$.

4) (Rohlin's signature theorem). The signature of a closed oriented $4k$ -dimensional manifold is an oriented cobordism invariant, i.e., if $\partial W = M \sqcup N$, it follows that $\sigma(M) = \sigma(N) \in \mathbb{Z}$. More precisely, the signature for oriented boundary $4k$ -dimensional manifolds is zero and it defines a linear form $\sigma : {}^+\Omega_{4k} \rightarrow \mathbb{Z}$.

Furthermore, let M and N be $4k$ -dimensional manifolds with differentiable boundaries: $\partial M = \bigcup_j X_j$, $\partial N = \bigcup_i Y_i$, such that $X_1 = Y_1$. Then one has the formula (39).

$$(39) \quad \sigma(M \bigcup_{X_1=Y_1} N) = \sigma(M) + \sigma(N).$$

5) (Hirzebruch's signature theorem (1952)). The signature of a closed oriented $4k$ -dimensional manifold M is given by

$$(40) \quad \sigma(M) = \langle \mathcal{L}_k(p_1, \dots, p_k), [M] \rangle \in \mathbb{Z}$$

with $\mathcal{L}_k(p_1, \dots, p_k)$ polynomial in the Pontrjagin classes p_j of M , i.e., $p_j(M) \equiv p_j(TM) \in H^{4j}(M)$, representing the \mathcal{L} genus, i.e., the genus of the formal power

TABLE 6. \mathcal{L}_k -polynomials for $\mathbb{C}P^{2k}$, with $k = 1, 2, 3, 4$ and related objects.

k	\mathcal{L}_k	$p_j(\mathbb{C}P^{2k})$	$\sigma(\mathbb{C}P^{2k})$
1	$\mathcal{L}_1(p_1) = \frac{1}{3}$	$p_1(\mathbb{C}P^2) = 3$	$\sigma(\mathbb{C}P^2) = \frac{1}{3} \times 3 = 1$
2	$\mathcal{L}_2(p_1, p_2) = \frac{1}{45}(7p_2 - p_1^2)$	$p_1(\mathbb{C}P^4) = 5$ $p_2(\mathbb{C}P^4) = 10$	$\sigma(\mathbb{C}P^4) = \frac{1}{45}(7 \times 10 - 25) = 1$
3	$\mathcal{L}_3(p_1, p_2, p_3) = \frac{1}{945}(62p_3 - 13p_2p_1 + 2p_1^3)$	$p_1(\mathbb{C}P^6) = 7$ $p_2(\mathbb{C}P^6) = 21$ $p_3(\mathbb{C}P^6) = 35$	$\sigma(\mathbb{C}P^6) = \frac{1}{945}(62 \times 35 - 13 \times 21 \times 7 + 2 \times 343) = 1$
4	$\mathcal{L}_4(p_1, p_2, p_3, p_4) = \frac{1}{14175}(381p_4 - 71p_3p_1 - 19p_2^2 + 22p_2p_1^2 - 3p_1^4)$	$p_1(\mathbb{C}P^8) = 9$ $p_2(\mathbb{C}P^8) = 36$ $p_3(\mathbb{C}P^8) = 84$ $p_4(\mathbb{C}P^8) = 126$	$\sigma(\mathbb{C}P^8) = \frac{1}{14175}(4806 - 53676 - 24624 + 64152 - 19683) = 1$

series given in (41).²⁵

$$(41) \quad \frac{\sqrt{z}}{\tanh(\sqrt{z})} = \sum_{k \geq 0} \frac{2^{2k} B_{2k} z^k}{(2k)!} = 1 + \frac{z}{3} - \frac{z^2}{45} + \dots$$

where the numbers B_{2k} are the Bernoulli numbers.²⁶

6) The intersection form of a $4k$ -dimensional manifold M has a 1-quadratic function iff it has $2k^{\text{th}}$ Wu class $v_{2k}(M) = 0 \in H^{2k}(M; \mathbb{Z}_2)$, in which case $\sigma(M) \equiv 0 \pmod{8}$.

²⁵A genus for closed smooth manifolds with some X -structure, is a ring homomorphism $\Omega_{\bullet}^X \rightarrow R$, where R is a ring. For example, if the X -structure is that of oriented manifolds, i.e., $X = SO$, then the signature of these manifolds just identifies a genus $\sigma : {}^+\Omega_{\bullet} = \Omega_{\bullet}^{SO} \rightarrow \mathbb{Z}$, such that $\sigma(1) = 1$, and $\sigma : {}^+\Omega_p \rightarrow 0$ if $p \neq 4q$. Therefore the genus identifies also a \mathbb{Q} -algebra homomorphism $\Omega_{\bullet}^{SO} \otimes_{\mathbb{Q}} \mathbb{Q} \rightarrow \mathbb{Q}$. More precisely, let $A \equiv \mathbb{Q}[t_1, t_2, \dots]$ be a graded commutative algebra, where t_i has degree i . Set $\mathcal{A} \equiv A[[a_0, a_1, \dots]]$, where $a_i \in A$ is homogeneous of degree i , i.e., the elements of \mathcal{A} are infinite formal sums $a \equiv a_0 + a_1 + a_2 + \dots$. Let $\mathcal{A}^{\bullet} \subset \mathcal{A}$ denote the subgroup of the multiplicative group of \mathcal{A} of elements with leading term 1. Let $K_1(t_1), K_2(t_1, t_2), K_3(t_1, t_2, t_3), \dots \in \mathcal{A}$, be a sequence of polynomials of A , where K_n is homogeneous of degree n . For $a = a_0 + a_1 + a_2 + \dots \in \mathcal{A}^{\bullet}$, we define $K(a) \in \mathcal{A}^{\bullet}$ by $K(a) = 1 + K_1(a_1) + K_2(a_1, a_2) + \dots$. We say that K_n form a multiplicative sequence if $K(ab) = K(a)K(b), \forall a, b \in \mathcal{A}^{\bullet}$. An example is with $K_n(t_1, \dots, t_n) = \lambda^n t_n, \lambda \in \mathbb{Q}$. Another example is given by the formal power series given in (41) with $\lambda_k = (-1)^{k-1} \frac{2^{2k} B_{2k}}{(2k)!}$. For any partition $I = (i_1, i_2, \dots, i_k)$ of n , set $\lambda_I = \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_k}$. Now define polynomials $\mathcal{L}_n(t_1, \dots, t_n) \in A$ by $\mathcal{L}_n(t_1, \dots, t_n) = \sum_I \lambda_I s_I(t_1, \dots, t_n)$, where the sum is over all partitions of n and s_I is the unique polynomial belonging to $\mathbb{Z}[t_1, \dots, t_n]$ such that $s_I(\sigma_1, \dots, \sigma_n) = \sum t^I$, where $\sigma_1, \dots, \sigma_n$ are the elementary symmetric functions that form a polynomial basis for the ring \mathcal{S}_n of symmetric functions in n variables. (\mathcal{S}_n is the graded subring of $\mathbb{Z}[t_1, \dots, t_n]$ of polynomials that are fixed by every permutation of the variables. Therefore we can write $\mathcal{S}_n = \mathbb{Z}[\sigma_1, \dots, \sigma_n]$, with σ_i of degree i . In Tab. 5 are reported the polynomials $s_I(\sigma_1, \dots, \sigma_n)$, for $0 \leq n \leq 4$.) \mathcal{L}_n form a multiplicative sequence. In fact $\mathcal{L}(ab) = \sum_I s_I(ab) = \sum_I \lambda_I \sum_{I_1 I_2 = I} s_{I_1}(a) s_{I_2}(b) = \sum_{I_1 I_2 = I} \lambda_{I_1} s_{I_1}(a) \lambda_{I_2} s_{I_2}(b) = \mathcal{L}(a)\mathcal{L}(b)$. Then for an n -dimensional manifold M one defines \mathcal{L} -genus, $\mathcal{L}[M] = 0$ if $n \neq 4k$, and $\mathcal{L}[M] = \langle K_k(p_1(TM), \dots, p_k(TM)), \mu_M \rangle$ if $n = 4k$, where μ_M is the rational fundamental class of M and $K_k(p_1(TM), \dots, p_k(TM)) \in H^n(M; \mathbb{Z})$.

²⁶Formula (40) is a direct consequence of Thom's computation of ${}^+\Omega_{\bullet} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}[y_{4k} | k \geq 1]$, with $y_{4k} = [\mathbb{C}P^{2k}]$. (${}^+\Omega_j \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ for $j \neq 4k$.) In Fact, one has $p_j(\mathbb{C}P^n) = p_j(\mathbb{C}P^n) = (n+1) \in H^{4j}(\mathbb{C}P^n) = \mathbb{Z}, 0 \leq j \leq \frac{n}{2}$. For $n = 2k$ the evaluation $\langle \mathcal{L}_k, [\mathbb{C}P^{2k}] \rangle = 1 \in \mathbb{Z}$ coincides with the signature of $\mathbb{C}P^{2k}$: $\sigma(\mathbb{C}P^{2k}) = \sigma(H^{2k}(\mathbb{C}P^{2k}), \lambda) = \sigma(\mathbb{Z}, 1) = 1 \in \mathbb{Z}$. Therefore, the signature identifies a \mathbb{Q} -algebra homomorphism $\Omega_{\bullet}^{SO} \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \mathbb{Q}$. So the Hirzebruch signature theorem states that this last homomorphism induced by the signature, coincides with the one induced by the genus. In Tab. 6 are reported some Hirzebruch's polynomials for $\mathbb{C}P^{2k}$. In Tab. 7 are reported also the Bernoulli numbers B_n , with the Kronecker's formula, and explicitly calculated for $0 \leq n \leq 18$.

TABLE 7. Bernoulli numbers $B_n = -\sum_{1 \leq k \leq n+1} \frac{(-1)^k}{k} \binom{n+1}{k} \sum_{1 \leq j \leq k} j^n \in \mathbb{Q}$, $n \geq 0$.

n	Numerator	Denominator
0	1	1
1	-1	2
2	1	6
4	-1	30
6	1	42
8	-1	30
10	5	66
12	-691	2730
14	7	6
16	-3617	510
18	43862	798

$B_n = 0$, $n = \text{odd} > 1$.

Definition 4.32. An n -dimensional manifold M is parallelizable if its tangent n -plane bundle $\tau_M : M \rightarrow BO(n)$ is trivial, i.e., isomorphic to $M \times \mathbb{R}^n \rightarrow M \cong \epsilon^n$. Two vector bundles ξ_1 and ξ_2 over a same base M are stably isomorphic if $\xi_1 \oplus \epsilon_1 \cong \xi_2 \oplus \epsilon_1$, where ϵ_i , $i = 1, 2$ are vector bundles over M with dimensions such that if M is a complex of dimension r , the total fiber dimensions of the Whitney sums exceeds r . Such bundles are said to be in stable range.

Proposition 4.33. A connected compact n -manifold M with non-trivial boundary, is parallelizable iff it is stably parallelizable.

Proof. In fact M has the homotopy type of an $(n-1)$ -complex and thus TM is in the stable range. \square

Proposition 4.34. The set of framed n -manifolds properly contains the set of parallelizable n -manifolds.

Example 4.35 (Bott-Milnor [4]). 1) The sphere S^n is framed with $S^n \times \mathbb{R} \subset \mathbb{R}^{n+1}$, $TS^n \oplus \epsilon \cong \epsilon^{n+1}$, but not necessarily parallelizable. (See Tab. 8.)

TABLE 8. Parallelizable S^n .

n	1	2	3	4	5	6	7	> 7
parallelizable	YES	NO	YES	NO	NO	NO	YES	NO

- 2) All spheres are stably parallelizable.
 3) Every homotopy sphere Σ^n is stably parallelizable.

Definition 4.36 (Pontrjagin-Thom construction framed-cobordism). If (M, φ) is any n -manifold with framing $\varphi : \nu(M) \cong M \times \mathbb{R}^{k-n}$ of the normal bundle in \mathbb{R}^k , the same definition yields a map $p(M, \varphi) \in \pi_n^s$. If $(M_1, \varphi_1) \sqcup (M_2, \varphi_2) \subset \mathbb{R}^k$ form the framed boundary of a $(n+1)$ -manifold $(W, \partial W, \Phi) \subset (\mathbb{R}^k \times [0, \infty), \mathbb{R}^k \times \{0\})$, we say that are framed cobordant. The framed cobordism is an equivalence relation and the corresponding set of framed cobordism classes is denoted by Ω_n^{fr} . This is an abelian additive group with respect the operation of disjoint union \sqcup . The class $[\emptyset]$ is the zero of Ω_n^{fr} . Then one has the canonical isomorphism given in (42).²⁷

²⁷In Tab. 4 are reported the n -stems for $0 \leq n \leq 17$.

$$(42) \quad \begin{cases} \Omega_n^{fr} \cong \pi_n^s \\ M \mapsto f_M : S^{n+k} \rightarrow S^{n+k}/(S^{n+k} \setminus (M \times \mathbb{R}^k)) = \Sigma^k M^+ \rightarrow S^k \end{cases}$$

where $M^+ \equiv M \cup \{pt\}$, and Σ^k is the k -suspension functor for spectra.

Example 4.37 (Smale's paradox and framed cobordism). *Let us consider the so-called Smale's paradox turning a sphere $S^2 \subset \mathbb{R}^3$ inside out. (See also [52].) Let us denote by $-S^2$ the sphere S^2 with reversed orientation. Let us first note that these surfaces are characterized by the same generalized curvature integra. It is useful to recall here some definitions and results about this topological invariant. The generalized Gauss map of an n -dimensional framed manifold M with $f : M \hookrightarrow \mathbb{R}^{n+k}$, $\nu_f \cong \epsilon^k$, is the map $c : M \rightarrow V_{n+k,k}$, $x \mapsto ((\nu_f)_x = \mathbb{R}^k \hookrightarrow T_{f(x)}\mathbb{R}^{n+k} = \mathbb{R}^{n+k})$ and classifies the tangent n -planes bundle $\tau_M : M \rightarrow BO(n)$, with the k -stable trivialization $TM \oplus \epsilon^k \cong TM \oplus \nu_f = \epsilon^{n+k} = T\mathbb{R}^{n+k}|_M$.²⁸ The generalized curvatura integra of M is the degree of the generalized Gauss map.*

$$(43) \quad c_*[M] \in H_n(V_{n+k,k}) = \begin{cases} \mathbb{Z}, & \text{if } n \equiv 0 \pmod{2} \\ \mathbb{Z}_2, & \text{if } n \equiv 1 \pmod{2} \end{cases} \quad (k > 1).$$

The curvatura integra of an n -dimensional framed manifold M can be expressed with the Kervaire's formula given in (44).

$$(44) \quad c_*[M] = \text{Hopf}(M) + \begin{cases} \chi(M)/2 \in \mathbb{Z}, & \text{if } n \equiv 0 \pmod{2} \\ \chi_{1/2}(M) \in \mathbb{Z}_2, & \text{if } n \equiv 1 \pmod{2} \end{cases}$$

where

$$(45) \quad \chi_{1/2}(M) = \sum_{0 \leq j \leq (n-1)/2} \dim_{\mathbb{Z}_2} H_j(M; \mathbb{Z}) \in \mathbb{Z}_2$$

is called the Kervaire semicharacteristic, and

$$(46) \quad \text{Hopf}[M] = \begin{cases} 0 \in \mathbb{Z}, & \text{if } n \equiv 0 \pmod{2} \\ H_2(F) \in \mathbb{Z}_2, & \text{if } n \equiv 1 \pmod{2} \end{cases}$$

with $F : S^{n+k} \rightarrow S^k$ the Pontrjagin-Thom map, and $H_2(F)$ determined by the mod 2 Hopf invariant. This is the morphism

$$(47) \quad \begin{cases} H_2 : \pi_{n+k}(S^k) \rightarrow \mathbb{Z} \\ (F : S^{n+k} \rightarrow S^k) \mapsto H_2(F), \quad (m \geq 1) \end{cases}$$

determined by the Steenrod square in the mapping cone $X = S^k \cup_F D^{n+k+1}$. If $a = 1 \in H^k(X; \mathbb{Z}_2) = \mathbb{Z}_2$, $b = 1 \in H^{n+k+1}(X; \mathbb{Z}_2) = \mathbb{Z}_2$, then $S_q^{n+1}(a) = H_2(F)b \in H^{n+k+1}(X; \mathbb{Z}_2)$. One has (Adams) that $H_2 = 0$ for $n \neq 1, 3, 7$. $\text{Hopf}(M)$ is a framed cobordism invariant.

Taking into account that $\chi(S^2) = \chi(-S^2) = 2$, and that $\text{Hopf}(S^2) = \text{Hopf}(-S^2) = 0$, we get $c_*[S^2] = c_*[-S^2] = 1$. Furthermore one has $\Omega_2^{fr} \cong \pi_2^s = \mathbb{Z}_2 \cong \Omega_2$. In order to see that S^2 is cobordant with $-S^2$ it is enough to prove that $-S^2$ can be

²⁸ $V_{n+k,k} \cong O(n+k)/O(n)$ is the Stiefel space of orthonormal k -frames in \mathbb{R}^{n+k} , or equivalently of isometries $\mathbb{R}^k \rightarrow \mathbb{R}^{n+k}$. $V_{n+k,k}$ is $(n-1)$ -connected with $H_n(V_{n+k,k}) = \mathbb{Z}$, if $n \equiv 0 \pmod{2}$ or if $k=1$, and $H_n(V_{n+k,k}) = \mathbb{Z}_2$, if $n \equiv 1 \pmod{2}$ and $k > 1$. One has $G_{n+k,k} = V_{n+k,k}/O(k)$, where $G_{n+k,k}$ is the Grassmann space of k -dimensional subspaces of \mathbb{R}^{n+k} . Then the classifying space for n -planes is $BO(n) = \varinjlim_k G_{n+k,k}$, and the corresponding stable classifying space is

$$BO = \varinjlim_n BO(n).$$

obtained by S^2 by a sequence of surgeries. (See Theorem 4.12.) In fact, we can write the oriented S^2 as $S^2 = D_W^2 \cup_{S^1} D_E^2$, where D_W^2 and D_E^2 are two oriented discs in such a way that S^2 is oriented with outgoing normal unitary vector field. (Fig. 3(a).) By a surgery we can remove D_E^2 and smoothly add another D_E^2 on the left of D_W^2 . (Fig. 3(b).) Next by an orientation preserving diffeomorphisms, we get $-S^2$, the surface represented in Fig. 3(c).

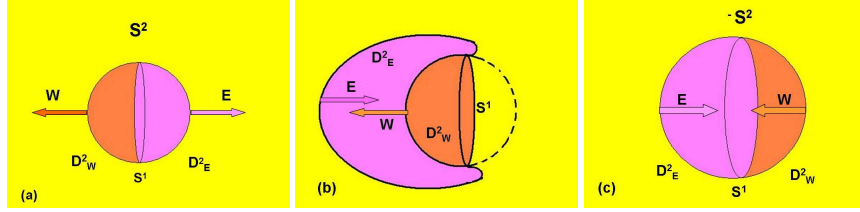


FIGURE 3. Surgery and Smale's paradox turning a sphere $S^2 \subset \mathbb{R}^3$ inside out.

In conclusion $S^2 \sqcup -S^2 = \partial W$, where $W \cong (S^2 \times \{0\}) \times I \cong (-S^2 \times \{1\}) \times I \subset \mathbb{R}^3 \times [0, \infty)$. Therefore, S^2 is framed cobordant with $-S^2$. Furthermore $S^2 \cong -S^2$ (diffeomorphism reversing orientation), that agrees with the well known result in differential topology that two connected, compact, orientable surfaces are diffeomorphic iff they have the same genus, the same Euler characteristics and the same number of boundaries. (See, e.g. [23].)

Another proof that $S^2 \sqcup -S^2$ are cobordant can be obtained by the Arf invariant. Let us recall that if M is a framed surface $M \times \mathbb{R}^k \subset \mathbb{R}^{k+2}$, the intersection form $(H^1(M), \lambda)$ has a canonical (-1) -quadratic function $\mu : H^1(M) = H_1(M) \rightarrow Q_{-1}(\mathbb{Z}) = \Omega_1^{fr} = \mathbb{Z}_2$, given by $x \mapsto (x : S^1 \hookrightarrow M)$, sending each $x \in H^1(M)$ to an embedding $x : S^1 \hookrightarrow M$ with a corresponding framing $S^1 \times \mathbb{R}^{k+1} \subset \mathbb{R}^{k+2}$, $\delta\nu_x : \nu_x \oplus \epsilon^k \cong \epsilon^{k+1}$. Then one has the isomorphism $Arf : \Omega_2^{fr} = \pi_2^s \cong \mathbb{Z}_2$, $[M] \mapsto Arf(H^1(M), \lambda, \mu)$. In the particular case that $M = S^2 \sqcup -S^2$, we get $H^1(M) = 0$ and $Arf(H^1(M), \lambda, \mu) = 0$, hence must necessarily be $[M = S^2 \sqcup -S^2] = 0 \in \Omega_2^{fr}$. This agrees with the fact that $\Omega_2^{fr} = \mathbb{Z}_2 = \Omega_2$, and that both surfaces S^2 and $-S^2$ belong to $0 \in \Omega_2$ since are orientable ones.

Definition 4.38. An n -dimensional manifold V with boundary ∂V , is almost framed if the open manifold $V \setminus \{pt\}$ framed: $(V \setminus \{pt\}) \times \mathbb{R}^k \subset \mathbb{R}^{n+k}$ (for k large enough).

Theorem 4.39 (Properties of almost framed manifold). 1) An almost framed manifold V , with $\partial V \neq \emptyset$ is a framed manifold and a parallelizable manifold.

2) If V is an almost framed manifold with $\partial V = \emptyset$, then there is a framing obstruction

$$(48) \quad \mathfrak{o}(V) \in \ker(J : \pi_{n-1}(O) \rightarrow \pi_{n-1}^s)$$

in the sense that V is framed iff $\mathfrak{o}(V) = 0$.

3) (Kervaire invariant for almost framed manifolds). Let $(M, \partial M)$ be a $(4k+2)$ -dimensional almost framed manifold with boundary such that either $\partial M = \emptyset$ or $H_\bullet(M) = H_\bullet(S^{4k+1})$, so that $(H^{2k+1}(M; \mathbb{Z}_2), \lambda \mu)$ is a nonsingular quadratic form

over \mathbb{Z}_2 . The Kervaire of M is defined in (49).

$$(49) \quad \text{Kervaire}(M) = \text{Arf}(H^{2k+1}(M; \mathbb{Z}_2), \lambda \mu).$$

One has the following propositions.

(i) If $\partial M = \emptyset$ and $M = \partial N$ is the boundary of a $(4k + 3)$ -dimensional almost framed manifold N , then $\text{Kervaire}(M) = 0 \in \mathbb{Z}_2$.

(ii) The Kervaire of a manifold identifies a framed cobordism invariant, i.e., it defines a map $\text{Kervaire} : \Omega_{4k+2}^{\text{fr}} = \pi_{4k+2}^s \rightarrow \mathbb{Z}_2$ that is 0 if $k \neq 2^i - 1$.

(iii) There exist $(4k + 2)$ -dimensional framed manifolds M with $\text{Kervaire}(M) = 1$, for $k = 0, 1, 3, 7$. For $k = 0, 1, 3$ can take $M = S^{2k+1} \times S^{2k+1}$.

4) (Kervaire-Milnor's theorem on almost framed manifolds). Let us denote by Ω_n^{afr} the cobordism group of closed n -dimensional almost framed manifolds. One has the exact sequence in (50).

$$(50) \quad \Omega_n^{\text{afr}} \xrightarrow{\circ} \pi_{n-1}(O) \xrightarrow{J} \pi_{n-1}^s$$

• For a $4k$ -dimensional almost framed manifold V one has the framing obstruction reported in (51).²⁹

$$(51) \quad \left\{ \begin{array}{l} \circ(V) = p_k(V)/(a_k(2k-1)!) \\ \ker(J : \pi_{4k-1}(O) \rightarrow \pi_{4k-1}^s) \\ = j_k \mathbb{Z} \subset \pi_{4k-1}(O) = \mathbb{Z} \end{array} \right\} \quad p_k(V) \in H^{4k}(V) = \mathbb{Z} \text{ (Pontryagin class)} \\ a_k = \left\{ \begin{array}{l} 1 \text{ for } k \equiv 0 \pmod{2} \\ 2 \text{ for } k \equiv 1 \pmod{2} \end{array} \right\} \quad j_k = \text{den} \left(\frac{B_k}{4k} \right)$$

5) (Kervaire-Milnor's theorem on almost framed manifolds-2). Let P_n be the cobordism group of n -dimensional framed manifolds with homotopy sphere boundary. (P_n is called the n -dimensional simply-connected surgery obstruction group.) For $n \geq 4$ Θ_n is finite, with the short exact sequence given in (52).

$$(52) \quad 0 \longrightarrow \text{coker}(a : \Omega_{n+1}^{\text{afr}} \rightarrow P_{n+1}) \longrightarrow \Theta_n \longrightarrow \ker(a : \Omega_n^{\text{afr}} \rightarrow P_n) \longrightarrow 0$$

and

$$\ker(a) \subseteq \text{coker}(J : \pi_n \rightarrow \pi_n^s) = \ker(\circ : \Omega_n^{\text{afr}} \rightarrow \pi_{n-1}(O)).$$

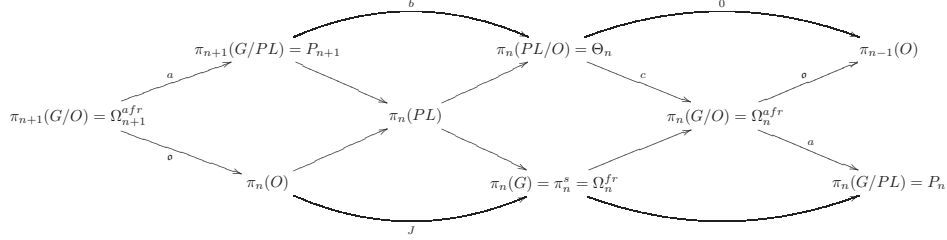
• In (53) are reported the calculated groups P_n .

$$(53) \quad \left\{ \begin{array}{l} P_{2n+1} = 0 \\ P_n = \left\{ \begin{array}{l} \mathbb{Z} \text{ if } n \equiv 0 \pmod{4} \\ 0 \text{ if } n \equiv 1 \pmod{4} \\ \mathbb{Z}_2 \text{ if } n \equiv 2 \pmod{4} \\ 0 \text{ if } n \equiv 3 \pmod{4} \end{array} \right\} \end{array} \right.$$

²⁹For $k = 1$ one has $j_1 = 24$; $\circ(V) = p_1(V)/2 \in 24\mathbb{Z} \subset \pi_3(O) = \mathbb{Z}$.

6) (Kervaire-Milnor's braid $n \geq 5$). For $n \geq 5$ there is the exact commutative braid diagram given in (54).

(54)



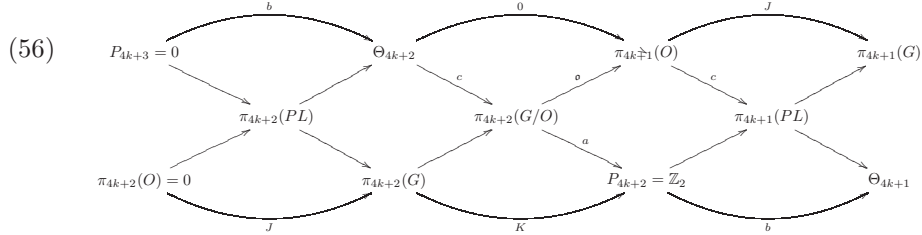
In (54) the mappings a , b and c are defined in (55).

(55)

$$\left\{ \begin{array}{l} a : \Omega_{2n}^{afr} \rightarrow P_{2n}, a(M) = \left\{ \begin{array}{l} \frac{1}{8}\sigma(M) \in \mathbb{Z} \text{ if } n \equiv 0 \pmod{2} \\ Kervaire(M) \in \mathbb{Z}_2 \text{ if } n \equiv 1 \pmod{2} \end{array} \right\} \in P_{2n}. \\ b : P_{2n} \rightarrow \Theta_{2n-1}, b(M) = \text{plumbing construction } \Sigma = \partial M. \\ c : \Theta_n \rightarrow \Omega_n^{afr}, c(\Sigma) = [\Sigma] \in \Omega_n^{afr} \end{array} \right.$$

The image of b is denoted $bP_n \triangleleft \Theta_{n-1}$. Then if $\Sigma \in bP_n$, then $\Sigma = \partial V$, where V is a n -dimensional framed differentiable manifold. Furthermore, by considering the mapping c as $c : \Theta_n \rightarrow \pi_n(G/O)$, it sends an n -dimensional exotic sphere Σ to its fibre-homotopy trivialized stable normal bundle.

7) (Kervaire-Milnor's braid $n = 4k + 2 \geq 5$). For $n = 4k + 2 \geq 5$ the exact commutative braid diagram given in (54) becomes the one reported in (56).



K is the Kervaire invariant on the $(4k + 2)$ -dimensional stable homotopy group of spheres

$$(57) \quad \left\{ \begin{array}{l} K : \pi_{4k+2}(G) = \pi_{4k+2}^s = \lim_{\substack{\longrightarrow \\ j}} \pi_{j+4k+2}(S^j) \\ = \Omega_{4k+2}^{afr} \rightarrow P_{4k+2} = \mathbb{Z}_2. \end{array} \right.$$

- K is the surgery obstruction: $K = 0$ iff every $(4k + 2)$ -dimensional framed differentiable manifold is framed cobordant to a framed exotic sphere.
- The exotic sphere group Θ_{4k+2} fits into the exact sequence (58).

$$(58) \quad 0 \longrightarrow \Theta_{4k+2} \longrightarrow \pi_{4k+2}(G) \xrightarrow{K} \mathbb{Z}_2 \longrightarrow \ker(\pi_{4k+1}(PL) \rightarrow \pi_{4k+1}(G)) \longrightarrow 0$$

- $a : \pi_{4k+2}(G/O) \rightarrow \mathbb{Z}_2$ is the surgery obstruction map sending a normal map $(f, b) : M \rightarrow S^{4k+2}$ to the Kervaire invariant of M .
- $b : P_{4k+2} = \mathbb{Z}_2 \rightarrow \Theta_{4k+1}$ sends the generator $1 \in \mathbb{Z}_2$ to the boundary $b(1) = \Sigma^{4k+1} = \partial W$ of the Milnor plumbing W of two copies of TS^{2k+1} using the standard rank 2 quadratic form $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ over \mathbb{Z} with Arf invariant 1. The subgroup $bP_{4k+2} \triangleleft \Theta_{4k+1}$

represents the $(4k+1)$ -dimensional exotic spheres $\Sigma^{4k+1} = \partial V$ that are boundaries of framed $(4k+2)$ -dimensional differentiable manifolds V .

If k is such that $K = 0$ (e.g. $k = 2$) then $bP_{4k+2} = \mathbb{Z}_2 \triangleleft \Theta_{4k+1}$ and if $\Sigma^{4k+1} = 1 \in bP_{4k+2}$, then the $(4k+2)$ -dimensional manifold $M = V \cup_{\Sigma^{4k+1}} D^{4k+2}$ is a PL manifold without a differentiable structure.

• For any $k \geq 1$ the following propositions are equivalent.

- (i) $K : \pi_{4k+2}(G) = \pi_{4k+2}^s \rightarrow \mathbb{Z}_2$ is $K = 0$.
- (ii) $\Theta_{4k+2} \cong \pi_{4k+2}(G)$.
- (iii) $\ker(\pi_{4k+1}(PL) \rightarrow \pi_{4k+1}(G)) \cong \mathbb{Z}_2$.
- (iv) Every simply-connected $(4k+2)$ -dimensional Poincaré complex X with a vector bundle reduction $\tilde{\nu}_x : X \rightarrow BO$ of the Spivak normal fibration $\nu_x : X \rightarrow BG$ is homotopy equivalent to a closed $(4k+2)$ -dimensional differentiable manifold.

Theorem 4.40 (Pontrjagin, Thom, Kervaire-Milnor). 1) Let bP_{n+1} denote the set of those h -cobordism classes of homotopy spheres which bound parallelizable manifolds.³⁰ For $n \neq 3$, there is a short exact sequence

$$(59) \quad 0 \longrightarrow bP_{n+1} \longrightarrow \Theta_n \longrightarrow \Theta_n/bP_{n+1} \longrightarrow 0$$

where the left hand group is finite cyclic. Furthermore, there exists an homomorphism $J : \pi_n(SO) \rightarrow \pi_n^s$ such that Θ_n/bP_{n+1} injects into $\pi_n^s/J(\pi_n(SO))$ via the Pontrjagin-Thom construction. When $n \neq 2^j - 2$, the right hand group is isomorphic to $\pi_n^s/J(\pi_n(SO))$.

- 2) If Σ^n bounds a parallelizable manifold, it bounds a parallelizable manifold W such that $\pi_j(W) = 0$, $j < n/2$.
- 3) For any $k \geq 1$, $bP_{2k+1} = 0$.

Proof. For any manifold M with stably trivial normal bundle with framing φ , there is a homotopy class $p(M, \varphi)$, depending on the framed cobordism class of (M, φ) . If $p(M) \subset \pi_n^s$ is the set of all $p(M, \varphi)$ where φ ranges over framings of the normal bundle, it follows that $0 \in p(M)$ iff M bounds a parallelizable manifold. (This is a result by Pontrjagin and Thom.) In particular, the set $p(S^n)$ has an explicit description. More precisely, the Whitehead J -homomorphism $: \pi_n(SO(r)) \rightarrow \pi_{n+r}(S^r)$ is defined by $J : (\alpha : S^n \rightarrow SO(r)) \mapsto (J(\alpha) : S^{n+r} \rightarrow S^r)$ that is the Pontrjagin-Thom map of $S^n \subset S^{n+r}$, with the framing $b_\alpha : S^n \times D^r \subset S^{n+r} = S^n \times D^r \cup D^{n+1} \times S^{r-1}$, $(x, y) \mapsto (x, \alpha(x)(y))$. Therefore, the map $J(\alpha) : S^{n+r} \rightarrow S^r$, is obtained by considering $S^{n+r} = (S^n \times D^r) \cup (D^{n+1} \times S^{r-1})$ and sending $(x, y) \in D^n \times D^r$ to $\alpha(x)y \in D^r/\partial D^r = S^r$ and $D^{n+1} \times S^{r-1}$ to the collapsed ∂D^r . Then $J : \pi_n(SO) = \varinjlim_{\vec{r}} \pi_n(SO(r)) \rightarrow \varinjlim_{\vec{r}} \pi_{n+r}(S^r) = \pi_n^s$ is the stable limit of the maps $J(\alpha)$ as $r \rightarrow \infty$, and $p(S^n)$ is the image $J(\pi_n(SO)) \subset \pi_n^s = \Omega_n^{fr}$, hence one has that to α there corresponds the framed cobordism class (S^n, b_α) . \square

The characterization of global solutions of a PDE $E_k \subseteq J_n^k(W)$, in the category \mathfrak{M}_∞ , can be made by means of its integral bordism groups $\Omega_p^{E_k}$, $p \in \{0, 1, \dots, n-1\}$. Let us shortly recall some fundamental definitions and results about.

³⁰ bP_{n+1} is a subgroup of Θ_n . If $\Xi_1, \Xi_2 \in bP_{n+1}$, with bounding parallelizable manifolds W_1 and W_2 respectively, then $\Xi_1 \# \Xi_2$ bounds the parallelizable manifold $W_1 \# W_2$, (commutative sum along the boundary).

Definition 4.41. Let $f_i : X_i \rightarrow E_k$, $f_i(X_i) \equiv N_i \subset E_k$, $i = 1, 2$, be p -dimensional admissible compact closed smooth integral manifolds of E_k . The admissibility requires that N_i should be contained into some solution $V \subset E_k$, identified with a n -chain, with coefficients in A . Then, we say that they are E_k -bordant if there exists a $(p+1)$ -dimensional smooth manifolds $f : Y \rightarrow E_k$, such that $\partial Y = X_1 \sqcup X_2$, $f|_{X_i} = f_i$, $i = 1, 2$, and $V \equiv f(Y) \subset E_k$ is an admissible integral manifold of E_k of dimension $(p+1)$. We say that N_i , $i = 1, 2$, are E_k -bordant if there exists a $(p+1)$ -dimensional smooth manifolds $f : Y \rightarrow J_{m|n}^k(W)$, such that $\partial Y = X_1 \sqcup X_2$, $f|_{X_i} = f_i$, $i = 1, 2$, and $V \equiv f(Y) \subset J_n^k(W)$ is an admissible integral manifold of $J_n^k(W)$ of dimension $(p+1)$. Let us denote the corresponding bordism groups by $\Omega_p^{E_k}$ and $\Omega_p(E_k)$, $p \in \{0, 1, \dots, n-1\}$, called respectively p -dimensional integral bordism group of E_k and p -dimensional quantum bordism group of E_k . Therefore these bordism groups work, for $p = (n-1)$, in the category of manifolds that are solutions of E_k , and $(J_n^k(W), E_k)$. Let us emphasize that singular solutions of E_k are, in general, (piecewise) smooth manifolds into some prolongation $(E_k)_{+s} \subset J_n^{k+s}(W)$, where the set, $\Sigma(V)$, of singular points of a solution V is a non-where dense subset of V . Here we consider Thom-Boardman singularities, i.e., $q \in \Sigma(V)$, if $(\pi_{k,0})_*(T_q V) \not\cong T_q V$. However, in the case where E_k is a differential equation of finite type, i.e., the symbols $g_{k+s} = 0$, $s \geq 0$, then it is useful to include also in $\Sigma(V)$, discontinuity points, $q, q' \in V$, with $\pi_{k,0}(q) = \pi_{k,0}(q') = a \in W$, or with $\pi_k(q) = \pi_k(q') = p \in M$, where $\pi_k = \pi \circ \pi(k, 0) : J_n^k(W) \rightarrow M$. We denote such a set by $\Sigma(V)_S$, and, in such cases we shall talk more precisely of singular boundary of V , like $(\partial V)_S = \partial V \setminus \Sigma(V)_S$. Such singular solutions are also called weak solutions.

Remark 4.42. Let us emphasize that weak solutions are not simply exotic solutions, introduced in Mathematical Analysis in order to describe "non-regular phenomena". But their importance is more fundamental in a theory of PDE's. In fact, by means of such solutions we can give a full algebraic topological characterization of PDE's. This can be well understood in Theorem 4.44 below, where it is shown the structural importance played by weak solutions. In this respect, let us, first, define some notation to distinguish between some integral bordisms group types.

$$(60) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & K_{n-1,w}^{E_k}/(s,w) & \longrightarrow & K_{n-1,w}^{E_k} & \longrightarrow & K_{n-1,s,w}^{E_k} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & K_{n-1,s}^{E_k} & \longrightarrow & \Omega_{n-1}^{E_k} & \longrightarrow & \Omega_{n-1,s}^{E_k} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & \longrightarrow & \Omega_{n-1,w}^{E_k} & \longrightarrow & \Omega_{n-1,w}^{E_k} \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

Definition 4.43. Let $\Omega_{n-1}^{E_k}$, (resp. $\Omega_{n-1,s}^{E_k}$, resp. $\Omega_{n-1,w}^{E_k}$), be the integral bordism group for $(n-1)$ -dimensional smooth admissible regular integral manifolds contained in E_k , borded by smooth regular integral manifold-solutions, (resp. piecewise-smooth or singular solutions, resp. singular-weak solutions), of E_k .

Theorem 4.44. Let $E_k \subset J_n^k(W)$ be a PDE on the fiber bundle $\pi : W \rightarrow M$, with $\dim(W) = m + n$ and $\dim M = n$.

1) One has the exact commutative diagram (60). Therefore, one has the canonical isomorphisms:

$$(61) \quad \begin{cases} K_{n-1,w/(s,w)}^{E_k} \cong K_{n-1,s}^{E_k}; & \Omega_{n-1}^{E_k}/K_{n-1,s}^{E_k} \cong \Omega_{n-1,s}^{E_k}; \\ \Omega_{n-1,s}^{E_k}/K_{n-1,s,w}^{E_k} \cong \Omega_{n-1,w}^{E_k}; & \Omega_{n-1}^{E_k}/K_{n-1,w}^{E_k} \cong \Omega_{n-1,w}^{E_k}. \end{cases}$$

If E_k is formally integrable, then one has the following isomorphisms:

$$(62) \quad \Omega_{n-1}^{E_k} \cong \Omega_{n-1}^{E_\infty} \cong \Omega_{n-1,s}^{E_\infty}; \quad \Omega_{n-1,w}^{E_k} \cong \Omega_{n-1,w}^{E_\infty}.$$

2) Let $E_k \subset J_n^k(W)$ be a quantum super PDE that is formally integrable, and completely integrable. We shall assume that the symbols $g_{k+s} \neq 0$, $s = 0, 1$. (This excludes the case $k = \infty$.) Then one has the following isomorphisms: $\Omega_{p,s}^{E_k} \cong \Omega_{p,w}^{E_k} \cong \Omega_p(E_k)$, with $p \in \{0, \dots, n-1\}$.

3) Let $E_k \subset J_n^k(W)$ be a PDE, that is formally integrable and completely integrable. One has the following isomorphisms: $\Omega_{n-1,w}^{E_k} \cong \Omega_{n-1}(E_k) \cong \Omega_{n-1,w}^{E_{k+h}} \cong \Omega_{n-1,w}^{E_\infty} \cong \Omega_{n-1,w}(E_{k+h}) \cong \Omega_{n-1}(E_\infty)$.

Proof. See [40, 50]. □

In order to distinguish between manifolds V representing singular solutions, where $\Sigma(V)$ has no discontinuities, and integral manifolds where $\Sigma(V)$ contains discontinuities, we can also consider "conservation laws" valued on integral manifolds N representing the integral bordism classes $[N]_{E_k} \in \Omega_p^{E_k}$.

Definition 4.45. Set

$$(63) \quad \begin{cases} \mathfrak{J}(E_k) & \equiv \bigoplus_{p \geq 0} \frac{\Omega^p(E_k) \cap d^{-1}(C\Omega^{p+1}(E_k))}{d\Omega^{p-1}(E_k) \oplus \{C\Omega^p(E_k) \cap d^{-1}(C\Omega^{p+1}(E_k))\}} \\ & \equiv \bigoplus_{p \geq 0} \mathfrak{J}(E_k)^p. \end{cases}$$

Here $C\Omega^p(E_k)$ denotes the space of all Cartan quantum p -forms on E_k . Then we define integral characteristic numbers of N , with $[N]_{E_k} \in \Omega_p^{E_k}$, the numbers $i[N] \equiv \langle [N]_{E_k}, [\alpha] \rangle \in \mathbb{R}$, for all $[\alpha] \in \mathfrak{J}(E_k)^p$.

Then, one has the following theorems.

Theorem 4.46. Let us assume that $\mathfrak{J}(E_k)^p \neq 0$. One has a natural homomorphism:

$$(64) \quad \begin{cases} \underline{j}_p : \Omega_p^{E_k} \rightarrow \text{Hom}(\mathfrak{J}(E_k)^p; \mathbb{R}), & [N]_{E_k} \mapsto \underline{j}_p([N]_{E_k}), \\ \underline{j}_p([N]_{E_k})([\alpha]) = \int_N \alpha \equiv \langle [N]_{E_k}, [\alpha] \rangle. \end{cases}$$

Then, a necessary condition that $N' \in [N]_{E_k}$ is the following: $i[N] = i[N']$, $\forall [\alpha] \in \mathfrak{J}(E_k)^p$. Furthermore, if N is orientable then above condition is sufficient also in order to say that $N' \in [N]_{E_k}$.

Proof. See [36, 37, 40, 41]. \square

Corollary 4.47. *Let $E_k \subseteq J_n^k(W)$ be a PDE. Let us consider admissible p -dimensional, $0 \leq p \leq n-1$, orientable integral manifolds. Let $N_1 \in [N_2]_{E_k} \in \Omega_p^{E_k}$, then there exists a $(p+1)$ -dimensional admissible integral manifold $V \subset E_k$, such that $\partial V = N_1 \sqcup N_2$, where V is without discontinuities iff the integral numbers of N_1 and N_2 coincide.*

Above considerations can be generalized to include more sophisticated solutions of PDEs.

Definition 4.48. *Let $E_k \subset J_n^k(W)$ be a PDE and let B be an algebra. Let us consider the following chain complex (bar chain complex of E_k): $\{\bar{C}_\bullet(E_k; B), \partial\}$, induced by B on the corresponding bar chain complex of E_k , i.e., $\{\bar{C}_\bullet(E_k; B), \partial\}$. (See refs.[36, 37, 40].) More precisely $\bar{C}_p(E_k; B)$ is the free two-sided B -module of formal linear combinations with coefficients in B , $\sum \lambda_i c_i$, where c_i is a singular p -chain $f : \Delta^p \rightarrow E_k$, that extends on a neighborhood $U \subset \mathbb{R}^{p+1}$, such that f on U is differentiable and $Tf(\Delta^p) \subset \mathbf{E}_n^k$, where \mathbf{E}_n^k is the Cartan distribution of E_k .*

Theorem 4.49. *The homology $\bar{H}_\bullet(E_k; B)$ of the bar chain complex of E_k is isomorphic to (closed) bar integral singular (p) -bordism groups, with coefficients in B , of E_k : ${}^B\bar{\Omega}_{p,s}^{E_k} \cong \bar{H}_q(E_k; B) \cong (\bar{\Omega}_{p,s}^{E_k} \otimes_{\mathbb{R}} B)$, $p \in \{0, 1, \dots, n-1\}$. (If $B = \mathbb{R}$ we omit the apex B). The relation between closed bordism and bordism, is given by the following unnatural isomorphism.³¹*

$$(65) \quad \text{Bor}_\bullet(E_k; B) \cong {}^B\bar{\Omega}_{\bullet|s}(E_k) \bigoplus \text{Cyc}_\bullet(E_k; B).$$

Proof. It follows from above results, and the following exact commutative diagram naturally associated to the bar quantum chain complex of E_k .

$$(66) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \bar{B}_\bullet(E_k; B) & \longrightarrow & \bar{Z}_\bullet(E_k; B) & \longrightarrow & \bar{H}_\bullet(E_k; B) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & \bar{C}_\bullet(E_k; B) & \longleftarrow & \bar{C}_{\bullet|s}(E_k; B) & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & {}^B\bar{\Omega}_{\bullet,s}(E_k) & \longrightarrow & \bar{\text{Cyc}}_\bullet(E_k; B) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

³¹Note that if X is a compact space with boundary ∂X , the boundary of $X \times I$, $I \equiv [0, 1] \subset \mathbb{R}$, is $\partial(X \times I) = (X \times \{0\}) \cup (\partial X \times I) \cup (X \times \{1\}) \equiv X_0 \cup P \cup X_1$, with $X_0 \equiv X \times \{0\}$, $X_1 \equiv X \times \{1\}$, $P \equiv \partial X \times I$. One has $\partial P = (\partial X \times \{0\}) \cup (\partial X \times \{1\}) = \partial X_0 \cup \partial X_1$. On the other hand, whether X is closed, then $\partial(X \times I) = X_0 \cup X_1$. Furthermore we shall denote by $[N]_{E_k}$ the equivalence class of the integral admissible bordism of $N \subset E_k$, even if N is not necessarily closed. So, if N is closed one has $[N]_{E_k} \in {}^B\bar{\Omega}_{\bullet,s}^{E_k}$, and if N is not closed one has $[N]_{E_k} \in \bar{B}_\bullet(E_k; B)$.

where $\bar{B}_\bullet(E_k; B) = \ker(\partial|_{\bar{C}_\bullet(E_k; B)})$, $\bar{Z}_\bullet(E_k; B) = \text{im}(\partial|_{\bar{C}_\bullet(E_k; B)})$, $\bar{H}_\bullet(E_k; B) = \bar{Z}_\bullet(E_k; B)/\bar{B}_\bullet(E_k; B)$. Furthermore,

$$\left\{ \begin{array}{l} b \in [a] \in \bar{B}or_\bullet(E_k; B) \Rightarrow a - b = \partial c, \quad c \in \bar{C}_\bullet(E_k; B), \\ b \in [a] \in \bar{C}yc_\bullet(E_k; B) \Rightarrow \partial(a - b) = 0, \\ b \in [a] \in {}^A\bar{\Omega}_{\bullet,s}(E_k) \Rightarrow \left\{ \begin{array}{l} \partial a = \partial b = 0 \\ a - b = \partial c, \quad c \in \bar{C}_\bullet(E_k; B) \end{array} \right\}. \end{array} \right.$$

Furthermore, one has the following canonical isomorphism: ${}^B\bar{\Omega}_{\bullet,s}(E_k) \cong \bar{H}_\bullet(E_k; B)$. As $\bar{C}_\bullet(E_k; B)$ is a free two-sided projective B -module, one has the unnatural isomorphism: $\bar{B}or_\bullet(E_k; B) \cong {}^B\bar{\Omega}_{\bullet,s}(E_k) \oplus \bar{C}yc_\bullet(E_k; B)$. \square

The spaces of conservation laws of PDEs, identify *Hopf algebras*. (Hopf algebras considered here are generalizations of usual Hopf algebras [37].)

Definition 4.50. *The full space of p -conservation laws, (or full p -Hopf algebra), of E_k is the following one: $\mathbf{H}_p(E_k) \equiv \mathbb{R}^{\Omega_p^{E_k}}$. We call full Hopf algebra, of E_k , the following: $\mathbf{H}_{n-1}(E_\infty) \equiv \mathbb{R}^{\Omega_{n-1}^{E_\infty}}$.*

Definition 4.51. *The space of (differential) conservation laws of $E_k \subset J_n^k(W)$, is $\mathbf{Cons}(E_k) = \mathfrak{J}(E_\infty)^{n-1}$.*

Theorem 4.52. *The full p -Hopf algebra of a PDE $E_k \subset J_n^k(W)$ has a natural structure of Hopf algebra (in extended sense). Furthermore, the space of conservation laws of E_k has a canonical representation in $\mathbf{H}_{n-1}(E_\infty)$.*

Proof. See [36, 37]. \square

Theorem 4.53. *Set: $\mathbf{H}_{n-1}(E_k) \equiv \mathbb{R}^{\Omega_{n-1}^{E_k}}$, $\mathbf{H}_{n-1,s}(E_k) \equiv \mathbb{R}^{\Omega_{n-1,s}^{E_k}}$, $\mathbf{H}_{n-1,w}(E_k) \equiv \mathbb{R}^{\Omega_{n-1,w}^{E_k}}$. One has the exact and commutative diagram reported in (67), that define the following spaces: $\mathbf{K}_{n-1,w/(s,w)}^{E_k}$, $\mathbf{K}_{n-1,w}^{E_k}$, $\mathbf{K}_{n-1,s,w}^{E_k}$, $\mathbf{K}_{n-1,s}^{E_k}$.*

$$(67) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longleftarrow & \mathbf{K}_{n-1,w/(s,w)}^{E_k} & \longleftarrow & \mathbf{K}_{n-1,w}^{E_k} & \longleftarrow & \mathbf{K}_{n-1,s,w}^{E_k} & \longleftarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longleftarrow & \mathbf{K}_{n-1,s}^{E_k} & \longleftarrow & \mathbf{H}_{n-1}(E_k) & \longleftarrow & \mathbf{H}_{n-1,s}(E_k) & \longleftarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & 0 & \longleftarrow & \mathbf{H}_{n-1,w}(E_k) & \longleftarrow & \mathbf{H}_{n-1,w}(E_k) & \longleftarrow & 0 \\ & & & & \uparrow & & \uparrow & & \\ & & & & 0 & & 0 & & \end{array}$$

More explicitly, one has the following canonical isomorphisms:

$$(68) \quad \begin{cases} \mathbf{K}_{n-1,w/(s,w)}^{E_k} \cong \mathbf{K}_{n-1,s}^{K^{E_k}}; \\ \mathbf{K}_{n-1,w}^{E_k} / \mathbf{K}_{n-1,s,w}^{E_k} \cong \mathbf{K}_{n-1,w/(s,w)}^{K^{E_k}}; \\ \mathbf{H}_{n-1}(E_k) / \mathbf{H}_{n-1,s}(E_k) \cong \mathbf{K}_{n-1,s}^{E_k}; \\ \mathbf{H}_{n-1}(E_k) / \mathbf{H}_{n-1,w}(E_k) \cong \mathbf{K}_{n-1,w}^{E_k} \\ \cong \mathbf{H}_{n-1,s}(E_k) / \mathbf{H}_{n-1,w}(E_k) \cong \mathbf{K}_{n-1,s,w}^{E_k}. \end{cases}$$

Furthermore, under the same hypotheses of Theorem 4.44(2) one has the following canonical isomorphism: $\mathbf{H}_{n-1,s}(E_k) \cong \mathbf{H}_{n-1,w}(E_k)$. Furthermore, we can represent differential conservation laws of E_k in $\mathbf{H}_{n-1,w}(E_k)$.

Proof. The proof follows directly for duality from the exact commutative diagram (60). \square

Definition 4.54. We define crystal obstruction of E_k the following quotient algebra: $\text{cry}(E_k) \equiv \mathbf{H}_n((E_k)_\infty) / \mathbb{R}^{\Omega_n}$. We say that E_k is a 0-crystal PDE if $\text{cry}(E_k) = 0$.

Remark 4.55. An extended 0-crystal equation $E_k \subset J_n^k(W)$ does not necessitate to be a 0-crystal PDE. In fact E_k is an extended 0-crystal PDE if $\Omega_{n,w}^{E_k} = 0$. This does not necessarily imply that $\Omega_n^{E_k} = 0$.

Corollary 4.56. Let $E_k \subset J_n^k(W)$ be a 0-crystal PDE. Let $N_0, N_1 \subset E_k$ be two closed compact $(n-1)$ -dimensional admissible integral manifolds of E_k such that $X \equiv N_0 \sqcup N_1 \in [0] \in \Omega_n$. Then there exists a smooth solution $V \subset E_k$ such that $\partial V = X$. (See also [50, 44, 45, 46, 47].)

Let us consider, now, the interaction between surgery and global solutions in PDE's of the category \mathfrak{M}_∞ . Since the surgery is a proceeding to obtain manifolds starting from other ones, or eventually from \emptyset , in any theory of PDE's, where we are interested to characterize nontrivial solutions, surgery is a fundamental tool to consider. We have just seen that integral bordism groups are the main structures able to characterize global solutions of PDE's. On the other hand surgery is strictly related to bordism groups, as it is well known in algebraic topology. Therefore, in this section, we shall investigate as integral surgery interacts with integral bordism groups.

Definition 4.57. Let $\pi : W \rightarrow M$ be a smooth fiber bundle of dimension $m+n$ over a n -dimensional manifold M . Let $E_k \subset J_n^k(W)$ be a PDE of order k for n -dimensional submanifolds of W . Let $N \subset E_k$ be an admissible integral manifold of dimension $p \in \{0, 1, \dots, n-1\}$. Therefore, there exists a solution $V \subset E_k$ such that $N \subset V$. An admissible integral i -surgery, $0 \leq i \leq n-1$, on N is the procedure of constructing a new p -dimensional admissible integral manifold N' :

$$(69) \quad N' \equiv \overline{N \setminus S^i \times D^{p-1-i}} \bigcup_{S^i \times S^{p-2-i}} D^{i+1} \times S^{p-2-i}$$

such that $D^{i+1} \times S^{p-2-i} \subset V$. Here \overline{Y} is the closure of the topological subspace $Y \subset X$, i.e., the intersection of all closed subsets $Z \subset X$, with $Y \subset Z$.

Theorem 4.58. Let $N_1, N_0 \subset E_k$ be two integral compact (non-necessarily closed) admissible p -dimensional submanifolds of the PDE $E_k \subset J_n^k(W)$, such that there

is an admissible $(p + 1)$ -dimensional integral manifold $V \subset E_k$, such that $\partial V = N_0 \sqcup N_1$. Then it is possible to identify an integral admissible manifold N'_1 , obtained from N_1 by means of an integral i -surgery, iff N'_1 is integral bording with N_0 , i.e., $N'_1 \in [N_0]_{E_k}$.

Proof. As it is well known N'_1 is bording with N_0 , i.e., there exists a $(p + 1)$ -dimensional manifold Y , with $\partial Y = N_0 \sqcup N'_1$. More precisely we can take $Y = N_0 \times I \cup D^{i+1} \times D^{p-1-i}$. By the way, in order N'_1 should be integral admissible, it is necessary that should be contained into a solution passing from N_1 . Then N'_1 is integral bording with N_1 , hence it is also integral bording with N_0 . \square

Theorem 4.59 (Integral h-cobordism in Ricci flow PDE). *The generalized Poincaré conjecture, for any dimension $n \geq 1$ is true, i.e., any n -dimensional homotopy sphere M is homeomorphic to S^n : $M \approx S^n$.*

For $1 \leq n \leq 6$ one has also that M is diffeomorphic to S^n : $M \cong S^n$. But for $n \geq 7$, it does not necessitate that M is diffeomorphic to S^n . This happens when the Ricci flow equation, under the homotopy equivalence full admissibility hypothesis, (see below for definition), becomes a 0-crystal.

Moreover, under the sphere full admissibility hypothesis, the Ricci flow equation becomes a 0-crystal in any dimension $n \geq 1$.

Proof. Let us first consider the following lemma.

Lemma 4.60. *Let $N_0, N_1 \subset (RF)$ be two space-like connected smooth compact Cauchy manifolds at two different instant $t_0 \neq t_1$, identified respectively with two different Riemannian structures (M, γ_0) and (M, γ_1) . Then one has $N_0 \cong N_1$.*

Proof. In fact this follows directly from the fact the diffeomorphisms $(M, \gamma_i) \cong N_i$, $i = 0, 1$, and from the fact that any Riemannian metric on M can be continuously deformed into another one. More precisely we shall prove that there exists two continuous functions $f : N_0 \rightarrow N_1$ and $h : N_1 \rightarrow N_0$, such that $h \circ f \simeq 1_{N_0}$ and $f \circ h \simeq 1_{N_1}$. Really we can always find homotopies $F, G : I \times M \rightarrow M$, that continuously deform γ_1 into γ_0 and viceversa. More precisely $F_0 = id_M$, $G_0 = id_M$, $F_1^* \gamma_1 = \gamma_0$, and $G_1^* \gamma_0 = \gamma_1$. Therefore, we get $G_1 \circ F_1 \simeq G_0 \circ F_0 = 1_M$ and $F_1 \circ G_1 \simeq F_0 \circ G_0 = 1_M$. Thus we can identify f with F_1 and h with G_1 . \square

Lemma 4.61. *Let $N_0, N_1 \subset (RF)_{+\infty}$ be two space-like, smooth, compact closed, homotopy equivalent Cauchy n -manifolds, corresponding to two different times $t_0 \neq t_1$. Then N_0 and N_1 , have equal all the the integral characteristic numbers.*

Proof. Since we have assumed $N_0 \cong N_1$, there are two mappings $f : N_0 \rightarrow N_1$ and $h : N_1 \rightarrow N_0$, such that $h \circ f \simeq 1_{N_0}$ and $f \circ h \simeq 1_{N_1}$. These mappings for functorial property induce canonical homomorphisms between the groups $\pi_p(N_i)$, $H_p(N_i)$, $H^p(N_i)$, $i = 0, 1$, that we shall simply denote by f_* and h_* or f^* and h^* according to respectively the direct or inverse character of functoriality. Then one has the following properties $f_* \circ h_* = 1$, $h_* \circ f_* = 1$ and similarly for the contravariant cases, i.e., $f^* \circ h^* = 1$, $h^* \circ f^* = 1$. These relations means that the induced morphisms f_* and f^* are isomorphisms with inverse h_* and h^* respectively. In other words one has the isomorphisms $\pi_p(N_0) \cong \pi_p(N_1)$, $H_p(N_0) \cong H_p(N_1)$, $H^p(N_0) \cong H^p(N_1)$. As

a by-product we get also the commutative diagram reported in (70).

$$(70) \quad \begin{array}{ccc} H_n(N_0; \mathbb{R}) \times H^n(N_0; \mathbb{R}) & & \\ \parallel & \searrow \langle, \rangle & \\ (f_*, (f^{-1})^*) \wr & & \mathbb{R} \\ \parallel & \nearrow \langle, \rangle & \\ H_n(N_1; \mathbb{R}) \times H^n(N_1; \mathbb{R}) & & \end{array}$$

Since $H_n(N_i; \mathbb{R}) \cong \mathbb{R} \cong H^n(N_i; \mathbb{R})$, $i = 0, 1$, let the isomorphism f_* be identified with a non-zero number $\lambda \in \mathbb{R} \setminus \{0\}$, then $(f^{-1})^* = 1/\lambda$, and we get that $\langle f_*[N_0], (f^{-1})^*\alpha \rangle = \langle \lambda, \mu/\lambda \rangle = \mu$ where μ is the number that represents the n -differential form α . On the other hand one has $\langle [N_0], \alpha \rangle = 1 \cdot \mu = \mu$. \square

Lemma 4.62. *Under the same hypotheses of Lemma 4.61, let us add only that we assume admissible only orientable Cauchy manifolds. Then $N_0 \in [N_1] \in \Omega_n^{(RF)+\infty}$. In other words, $N = \partial V$, where V is a smooth solution, iff $\langle [\alpha], [N] \rangle = 0$ for all the conservation laws α .*

Proof. If we assume admissible only orientable Cauchy manifolds, then the canonical homomorphism $j_n : \Omega_n^{(RF)+\infty} \rightarrow (\mathcal{I}(RF)_{+\infty}^n)^*$ is injective, (see [36]), therefore $N_0 \in [N_1] \in \Omega_n^{(RF)+\infty}$ iff N_0 and have equal all integral characteristic numbers. \square

Since Lemma 4.62 is founded on the assumption that the space of conservation laws of (RF) is not zero, in the following lemma we shall prove that such an assumption is true.

Lemma 4.63. *The space $(\mathcal{I}(RF)_{+\infty}^n \cong E_1^{0,n})$ of conservation laws of the (RF) is not zero. In fact any differential n -form given in (71) is a conservation law of (RF) .³²*

$$(71) \quad \omega = T dx^1 \wedge \cdots \wedge dx^n + X^p (-1)^p dt \wedge dx^1 \wedge \cdots \wedge \widehat{dx^p} \wedge \cdots \wedge dx^n$$

with

$$(72) \quad \left\{ \begin{array}{l} T = g_{ij} \varphi^{ij} \\ X^p = \kappa \int (R_{ij}(g) \varphi^{ij} - R^{ij}(\varphi) g_{ij}) dx^p + c^p \end{array} \right\} : \left\{ \begin{array}{l} \varphi^{ij},_t - \kappa R^{ij}(\varphi) = 0 \\ g_{ij,t} + \kappa R_{ij}(g) = 0 \end{array} \right\}.$$

$c^p \in \mathbb{R}$ are arbitrary constants and φ^{ij} , $1 \leq i, j \leq n$, are functions on $\mathbb{R} \times M$, symmetric in the indexes, solutions of the equation given in (72).

Proof. Let us prove that $d\omega|_V = 0$ for any solution V of (RF) . In fact, by a direct calculation we get

$$(73) \quad d\omega = [(g_{ij,t} + \kappa R_{ij}(g)) \varphi^{ij} + g_{ij} (\varphi^{ij},_t - \kappa R^{ij}(\varphi))] dt \wedge dx^1 \wedge \cdots \wedge dx^n.$$

Therefore, the conservation laws in (71) are identified with the solutions of the PDE given in (72) for φ_{ij} . This is an equation of the same type of the Ricci flow equation, hence its set of solutions is not empty. \square

³² $E_1^{0,n}$ is the spectral term, in the Cartan spectral sequence of a PDE $E_k \subset J_n^k(W)$, just representing the conservation laws space of E_k . (See e.g., [36, 37, 41].)

Now, let M belong to the same integral bordism class of S^n in (RF) : $M \in [S^n] \in \Omega_n^{(RF)}$. It follows from Theorem 3.6 and Theorem 3.1, that M is necessarily homeomorphic to S^n . However, if $n \geq 4$, the smooth solution V such that $\partial V = M \sqcup S^n$ does not necessitate to be a trivial h-cobordism. This happens iff the homotopy equivalence $f : M \cong S^n$ is such that $f \simeq 1_{S^n}$. (See Theorem 3.18.) This surely is the case at low dimensions $n = 1, 2, 3, 4, 5, 6$. But for $n \geq 7$ an homotopy sphere may have different structures with respect to this property. (See Tab. 4.) In fact it is well known that there are homotopy spheres characterized by rational Pontrjagin numbers. Since rational Pontrjagin classes $p_q \in H^{4q}(M; \mathbb{Q})$ are homeomorphic invariants, such manifolds cannot admit a differentiable structure, taking into account the fact that the signature is a topological invariant. Such homotopy spheres are obtained by gluing a disk D^n , along its boundary S^{n-1} , with the boundary of a disk- D^q -fiber bundle over a sphere S^m , $E \rightarrow S^m$, such that $q = n - m$. When the $(n - 1)$ -dimensional boundary ∂E is diffeomorphic to S^{n-1} , gives to $\tilde{E} \equiv E \cup D^n$ a differentiable structure. But whether $\partial E \approx S^{n-1}$, Σ^n cannot, in general, have a differentiable structure, since it is characterized by rational Pontrjagin numbers. Therefore there are exotic spheres, (for example $\Sigma^{n-1} \equiv \partial E$), that are homeomorphic, but not diffeomorphic to S^{n-1} . In such cases the solution V of the Ricci flow equation such that $\partial V = \Sigma^{n-1} \sqcup S^{n-1}$, cannot be, in general, a trivial h-cobordism. By conclusion we get that not all n -dimensional homotopy spheres M can, in general, belong to the same integral boundary class of $[S^n] \in \Omega_n^{(RF)}$, even if there exist singular solutions $V \subset (RF)$ such that $\partial V = M \sqcup S^n$. In fact, it does not necessitate, in general, that V should be a trivial h-cobordism, i.e., that the homotopy equivalence between M and S^n should be a diffeomorphism of S^n isotopic to the identity. This has, as a by-product, that in general M is only homeomorphic to S^n but not diffeomorphic to S^n . In order to better understand this aspect in the framework of PDE's algebraic topology, let us first show how solutions with neck-pinching singular points are related to smooth solutions. Recall the commutative diagram in Theorem 2.24 in [36], here adapted in (74) to $(RF)_{+\infty}$ and in dimension $p = n$.

$$(74) \quad \begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & & & \Omega_n^{(RF)_{+\infty}} & \xrightarrow{i_n} & \Omega_{n,s}^{(RF)_{+\infty}} \cong \bar{H}_n((RF)_{+\infty}; \mathbb{R}) \\ & & & & \downarrow & & \downarrow \\ \Omega_n^{(RF)_{+\infty}} & \xrightarrow{j_n} & \Omega_{n,s}^{(RF)_{+\infty}} & \xrightarrow{\cong} & \bar{H}_n((RF)_{+\infty}; \mathbb{R}) & & \\ 0 & \longrightarrow & (\mathcal{I}((RF)_{+\infty}))^* & \longrightarrow & (\bar{H}_n((RF)_{+\infty}; \mathbb{R}))^* & \longrightarrow & 0 \end{array}$$

There $\bar{H}^n((RF)_{+\infty}; \mathbb{R})$ is the n -dimensional bar de Rham cohomology of $(RF)_{+\infty}$. The isomorphism $\Omega_{n,s}^{(RF)_{+\infty}} \cong \bar{H}_n((RF)_{+\infty}; \mathbb{R})$ is a direct by-product of the exact commutative diagram in Definition 4.8(3) in [37]. Then, taking into account that in solutions of $(RF)_{+\infty}$ cannot be present Thom-Boardman singularities, it follows that solutions bording smooth Cauchy manifolds in the bordism classes of $\widehat{\Omega}_n^{(RF)_{+\infty}} \equiv i_n(\Omega_n^{(RF)_{+\infty}}) \triangleleft \Omega_{n,s}^{(RF)_{+\infty}}$, can have singularities of neck-pinching type. (See Fig. 4(a).)

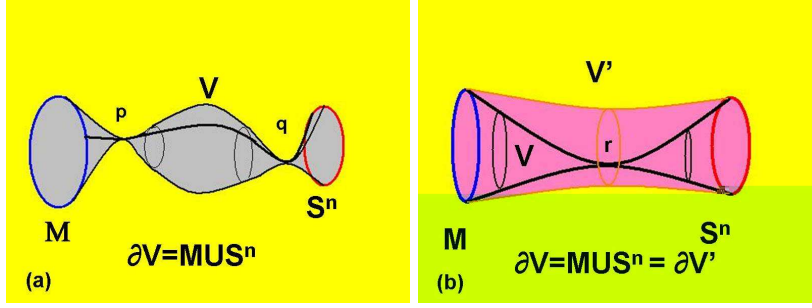


FIGURE 4. Neck-pinching singular solutions type, V , $\partial V = M \sqcup S^n$, in Ricci flow equations, with singular points p, q in (a) and r in (b). In (b) is reported also a smooth solution V' , bording M and S^n as well as a neck-pinching singular solution V bording the same manifolds.

From Corollary 2.5 in [36] it follows that if M is an homotopy sphere belonging to the integral bordism class $[S^n] \in \widehat{\Omega_n^{(RF)_{+\infty}}}$, one has surely $M \sqcup S^n = \partial V'$, for some smooth solution V' of (RF) , but can be also $M \sqcup S^n = \partial V$ for some solution $V \subset (RF)_{+\infty}$ having some neck-pinching singularity. (See Fig. 4(b).) In general V cannot be considered isotopic to V' . However M is diffeomorphic to S^n , (the diffeomorphism is that induced by the smooth solution V'), and all the singular points, in the neck-pinching singular solutions, bording M with S^n , are "solved" by the smooth bordism V' . Let us also emphasize that if an n -dimensional homotopy sphere $M \in [S^n] \in \widehat{\Omega_n^{(RF)_{+\infty}}}$, i.e., there exists a smooth solution $V \subset (RF)_{+\infty}$ such that $\partial V = M \sqcup S^n$, means that the characteristic flow on V is without singular points, hence from Theorem 3.1 it follows that $V \cong M \times I$, and $V \cong S^n \times I$, hence $M \cong S^n$. If this happens for all n -dimensional homotopy sphere, then $\Theta_n = 0$ and vice versa. However, it is well known that there are homotopy spheres of dimensions $n \geq 7$ for the which $\Theta_n \neq 0$. (For example the Milnor spheres.) This is equivalent to say that $\pi_0(Diff_+(S^{n-1})) \neq 0$, since $\Theta_n \cong \pi_0(Diff_+(S^{n-1}))$ (Smale). This happens when there are homotopy spheres that bound non-contractible manifolds. In fact, if there exists a trivial h-bordism V bording S^n with M , then $W = V \bigcup_{S^n} D^{n+1} \cong D^{n+1}$ and $\partial W = M$. However, since the conservation laws of (RF) depend on a finite derivative order (second order), the fact that all n -dimensional homotopy spheres have the same integral characteristic numbers of the sphere S^n , implies that there are smooth integral manifolds bording them at finite order. There the symbols of the Ricci flow equation, and its finite order prolongations, are not trivial ones, hence in general such solutions present Thom-Boardman singular points. As a by-product of Theorem 2.25 in [36], (see also [37]), and Theorem 2.1, Theorem 2.12 and Theorem 3.6 in [40] between such solutions, there are ones that are not smooth, but are topological solutions inducing the homeomorphisms between M and S^n : $M \approx S^n$. Therefore, if we consider admissible in (RF) only space-like Cauchy integral manifolds, corresponding to homotopy spheres, (*homotopy equivalence full admissibility hypothesis*), then one

has the short exact sequence (75).

$$(75) \quad 0 \longrightarrow K_{n,s}^{(RF)} \longrightarrow \Omega_n^{(RF)} \longrightarrow \Omega_{n,s}^{(RF)} = 0 \longrightarrow 0$$

We get

$$(76) \quad \Omega_n^{(RF)} \cong K_{n,s}^{(RF)} = \{[M] \mid M = \partial V, V = \text{singular solution of } (RF)\}$$

and $M \in [S^n] \in \Omega_n^{(RF)}$ iff $M \cong S^n$. Furthermore, two n -dimensional homotopy spheres Σ^n and Σ^n belong to the same bordism class in $\Omega_n^{(RF)}$ iff $\Sigma^n \cong \Sigma^n$. Therefore we get the following canonical mapping $\Theta_n \rightarrow \Omega_n^{(RF)}$, $[\Sigma^n]_{\Theta_n} \mapsto [\Sigma^n]_{\Omega_n^{(RF)}}$, such that $0 \in \{\Theta_n\} \mapsto [S^n]_{\Omega_n^{(RF)}}$. This mapping is not an isomorphism. Therefore, in the homotopy equivalence full admissibility hypothesis, and in the case that $\Theta_n = 0$, we get that the Ricci flow equation becomes a 0-crystal PDE, so all homotopy spheres are diffeomorphic to S^n . This is the case, for example, of $n = 3$, corresponding to the famous Poincaré conjecture. Finally, if we consider admissible in (RF) only space-like Cauchy integral manifolds, corresponding to manifolds diffeomorphic to spheres, (*sphere full admissibility hypothesis*), then $\Omega_n^{(RF)} \cong K_{n,s}^{(RF)} \cong \Omega_{n,s}^{(RF)} = 0$ and one has $cry(RF) = 0$, i.e. (RF) becomes a 0-crystal for any dimension $n \geq 1$.³³ \square

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³³From this theorem we get the conclusion that the Ricci flow equation for n -dimensional Riemannian manifolds, admits that starting from a n -dimensional sphere S^n , we can dynamically arrive, into a finite time, to any n -dimensional homotopy sphere M . When this is realized with a smooth solution, i.e., solution with characteristic flow without singular points, then $S^n \cong M$. The other homotopy spheres Σ^n , that are homeomorphic to S^n only, are reached by means of singular solutions. So the title reserved to this paper, and its companion [52], now it appears justified !

Results of this paper agree with previous ones by J. Cerf [6], M. Freedman [13], M. A. Kervaire and J. W. Milnor [25, 28], E. Moise [29, 30] and S. Smale [55, 56, 57], and with the recent proofs of the Poincaré conjecture by R. S. Hamilton [17, 18, 19, 20, 21], G. Perelman [33, 34], and A. Prástaro [50, 1].

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